Chapter 3

Addition of Penalty Costs

The next problem of interest will be to examine the implications of adding penalty costs to the model. These penalty costs can be considered as issuing costs or installation costs.

The removal of assumption (4) which states that there are no penalty costs is important not only because it is often the case in practical situations that there is an installation or work-stoppage cost but also because the optimal policy in the model without penalty costs may no longer be optimal when penalty costs are added. This latter point can be seen, for example, in the case where the optimal policy in the case of no penalty costs issues a large number of items whereas some suboptimal policy may issue only a few items. Then if the penalty cost is sufficiently large the policy which was optimal in the no penalty case could easily become the worst policy after subtracting the penalty costs.

Henceforth in this chapter it is assumed that there is a constant penalty cost, p , associated with the issuance of each item from the stockpile. Furthermore it is assumed that p is defined in the same units of measure as $L(S)$.

In previous sections Q_{A_i} was defined as the total field life obtained from the issuance of i items in accordance with policy A. This notation will be retained and in addition a return function R_{A_i} will be used where $R_{A_i} = Q_{A_i} - ip$. That is, R_{A_i} is the total return obtained from the issuance of i items in accordance with policy A; it equals the total field life less the total penalty cost. The objective will be to find a policy which will maximize R over all possible policies.

It is conceivable that in issuing an item which has positive field life, the net increase (if any) in the total field life may be more than offset by the penalty cost incurred. Because of this event, we will also remove the assumption that an item must be issued if it has positive field life. In its place we will merely assume that any item with zero field life will not be issued. Furthermore we will assume that there is no cost associated with the disposal of items which are not issued.

It should be noted that to start the process, it may no longer be optimal to issue ν items to $M_1, \ldots, M_\nu$. If the optimal policy calls for the issuance of only $i < \nu$ items, then the i items would be issued immediately and the process would terminate.

3.1 The Case for FIFO

It will be useful to define: for $j \leq n$

(i) $A_{j,\nu}$ is any policy of issuing j items to ν demand sources

(ii) $F_{j,\nu}$ is the policy of issuing the <u>same</u> j items as are issued in (i) to ν demand sources by FIFO

(iii) $F_{(j,\nu)*}$ is the policy of issuing the <u>youngest</u> j items to ν demand sources by FIFO.

In other words, if the FIFO issuance of any j items is

$$F_{j,\nu} = [S_{1_1}, S_{1_2}, \dots , S_{1_{i_1}} ; \dots ; S_{\nu_1}, \dots , S_{\nu_{i_\nu}}]$$

where $S_{k_j} > S_{k_{j+1}}$ for all $k = 1, \dots , \nu$ and $j = 1, \dots , i_k$

then

$A_{j,\nu}$ is any permutation of the above S_{k_j}'s.

Now $F_{(j,\nu)*}$ does not necessarily include any of the j items of policy $F_{j,\nu}$.

$F_{(j,\nu)*}$

$$= [S_j, S_{j-\nu}, \dots ; S_{j-1}, S_{j-\nu-1}, \dots ; \dots ; S_{j-\nu+1}, S_{j-2\nu+1}, \dots] .$$

$R_{A_{j,\nu}}$, $Q_{A_{j,\nu}}$, $R_{F_{j,\nu}}$, $Q_{F_{j,\nu}}$, $R_{F_{(j,\nu)*}}$ and $Q_{F_{(j,\nu)*}}$ are defined with respect to the above policies.

<u>Lemma 3.1</u>: Let $L(S)$ be a concave function with $L^-(S) \geq -1$ for $0 < S \leq S_0$ and $L^+(0) \leq 0$. Let $\nu \geq 1$. If FIFO is the issuing policy which maximizes the total field life for any j items in inventory, then

$$R_{F_{j,\nu}} \geq R_{A_{j,\nu}} \qquad \text{for any } j = 1, \ldots, n .$$

Proof of Lemma 3.1: If $j \leq \nu$ then $R_{F_{j,\nu}} = R_{A_{j,\nu}}$ since exactly j demand sources receive an item.

If $\nu < j \leq n$ then since FIFO maximizes the total field life for any j items (by hypothesis) we have

$$Q_{F_{j,\nu}} \geq Q_{A_{j,\nu}} .$$

Hence

$$R_{F_{j,\nu}} - R_{A_{j,\nu}} = Q_{F_{j,\nu}} - j \cdot p - [Q_{A_{j,\nu}} - j \cdot p]$$

$$= Q_{F_{j,\nu}} - Q_{A_{j,\nu}} \geq 0 .$$

And

$$R_{F_{j,\nu}} \geq R_{A_{j,\nu}} \qquad \text{for all } j = 1, \ldots, n$$

since j was arbitrary.

q.e.d.

Lemma 3.2: Let $L(S)$ be a concave function with $L^{-}(S) \geq -1$ for $0 < S \leq S_o$ and $L^{+}(0) \leq 0$. Let $\nu \geq 1$. Then

$$R_{F_{(j,\nu)*}} \geq R_{F_{j,\nu}} \qquad \text{for any } j = 1, \ldots, n .$$

Proof of Lemma 3.2: If $F_{(j,\nu)*} \approx F_{j,\nu}$ (where $\approx$ means "is the same as") then $R_{F_{(j,\nu)*}} = R_{F_{j,\nu}}$.

If $F_{(j,\nu)*} \not\approx F_{j,\nu}$ then $F_{j,\nu}$ must contain at least one item which has initial age greater than S_j (note that S_j is the oldest item in $F_{(j,\nu)*}$).

Now by lemma 2.5, $Q_{F_{(j,\nu)*}} \geq Q_{F_{j,\nu}}$ hence

$$R_{F_{(j,\nu)*}} - R_{F_{j,\nu}} = Q_{F_{(j,\nu)*}} - j \cdot p - [Q_{F_{j,\nu}} - j \cdot p]$$

$$= Q_{F_{(j,\nu)*}} - Q_{F_{j,\nu}} \geq 0 \ .$$

And since j was arbitrary $R_{F_{(j,\nu)*}} \geq R_{F_{j,\nu}}$ for any $j = 1, \ldots , n$.

q.e.d.

Using these two lemmas, we obtain the following interesting theorem.

Theorem 3.1: Let $L(S)$ be a concave function with $L^{-}(S) \geq -1$ for $0 < S \leq S_o$ and $L^{+}(0) \leq 0$. Let $\nu \geq 1$. If FIFO is the issuing policy which maximizes the total field life for any j items in inventory, then the optimal issuing policy must be one of the n policies

$$F_{(1,\nu)*}, F_{(2,\nu)*}, \ldots , F_{(n,\nu)*} \ .$$

Proof of Theorem 3.1: The proof is immediate from lemmas 3.1 and 3.2.

q.e.d.

Corollary 3.1: If $L(S) = aS + b$ where $b > 0 > a > -1$, then the optimal issue policy must be one of the n policies

$$F_{(1,\nu)*}, F_{(2,\nu)*}, \cdots, F_{(n,\nu)*} .$$

Proof of Corollary 3.1: By Zehna [11] Theorems 4.1 and 4.3 and Lieberman [9] Theorem 3, FIFO is optimal for all n and ν, thus by Theorem 3.1 the above result follows.

q.e.d.

Corollary 3.2: Let $L(S)$ be concave with $L^-(S) \geq -1$ for $0 < S \leq S_o$ and $L^+(0) \leq 0$. Let $\nu = 1$. Then the optimal policy is one of the n policies

$$F_{(1,1)*}, F_{(2,1)*}, \cdots, F_{(n,1)*} .$$

Proof of Corollary 3.2: Again by Lieberman [9] Theorem 3, FIFO is optimal for all n; hence by Theorem 3.1 the result follows.

q.e.d.

