Chapter 2

Multiple Demands on the Stockpile

2.1 Modification of Assumption (6)

The model, as previously defined, contains the implicit hypothesis that there is only one demand source withdrawing items from the stockpile. Except for Zehna [11], Chapter 4, all of the previous work done on the deterministic inventory depletion model necessarily requires this single demand source assumption. Zehna, however, proved that when $L(S) = aS + b \quad (b > 0 > a > -1)$ then FIFO is optimal for one or more demand sources. In addition, he showed that if $L(S)$ is either a convex or a concave differentiable function with $L'(S) < -1$, then LIFO is optimal for one or more demand sources.

One of the objectives of the present work, is to remove the assumption of a single demand source. We will denote the number of demand sources requesting items from the stockpile by the letter "ν." ν is an integer and is bounded by $1 \leq \nu \leq n$ where n is the number of items initially in the stockpile. We do not consider $\nu > n$ since the policy of issuing the n items to the n demand sources cannot be improved upon in terms of maximizing the total field life of the stockpile. The demand sources will be denoted by $M_1, M_2, \ldots, M_\nu$.

Since assumption (6) contains the implicit assumption of a single demand source we will modify assumption (6) as follows:

(6)' An item is issued from the stockpile whenever any demand source has consumed the entire useful field life of the item previously issued to it. If two or more demand sources request a new item at the same time, the new items will be issued to them in the same order as they received their last previously issued items.

A policy is said to be feasible if a demand on the stockpile is always satisfied, provided (i) the stockpile is not empty and (ii) the remaining items in the stockpile have positive field life. In seeking the optimal policy we will only be concerned with the optimal policy which belongs to the class of feasible policies.

Before proceeding further, it will be useful to define the notation which is used to describe a policy. An issuing policy for ν demand sources:

(1) List the items assigned to a particular demand source in their order of use from the first item used until the last item used, and

(2) separate the items for different demand sources by a semicolon.

For example, a policy A can be described as follows:

$$A = [S_{11}, S_{12}, \dots , S_{1i_1}; S_{21}, \dots , S_{2i_2}; \dots ; S_{\nu 1}, \dots , S_{\nu i_\nu}]$$

Thus A is the issuing policy which assigns

items $S_{11}, S_{12}, \dots, S_{1i_1}$ to demand source M_1 in that order ,
items $S_{21}, S_{22}, \dots, S_{2i_2}$ to demand source M_2 in that order , ...
items $S_{\nu 1}, S_{\nu 2}, \dots, S_{\nu i_\nu}$ to demand source M_ν in that order .

Note that $\sum_{j=1}^{\nu} i_j = n$ if all items are assigned. It is obvious that the choice of $M_1, M_2, \dots, M_\nu$ for the particular assignment of items above was arbitrary. Hence the $\nu!$ policies obtained by permuting the M_i's are equivalent policies in the sense that the total field life obtained from the n items is unchanged regardless of how the demand sources are labelled.

It is assumed that the process begins by issuing ν items, one to each $M_1, M_2, \dots, M_\nu$.

2.2 General Relationships

Among the items which have a deteriorating field life function there are several interesting relationships which will be useful at various times throughout the subsequent chapters. For this reason these relationships have been gathered together and stated as lemmas in this section.

Lemma 2.1: Let $L(S)$ be a continuous nonincreasing function with $L^-(S) \geq -1$ for $0 < S \leq S_o$. Let $\nu = 1$. If the items in the stockpile are issued according to FIFO, the field life of any item at the time of issue is strictly positive.

Proof of Lemma 2.1: If $S_o = +\infty$, the lemma is trivially true. Hence assume $S_o < +\infty$. By FIFO S_n is the first item issued and by assumption (7) of the model $L(S_n) > 0$. Now assume the lemma is true for the first k items issued and it will be proved true for the first $k + 1$ items issued.

Since items are withdrawn from the stockpile in decreasing order of their index numbers (under FIFO), let the k^{th} item issued be denoted by S_j and the $k + 1^{st}$ item issued by S_{j-1}. Let x denote the total field life of the first $k - 1$ items (under FIFO). Then the inductive hypothesis is:

$$L(S_j + x) > 0 \tag{2.2.1}$$

which implies that $S_j + x < S_o$. Now it must be proved that

$$L(S_{j-1} + x + L(S_j + x)) > 0 \tag{2.2.2}$$

or in other words that

$$S_{j-1} + x + L(S_j + x) < S_o .$$

Now by hypothesis $L^-(S) \geq -1$ for all S with $0 < S \leq S_o$ and since $L(\cdot)$ is continuous and since $S_j + x < S_o$ by (2.2.1) we can form

$$\frac{L(S_j + x) - L(S_o)}{S_j + x - S_o} \geq -1 \tag{2.2.3}$$

hence

$$L(S_j + x) - L(S_o) \leq S_o - S_j - x$$

and since $L(S_o) = 0$ we obtain

$$L(S_j + x) + x + S_j \leq S_o \ . \qquad (2.2.4)$$

But $S_{j-1} + x < S_j + x$, hence in (2.2.4)

$$S_{j-1} + x + L(S_j + x) < S_j + x + L(S_j + x) \leq S_o$$

which proves (2.2.2). Therefore by induction the lemma is proved.

q.e.d.

The next lemma is concerned with the effect on total field life when M items of arbitrary ages are combined with the inventory of $n = N$ items and the process then starts. We assume a FIFO issuing policy is used.

Lemma 2.2: Let $L(S)$ be a continuous nonincreasing function with $L^-(S) \geq -1$ for all S with $0 < S \leq S_o$. Let $\nu = 1$. Denote by Q_{F_N} , the total field life obtained by issuing the $n = N$ items according to FIFO. Let $M \geq 1$ additional items of initial ages $S_1^* < S_2^* < \cdots < S_M^* < S_o$ be combined with the original N items. Let the $N + M$ items be issued by FIFO and denote the total field life by $Q_{F_{N+M}}$. Let $S_i^* \neq S_j$ for all i, j .

