Chapter 4

Field Life Functions Which Are Not Convex or Concave

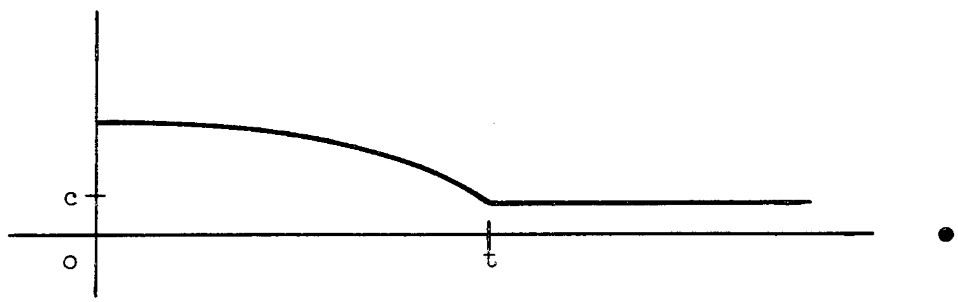
It may be the case that for a certain type of inventory item, the actual field life function may not be convex or concave but would be an S-shaped type of function. For example,

$$L(S) = \begin{cases} (-S + 3)^{\frac{1}{3}} + 2 & \text{for} \quad 0 \leq S \leq 11 \\ 0 & \text{otherwise} \end{cases}$$

yields an S-shaped function of this type

Unfortunately, it is not possible to find a specific policy which is optimal for the general S-shaped function. It is possible, however, to define a particular type of S-shaped function which has the property that when there are n items in inventory, the optimal policy must be one of n policies. It has the added property that it can be used as an approximation to the general S-shaped function. The particular S-shaped function referred to above is: $L(S)$ is concave nonincreasing

for all $S \epsilon [0, t]$ where $t > 0$ and $L(S) = c$ for all $S = [t, \infty)$.
In addition will usually be assumed that $L^{-}(S) \geq -1$ for all $S \epsilon (0, t]$.
Diagrammatically,

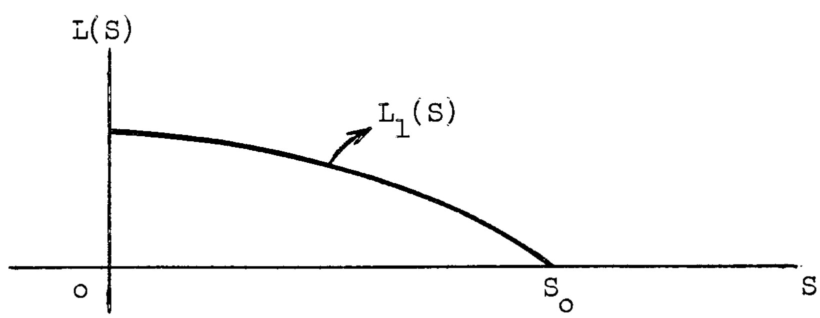


The more specialized field life function $L(S) = aS + b$ for all $S \epsilon [0, t]$ and $L(S) = c$ for all $S \epsilon [t, \infty)$ where $b > c > 0 > a > -1$ is also examined. In this case specific statements about the optimal policy can be made.

Because it will continually be of interest in this chapter, we will define two models for the field life function, $L(S)$.

<u>Model I</u>

$L_1(S)$ is concave nonincreasing for all $S \epsilon [0, S_o]$,
$L(S) = 0$ for all $S \epsilon [S_o, \infty)$ and
$L_1^{-}(S) \geq -1$ for all $S \epsilon (0, S_c]$.

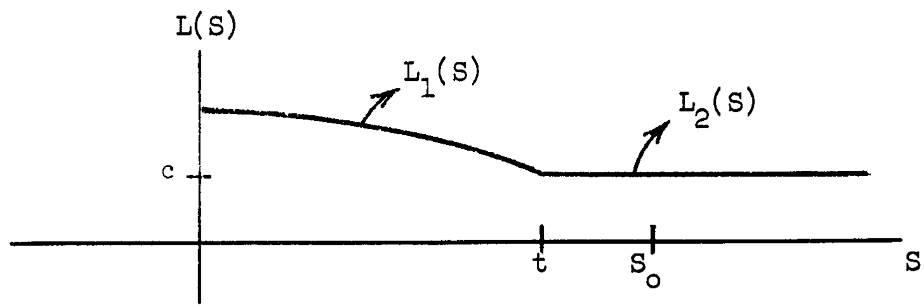


Model II

$L_1(S)$ coincides with $L_1(S)$ in Model I for all $S\epsilon[0, t]$ where

$t < S_o$ and

$L_2(S) = c$ for all $S\epsilon[t, \infty)$



The results concerning the optimal policies for Model II are presented in Section 4.2. Section 4.1 contains a series of lemmas which aid in the proofs of the theorems of Section 4.2.

4.1 Lemmas

Lemma 4.1: Let $\nu = 1$. If a FIFO issuing policy is used in both Model I and Model II, then

$$Q_{F_{II}} \geq Q_{F_I} .$$

Proof of Lemma 4.1: The proof will be by induction. $n = 1$ is trivially true. Now assume the lemma is true for $n = k$ and it will be proved for $n = k + 1$.

Let x and y denote the total field lives from the first k items issued by FIFO in Model II and Model I respectively. Then by the inductive assumption $x \geq y$. We consider the five mutually exclusive and exhaustive cases:

<u>Case 1</u> $\quad t \leq S_1 + y \leq S_1 + x$

Then $Q_{F_{II}} = x + L_2(S_1 + x) = x + c \geq y + L_1(S_1 + y) = Q_{F_I}$

<u>Case 2</u> $\quad S_1 + y < t \leq S_1 + x \qquad$ (hence $x > y$)

Then since $L_1^-(S) \geq -1$ for all $S \epsilon [0, S_o]$

$$\frac{L_1(S_1 + y) - L_2(S_1 + x)}{y - x} \geq -1$$

implies $Q_{F_{II}} = x + L_2(S_1 + x) = x + c \geq L_1(S_1 + y) + y = Q_{F_I}$

<u>Case 3</u> $\quad S_1 + y < S_1 + x \leq t \qquad$ (hence $x > y$)

Then

$$\frac{L_1(S_1 + x) - L_1(S_1 + y)}{x - y} \geq -1$$

implies $Q_{F_{II}} = x + L_1(S_1 + x) \geq y + L_1(S_1 + y) = Q_{F_I}$

<u>Case 4</u> $\quad S_1 + y = S_1 + x \leq t \qquad$ (hence $x = y$)

Then

$$Q_{F_{II}} = x + L_1(S_1 + x) = y + L_1(S_1 + y) = Q_{F_I}$$

Case 5 $\qquad S_1 + y \leq t < S_1 + x \qquad$ (hence $x > y$)

Then

$$\frac{L_1(S_1 + y) - L_2(S_1 + x)}{y - x} \geq -1$$

implies

$$Q_{F_{II}} = x + L_2(S_1 + x) = x + c \geq y + L_1(S_1 + y) = Q_{F_I}$$

And in all cases $Q_{F_{II}} \geq Q_{F_I}$.

q.e.d.

Lemma 4.2: (Generalization of Lemma 4.1 to $\nu \geq 1$) Let $\nu \geq 1$. If a FIFO issuing policy is used in both Model I and Model II, then $Q_{F_{II}} \geq Q_{F_I}$.

Proof of Lemma 4.2: By lemma 2.3 each demand source receives the same indexed items (and in the same order) under both models. Hence we can consider each demand source separately. Then $Q_{F_{II}} = \sum_{i=1}^{\nu} Q_{II_{M_i}}$ and $Q_{F_I} = \sum_{i=1}^{\nu} Q_{I_{M_i}}$. But by lemma 4.1 $Q_{II_{M_i}} \geq Q_{I_{M_i}}$ for all $i = 1, \ldots, \nu$; therefore $Q_{F_{II}} \geq Q_{F_I}$.

q.e.d.