Thus Theorem 3.1 and Corollaries 3.1 and 3.2 state that is is only necessary to search n policies until the optimal $F_{(j,\nu)*}$ is found; then issue the j newest items by FIFO and discard the remaining $n - j$ items without issuing them even if they have positive field life. Their positive field life is offset by the penalty cost of installation.

In certain cases it is possible to select analytically the optimal policy from the n policies $F_{(1,\nu)*}, \cdots, F_{(n,\nu)*}$. These cases involve placing special restrictions on the derivative of $L(S)$ and relating these restrictions to the penalty cost.

Theorem 3.2: Let $L(S)$ be a concave function with $L^-(S) \geq -1 + \frac{1}{K}$ for $0 < S \leq S_o$ where $K > 1$ is any finite real number and with $L^+(0) \leq 0$. Let $\nu \geq 1$. If

$$0 < p \leq (\tfrac{1}{K})^{[\frac{n-1}{\nu}]} L(S_n)$$

and if FIFO maximizes the total field life $Q_{F_{(i,\nu)*}}$ for all i and ν, then $F_{(n,\nu)*}$ is the optimal policy. Furthermore

$$R_{F_{(n,\nu)*}} \geq R_{F_{(n-1,\nu)*}} \geq \cdots \geq R_{F_{(1,\nu)*}} .$$

The proof of Theorem 3.2 will be aided by the following lemma.

Lemma 3.3: Let $L(S)$ be a concave function with $L^-(S) \geq -(1 - \frac{1}{K})$ for $0 \leq S \leq S_o$ where $K > 1$ is some real number and $L^+(0) \leq 0$. Let $S_{j_1}, \ldots, S_{j_i}$ be any i items with $S_{j_k} < S_{j_{k+1}}$ for all $k = 1, \ldots, i-1$ and $S_{j_i} < S_o$. Then if

$$F_i = [S_{j_i}, S_{j_{i-1}}, \ldots, S_{j_1}]$$

and $$F_{i-1} = [\qquad S_{j_{i-1}}, \ldots, S_{j_1}]$$

are two FIFO policies for issuing the i and $i-1$ youngest items respectively then

$$Q_{F_i} - Q_{F_{i-1}} - (\tfrac{1}{K})^{i-1} L(S_{j_i}) \geq 0 .$$

Proof of Lemma 3.3: The proof will be by induction. Consider $i = 2$, it must be shown that

$$Q_{F_2} - Q_{F_1} - (\tfrac{1}{K})L(S_{j_2}) \geq 0$$

where
$$Q_{F_2} = L(S_{j_2}) + L(S_{j_1} + L(S_{j_2}))$$

and
$$Q_{F_1} = L(S_{j_1}) \ .$$

Now by lemma 2.1 $S_{j_1} + L(S_{j_2}) < S_o$ hence

$$\frac{L(S_{j_1} + L(S_{j_2})) - L(S_{j_1})}{L(S_{j_2})} \geq -1 + \frac{1}{K}$$

$$\Rightarrow L(S_{j_1} + L(S_{j_2})) + L(S_{j_2}) - L(S_{j_1}) - \frac{1}{K} L(S_{j_2}) \geq 0$$

as required.

Now assume the lemma is true for $i - 1$ and it will be shown to be true for i. Let x be the total field life from the issuance of the $i - 1$ oldest items by FIFO i.e., $A_x = [S_{j_i}, S_{j_{i-1}}, \ldots, S_{j_2}]$ and let y be the total field life from the FIFO issuance of $A_y = [S_{j_{i-1}}, \ldots, S_{j_2}]$. Then by lemma 2.1 $S_{j_1} + x < S_o$ and $S_{j_1} + y < S_o$ and by lemma 2.2, $x \geq y$ and actually $x > y$ since $L^{-}(S) > -1$,

$$\frac{L(S_{j_1} + x) - L(S_{j_1} + y)}{x - y} \geq -1 + \frac{1}{K}$$

$$\Rightarrow \; L(S_{j_1} + x) + x - [L(S_{j_1} + y) + y] \geq \frac{1}{K} (x - y)$$

but $$L(S_{j_1} + x) + x = Q_{F_i}$$

and $$L(S_{j_1} + y) + y = Q_{F_{i-1}}$$

thus $$Q_{F_i} - Q_{F_{i-1}} \geq \frac{1}{K} (x - y) \; ;$$

now subtract $(\frac{1}{K})^{i-1} L(S_{j_i})$ from both sides to obtain

$$Q_{F_i} - Q_{F_{i-1}} - (\tfrac{1}{K})^{i-1} L(S_{j_i}) \geq \frac{1}{K} \left[x - y - (\tfrac{1}{K})^{i-2} L(S_{j_i}) \right] ,$$

but by the inductive assumption on $i - 1$

$$x - y - (\tfrac{1}{K})^{i-2} L(S_{j_i}) \geq 0$$

and since $\frac{1}{K} > 0$ then

$$Q_{F_i} - Q_{F_{i-1}} - (\tfrac{1}{K})^{i-1} L(S_{j_i}) \geq 0 \; .$$

q.e.d.

Proof of Theorem 3.2: To show $F_{(n,\nu)*}$ is the optimal policy, it is sufficient to show

$$R_{F_{(n,\nu)*}} \geq R_{F_{(n-1,\nu)*}} \geq \cdots \geq R_{F_{(1,\nu)*}}$$

since by Theorem 3.1 it is only necessary to consider $F_{(n,\nu)*}, \ldots, F_{(1,\nu)*}$.

It will be shown that $R_{F_{(j+1,\nu)*}} \geq R_{F_{(j,\nu)*}}$ for any $j = 1, \ldots, n-1$.

If $j \leq \nu - 1$ then

$$R_{F_{(j+1,\nu)*}} - R_{F_{(j,\nu)*}} = L(S_{j+1}) - p$$

$$\geq L(S_{j+1}) - \left(\frac{1}{K}\right)^{[\frac{n-1}{\nu}]} L(S_n)$$

$$\text{but } 1 > \frac{1}{K} > 0$$

$$\geq L(S_{j+1}) - L(S_n) \geq 0$$

since $L(\cdot)$ is nonincreasing and $n \geq \nu \geq j+1$.

Hence

$$R_{F_{(j+1,\nu)*}} \geq R_{F_{(j,\nu)*}} \quad \text{for all } j = 1, \ldots, \nu - 1. \tag{3.1.1}$$

If $n - 1 \geq j > \nu - 1$ then consider policies $F_{(j+1,\nu)*}$ and $F_{(j,\nu)*}$ which by lemma 2.3 can be written as

$$\begin{array}{lll}
 & \underline{F_{(j+1,\nu)*}} & \underline{F_{(j,\nu)*}} \\
M_1 & [S_{j+1}, S_{j-\nu+1}, S_{j-2\nu+1}, \dots] & [S_j, S_{j-\nu}, S_{j-2\nu}, \dots] \\
M_2 & [S_j, S_{j-\nu}, S_{j-2\nu}, \dots] & [S_{j-1}, S_{j-\nu-1}, \dots] \\
\vdots & & \\
M_{\nu-1} & [S_{j-\nu+3}, S_{j-2\nu+3}, \dots] & [S_{j-\nu+2}, S_{j-2\nu+2}, \dots] \\
M_\nu & [S_{j-\nu+2}, S_{j-2\nu+2}, \dots] & [S_{j-\nu+1}, S_{j-2\nu+1}, \dots]
\end{array}$$