Lemma 4.3: Let $L(S) = aS + b$ for all $S \epsilon [0, S_o]$ where $b > 0 > a > -1$ and $S_o = -\frac{b}{a}$. Let $\nu = 1$. If a FIFO issuing policy is used then the total field life, Q_{F_n} , is

$$Q_{F_n} = a \sum_{i=1}^{n} (1 + a)^{i-1} S_i + \frac{b}{a} [(1 + a)^n - 1] \ .$$

Proof of Lemma 4.3: By lemma 2.1, the field life of any item on issuance is positive. The proof proceeds by induction. Let $n = 1$ then

$$Q_{F_1} = aS_1 + \frac{b}{a} (1 + a - 1) = aS_1 + b$$

as required. Assume the lemma is true for $n = k$. Then

$$Q_{F_{k+1}} = aS_{k+1} + b + a \sum_{i=1}^{k} (1 + a)^{i-1} (S_i + aS_{k+1} + b) + \frac{b}{a} [(1 + a)^k - 1]$$

$$= aS_{k+1} \left[1 + a \sum_{i=1}^{k} (1 + a)^{i-1}\right] + a \sum_{i=1}^{k} (1 + a)^{i-1} S_i + b$$

$$+ ba \sum_{i=1}^{k} (1 + a)^{i-1} + b \sum_{i=1}^{k} (1 + a)^{i-1}$$

$$= aS_{k+1} \left[1 + a \left[\frac{(1 + a)^k - 1}{1 + a - 1}\right]\right] + a \sum_{i=1}^{k} (1 + a)^{i-1} S_i + b$$

$$+ b \sum_{i=1}^{k} (1 + a)^i$$

$$= aS_{k+1} (1 + a)^k + a \sum_{i=1}^{k} (1 + a)^{i-1} S_i + b \sum_{i=1}^{k+1} (1 + a)^{i-1}$$

$$= a \sum_{i=1}^{k+1} (1 + a)^{i-1} S_i + \frac{b}{a} [(1 + a)^{k+1} - 1] \ .$$

And by induction the lemma is proved.

q.e.d.

Lemma 4.4: Let c, b, a be given real numbers such that $b \geq c > 0 > a > -1$. Then the function

$$L_i = \frac{c - b(1 + a)^{i-1}}{a(1 + a)^{i-1}} \qquad \text{for } i = 1, 2, 3, \ldots$$

is a strictly decreasing function of i.

Proof of Lemma 4.4: Consider $i = k$ and $i = k + 1$ and form

$$L_k - L_{k+1} = \frac{c - b(1 + a)^{k-1}}{a(1 + a)^{k-1}} - \frac{c - b(1 + a)^k}{a(1 + a)^k}$$

$$= \frac{c}{a(1 + a)^{k-1}} - \frac{c}{a(1 + a)^k} - \frac{b}{a} + \frac{b}{a}$$

$$= \frac{c}{a}\left[\frac{1}{(1 + a)^{k-1}} - \frac{1}{(1 + a)^k}\right] = \frac{c}{a}\left[\frac{1 + a - 1}{(1 + a)^k}\right]$$

$$= \frac{c}{(1 + a)^k} > 0 \quad \text{since } 1 + a > 0 \text{ and } c > 0 .$$

Therefore $L_k > L_{k+1}$ and since k was arbitrary the lemma is proved.

q.e.d.

Lemma 4.5: Let $L(S) = aS + b$ for all $S \epsilon [0, t]$ and $L(S) = c$ for all $S \epsilon [t, \infty)$ where $b > c > 0 > a > -1$. [Thus $t = \frac{c - b}{a}$.] Let $\nu = 1$.

(i) If $S_i \leq \frac{c - b(1 + a)^{i-1}}{a(1 + a)^{i-1}}$ for some $i = 2, \ldots, n$ and if a FIFO issuing policy is used for $S_i, S_{i-1}, \ldots, S_1$ then the field life on issuance of each of the items $S_i, S_{i-1}, \ldots, S_1$ is strictly greater than c.

(ii) If $S_1 < \frac{c - b}{a}$ then the field life of S_1 is strictly greater than c and if $S_1 \geq \frac{c - b}{a}$ then the field lives of all items on issuance are equal to c.

Proof of Lemma 4.5: We first prove part (ii). $S_1 < \frac{c - b}{a} = t$ implies $L(S_1) > a\left(\frac{c - b}{a}\right) + b = c$. $S_1 \geq \frac{c - b}{a} = t$ implies $S_i > \frac{c - b}{a} = t$, therefore $L(S_i) = c$ for all i. Hence if S_i is issued x_i time units after the process starts

$$L(x_i + S_i) = L(S_i) = c .$$

We now prove part (i). By lemma 4.4 for $i = 2, \ldots, n$

$$S_i \leq \frac{c - b(1 + a)^{i-1}}{a(1 + a)^{i-1}} < \frac{c - b}{a} = t \text{ ; hence } L(S_i) > c .$$

Assume item S_{i-j} has field life $> c$ on issuance. It will be proved that item S_{i-j-1} has field life $> c$ on issuance ($j = 1, \ldots, i - 2$). Let x_{i-j} be the total field life from the FIFO issuance of $S_i, S_{i-1}, \ldots, S_{i-j}$. Then we must prove $L(S_{i-j-1} + x_{i-j}) > c$. It suffices to show that

$$S_{i-j-1} + x_{i-j} < t . \tag{4.1.1}$$

Now by lemma 4.3 and the inductive assumption

$$x_{i-j} = a \sum_{k=1}^{j+1} (1 + a)^{k-1} S_{i+k-j-1} + \frac{b}{a} [(1 + a)^{j+1} - 1] .$$

Since $0 > a > -1$ and since $S_{i-j-1} < S_{i+k-j-1}$ for all $k = 1, \ldots, j+1$

$$x_{i-j} < aS_{i-j-1} \sum_{k=1}^{j+1} (1+a)^{k-1} + \frac{b}{a} [(1+a)^{j+1} - 1]$$

$$= S_{i-j-1}[(1+a)^{j+1} - 1] + \frac{b}{a} [(1+a)^{j+1} - 1] .$$

Then

$$S_{i-j-1} + x_{i-j} < S_{i-j-1}(1+a)^{j+1} + \frac{b}{a} [(1+a)^{j+1} - 1]$$

$$< S_i(1+a)^{j+1} + \frac{b}{a} [(1+a)^{j+1} - 1] . \qquad (4.1.2)$$

Now by lemma 4.4 and the hypothesis of this lemma

$$S_i \leq \frac{c - b(1+a)^{i-1}}{a(1+a)^{i-1}} = L_i < L_{i-1} < \cdots < L_{j+2} = \frac{c - b(1+a)^{j+1}}{a(1+a)^{j+1}}$$

where $j = 0, 1, \ldots, i-2$ then from (4.1.2)

$$S_{i-j-1} + x_{i-j} < \left(\frac{c - b(1+a)^{j+1}}{a(1+a)^{j+1}}\right)(1+a)^{j+1} + \frac{b}{a} [(1+a)^{j+1} - 1]$$

$$= \frac{c}{a} - \frac{b}{a} (1+a)^{j+1} + \frac{b}{a} (1+a)^{j+1} - \frac{b}{a}$$

$$= \frac{c-b}{a} = t ,$$

Hence (4.1.1) holds and by induction the lemma is proved.

q.e.d.

Lemma 4.6: Let $L(S) = aS + b$ for all $S \epsilon [0, t]$ and $L(S) = c$ for $S \epsilon [t, \infty)$ where $b > c > 0 > a > -1$. Let $\nu = 1$. If

$$S_i \leq \frac{c - b(1 + a)^{i-1}}{a(1 + a)^{i-1}} = L_i$$

and

$$S_{i+1} \geq \frac{c - b(1 + a)^{i}}{a(1 + a)^{i}} = L_{i+1}$$

for some $i = 1, \ldots, n - 1$ and if a FIFO policy is used on $S_i, S_{i-1}, \ldots, S_1$ then the age of item S_{i+1} (and hence all $S_k > S_{i+1}$) is greater than or equal to t after these first i items $S_i, \ldots, S_1$ are issued by FIFO.