It should now be noted that except for demand sources M_1 in $F_{(j+1,\nu)*}$ and M_ν in $F_{(j,\nu)*}$, the items assigned to the other demand sources $M_2, \dots, M_\nu$ in $F_{(j+1,\nu)*}$ and $M_1, \dots, M_{\nu-1}$ in $F_{(j,\nu)*}$ are the same except for the indexing of the M's. Let the total field life for M_i be denoted by Q_{M_i}, then $\{Q_{M_i}|F_{(j+1,\nu)*}\} = \{Q_{M_{i-1}}|F_{(j,\nu)*}\}$ for all $i = 2, \dots, \nu$ and

$$Q_{F_{(j+1,\nu)*}} - \{Q_{M_1}|F_{(j+1,\nu)*}\} = Q_{F_{(j,\nu)*}} - \{Q_{M_\nu}|F_{(j,\nu)*}\} ,$$

and

$$\begin{aligned}
R_{F_{(j+1,\nu)*}} - R_{F_{(j,\nu)*}} &= Q_{F_{(j+1,\nu)*}} - Q_{F_{(j,\nu)*}} - p \\
&= \{Q_{M_1}|F_{(j+1,\nu)*}\} - \{Q_{M_\nu}|F_{(j,\nu)*}\} - p. \quad (3.1.2)
\end{aligned}$$

For simplicity let policy A be the items issued to M_1 under $F_{(j+1,\nu)*}$ and let policy B be the items issued to M_ν under $F_{(j,\nu)*}$. Then

$$A = [S_{j+1}, S_{j-\nu+1}, S_{j-2\nu+1}, \ldots]$$

and
$$B = [\quad S_{j-\nu+1}, S_{j-2\nu+1}, \ldots] .$$

Also let $Q_A = \{Q_{M_1} | F_{(j+1,\nu)*}\}$ and $Q_B = \{Q_{M_\nu} | F_{(j,\nu)*}\}$. Now by Corollary 2.3.1 demand source M_1 in $F_{(j+1,\nu)*}$ receives $\left[\frac{j+1-1}{\nu}\right] + 1 = \left[\frac{j}{\nu}\right] + 1$ items hence by lemma 3.3

$$Q_A - Q_B - \left(\frac{1}{K}\right)^{[\frac{j}{\nu}]} L(S_{j+1}) \geq 0$$

but
$$\left(\frac{1}{K}\right)^{[\frac{j}{\nu}]} L(S_{j+1}) \geq \left(\frac{1}{K}\right)^{[\frac{j}{\nu}]} L(S_n)$$

$$\geq \left(\frac{1}{K}\right)^{[\frac{n-1}{\nu}]} L(S_n) \geq p$$

since $L(\cdot)$ is nonincreasing, since $n - 1 \geq j \geq \nu$ and since $1 > \frac{1}{K} > 0$.

Thus

$$Q_A - Q_B \geq \left(\frac{1}{K}\right)^{[\frac{j}{\nu}]} L(S_{j+1}) \geq p .$$

Hence

$$Q_A - Q_B - p \geq 0$$

and by (3.1.2)

$$R_{F_{(j+1,\nu)*}} - R_{F_{(j,\nu)*}} \geq 0 \quad \text{for any} \quad j = \nu, \ldots , n-1 .$$

$$(3.1.3)$$

Combining (3.1.1) and (3.1.3)

$$R_{F_{(j+1,\nu)*}} \geq R_{F_{(n,\nu)*}} \qquad \text{for any} \quad j = 1, \ldots , n-1$$

$\therefore \quad F_{(n,\nu)*}$ is optimal.

q.e.d.

The preceding theorem placed restrictions on $L^-(S)$ which kept $L^-(S) > -1$. Theorem 3.3 allows $L^-(S) \geq -1$ but restricts $L^+(S)$. In this case, it is not possible to say precisely what the optimal policy is for general concave nonincreasing $L(S)$. However, it is possible to eliminate some of the n policies which cannot be optimal.

<u>Theorem 3.3</u>: Let $L(S)$ be a concave function with $L^-(S) \geq -1$ for $0 < S \leq S_o$ and $L^+(S) \leq -\frac{1}{K}$ for $0 \leq S < S_o$ where $K > 1$ is any finite real number. Let $\nu = 1$. If

$$p \geq \left(\frac{K-1}{K}\right)^j L(S_{j+1}) > 0 \quad \text{for some} \quad j = 1, \ldots , n-1 ,$$

then

$$R_{F_{(j,1)*}} \geq R_{F_{(j+i,1)*}} \quad \text{for all} \quad i = 1, \ldots , n-j .$$

Here again the proof will be made easier with the use of the following lemma.

Lemma 3.4: Let $L(S)$ be a concave function with $L^-(S) \geq -1$ for $0 < S \leq S_o$ and $L^+(S) \leq -\frac{1}{K}$ for $0 \leq S < S_o$ where $K > 1$ is any finite real number. Let $S_{j_1}, \ldots, S_{j_i}$ be any i items with $S_{j_k} < S_{j_{k+1}}$ for all $k = 1, \ldots, i-1$ and $S_{j_i} < S_o$. Then if

$$F_i = [S_{j_i}, S_{j_{i-1}}, \ldots, S_{j_1}]$$

and

$$F_{i-1} = [\quad S_{j_{i-1}}, \ldots, S_{j_1}]$$

are two FIFO policies for issuing the i and $i-1$ youngest items respectively, then

$$Q_{F_{i-1}} - Q_{F_i} + \left(\frac{K-1}{K}\right)^{i-1} L(S_{j_i}) \geq 0 .$$

Proof of Lemma 3.4: The proof will be by induction. Consider $i = 2$, it must be shown that

$$Q_{F_1} - Q_{F_2} + \left(\frac{K-1}{K}\right) L(S_{j_2}) \geq 0$$

where $Q_{F_1} = L(S_{j_1})$ and $Q_{F_2} = L(S_{j_2}) + L(S_{j_1} + L(S_{j_2}))$.

Now by lemma 2.1 $S_{j_1} + L(S_{j_2}) < S_o$ hence

$$\frac{L(S_{j_1} + L(S_{j_2})) - L(S_{j_1})}{L(S_{j_2})} \leq -\frac{1}{K}$$

$$\Rightarrow L(S_{j_1} + L(S_{j_2})) + \frac{1}{K} L(S_{j_2}) \leq L(S_{j_1}). \qquad (3.1.4)$$

But

$$L(S_{j_1} + L(S_{j_2})) + \frac{1}{K} L(S_{j_2}) = L(S_{j_1} + L(S_{j_2})) + \frac{1}{K} L(S_{j_2}) - L(S_{j_2}) + L(S_{j_2})$$

$$= Q_{F_2} - \left(\frac{K-1}{K}\right) L(S_{j_2})$$

hence in (3.1.4)

$$Q_{F_2} - \left(\frac{K-1}{K}\right) L(S_{j_2}) \leq L(S_{j_1}) = Q_{F_1}$$

$$\Rightarrow Q_{F_1} - Q_{F_2} + \left(\frac{K-1}{K}\right) L(S_{j_2}) \geq 0$$

as required. Now assume the lemma is true for $i - 1$ and it will be shown to be true for i.

Let x be the total field life from the FIFO issuance of the $i - 2$ items $[S_{j_{i-1}}, \ldots, S_{j_2}]$ and let y be the total field life from the FIFO issuance of the $i - 1$ items $[S_{j_i}, S_{j_{i-1}}, \ldots, S_{j_2}]$. Then by the inductive assumption

$$x - y + \left(\frac{K-1}{K}\right)^{i-2} L(S_{j_i}) \geq 0 . \qquad (3.1.5)$$

By lemma 2.2,

$$y - x \geq 0 .$$

Now by lemma 2.1 $S_{j_1} + x < S_c$ and $S_{j_2} + y < S_o$ thus if $y > x$

$$\frac{L(S_{j_1} + y) - L(S_{j_1} + x)}{y - x} \leq -\frac{1}{K}$$

$$\Rightarrow L(S_{j_1} + y) - L(S_{j_1} + x) \leq -\frac{1}{K}(y - x) .$$

Add $y - x - \left(\frac{K-1}{K}\right)^{i-1} L(S_{j_i})$ to both sides; then

$$\begin{aligned} L(S_{j_1} + y) + y - [L(S_{j_1} + x) + x] - \left(\frac{K-1}{K}\right)^{i-1} L(S_{j_i}) & \\ \leq -\frac{1}{K}(y - x) + y - x - \left(\frac{K-1}{K}\right)^{i-1} L(S_{j_i}) & \\ = \left(\frac{K-1}{K}\right)\left[y - x - \left(\frac{K-1}{K}\right)^{i-2} L(S_{j_i})\right] & \\ \leq 0 \qquad (3.1.6) & \end{aligned}$$

by (3.1.5) since $\frac{K-1}{K} > 0$. If $y = x$ then (3.1.6) still holds.