Proof of Lemma 4.6: Note in the hypothesis and by lemma 4.4 that $S_i < S_{i+1}$ and $L_{i+1} < L_i$. These relationships are often used in this proof.

Case 1 $\qquad S_i \leq L_{i+1}$

By lemma 4.3 and lemma 4.5 the total field life for the first i items issued by FIFO is

$$Q_{F_i} = a \sum_{k=1}^{i} (1 + a)^{k-1} S_k + \frac{b}{a} [(1 + a)^i - 1]$$

and since $0 > a > -1$ and $1 + a > 0$ and $S_k \leq L_{i+1} \; \forall \, k \leq i$

$$Q_{F_i} \geq a \sum_{k=1}^{i} (1 + a)^{k-1} \left[\frac{c - b(1 + a)^i}{a(1 + a)^i}\right] + \frac{b}{a} [(1 + a)^i - 1]$$

$$= a\left[\frac{(1 + a)^i - 1}{1 + a - 1}\right] \cdot \left[\frac{c - b(1 + a)^i}{a(1 + a)^i}\right] + \frac{b}{a} [(1 + a)^i - 1]$$

$$= [(1 + a)^i - 1]\left[\frac{c}{a(1 + a)^i} - \frac{b}{a} + \frac{b}{a}\right]$$

$$= \frac{c}{a} - \frac{c}{a(1 + a)^i} .$$

Now

$$S_{i+1} + Q_{F_i} \geq \frac{c - b(1 + a)^i}{a(1 + a)^i} + Q_{F_i}$$

$$\geq \frac{c - b(1 + a)^i}{a(1 + a)^i} + \frac{c}{a} - \frac{c}{a(1 + a)^i}$$

$$= \frac{c}{a(1 + a)^i} - \frac{b}{a} + \frac{c}{a} - \frac{c}{a(1 + a)^i}$$

$$= \frac{c - b}{a} = t$$

$\therefore$ for $S_i < \dfrac{c - b(1 + a)^i}{a(1 + a)^i}$

$$S_{i+1} + Q_{F_i} \geq t$$

as required.

Case 2

$$L_{i+1} = \frac{c - b(1 + a)^i}{a(1 + a)^i} \leq S_i \leq \frac{c - b(1 + a)^{i-1}}{a(1 + a)^{i-1}} = L_i$$

Let $0 \leq \beta \leq 1$ and let

$$S_i = P_o = \beta\, L_i + (1 - \beta)L_{i+1}$$

for some β. Then

$$P_o = \beta\left[\frac{c}{a(1 + a)^{i-1}} - \frac{b}{a}\right] + (1 - \beta)\left[\frac{c}{a(1 + a)^i} - \frac{b}{a}\right]$$

$$= \frac{\beta c}{a(1 + a)^{i-1}} + \frac{c}{a(1 + a)^i} - \frac{\beta c}{a(1 + a)^i} - \frac{b}{a}$$

$$= \frac{\beta c}{(1 + a)^i} + \frac{c}{a(1 + a)^i} - \frac{b}{a} \qquad (4.1.3)$$

since

$$\frac{\beta c}{a(1 + a)^{i-1}}\left[1 - \frac{1}{1 + a}\right] = \frac{\beta c}{a(1 + a)^{i-1}}\left(\frac{1 + a - 1}{1 + a}\right).$$

Again by lemma 4.3 and 4.5

$$Q_{F_i} = a \sum_{k=1}^{i} (1 + a)^{k-1} S_k + \frac{b}{a}\,[(1 + a)^i - 1]$$

and again since $0 > a > -1$ and $1 + a > 0$ and $S_k \leq P_o \;\; \forall\, k \leq i$

$$Q_{F_i} \geq a \sum_{k=1}^{i} (1 + a)^{k-1} \left[\frac{\beta c}{(1 + a)^i} + \frac{c}{a(1 + a)^i} - \frac{b}{a} \right] + \frac{b}{a} [(1 + a)^i - 1]$$

$$= [(1 + a)^i - 1] \left[\frac{\beta c}{(1 + a)^i} + \frac{c}{a(1 + a)^i} - \frac{b}{a} \right] + \frac{b}{a} [(1 + a)^i - 1]$$

$$= \beta c + \frac{c}{a} - \frac{\beta c}{(1 + a)^i} - \frac{c}{a(1 + a)^i} . \tag{4.1.4}$$

Now $S_{i+1} > S_i \Rightarrow S_{i+1} > P_o$ hence

$S_{i+1} + Q_{F_i} > P_o + Q_{F_i}$ and using (4.1.3) and (4.1.4)

$$\geq \frac{\beta c}{(1 + a)^i} + \frac{c}{a(1 + a)^i} - \frac{b}{a} + \beta c + \frac{c}{a} - \frac{\beta c}{(1 + a)^i} - \frac{c}{a(1 + a)^i}$$

$$= \frac{c - b}{a} + \beta c$$

$$= t + \beta c$$

$$\geq t$$ since $1 \geq \beta \geq 0$ and $c > 0$.

Thus for all $S_i \leq L_i$ we have

$$S_{i+1} + Q_{F_i} \geq t$$

for any $i = 1, \ldots , n - 1$ satisfying the hypothesis.

q.e.d.

4.2 Optimal Policies

In this section the preceding lemmas will be used to obtain (i) the set of n policies which contains the optimal policy for Model II and (ii) the specific optimal policy when $L_1(S)$ is linear.

Let us define F_iA as the policy which issues the i youngest items by FIFO first and then the remaining $n - i$ items by any arbitrary policy A.

Theorem 4.1: Let $L(S)$ be concave nonincreasing for all $S \epsilon [0, t]$ and $L(S) = c$ for all $S \epsilon [t, \infty)$. Let $L^-(S) \geq -1$ for all $S \epsilon (0, t]$. Let $\nu \geq 1$. If B is any arbitrary policy which results in exactly i items having field life $> c$ on issuance and the remaining $n - i$ items having field life $= c$ on issuance and if FIFO is optimal for Model I, then in Model II

$$Q_{F_iA} \geq Q_B .$$

Proof of Theorem 4.1: Denote the i items in policy B which have field life $> c$ on issuance and the demand sources to which the i items are assigned by (abbreviate the words "field life" by "f.l.")

Demand Source	Items with f.l. > c
M_1	$S_{11}, S_{12}, \ldots, S_{1k_1}$
M_2	$S_{21}, S_{22}, \ldots, S_{2k_2}$
$\vdots$	$\vdots$
M_ν	$S_{\nu 1}, S_{\nu 2}, \ldots, S_{\nu k_\nu}$

where $i = \sum_{j=1}^{\nu} k_j$.

Now for any M_j we can locate k_j ages $\overline{S}_{j1}, \overline{S}_{j2}, \ldots, \overline{S}_{jk_j}$ in Model I such that the field life from each of the k_j items in Model I is the same as the field life of each of the k_j items in Model II under policy B and in the same order. This is done by: If η_1 items with field life c are issued first to M_j and then item S_{j1} is issued we define $\overline{S}_{j1} = S_{j1} + \eta_1 c$; if η_2 items of field life c are issued between S_{j1} and S_{j2} then $\overline{S}_{j2} = L_1(S_{j1}) + (\eta_1 + \eta_2)c + S_{j2}$; etc. for the other $\overline{S}_{j3}, \ldots, \overline{S}_{jk_j}$. Now since this relocating of items in Model I can be done for all M_j's we have all i items so relocated.

Denote the total field life of the i items in Model II under policy B by x_{B_i} . Denote the total field life of the i relocated items in Model I by Q_i . By the construction above $x_{B_i} = Q_i$.