But $$L(S_{j_1} + y) + y = Q_{F_i}$$

and $$L(S_{j_1} + x) + x = Q_{F_{i-1}}$$

and (3.1.6) becomes

$$Q_{F_i} - Q_{F_{i-1}} - \left(\frac{K-1}{K}\right)^{i-1} L(S_{j_i}) \leq 0$$

and by induction the lemma is proved.

q.e.d.

Proof of Theorem 3.3: It will be shown that

$$R_{F_{(j+i,1)*}} \geq R_{F_{(j+i+1,1)*}} \quad \text{for any } i = 0, 1, \ldots, n-j-1 .$$

$$\begin{aligned}
R_{F_{(j+i,1)*}} - R_{F_{(j+i+1,1)*}} \\
&= Q_{F_{(j+i,1)*}} - (j+i)p - [Q_{F_{(j+i+1,1)*}} - (j+i+1)p] \\
&= Q_{F_{(j+i,1)*}} - Q_{F_{(j+i+1,1)*}} + p \\
&\geq Q_{F_{(j+i,1)*}} - Q_{F_{(j+i+1,1)*}} - \left(\frac{K-1}{K}\right)^{j+i} L(S_{j+i+1}) \\
&\geq 0 \quad \text{by lemma 3.3, for any } i = 0, 1, \ldots, n-j-1.
\end{aligned}$$

The above inequalities follow from

$$\begin{aligned}
p \geq \left(\frac{K-1}{K}\right)^{j} L(S_{j+1}) &\geq \left(\frac{K-1}{K}\right)^{j+i} L(S_{j+1}) \\
&\geq \left(\frac{K-1}{K}\right)^{j+i} L(S_{j+1+i}) \quad \text{since } L(\cdot) \text{ is}
\end{aligned}$$

nonincreasing and since $1 > \frac{K-1}{K} > 0$. Thus

$$R_{F_{(j,1)*}} \geq R_{F_{(j+1,1)*}} \geq \cdots \geq R_{F_{(n,1)*}} .$$

q.e.d.

Thus Theorem 3.3 states that all items which have initial age greater than S_j may be discarded immediately because by Theorem 3.1 and Theorem 3.3 the optimal policy must be one of the j policies $F_{(1,1)*}, F_{(2,1)*}, \cdots, F_{(j,1)*}$. It is now possible to generalize Theorem 3.3 to the case of $\nu \geq 1$ demand sources. But before so doing, some useful notation will be presented.

By using corollary 2.3.1 and lemma 2.3 it is possible to re-index the FIFO assigned items to the ν demand sources in the following manner:

Demand Source	Items assigned to the demand source using lemma 2.3		New indexing of the items using corollary 2.3.1
M_1	$[S_n, S_{n-\nu}, \cdots]$	$\equiv$	$[S^{(1)}_{[\frac{n-1}{\nu}]+1}, S^{(1)}_{[\frac{n-1}{\nu}]}, \cdots, S^{(1)}_1]$
$\vdots$			
M_j	$[S_{n-j+1}, S_{n-j-\nu+1}, \cdots]$	$\equiv$	$[S^{(j)}_{[\frac{n-j}{\nu}]+1}, S^{(j)}_{[\frac{n-j}{\nu}]}, \cdots, S^{(j)}_1]$
$\vdots$			
M_ν	$[S_{n-\nu+1}, S_{n-2\nu+1}, \cdots]$	$\equiv$	$[S^{(\nu)}_{[\frac{n-\nu}{\nu}]+1}, S^{(\nu)}_{[\frac{n-\nu}{\nu}]}, \cdots, S^{(\nu)}_1]$

That is, re-index the items assigned to M_j by FIFO from $\left[\frac{n-j}{\nu}\right] + 1$ down to 1 and attach the superscript (j) to indicate that the items are assigned to M_j.

Theorem 3.4: Let $L(S)$ be a concave function with $L^-(S) \geq -1$ for $0 < S \leq S_o$ and $L^+(S) \leq -\frac{1}{K}$ for $0 \leq S < S_o$ where $K > 1$ is any finite real number. Let $\nu \geq 1$.

If FIFO maximizes the total field life $Q_{F(i,\nu)^*}$ for all i and ν $(i \geq \nu)$ and if

$$p \geq \left(\frac{K-1}{K}\right)^t L(S_{j+1}) > 0 \quad \text{for some} \quad j = 1, \ldots, n-1$$

where S_{j+1} is the $t+1^{st}$ item remaining to be issued to some demand source, say M_k, (hence $t \epsilon \left\{0, 1, \ldots, \left[\frac{n-k}{\nu}\right]\right\}$), then

$$R_{F(j,\nu)^*} \geq R_{F(j+i,\nu)^*} \quad \text{for all} \quad i = 1, 2, \ldots, n-j.$$

Proof of Theorem 3.4: Using the notation developed above

$$S_{j+1} \equiv S_{t+1}^{(k)}$$

and writing the entire array of items in the new notation:

$$M_1 \qquad \left[S^{(1)}_{[\frac{n-1}{\nu}]+1}, \ldots, S^{(1)}_{t+1}, \ldots, S^{(1)}_{1}\right]$$

$$M_2 \qquad \left[S^{(2)}_{[\frac{n-2}{\nu}]+1}, \ldots, S^{(2)}_{t+1}, \ldots, S^{(2)}_{1}\right]$$

$$\vdots \qquad \vdots$$

$$M_k \qquad \left[S^{(k)}_{[\frac{n-k}{\nu}]+1}, \ldots, S^{(k)}_{t+1}, \ldots, S^{(k)}_{1}\right]$$

$$\vdots \qquad \vdots$$

$$M_\nu \qquad \left[S^{(\nu)}_{[\frac{n}{\nu}]}, \ldots, S^{(\nu)}_{t+1}, \ldots, S^{(\nu)}_{1}\right]$$

But since $L(\cdot)$ is nonincreasing and $0 < \frac{K-1}{K} < 1$

$$p \geq \left(\frac{K-1}{K}\right)^t L(S_{j+1}) \geq \left(\frac{K-1}{K}\right)^t L(S_{j+1+i}) \quad \text{for all} \quad i = 1, \ldots, n-j-1$$

$$\geq \left(\frac{K-1}{K}\right)^{t+u} L(S_{j+1+i}) \quad \text{for all} \quad u = 0, 1, \ldots .$$

Thus

$$p \geq \left(\frac{K-1}{K}\right)^t L\left(S^{(k)}_{t+1}\right) \geq \left(\frac{K-1}{K}\right)^t L\left(S^{(r)}_{t+1}\right) \quad \text{for} \quad r = 1, \ldots, k$$

$$\geq \left(\frac{K-1}{K}\right)^{t+1} L\left(S^{(r)}_{t+1}\right)$$

$$\geq \left(\frac{K-1}{K}\right)^{t+1} L\left(S^{(s)}_{t+2}\right) \quad \text{for} \quad s = k+1, \ldots, \nu .$$

Using these inequalities then Theorem 3.3 can be applied to each demand source separately. Hence all items older than S_j may be discarded immediately and

$$R_{F_{(j,\nu)*}} \geq R_{F_{(j+i,\nu)*}} \quad \text{for all} \quad i = 1, \ldots, n - j .$$

q.e.d.

3.2 The Case for FIFO When $L(S)$ is Linear

In the case where $L(S)$ is a linearly decreasing function with slope > -1, precise statements can be made concerning the optimality of $F_{(j,\nu)*}$ as the issuing policy which maximizes the total return.

The linear function $L(S)$ allows us to combine the knowledge gained from Theorems 3.2, 3.3, and 3.4 since we now have the condition that $L^+(S) = L^-(S) \doteq -\frac{1}{K}$ for all $0 < S < S_o$.

Changing the notation slightly let $L(S) = aS + b$ for $0 \leq S \leq S_o$ where $b > 0 > a > -1$. Hence $a \equiv -\frac{1}{K}$ in the preceding results.

<u>Lemma 3.5</u>: Let $L(S) = aS + b$ for $0 \leq S \leq S_o$ and $b > 0 > a > -1$. Let $S_{j_1}, \ldots, S_{j_i}$ be any i items with $S_{j_k} < S_{j_{k+1}}$ for all $k = 1, \ldots, i - 1$ and $S_{j_i} < S_o$. Then if

$$F_i = [S_{j_i}, S_{j_{i-1}}, \ldots, S_{j_1}]$$

and

$$F_{i-1} = [\quad S_{j_{i-1}}, \ldots, S_{j_1}]$$

are two FIFO policies for issuing the i and $i - 1$ youngest items respectively, then

$$Q_{F_i} - Q_{F_{i-1}} - (1 + a)^{i-1}L(S_{j_i}) = 0 .$$

Proof of Lemma 3.5: The proof is similar to the proof of lemma 3.4. For completeness it is given below. The proof is by induction. Consider $i = 2$.

$$Q_{F_1} = L(S_{j_1}) \quad \text{and} \quad Q_{F_2} = L(S_{j_2}) + L(S_{j_1} + L(S_{j_2}))$$

By lemma 2.1 $S_{j_1} + L(S_{j_2}) < S_o$ hence

$$\frac{L(S_{j_1} + L(S_{j_2})) - L(S_{j_1})}{L(S_{j_2})} = a$$

$$\Rightarrow L(S_{j_1} + L(S_{j_2})) - L(S_{j_1}) = aL(S_{j_2})$$

$$\Rightarrow L(S_{j_1} + L(S_{j_2})) + L(S_{j_2}) - L(S_{j_1}) - (1 + a)L(S_{j_2}) = 0$$

as required.

Now assume the lemma is true for $i - 1$ and it will be shown to be true for i . Let x be the total field life from the FIFO issuance of the $i - 2$ items $[S_{j_{i-1}}, \ldots , S_{j_2}]$ and let y be the total field life from the FIFO issuance of the $i - 1$ items $[S_{j_1}, \ldots , S_{j_2}]$. Then by the inductive assumption

$$y - x - (1 + a)^{i-2}L(S_{j_i}) = 0 \tag{3.2.1}$$

$$\Rightarrow \quad y - x = (1 + a)^{i-2}L(S_{j_i}) > 0 \; .$$

By lemma 2.1, $S_{j_1} + x < S_o$ and $S_{j_1} + y < S_o$ thus

$$\frac{L(S_{j_1} + y) - L(S_{j_1} + x)}{y - x} = a$$

$$\Rightarrow L(S_{j_1} + y) - L(S_{j_1} + x) = a(y - x)$$

$$\Rightarrow L(S_{j_1} + y) + y - L(S_{j_1} + x) + x - (1 + a)(y - x) = 0$$

but $L(S_{j_1} + y) + y = Q_{F_i}$ and $L(S_{j_1} + x) + x = Q_{F_{i-1}}$ hence

$$Q_{F_i} - Q_{F_{i-1}} - (1 + a)(y - x) = 0$$

$$\Rightarrow Q_{F_i} - Q_{F_{i-1}} - (1 + a)^{i-1}L(S_{j_i}) + (1 + a)^{i-1}L(S_{j_i}) - (1 + a)(y - x) = 0 \tag{3.2.2}$$

but

$$(1 + a)^{i-1}L(S_{j_i}) - (1 + a)(y - x) = - (1 + a)[y - x - (1 + a)^{i-2}L(S_{j_i})]$$

$$= 0$$

by (3.2.1) and using this fact in (3.2.2) we obtain

$$Q_{F_i} - Q_{F_{i-1}} - (1 + a)^{i-1} L(S_{j_i}) = 0$$

as required, and by induction the lemma is proved.

q.e.d.

We are now prepared to determine the optimal issuing policy in the linear case. Theorem 3.5 gives the optimal result when $\nu = 1$ and Theorem 3.6 for $\nu \geq 1$. Since Theorem 3.5 is used in the proof of Theorem 3.6 the proof of Theorem 3.5 is presented also.

Theorem 3.5: Let $L(S) = aS + b$ for $0 \leq S \leq S_o$ and $b > 0 > a > -1$. Let $\nu = 1$.

If (i) $p \geq (1 + a)^j L(S_{j+1})$

and (ii) $p < (1 + a)^{j-1} L(S_j)$ for some $j = 1, \ldots, n - 1$

then $F_{(j,1)*}$ is the optimal policy. That is,

$$R_{F_{(j,1)*}} \geq R_{F_{(j+i,1)*}} \quad \text{for } i = 1, \ldots, n - j \tag{3.2.3}$$

$$R_{F_{(j,1)*}} \geq R_{F_{(j-i,1)*}} \quad \text{for } i = 1, \ldots, j - 1. \tag{3.2.4}$$

Furthermore,

$$R_{F_{(j,1)*}} > R_{F_{(j-1,1)*}} > \cdots > R_{F_{(1,1)*}} . \tag{3.2.5}$$

Proof of Theorem 3.5: Inequality (3.2.3) holds by Theorem 3.3. Hence it is only necessary to show (3.2.5); (3.2.4) then follows immediately.

Consider

$$R_{F_{(j-i,1)*}} - R_{F_{(j-i-1,1)*}} \quad \text{for any} \quad i = 0, 1, \ldots, j-2$$

then

$$\begin{aligned} R_{F_{(j-i,1)*}} - R_{F_{(j-i-1,1)*}} &= Q_{F_{(j-i,1)*}} - Q_{F_{(j-i-1,1)*}} - p \\ &> Q_{F_{(j-i,1)*}} - Q_{F_{(j-i-1,1)*}} - (1+a)^{j-1}L(S_j) \\ &\geq Q_{F_{(j-i,1)*}} - Q_{F_{(j-i-1,1)*}} - (1+a)^{j-i-1}L(S_{j-i}) \end{aligned}$$

since $L(\cdot)$ is nonincreasing and $1 > 1 + a > 0$, but

$$Q_{F_{(j-i,1)*}} - Q_{F_{(j-i-1,1)*}} - (1+a)^{j-i-1}L(S_{j-i}) = 0$$

by lemma 3.5. Thus $R_{F_{(j-i,1)*}} > R_{F_{(j-i-1,1)*}}$ for all $i = 0, 1, \ldots, j-2$.

q.e.d.

An obvious consequence of Theorem 3.5 is when $p < (1+a)^{n-1}L(S_n)$, then $F_{(n,1)*}$ is optimal.