Furthermore, denote by Q_{F_i} and $Q^*_{F_i}$ the total field lives of the i relocated items in Model I and the i youngest items $(S_i, \ldots, S_1)$ respectively where in both cases FIFO is used. [We know that the i youngest items must have $S_i < S_o$ or else in Model II under policy B there could not be i items with f.l. $> c$.]

Since FIFO is optimal in Model I then

$$Q_{F_i} \geq Q_i$$

and by lemma 2.5

$$Q^*_{F_i} \geq Q_{F_i} .$$

But $Q^*_{F_i} \equiv Q_{F_I}$ of lemma 4.2; thus

$$Q_{F_{II}} \geq Q_{F_I} \equiv Q^*_{F_i} \geq Q_{F_i} \geq Q_i = x_{B_i} ,$$

where $Q_{F_{II}}$ is the total f.l. from the FIFO issuance of the i youngest items in Model II. If we denote the total field life from the remaining $n - i$ items in policy F_iA by $Q_{A_{n-i}}$ then

$$Q_{F_iA} = Q_{F_{II}} + Q_{A_{n-i}} \geq x_{B_i} + (n - i)c = Q_B .$$

And since B was any arbitrary policy with exactly i items with f.l. $> c$ then F_iA dominates any policy with this characteristic.

q.e.d.

Theorem 4.1 reduces the search for the optimal policy to the policies $F_1A, \ldots, F_nA$. But when $\nu = 1$, then F_iA, itself, consists of $(n - i)!$ policies and it appears that Theorem 4.1 has not greatly reduced the set of possible policies. But we need only to apply Theorem 4.1 over and over again to see that the optimal policy must have the property that the $n - i$ items issued by A all have field life $= c$ on issuance. For example, let F_jA be a policy with $j + k$ items with f.l. $> c$ on issuance for some $k = 1, 2, \ldots, n - j$. Now none of the first j items issued can have f.l. $= c$ on issuance or else all of the items initially older than S_j <u>viz.</u> $S_{j+1}, S_{j+2}, \ldots, S_n$ would also have f.l. $= c$ and then less than j of the n items would have f.l. $> c$ on issuance. By Theorem 4.1 we then have that $Q_{F_{j+k}A} \geq Q_{F_jA}$. Thus by repeated application of the theorem we see that the optimal policy must have the property that all of the items issued by A have f.l. $= c$ on issuance. But then A no longer need consist of the

$(n - i)!$ policies but can be reduced to any fixed policy. Hence we arbitrarily let $A = \text{LIFO}$ and we only need to search the n policies $F_1L, F_2L, \ldots, F_nL$ (actually we only need to search the less than n policies such that the LIFO issued items have f.l. = c on issuance).

The next theorem reduces the search even further. It states that the optimal policy can be found among the F_iL's which have the additional property that all of the i items issued by FIFO have field life $> c$ on issuance. That the search cannot be narrowed even further is shown by an example following Theorem 4.2.

Theorem 4.2: Let $L(S)$ be a concave nonincreasing function for all $S \epsilon [0, t]$ and $L(S) = c$ for all $S \epsilon [t, \infty)$. Let $L^-(S) \geq -1$ for all $S \epsilon (0, t]$. Let $\nu \geq 1$. Assume FIFO is optimal for Model I.

If the policy F_kA $(k = 2, \ldots, n)$ has the property that only j items (where $1 \leq j < k$) have field life $> c$, then there exists a policy F_iA with $i \leq j < k$ and with the property that all i of the first items issued (by FIFO) have field life $> c$ on issuance and such that $Q_{F_iA} \geq Q_{F_jA} \geq Q_{F_kA}$.

Proof of Theorem 4.2: Note that $j \geq 1$ implies $S_1 < t$ which implies $L(S_1) > c$. Therefore the set of F_iA policies such that the first i items have field life $> c$ on issuance is not the empty set.

Let $F_kA \equiv B$ in Theorem 4.1 then by application of Theorem 4.1 $Q_{F_jA} \geq Q_{F_kA} \equiv Q_B$. Now if F_jA has the first j items with field life $> c$, this theorem is proved. Therefore assume $j_1 < j$ of the first j items have f.l. $> c$; we will show only j_1 of all the n items issued by F_jA have f.l. $> c$:

Case 1 $\qquad S_j \geq t$

Then $S_{j+p} \geq t$ for all $p = 0, 1, \ldots, n - j$ hence there are only j_1 items with f.l. $> c$.

Case 2 $\qquad S_j < t$ and $j \leq \nu$

Then $L(S_{j-p}) > c$ for all $p = 0, \ldots, j - 1$ but since $j \leq \nu$ all the j items are issued immediately to start the process; hence all j items have f.l. $> c$ on issuance contrary to our assumption that $j_1 < j$.

Case 3 $\qquad S_j < t$ and $j > \nu$

We must show that when $S_{j+1}, \ldots, S_n$ are issued, they will have f.l. $= c$.

Let S_{j-p} for some $p = 1, \ldots, j - 1$ be the oldest item among the $S_j, \ldots, S_1$ such that when S_{j-p} is issued it has f.l. $= c$. We know S_{j-p} exists since $j_1 < j$. Let S_{j-p} be assigned to demand source M_α, and let the total f.l. of all the items assigned to M_α up to but not including item S_{j-p} be denoted by x_α. Thus $S_{j-p} + x_\alpha \geq t$ since $L(S_{j-p} + x_\alpha) = c$.

But $S_{j-p} < S_{j+1} < \cdots < S_n$ hence $t \leq S_{j-p} + x_\alpha < S_{j+1} + x_\alpha < \cdots < S_n + x_\alpha$. Thus if any of the items $S_{j+1}, \ldots, S_n$ are issued to M_α in the A stage of F_jA, they will have f.l. $= c$ on issuance.

Now denote the f.l. of the other demand sources at the time of issuance of S_{j-p} to M_α by $x_1, x_2, \ldots, x_{\alpha-1}, x_{\alpha+1}, \ldots, x_\nu$. Now since each demand source is busy from the time the process starts then

$x_1 = x_2 = \cdots = x_{\alpha-1} = x_{\alpha+1} = \cdots = x_\nu = x_\alpha$ at the instant S_{j-p} is issued to M_α . But then $t < S_{j+1} + x_\alpha = S_{j+1} + x_1 = \cdots = S_{j+1} + x_\nu$, which implies that all items $S_{j+1}, \ldots , S_n$ have f.l. = c on issuance.

Therefore in all cases there are only j_1 of the n items which have f.l. > c on issuance and these j_1 items belong to the first j items.

But then application of Theorem 4.1 again gives

$$Q_{F_{j_1}A} \geq Q_{F_jA} .$$

We repeatedly use Theorem 4.1 until we achieve an F_iA policy $(i \leq j < k)$ with all of the first i items issued (by FIFO) having f.l. > c . This F_iA is achieved since

(i) at least one such policy exists viz. F_1A and

(ii) the number of possible F_iA's is finite.

q.e.d.

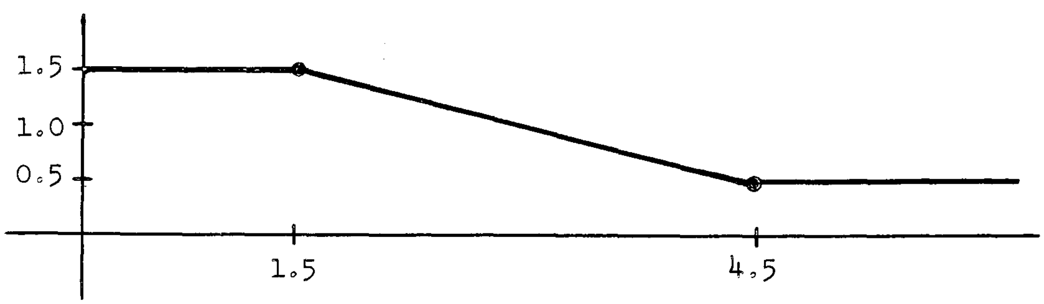
At this point is is worth noting that if $\nu = 1$ or if $L_1(S)$ is linear then the assumption that FIFO is optimal for Model I can be removed in both theorems 4.1 and 4.2.