As mentioned previously Theorem 3.6 generalizes Theorem 3.5 to the case $\nu \geq 1$. An algorithm for obtaining the optimal policy when $\nu \geq 1$ is presented. For ease of defining the algorithm and for ease of proving that an optimal policy results, it is useful to define an augmented set of items and a search procedure. Recall that assumption (1) of the model states that the process starts initially with n items of initial ages $0 < S_1 < S_2 < \cdots < S_n$. The augmented set of items is the set of $n + \nu$ items $0 < S_1 < S_2 < \cdots < S_n < S_{n+1} < \cdots < S_{n+\nu}$ where $L(S_i) > 0$ for all $i = 1, \ldots, n + \nu$ and where the penalty cost p has

$$p \geq (1 + a)^{[\frac{n-\nu}{\nu}]+1} L(S_{n+1}) = (1 + a)^{[\frac{n}{\nu}]} L(S_{n+1})$$

Note that it is always possible to find items $S_{n+1}, \ldots, S_{n+\nu}$ which satisfy the augmented system since $S_n < S_o$ and p is a fixed positive constant; i.e., for $L(S) = aS + b$ (with $b > 0 > a > -1$) S_o exists and $L(S) \to 0$ as $S \to S_o$. It will be shown in the proof of Theorem 3.6 that the optimal policy for the augmented set is the same as for the original set. We now define the search procedure.

Search Procedure: Using the re-indexed method of labelling the items issued to each demand source, consider all adjacent pairs of items for each demand source starting with the oldest adjacent pair for M_1, viz. $S^{(1)}_{[\frac{n-1}{\nu}]+2}$ and $S^{(1)}_{[\frac{n-1}{\nu}]+1}$, then the oldest pair for M_2 viz. $S^{(2)}_{[\frac{n-2}{\nu}]+2}$

and $S^{(2)}_{[\frac{n-2}{\nu}]+1}$, etc. for M_3 through M_ν. Then consider the second oldest adjacent pair for M_1 viz. $S^{(1)}_{[\frac{n-1}{\nu}]+1}$ and $S^{(1)}_{[\frac{n-1}{\nu}]}$, etc. for M_2 through M_ν. Continue in this manner searching all adjacent pairs in order of their age from oldest pair to the youngest pair.

We can now state and prove Theorem 3.6.

Theorem 3.6: Let $L(S) = aS + b$ for $0 \leq S \leq S_o$ and $b > 0 > a > -1$. Let $\nu \geq 1$.

Two cases are possible:

(i) if $p \geq L(S_1)$ then the penalty cost is greater than the value received from any item hence it does not pay to issue any item but if ν items must be issued then $F_{(\nu,\nu)*}$ is optimal.

(ii) if $p \not\geq L(S_1)$ then apply the Search Procedure to the augmented set of items. If for some demand source M_i, it is the first demand source such that for some $j = 1, \ldots, [\frac{n-j}{\nu}] + 1$

$$(1 + a)^{j-1}L(S_j^{(i)}) > p \geq (1 + a)^j L(S_{j+1}^{(i)}) \qquad (3.2.6)$$

then the FIFO policy which issues all items of initial age less than or equal to $S_j^{(i)}$ and discards all items strictly older than $S_j^{(i)}$ is the optimal issuing policy. That is, if $S_j^{(i)} \equiv S_t$ then $F_{(t,\nu)*}$ is optimal.

Before beginning the proof of this theorem, it will be enlightening to consider the effect of the theorem on a simple example. Let $n = 10$ and $\nu = 3$, then the augmented set of items is $S_{13}, S_{12}, S_{11}, S_{10}, S_9, S_8, S_7, S_6, S_5, S_4, S_3, S_2, S_1$ and the policy $F_{(13,3)*}$ assigns:

$$M_1 \text{ receives } [S_{13}, S_{10}, S_7, S_4, S_1]$$

$$M_2 \text{ receives } [S_{12}, S_9, S_6, S_3]$$

$$M_3 \text{ receives } [S_{11}, S_8, S_5, S_2]$$

where in the process of augmentation, $[\frac{n}{\nu}] = 3$ then

$$p \geq (1 + a)^3 L(S_{11}) \geq (1 + a)^3 L(S_{12}) \geq (1 + a)^3 L(S_{13}) > (1 + a)^4 L(S_{13}) .$$

Now we look at all adjacent pairs from oldest to youngest i.e., we look at

$$(S_{13}, S_{10}), (S_{12}, S_9), (S_{11}, S_8), (S_{10}, S_7), (S_9, S_6),$$

$$(S_8, S_5), (S_7, S_4), (S_6, S_3), (S_5, S_2), (S_4, S_1)$$

Let us say that (S_{10}, S_7) is the oldest adjacent pair which satisfies (3.2.6). Then

$$(1 + a)^2 L(S_7) > p \geq (1 + a)^3 L(S_{10}) .$$

Furthermore the following relationships hold:

$$p < (1 + a)^2 L(S_7) \leq (1 + a)^2 L(S_6) \leq (1 + a)^2 L(S_5)$$

and $$p \geq (1 + a)^3 L(S_8) \geq (1 + a)^3 L(S_9) \geq (1 + a)^3 L(S_{10}) \ .$$

Thus if we apply Theorem 3.5 to maximize the total field life from each demand source, we find that for M_1 we use S_7, S_4, S_1 and discard S_{10} and S_{13}, for M_2 we use S_6, S_3 and discard S_9 and S_{12}, for M_3 we use S_5, S_2 and discard S_8 and S_{11}. But using S_7, S_4, S_1, S_6, S_3, S_5 and S_2 and discarding S_{13}, S_{12}, S_{11}, S_{10}, S_9, and S_8 is just $F_{(7,3)*}$. Now Theorem 3.6 says $F_{(7,3)*}$ is optimal hence it is only necessary at this point to show that the sum of the individual maxima results in the maximum total field life. This last step is immediate however, since lemma 2.3 shows that the same items are always grouped together for all $F_{(i,\nu)*}$ i.e., item S_1 is issued after item $S_{1+\nu} = S_4$ and S_4 is issued after item $S_{1+2\nu} = S_7$ etc. Likewise S_2 is issued after item $S_{2+\nu} = S_5$ etc. Hence we can do no better for the total set of items than to maximize the field life from each individual set.

The proof of Theorem 3.6 follows in the same manner as the example above.

<u>Proof of Theorem 3.6</u>:

<u>Part (i)</u>: $p \geq L(S_1)$ implies $p \geq (1 + a)^{j-1} L(S_j^{(i)})$ for all $j = 1, \ldots, [\frac{n-i}{\nu}]$ and $i = 1, \ldots, \nu$ since $L(\cdot)$ is nonincreasing.

Part (ii): Assume there exists an $S_j^{(i)}$ and $S_{j+1}^{(i)}$ with the property that upon application of the Search Procedure these two items are the first adjacent pair found to satisfy (3.2.6). Since $S_j^{(i)}$ is the first item with $p < (1+a)^{j-1}L(S_j^{(i)})$ then for all items $S_t^{(k)}$ strictly older than $S_j^{(i)}$ i.e., $S_j^{(i)} < S_t^{(k)}$ we have

$$(1+a)^{t-1}L(S_t^{(k)}) \leq p \; . \tag{3.2.7}$$

Now at least ν such $S_t^{(k)}$'s exist since the augmented set of items has $p \geq (1+a)^{[\frac{n-\nu}{\nu}]+1} L(S_{n+1})$

$$\geq (1+a)^{[\frac{n-j}{\nu}]+1} L(S_{n+\nu-j+1}) \text{ for all } j = 1, \ldots, \nu \; .$$

Furthermore for all items $S_u^{(k)} < S_j^{(i)}$,

$$\begin{aligned} p &< (1+a)^{j-1}L(S_j^{(i)}) \\ &\leq (1+a)^{j-1}L(S_u^{(k)}) \\ &\leq (1+a)^{u-1}L(S_u^{(k)}) \end{aligned} \tag{3.2.8}$$

since $u \leq j$ and $0 < 1+a < 1$.

Using (3.2.7) and (3.2.8) we see that for each demand source M_k if we let u_k be the subscript of the first item satisfying (3.2.8) then for all $k = 1, 2, \ldots, \nu$ $S_{u_k}^{(k)} \leq S_j^{(i)}$ and $S_{u_k+1}^{(k)} > S_j^{(i)}$ then

$$(1 + a)^{u_k} L(S^{(k)}_{u_k+1}) \leq p < (1 + a)^{u_k - 1} L(S^{(k)}_{u_k}) \qquad (3.2.9)$$

Hence for each demand source we can apply Theorem 3.5 to maximize the total return for that demand source.

Therefore, if we let $R^{(k)}_{F_{u_k}}$ be the return to M_k starting with item $S^{(k)}_{u_k}$ and following a FIFO policy, then by Theorem 3.5

$$R^{(k)}_{F_{u_k}} \geq R^{(k)}_{F_{u_k+w}} \quad \text{for all} \quad k = 1, \ldots, \nu \quad \text{and}$$
$$w = 1, \ldots, [\tfrac{n-k}{\nu}] + 2 - u_k$$

and

$$R^{(k)}_{F_{u_k}} \geq R^{(k)}_{F_{u_k-y}} \quad \text{for all} \quad k = 1, \ldots, \nu \quad \text{and}$$
$$y = 1, \ldots, u_k - 1 .$$

Hence to maximize the total return for each M_k we merely discard all items older than $S^{(k)}_{u_k}$ and follow a FIFO policy for the remaining items. But by so doing we are precisely following policy $F_{(t,\nu)*}$ since all items $S_u > S^{(i)}_j$ are discarded and all items $S_u \leq S^{(i)}_j$ are issued.

We must now show that maximizing the total return for each M_k is the same as maximizing the total return for all M_k's put together. But this is immediately apparent since by lemma 2.3, the relative item assignment by FIFO for any two policies $F_{(f,\nu)*}$ and $F_{(g,\nu)*}$ (for say $g < f$) is the same for all items common to each policy except that the demand sources are rotated as to the specific set of items assigned to them. That is under any $F_{(k,\nu)*}$ policy S_1 is assigned to the same demand source as are items $S_{1+\nu}$, $S_{1+2\nu}$, ... ; S_2 is assigned to the same demand source as are items $S_{2+\nu}$, $S_{2+2\nu}$, ... ; etc. Thus maximizing the individual returns also maximizes the total return. Hence $F_{(t,\nu)*}$ is optimal since this policy maximizes all of the individual returns to each M_k.

q.e.d.

Before proceeding to the next section it should be pointed out that if the assumption that the process begins with the issuance of ν items to $M_1, \ldots, M_\nu$ is retained, then it is only necessary to find the optimal policy among the $n - \nu + 1$ policies $F_{(\nu,\nu)*}, \ldots, F_{(n,\nu)*}$. This would be the case for the theorems of section 3.1 as well as for the theorems of this section. In addition if we insist that all items with positive field life remaining must be issued, then the optimal policy will be found among the $n - \nu + 1$ policies $F_{(\nu,\nu)*}, \ldots, F_{(n,\nu)*}$ also.

3.3 The Case for LIFO

The results of this section are very similar to the results of the last two sections in that we reduce the search for the optimal policy to the case of searching only n policies. Indeed these n policies bear strong resemblance to the n policies of the previous sections. Here we will be concerned with the n LIFO policies $L_{(1,\nu)*}, \ldots, L_{(n,\nu)*}$

where $L_{(i,\nu)*}$ is the LIFO issuance of the i youngest items in the stockpile to the ν demand sources. Defining $Q_{L_{(i,\nu)*}}$ and $R_{L_{(i,\nu)*}}$ as the total field life and the total return, respectively, when following policy $L_{(i,\nu)*}$ $(R_{L_{(i,\nu)*}} = Q_{L_{(i,\nu)*}} - i \cdot p)$, we can state

Theorem 3.7: Let $L(S)$ be a convex nonincreasing function. Let $\nu \geq 1$. If LIFO is the issuing policy which maximizes the total field life for any i items in inventory, then the optimal issuing policy which maximizes the total return must be one of the n policies

$$L_{(1,\nu)*}, L_{(2,\nu)*}, \cdots, L_{(n,\nu)*} .$$

We defer the proof of Theorem 3.7 until we have stated and proved the following three lemmas.

Lemma 3.6: Let $L(S)$ be a nonnegative convex nonincreasing function defined on $[0, \infty)$. Let $S_1 < S_2$ be any two points on $[0, \infty)$. Then $L^+(S_2 + L(S_1)) \geq -1$.

Proof of Lemma 3.6:

(i) If $L^+(S_2) \geq -1$, then since $L(\cdot)$ is convex nonincreasing $L^+(S_2 + L(S_1)) \geq -1$.

(ii) If $L^+(S_2) < -1$, then let $\bar{S}$ be the point where $L^-(\bar{S}) \leq -1$ and $L^+(\bar{S}) \geq -1$. Now $\bar{S} < \infty$ since $L(S)$ is a nonnegative convex nonincreasing function. Furthermore $S_2 < \bar{S}$ and

$$\frac{L(\bar{S}) - L(S_2)}{\bar{S} - S_2} < -1$$

which implies

$$L(S_2) + S_2 > L(\bar{S}) + \bar{S} \geq \bar{S} .$$

Then $L(S_1) + S_2 \geq L(S_2) + S_2 > \bar{S}$ since $L(\cdot)$ is nonincreasing. We finally obtain

$$-1 \leq L^+(\bar{S}) \leq L^-(S_2 + L(S_1)) \leq L^+(S_2 + L(S_1))$$

since $L(S)$ is convex nonincreasing.

q.e.d.

Lemma 3.7: Let $L(S)$ be a convex nonincreasing function on $[0, \infty)$. Let $\nu = 1$. Let two sets of items with the following characteristics be given:

$$I = \{S_1, \ldots , S_n | S_i < S_{i+1} \text{ and } S_n < S_o\}$$

$$II = \{\hat{S}_1, \ldots , \hat{S}_n | \hat{S}_i < \hat{S}_{i+1} \text{ and } \hat{S}_n < S_o\}$$

and $S_i \leq \hat{S}_i$ for all $i = 1, \ldots , n$. Denote by Q_L and $\hat{Q}_L$ the total field life by LIFO issuance of the items of Set I and Set II respectively. Then $Q_L \geq \hat{Q}_L$.

Proof of Lemma 3.7: The proof will be by induction. Let $n = 1$. Since $L(\cdot)$ is nonincreasing and $S_1 \leq \hat{S}_1$ then $L(S_1) \geq L(\hat{S}_1)$ as required. Assume the lemma is true for $n = k$ and it will be proved true for $n = k + 1$.

Let x and y denote the total field lives from the LIFO issuance of the first k items in Sets I and II respectively. Now $x \geq y$ by the inductive assumption and $x > 0$ and $y > 0$ since $L(S_1) > 0$ and $L(\hat{S}_1) > 0$. We must show

$$Q_L = x + L(S_{k+1} + x) \geq y + L(\hat{S}_{k+1} + y) = \hat{Q}_L \qquad (3.3.1)$$

If $x = y$, then (3.3.1) obviously holds. If $x > y$, then by lemma 3.6 $L^+(\hat{S}_{k+1} + y) \geq -1$ and $L^+(S_{k+1} + x) \geq -1$. Now since $\hat{S}_{k+1} + x > \hat{S}_{k+1} + y$ and $L(S)$ is convex nonincreasing,

$$L^-(\hat{S}_{k+1} + x) \geq L^+(\hat{S}_{k+1} + y) \geq -1 .$$

We now obtain

$$\frac{L(\hat{S}_{k+1} + x) - L(\hat{S}_{k+1} + y)}{x - y} \geq -1$$

and

$$\begin{aligned} L(\hat{S}_{k+1} + y) + y &\leq L(\hat{S}_{k+1} + x) + x \\ &\leq L(S_{k+1} + x) + x \end{aligned}$$

and (3.3.1) holds. By induction the lemma is proved. Note that if $L(\hat{S}_{k+1} + x) = 0$ and/or $L(\hat{S}_{k+1} + y) = 0$ the proof still holds.

q.e.d.