Now the results of Theorem 4.2 do not imply that there does not exist an F_rA policy such that the first r items have f.l. > c on issuance and $r > i$. But if such an F_rA policy does exist then under the conditions of Theorem 4.2, we must have $r < k$. This last statement is proved as follows: assume r exists and $r \geq k$. Clearly $r \neq k$ since the hypothesis of Theorem 4.2 is then violated. Hence consider $r > k$. But by lemma 2.2 applied to each demand source, each of the first k items of F_kA has age less than or equal to the age of these same k items upon issuance under F_rA. But under F_rA these k items have f.l. $> c$ on issuance; hence under F_kA these k items must also have f.l. $> c$ on issuance. We have achieved a contradiction to the hypothesis that $j < k$; hence it must be true that $r < k$.

As mentioned before, Theorems 4.1 and 4.2 reduce the search for the optimal policy to those F_iL's with the property that the first i items have f.l. $> c$ and the last $n - i$ items have f.l. $= c$ on issuance. That we cannot go further is shown by the following example:

$$L(S) = \begin{cases} 1.5 & \text{for } 0 \leq S \leq 1.5 \\ -\frac{1}{3} S + 2 & \text{for } 1.5 \leq S \leq 4.5 \\ 0.5 & \text{for } 4.5 \leq S \end{cases}$$

$$\nu = 1$$



For

$S_1 = 2.0$ $\quad Q_{F_1L} = 2.833$

$S_2 = 4.0$ $\quad Q_{F_2L} = 2.777$

$S_3 = 5.0$ $\quad Q_{F_3L} = 2.500$

$S_4 = 6.0$ $\quad Q_{F_4L} = 2.333$

and $F_1L = [S_1, S_2, S_3, S_4]$ is optimal. But both F_1L <u>and</u> F_2L have the property that the FIFO issued items have f.l. $> c$ and the LIFO issued items have f.l. $= c$ on issuance. Hence we cannot always locate in the set of $\{F_iL\}$ policies, a unique F_iL with the requisite properties. However, in Model II when we let $L_1(S) = aS + b$ with $b > c > 0 > a > -1$, we are able to isolate the unique optimal F_iL policy. In addition we are able to show that if $L_1(S)$ is concave or convex and $L_1'(S) \leq -1$, then F_1L is optimal. But before doing either we will prove the following lemma.

<u>Lemma 4.7</u>: Let $L(S) = aS + b$ for all $S\epsilon[0, t]$ and $L(S) = c$ for all $S\epsilon[t, \infty)$ where $b > c > 0 > a > -1$. Let $\nu = 1$. If

$$S_i \leq \frac{c - b(1 + a)^{i-1}}{a(1 + a)^{i-1}} = L_i$$

and

$$S_{i+1} > \frac{c - b(1 + a)^{i}}{a(1 + a)^{i}} = L_{i+1} \text{ for some } i = 1, \ldots, n-1$$

then for any F_jL policy with $j \leq i$, the age of item S_{i+1} when it is issued is $\geq t$. Hence item S_{i+1} has field life $= c$ on issuance. Consequently all S_{i+1+k} for $k = 0, 1, \ldots, n - (i + 1)$ have field life $= c$ on issuance.

<u>Proof of Lemma 4.7</u>: If $i = j$ then by lemma 4.6, this lemma holds. If $j < i$, then we consider two cases:

<u>Case 1</u>: Some item S_k, where $j < k \leq i$, under policy F_jL has f.l. $= c$ on issuance. But then all items S_{k+p} for $p = 1, \ldots, n - k$

must have f.l. = c on issuance since their initial ages are $> S_k$ and they are issued after S_k is issued. Therefore S_{i+1} has f.l. = c on issuance since $i + 1 > k$.

<u>Case 2</u>: All items S_k for $j < k \leq i$ under policy F_jL have f.l. > c on issuance.

First note that all of the first j items issued by FIFO have f.l. > c on issuance since by lemma 4.5 F_iL has its first i items with f.l. > c. Then by application of lemma 2.2 each of the first j items of F_jL must have f.l. > c on issuance.

Now by lemma 4.3 the first j items issued have total field life

$$B_j = a \sum_{p=1}^{j} (1 + a)^{p-1} S_p + \frac{b}{a} [(1 + a)^j - 1] . \qquad (4.2.1)$$

Since F_jL says to issue in the order $S_j, S_{j-1}, \ldots, S_2, S_1, S_{j+1}, S_{j+2}, \ldots, S_i, \ldots, S_n$ then by induction we will show that the total field life for items $S_{j+1}, \ldots, S_i$ is given by

$$C_i = a \sum_{p=1}^{i-j} (1 + a)^{p-1} S_{i-p+1} + B_j[(1 + a)^{i-j} - 1] + \frac{b}{a} [(1 + a)^{i-j} - 1] . \qquad (4.2.2)$$

First note that for $i = j + 1$

$$C_{j+1} = L(S_{j+1} + B_j) = a(S_{j+1} + B_j) + b = aS_{j+1} + aB_j + b$$

as required. Now assume (4.2.2) holds for $i = k - 1$, then

$$C_k = L(S_k + C_{k-1} + B_j) + C_{k-1} = a(S_k + C_{k-1} + B_j) + b + B_{k-1}$$

$$= aS_k + (1 + a)C_{k-1} + aB_j + b$$

$$= aS_k + (1 + a)\left[a \sum_{p=1}^{k-j-1} (1 + a)^{p-1} S_{k-p} + \left(B_j + \frac{b}{a}\right)\left\{(1 + a)^{k-j-1} - 1\right\}\right] + aB_j + b$$

$$= a \sum_{p=1}^{k-j} (1 + a)^{p-1} S_{k-p+1} + B_j[(1 + a)^{k-j} - 1] + \frac{b}{a} [(1 + a)^{k-j} - 1]$$

which is (4.2.2) as required. Combining (4.2.1) and (4.2.2) we obtain the total field life of the first i items issued by $F_j L$.

$$B_j + C_i = a \sum_{p=1}^{i-j} (1 + a)^{p-1} S_{i-p+1} + B_j(1 + a)^{i-j} + \frac{b}{a} [(1 + a)^{i-j} - 1]$$

$$= a \sum_{p=1}^{i-j} (1 + a)^{p-1} S_{i-p+1} + \frac{b}{a} [(1 + a)^{i-j} - 1] + \left[a \sum_{p=1}^{j} (1 + a)^{p-1} S_p + \frac{b}{a} \left\{(1 + a)^j - 1\right\}\right](1 + a)^{i-j}$$

$$= a \sum_{p=1}^{i-j} (1 + a)^{p-1} S_{i-p+1} + a(1 + a)^{i-j} \sum_{p=1}^{j} (1 + a)^{p-1} S_p + \frac{b}{a} [(1 + a)^i - 1]. \qquad (4.2.3)$$