Denote by $L_{i,\nu}$ the policy of issuing any i items to ν demand sources by LIFO.

Lemma 3.8: Let $L(S)$ be a convex nonincreasing function. Let $\nu \geq 1$. If LIFO is the policy which maximizes the total field life for any i items in inventory, then $Q_{L_{(i,\nu)^*}} \geq Q_{L_{i,\nu}}$.

This lemma is a direct consequence of lemma 3.7 when $\nu = 1$. However for $\nu > 1$ the proof becomes somewhat more complicated.

Proof of Lemma 3.8: Choose any i items from the n possible items and denote them by $S_{t_1} < S_{t_2} < \cdots < S_{t_i}$. Assume that the i items are not the i newest items. Let $Q_{L_{i,\nu}}(S_{t_1}, \ldots, S_{t_i})$ be the total field life obtained from issuing these i items by LIFO. We will show that there is a policy which yields a greater total field life, call this policy A.

Policy A: Let S_{t_j} be the youngest item of the i items such that $S_{t_i} \neq S_i$. If M_j is the demand source which receives item S_{t_j} under $L_{i,\nu}(S_{t_1}, \ldots, S_{t_i})$ then instead of issuing S_{t_j} to M_j issue S_j in its place. Do not change any of the other items, their order of issue or the demand source to whom they are issued.

By lemma 3.7 the field life contributed by M_j under policy A is greater than or equal to the field life under policy $L_{i,\nu}(S_{t_1}, \dots, S_{t_i})$. And the total field life of the other M_k's $(k \neq j)$ is unchanged. Thus

$$Q_A = \sum_{\substack{k=1 \\ k\neq j}}^{\nu} Q_{M_k} + Q_{M_j}^{(A)} \geq \sum_{k=1}^{\nu} Q_{M_k} = Q_{L_{i,\nu}}(S_{t_1}, \dots, S_{t_i}) .$$

But LIFO is optimal for any i items. Hence

$$Q_{L_{i,\nu}}(S_1, \dots, S_j, S_{t_{j+1}}, \dots, S_{t_i})$$

$$\geq Q_A \geq Q_{L_{i,\nu}}(S_1, \dots, S_{t_j}, S_{t_{j+1}}, \dots, S_{t_i})$$

Now apply policy A again to the new set of items $S_1, S_2, \dots, S_j, S_{t_{j+1}}, \dots, S_{t_i}$ and continue this process until $Q_{L_{(i,\nu)*}} \geq Q_{L_{i,\nu}}(S_{t_1}, \dots, S_{t_i})$ as required. The process terminates at policy $L_{(i,\nu)*}$ since there are only a finite number of items (at most n) to replace.

q.e.d.

We are now ready to prove Theorem 3.7.

Proof of Theorem 3.7: Let $A_{i,\nu}$ be any arbitrary policy of issuing any i items to the ν demand sources. Now since LIFO is optimal for any i then $Q_{L_{i,\nu}} \geq Q_{A_{i,\nu}}$ and by lemma 3.8 we have $Q_{L_{(i,\nu)*}} \geq Q_{L_{i,\nu}} \geq Q_{A_{i,\nu}}$. Now $L_{(i,\nu)*}$ issues at most i items (less than i if some have zero field life). Thus,

$$R_{L_{(i,\nu)*}} \geq Q_{L_{(i,\nu)*}} - ip \geq Q_{A_{i,\nu}} - ip = R_{A_{i,\nu}} .$$

And since $A_{i,\nu}$ was any arbitrary policy, the theorem is proved.

q.e.d.

Corollary 3.3: Let $L(S)$ be linear on $[0, S_o]$ with $L'(S) = -1$ on $[0, S_o]$. Let $\nu \geq 1$. The optimal policy which maximizes the total return must be one of the n policies $L_{(1,\nu)*}, \ldots, L_{(n,\nu)*}$.

Proof of Corollary 3.3: By lemma 5.2 (ff.) LIFO maximizes the total field life for any i items assigned to the ν demand sources. Therefore, the application of Theorem 3.7 proves the corollary.

q.e.d.

Theorem 3.8: Let $L(S)$ be a convex or a concave differentiable function on $[0, S_o]$ with $L'(S) < -1$ on $[0, S_o]$. Let $\nu \geq 1$. The optimal policy which maximizes the total return must be one of the ν policies $L_{(1,\nu)*}, \ldots, L_{(\nu,\nu)*}$.

Proof of Theorem 3.8: By Zehna [11] Theorems 2.4, 2.6, 4.2 and 4.3, LIFO maximizes the total field life for any i items and ν demand sources. Since this condition satisfies the hypothesis of Theorem 3.7, we merely apply Theorem 3.7 and we have proved Theorem 3.8 in the convex case. However the following proof holds for both the convex and concave cases. If only i items where $i < \nu$ have the property that $L(S_j) > p$ for $j = 1, \ldots, i$ then it is never optimal to issue more than the i

newest items since if more than i items are issued the penalty costs exceed the value of the additional items or if the i items issued are not the i newest items then the total return can be improved by issuing the i newest items. Henceforth assume $i \geq \nu$ items have the property that $L(S_j) > p$ for $j = 1, \dots, i$. Now for any $S < S_o$ and since $L'(S) < -1$ we have

$$\frac{L(S_o) - L(S)}{S_o - S} < -1$$

which implies

$$S_o < L(S) + S \ . \tag{3.3.2}$$

As pointed out above Zehna's theorems prove that LIFO maximizes the total field life. Hence for any policy $A_{i,\nu}$ $Q_L \geq Q_{A_{i,\nu}}$. But under LIFO after the first ν items are issued to start the process, all other items $S_k > S_\nu$ have field life of zero when they are to be issued since

$$S_k + L(S_\nu) > S_\nu + L(S_\nu) > S_o$$

by (3.3.2). Hence LIFO issues only ν items. Therefore

$$R_L = Q_L - \nu \cdot p \geq Q_{A_i} - \nu \cdot p$$

$$\geq Q_{A_i} - i \cdot p = R_{A_i}$$

and since A_i was arbitrary, the theorem is proved.

q.e.d.

Just as in the case for FIFO, if we retain the assumption that ν items must be issued to start the process, then the optimal policy will be found among the $n - \nu + 1$ policies $L_{(\nu,\nu)*}, \cdots, L_{(n,\nu)*}$ in the general case of Theorem 3.7 and Corollary 3.3 and will be $L_{(\nu,\nu)*}$ in the case of Theorem 3.8. Furthermore it should be noted that if the assumption is retained that any item will be issued provided it has positive field life, then LIFO is optimal for Theorems 3.7 and 3.8 and Corollary 3.3. By LIFO it is meant that the newest item is always issued to a demand source provided the item has positive field life (in Theorem 3.8 LIFO $\equiv L_{(\nu,\nu)*}$).

It is also interesting to point out that if only i items where $i \leq \nu$ have the property that $L(S_j) > p$ for $j = 1, \ldots, i$ then $L_{(i,\nu)*}$ is the optimal policy for the general case of Theorems 3.7 and 3.8 and for Corollary 3.3. The proof of this statement is the same as is given in Theorem 3.8.