We now establish an inequality for (4.2.3) since $0 > a > -1$, $1 + a > 0$ and $S_{i+1} > S_p$ for all $p = 1, \ldots, i$ we have

$$B_j + C_i > a \sum_{p=1}^{i-j} (1 + a)^{p-1} S_{i+1} + a(1 + a)^{i-j} \sum_{p=1}^{j} (1 + a)^{p-1} S_{i+1}$$

$$+ \frac{b}{a} [(1 + a)^i - 1]$$

$$= aS_{i+1}\left[\sum_{p=1}^{i-j} (1 + a)^{p-1} + (1 + a)^{i-j} \sum_{p=1}^{j} (1 + a)^{p-1}\right]$$

$$+ \frac{b}{a} [(1 + a)^i - 1]$$

$$= S_{i+1}[(1 + a)^{i-j} - 1 + (1 + a)^i - (1 + a)^{i-j}] + \frac{b}{a} [(1 + a)^i - 1]$$

$$= \left(S_{i+1} + \frac{b}{a}\right)[(1 + a)^i - 1] \quad . \qquad (4.2.4)$$

Now since

$$S_{i+1} > \frac{c - b(1 + a)^i}{a(1 + a)^i}$$

we have

$$S_{i+1} + B_j + C_i > S_{i+1} + S_{i+1}[(1 + a)^i - 1] + \frac{b}{a} [(1 + a)^i - 1]$$

$$= S_{i+1}(1 + a)^i + \frac{b}{a} [(1 + a)^i - 1]$$

$$> \frac{c - b(1 + a)^i}{a(1 + a)^i} (1 + a)^i + \frac{b}{a}[(1 + a)^i - 1]$$

$$= \frac{c}{a} - \frac{b}{a} (1 + a)^i + \frac{b}{a} (1 + a)^i - \frac{b}{a}$$

$$= \frac{c - b}{a} = t \quad .$$

Therefore item S_{i+1} has f.l. = c on issuance.

q.e.d.

Theorem 4.3: Let $L(S) = aS + b$ for all $S \in [0, t]$ and $L(S) = c$ for all $S \in [t, \infty)$ where $b > c > 0 > a > -1$. Let $\nu \geq 1$. Using the item indexing notation of Chapter 3 (cf. Theorem 3.6)

(a) If

$$S_J \equiv S^{(j)}_{[\frac{n-j}{\nu}]+1} \leq \frac{c - b(1 + a)^{[\frac{n-j}{\nu}]}}{a(1 + a)^{[\frac{n-j}{\nu}]}}$$

and

$$S^{(j-1)}_{[\frac{n-j+1}{\nu}]+1} > \frac{c - b(1 + a)^{[\frac{n-j+1}{\nu}]}}{a(1 + a)^{[\frac{n-j+1}{\nu}]}}$$

for some $j = 1, \ldots, \nu$, then $F_J L$ is the optimal policy.

(b) If $S_J \equiv S_1^{(j)} \geq \frac{c - b}{a}$ for some $j = 1, \ldots, \nu$ then $F_J L$ is the optimal policy. [In this case $F_J L$ = LIFO.]

(c) If neither (a) nor (b) is satisfied then use the Search Procedure defined in Chapter 3 and consider all adjacent pairs of items for each demand source starting with the oldest adjacent pair and ending with the newest, then if M_j is the first demand source such that for two adjacent items $S_i^{(j)} \equiv S_I$ and $S_{i+1}^{(j)} \equiv S_{I+\nu}$ assigned to M_j

$$S_i^{(j)} \leq \frac{c - b(1 + a)^{i-1}}{a(1 + a)^{i-1}} \qquad (4.2.5)$$

and

$$S_{i+1}^{(j)} > \frac{c - b(1 + a)^{i}}{a(1 + a)^{i}} \qquad (4.2.6)$$

for some $I \in \{\nu + 1, \ldots, n - \nu - 1\}$ then $F_I L$ is the optimal policy.

Proof of Theorem 4.3: Note that by lemma 4.4 (a), (b) and (c) are mutually exclusive and exhaustive.

We defer the proof of (a) until after we have proved (b) and (c).

Part (b): $S_J \equiv S_1^{(j)} \geq \frac{c - b}{a} = t$ implies all $S_i > S_1^{(j)} \geq t$ and $L(S_i) = c$ for all $i \geq J$. But then less than ν items have initial field life $> c$ and all $n - \nu$ or more items have initial f.l. = c. It is optimal to issue immediately the $J - 1$ or less items with f.l. $> c$ and then issue the remaining items by any policy. But policy $F_J L$ does precisely this. Hence $F_J L$ is optimal.

Part (c): Since $S_i^{(j)}$ is the first (in the sense of oldest) item for which (4.2.5) and (4.2.6) hold, then for all $0 < j - k < j$ where $(k = 1, \ldots, j - 1)$

$$S_i^{(j-k)} > \frac{c - b(1 + a)^{i-1}}{a(1 + a)^{i-1}} \tag{4.2.7}$$

and for all $j + k > j$ where $(k = 1, \ldots, \nu - j)$

$$S_{i+1}^{(j+k)} > \frac{c - b(1 + a)^{i}}{a(1 + a)^{i}} . \tag{4.2.8}$$

In addition for all $S_p \leq S_i^{(j)}$ we have

$$S_p \leq \frac{c - b(1 + a)^{i-1}}{a(1 + a)^{i-1}} . \tag{4.2.9}$$

In (4.2.9) we consider in particular

$$S_i^{(j+k)} < \frac{c - b(1 + a)^{i-1}}{a(1 + a)^{i-1}} \text{ for all } k = 1, \ldots , \nu - j \tag{4.2.10}$$

and

$$S_{i-1}^{(j-k)} < \frac{c - b(1 + a)^{i-1}}{a(1 + a)^{i-1}} < \frac{c - b(1 + a)^{i-2}}{a(1 + a)^{i-2}} \tag{4.2.11}$$

for $k = 1, \ldots , j - 1$, by lemma 4.4. Hence combining (4.2.7) with (4.2.11) and (4.2.8) with (4.2.10) we have the case that all $\nu - 1$ pairs of items following the <u>first</u> pair, $S_i^{(j)}$ and $S_{i+1}^{(j)}$, also satisfy conditions (4.2.5) and (4.2.6) of the theorem. (Since $[\frac{n}{\nu}] > i > 1$ we know (4.2.7), (4.2.8), (4.2.10), and (4.2.11) exist for all $j = 1, \ldots , \nu$.) We will now show

$$Q_{F_I L} \geq Q_{F_{I-k} L} \quad \text{for all } k = 1, \ldots , I - 1 \tag{4.2.12}$$

and

$$Q_{F_I L} \geq Q_{F_{I+k} L} \quad \text{for all } k = 1, \ldots , n - I . \tag{4.2.13}$$

We first prove (4.2.12).

By lemma 4.5 the first I items issued under $F_I L$ have f.l. $> c$ since (4.2.5), (4.2.6), (4.2.7), (4.2.8), (4.2.10) and (4.2.11) hold for all M_q $(q = 1, \ldots , \nu)$. We wish to show that the remaining $n - I$ items $S_{I+1}, \ldots , S_n$ under $F_I L$ have f.l. $= c$ on issuance. Then by lemma 4.7 any $F_{I-k} L$ policy has $S_{I+1}, \ldots , S_n$ with f.l. $= c$ on issuance.

Consider item S_{I+1} in $F_I L$. Item S_{I+1} is the first item to be issued under the LIFO part of $F_I L$. We will show that item S_{I+1} has f.l. = c on issuance under $F_I L$. If $j = 1$ in (4.2.5) and (4.2.6) then by (4.2.8)

$$S_{I+1} \equiv S_{i+1}^{(\nu)} > \frac{c - b(1 + a)^i}{a(1 + a)^i}$$

but by (4.2.9)

$$S_{I+1-\nu} \equiv S_i^{(\nu)} \leq \frac{c - b(1 + a)^{i-1}}{a(1 + a)^{i-1}}$$

hence by lemma 4.7, S_{I+1} has age $\geq t$ on issuance to any M_q $(q = 1, \ldots, \nu)$. If $j > 1$ in (4.2.5) and (4.2.6) then S_{I+1} can be represented by (4.2.7) or (4.2.8). If S_{I+1} is represented by (4.2.7) then

$$S_{I+1} = S_i^{(j-k)} > \frac{c - b(1 + a)^{i-1}}{a(1 + a)^{i-1}} > \frac{c - b(1 + a)^i}{a(1 + a)^i} \qquad (4.2.14)$$

by lemma 4.4. If S_{I+1} is represented by (4.2.8) then

$$S_{I+1} = S_i^{(j+k)} > \frac{c - b(1 + a)^i}{a(1 + a)^i} . \qquad (4.2.15)$$

And in either case (4.2.9) still holds for $S_{I+1-\nu}$. Thus applying lemma 4.7 again the age of S_{I+1} on issuance is $\geq t$ for any M_q $(q = 1, \ldots, \nu)$. Hence for $F_I L$ item S_{I+1} (and all $S_k > S_{I+1}$) has f.l. = c on issuance. Now by repeated applications of lemma 4.7 to each demand source under $F_{I-k}L$ we have that regardless of which demand source receives item S_{I+1} , S_{I+1} will have f.l. = c on issuance. Thus all $S_{I+1}, \ldots, S_n$ will have f.l. = c on issuance. Hence policy $F_{I-k}L$ can have at most I items with f.l. > c on issuance. Now $F_{I-k}L$ cannot have less than I - k items with f.l. > c (namely the first I - k items). This last statement follows by applying lemma 4.5 and lemma 2.2 to each demand source and noting that the first I items issued by $F_I L$ have f.l. > c on issuance. Now since $L_1(S)$ is linear then by Zehna [11], FIFO is optimal for Model I. Thus we can apply the results of Theorem 4.1 and Theorem 4.2 which allow us to restrict our search for the optimal policy to those $F_K L$'s which have the properties (i) the first K items have f.l. > c on issuance and (ii) the remaining n - K items have f.l. = c on issuance.

Let $F_{I-k}L$ be any policy with these properties where $k = 0, 1, \ldots, I - 1$. Form

$$Q_{F_I L} - Q_{F_{I-k} L} \quad \text{for} \quad k > 0 \,. \tag{4.2.16}$$

If $k = 0$, then $F_I L$ is the only policy satisfying the above properties and by the argument given in the preceding paragraph, $F_I L$ is then optimal.

It will be convenient to change our notation in regard to the items assigned to M_q under any policy $F_f L$. By lemma 2.3 we stated that M_q receives items indexed by $(n - h\nu - q + 1)$, we could have relabelled the M_j's to say M_q receives items indexed by $q + h\nu$ for $h = 0, 1, 2, \ldots$ where $q + h\nu \leq n$. We will now use this second method. Then for any two policies say $F_f L$ and $F_{f+g} L$ demand source M_q receives the same indexed items except under $F_{f+g} L$, M_q perhaps receives more items of higher indexing.

Under $F_I L$, M_q receives i (or $i - 1$) items in the FIFO part of the policy and under $F_{I-k} L$, M_q receives, say $i - k_q$ items in the FIFO part of the policy where $\sum_{r=1}^{\nu} k_r = k$. Then if we denote by $Q_{M_{q,i}}$ and $Q_{M_{q,i-k_q}}$ the total field life of the first i items and the first $i - k_q$ items issued to M_q by FIFO under $F_I L$ and $F_{I-k} L$ respectively then we will show

$$Q_{M_{q,i}} - Q_{M_{q,i-k_q}} > k_q(1 + a)^{i-1} L(S_I) \tag{4.2.17}$$

as follows:

Apply lemma 3.5 to each pair in the right hand side of

$$Q_{M_{q,i}} - Q_{M_{q,i-k_q}} = Q_{M_{q,i}} - Q_{M_{q,i-1}} + Q_{M_{q,i-1}} - \cdots + Q_{M_{q,i-k_q+1}} - Q_{M_{q,i-k_q}}$$

$$= (1 + a)^{i-1} L(S_i^{(q)}) + (1 + a)^{i-2} L(S_{i-1}^{(q)}) + \cdots + (1 + a)^{i-k_q} L(S_{i-k_q+1}^{(q)}) \tag{4.2.18}$$

but $1 + a > 0$ and $L(S_{i-p}^{(q)}) \geq L(S_I)$ for all $p = 0, 1, \ldots, k_q - 1$

hence

$$Q_{M_{q,i}} - Q_{M_{q,i-k_q}} \geq (1 + a)^{i-1}L(S_I) + (1 + a)^{i-2}L(S_I) + \cdots + (1 + a)^{i-k_q}L(S_I)$$

$$> k_q(1 + a)^{i-1}L(S_I)$$

which is (4.2.17) since $(1 + a)^{i-1} < (1 + a)^{i-2} < \cdots < (1 + a)^{i-k_q}$.

But then in (4.2.16) we have

$$Q_{F_I L} - Q_{F_{I-k} L} > (1 + a)^{i-1}L(S_I) \sum_{r=1}^{\nu} k_r - kc$$

where $-kc$ appears since $Q_{F_{I-k} L}$ has k more items with f.l. = c than does $Q_{F_I L}$. Thus

$$Q_{F_I L} - Q_{F_{I-k} L} > (1 + a)^{i-1}L(S_I)k - kc$$

$$= k[(1 + a)^{i-1}L(S_I) - c]$$

$$= k[(1 + a)^{i-1}(aS_I + b) - c]$$

$$\geq k\left[(1 + a)^{i-1}\left\{a\left[\frac{c - b(1 + a)^{i-1}}{a(1 + a)^{i-1}}\right] + b\right\} - c\right]$$

since $aS_I k(1 + a)^{i-1} < 0$

and $0 < S_I \leq \dfrac{c - b(1 + a)^{i-1}}{a(1 + a)^{i-1}}$

$$= k[(1 + a)^{i-1}b + c - b(1 + a)^{i-1} - c]$$

$$= 0 .$$

Therefore $\quad Q_{F_I L} > Q_{F_{I-k} L}$ where $k = 1, \ldots, I-1$,

and (4.2.12) holds.

We now prove (4.2.13).

By Theorem 4.2 we only need to consider policies where the first $I + k$ items have f.l. $> c$ on issuance.

If none exists then since (4.2.12) holds and by Theorem 4.2, $F_I L$ is optimal. Let F_{I+k} be such a policy for $k > 0$. Now item S_{I+1} has f.l. $= c$ on issuance under $F_I L$. Let us look at item S_{I+k+1} under $F_{I+k} L$. It also has f.l. $= c$ on issuance since by lemma 2.2 the total field life of the first $I + k$ items issued is (if we rearrange the labelling of the M_q's) at least as large for each M_q as it is for the first I items issued under $F_I L$. Thus $S_{I+k+1}, \ldots, S_n$ have f.l. $= c$ on issuance. Now under the FIFO part of $F_{I+k} L$ each demand source will have k_q more items assigned than under $F_I L$, where $k_q \geq 0$ and $\sum_{i=1}^{\nu} k_i = k$. Then by applying lemma 3.5 to each pair on the right hand side of

$$Q_{M_{q,i+k_q}} - Q_{M_{q,i}} = Q_{M_{q,i+k_q}} - Q_{M_{q,i+k_q-1}} + Q_{M_{q,i+k_q-1}} - \cdots - Q_{M_{q,i}} \tag{4.2.19}$$

we obtain

$$Q_{M_{q,i+k_q}} - Q_{M_{q,i}} = (1 + a)^{i+k_q-1} L(S^{(q)}_{i+k_q}) + \cdots + (1 + a)^i L(S^{(q)}_{i+1})$$

$$\leq [(1 + a)^{i+k_q-1} + (1 + a)^{i+k_q-2} + \cdots + (1 + a)^i] L(S^{(q)}_{i+1})$$

since $L(S^{(q)}_{i+1}) \geq L(S^{(q)}_{i+j})$ for all $j = 1, \ldots, k_q$

$$\leq [(1 + a)^{i+k_q-1} + \cdots + (1 + a)^i] L(S_{I+1})$$

since $L(S_{I+1}) \geq L(S^{(q)}_{i+1})$

$$< k_q(1 + a)^i L(S_{I+1}) \ . \qquad (4.2.20)$$

That is

$$Q_{M_{q,i}} - Q_{M_{q,i+k_q}} > - k_q(1 + a)^i L(S_{I+1}) \qquad (4.2.21)$$

and applying this to

$$Q_{F_I}L - Q_{F_{I+k}}L > - \sum_{r=1}^{\nu} k_r(1 + a)^i L(S_{I+1}) + (I + k - I)c$$

$$= -k(1 + a)^i L(S_{I+1}) + kc$$

$$= -k(1 + a)^i (aS_{I+1} + b) + kc$$

$$= -ak(1 + a)^i S_{I+1} - kb(1 + a)^i + kc$$

but $S_{I+1} > \dfrac{c - b(1 + a)^i}{a(1 + a)^i}$ as shown in in (4.2.14) and (4.2.15). And since $-a > 0$, $k > 0$, and $(1 + a) > 0$

$$> -ak(1+a)^i\left[\frac{c - b(1+a)^i}{a(1+a)^i}\right] - kb(1+a)^i + kc$$

$$= -kc + kb(1+a)^i - kb(1+a)^i + kc$$

$$= 0 \ . \qquad (4.2.22)$$

Therefore $Q_{F_I L} > Q_{F_{I+k} L}$ hence (4.2.13) holds since $k > 0$ was arbitrary. Thus for part (c) $F_I L$ is optimal since (4.2.12) and (4.2.13) hold. We now prove part (a). But (a) is just a special case of the proof of (4.2.12) of part (c) above. Since if

$$S^{(j)}_{[\frac{n-j}{\nu}]+1} \leq \frac{c - b(1+a)^{[\frac{n-j}{\nu}]}}{a(1+a)^{[\frac{n-j}{\nu}]}}$$

is the first item and

$$S^{(j-1)}_{[\frac{n-j+1}{\nu}]+1} > \frac{c - b(1+a)^{[\frac{n-j+1}{\nu}]}}{a(1+a)^{[\frac{n-j+1}{\nu}]}}$$

and let $S^{(j)}_{[\frac{n-j}{\nu}]+1} \equiv S_I$ in the proof of (4.2.12) and $S^{(j-1)}_{[\frac{n-j+1}{\nu}]} \equiv S_{I+1}$ in the proof of (4.2.12) and then $F_J L = F_I L$ is optimal for part (a) (where $J = I$) .

q.e.d.

<u>Theorem 4.4</u>: Let $L(S)$ be a concave or convex decreasing function with $L^-(S) \leq -1$ and $L^+(0) \leq -1$ for all $S \epsilon [0, t]$. Let $L(S) = c$ for all $S \epsilon [t, \infty)$. Let $\nu \geq 1$. Then LIFO is the optimal policy.

Proof of Theorem 4.4: First, the theorem will be proved for $\nu = 1$.

Let A be any issue policy other than LIFO and let A have j items with f.l. $> c$ on issuance, $j = 0, 1, \ldots, n$. We will now show that once any item S_i has been issued all items $S_k > S_i$ which are still unissued have f.l. $= c$. For any $S_k > S_i$ and S_k issued after S_i let $S_i + x$ and $S_k + y$ be the age of items S_i and S_k respectively when they are issued to the field. Then

$$y \geq x + L(S_i + x) > 0$$

If $S_i + x \geq t$ then $L(S_i + x) = c$ and $L(S_k + y) = c$. If $S_i + x < t$ then $L(S_i + x) > c$

(i) if $x = 0$ then $\dfrac{L(S_i) - L(t)}{S_i - t} \leq -1$ implies

$$L(S_i) + S_i \geq L(t) + t$$

$$> t \qquad \text{since } L(t) = c > 0 ;$$

but then

$$y + S_k \geq L(S_i) + S_k > L(S_i) + S_i > t$$

and

$$L(S_k + y) = c .$$

(ii) if $x > 0$ then $\dfrac{L(S_i + x) - L(t)}{S_i + x - t} \leq -1$ implies

$$L(S_i + x) + S_i + x \geq L(t) + t > t$$

and

$$y + S_k > y + S_i \geq L(S_i + x) + x + S_i > t$$

and

$$L(S_k + y) = c .$$

We are now able to apply the above result to the following three cases concerning policy A.

<u>Case 1</u> $j = 0$

Then

$$Q_L = L(S_1) + (n - 1)c \geq nc = Q_A$$

since $L(S_i) \geq c$.

<u>Case 2</u> $j = 1$

Then

$$Q_L = L(S_1) + (n - 1)c$$

$$Q_A = L(S_i + x) + (n - 1)c$$

Now if

(i) $x > 0$ then $\dfrac{L(S_i) - L(S_i + x)}{-x} \leq -1$ implies

$$L(S_i) \geq L(S_i + x) + x > L(S_i + x)$$

but $L(S_1) \geq L(S_i)$ therefore $Q_L \geq Q_A$.

(ii) if $x = 0$ then $L(S_1) \geq L(S_i)$ and $Q_L \geq Q_A$.

Case 3 $\qquad 1 < j \leq n$

Let S_i denote the initial age of the youngest item issued under policy A such that upon issuance of S_i , it has f.l. $> c$. Then all items issued after S_i have f.l. $= c$, since for all $S_k > S_i$ issued after S_i , S_k has f.l. $= c$ on issuance and for all $S_k < S_i$ issued after S_i , S_k has f.l. $= c$ since S_i is youngest item with f. l. $> c$ on issuance. But this implies that S_i is the last item issued such that it has f.l. $> c$. Thus if $S_i + x$ is the age of S_i on issuance then all of the field life of the other $j - 1$ items with f.l. $> c$ is included in x . Now $j > 1$ hence there is at least one item with f.l. $> c$ issued before S_i hence $x > 0$. Thus

$$\frac{L(S_i) - L(S_i + x)}{-x} \leq -1$$

implies

$$L(S_i) \geq L(S_i + x) + x \ . \qquad (4.2.23)$$

But

$$Q_L = L(S_1) + (n - 1)c$$

and

$$Q_A = x + L(S_i + x) + (n - p)c \quad \text{where } p \geq j > 1 \ .$$

since x may include some items with f.l. $= c$.

Using (4.2.23) then $Q_L > Q_A$ since $(n - 1)c > (n - p)c$. Now consider $\nu = 2$. If LIFO is optimal for $\nu = 2$ also, then by Zehna [11] Theorem 4.3, LIFO is optimal for all $1 \le \nu \le n$, ν integer.

Let A be any policy, not LIFO, and let j items under policy A have f.l. $> c$ on issuance.

<u>Case 1</u> $j \le 2 = \nu$

Then

$$Q_L = L(S_1) + L(S_2) + (n - 2)c$$

$$Q_A = L(S_i + x) + L(S_j + y) + (n - 2)c$$

where $L(S_i + x) \ge c$, $L(S_j + y) \ge c$ $x, y \ge 0$ and without loss of generality assume $S_i < S_j$.

Then $L(S_1) \ge L(S_i + x)$ and $L(S_2) \ge L(S_j + y)$ hence $Q_L \ge Q_A$.

<u>Case 2</u> $2 < j \le n$

Let j_1 items issued to M_1 have f.l. $> c$ and j_2 items issued to M_2 have f.l. $> c$. Then $j = j_1 + j_2$ and $j_1 \ge 0$, $j_2 \ge 0$.

Let S_i and S_j denote the <u>youngest</u> items issued by A to M_1 and M_2 respectively such that upon issuance items S_i and S_j have f.l. $> c$. Assume $S_i < S_j$. Then by the same argument as in <u>Case 3</u> $\nu = 1$ above we have

$$L(S_1) \ge L(S_i + x) + x$$

$$L(S_2) \ge L(S_j + y) + y$$

and

$$Q_A = L(S_i + x) + x + L(S_j + y) + y + (n - p)c \quad \text{where} \quad p \geq j$$

and

$$Q_L \geq Q_A$$

as required.

q.e.d.