Then $Q_{F_{N+M}} \geq Q_{F_N}$ for any finite N, M .

Proof of Lemma 2.2: The proof will be by induction. Let $M = 1$ and $N \geq 1$. Three cases are possible.

Case 1 $\qquad S_1^* < S_1$

$$Q_{F_N} \leq Q_{F_N} + L(S_1^* + Q_{F_N}) = Q_{F_{N+1}}$$

Case 2 $\qquad S_N < S_1^*$

For this case we will use induction on N to show $Q_{F_{N+1}} \geq Q_{F_N}$. Let $N = 1$. We have since $L(S)$ is nonincreasing $L(S_1) \geq L(S_1^*)$ and since $L(\cdot)$ is continuous and $L^-(S) \geq -1$ for $0 < S < S_o$ then

$$\frac{L(S_1) - L(S_1 + L(S_1^*))}{-L(S_1^*)} \geq -1$$

$$L(S_1) - L(S_1 + L(S_1^*)) \leq L(S_1^*)$$

and $$Q_{F_1} = L(S_1) \leq L(S_1^*) + L(S_1 + L(S_1^*)) = Q_{F_2}$$

Now assume true for $N = j$ and prove true for $N = j + 1$. Let x be the total field life from issuing items $[S_1^*, S_{j+1}, S_j, \ldots, S_2]$ by FIFO and let y be the total field life from issuing $[S_{j+1}, S_j, \ldots, S_2]$ by FIFO. Then the inductive assumption states

$$x \geq y . \qquad (2.2.5)$$

We must show

$$Q_{F_{N+1}} = x + L(S_1 + x) \geq y + L(S_1 + y) = Q_{F_N} \tag{2.2.6}$$

(i) if $x = y$, then (2.2.6) holds with equality

(ii) if $x > y$, then by lemma 2.1, $S_1 + x < S_o$ and $S_1 + y < S_o$ and since $L(\cdot)$ is continuous and $L^-(S) \geq -1$ for $S \leq S_o$ we have

$$\frac{L(S_1 + x) - L(S_1 + y)}{x - y} \geq -1$$

$$L(S_1 + x) - L(S_1 + y) \geq -x + y$$

and

$$L(S_1 + x) + x \geq L(S_1 + y) + y$$

which proves (2.2.6); hence by induction the lemma is true for this case.

<u>Case 3</u> $S_1 < \cdots < S_i < S_1^* < S_{i+1} < \cdots < S_N$

Let x denote the total field life of the N^{th}, $N - 1^{st}$, ... , $i + 1^{st}$ items issued by FIFO (i.e., of items S_N, S_{N-1}, ... , S_{i+1}). Then $S_1^* + x > S_i + x > S_i$ and since $L(\cdot)$ is nonincreasing

$$L(S_1^* + x) \leq L(S_i + x) \ . \tag{2.2.7}$$

Now let

$$S_1^* + x = T_1^*$$

$$S_j + x = T_j \qquad \text{for all } j = 1, \ldots, i .$$

Then (2.2.7) becomes $L(T_1^*) \leq L(T_i)$ and $0 < S_1 + x < S_2 + x < \cdots <$ $S_i + x < S_1^* + x$ is rewritten as

$$0 < T_1 < T_2 < \cdots < T_i < T_1^* \tag{2.2.8}$$

but (2.2.8) shows that we now have <u>case 2</u> above with $N = i$ hence $x + Q_{F_{i+1}} \geq x + Q_{F_i}$ by <u>case 2</u>. But

$$Q_{F_{N+1}} = x + Q_{F_{i+1}} \geq x + Q_{F_i} = Q_{F_N} .$$

Therefore $Q_{F_{N+1}} \geq Q_{F_N}$ in all three cases and since the three cases exhaust all possibilities the lemma is proved for $M = 1$ and $N \geq 1$.
Let $M > 1$ and $N \geq 1$.
Assume the lemma is true for $M > 1$ and consider adding $M + 1$ items of initial ages $\{S_i^*\}_{i=1}^{M+1}$, $(S_i^* < S_{i+1}^*)$. Ignoring S_{M+1}^* temporarily, the total field life of the remaining items $Q_{F_{N+M}}$ satisfies $Q_{F_{N+M}} \geq Q_{F_N}$ by the inductive assumption. Then adding S_{M+1}^* can only increase the total field life by the case $M = 1$ i.e., $Q_{F_{N+M+1}} \geq Q_{F_{N+M}}$ $\geq Q_{F_N}$ and by induction the lemma is proved.

q.e.d.

It is very important to know the ordering of a FIFO assignment to $\nu > 1$ demand sources. Lemma 2.3 below gives such an assignment under a deteriorating field life function with slope ≥ -1.

Lemma 2.3: Let $L(S)$ be a continuous nonincreasing function with $L^{-}(S) \geq -1$ for all S such that $0 < S \leq S_o$. Let $\nu \geq 1$. Then starting from the oldest item S_n , FIFO assigns every ν^{th} item to the same demand source, i.e., without loss of generality we can arbitrarily let M_1 receive S_n , M_2 receive S_{n-1} etc. to start, then

demand source M_1 receives items indexed by $n - k\nu$

demand source M_2 receives items indexed by $n - k\nu - 1$

$\vdots$

demand source M_j receives items indexed by $n - k\nu - j + 1$

$\vdots$

demand source M_ν receives items indexed by $n - k\nu - \nu + 1$

for $k = 0, 1, 2, \cdots$ until all items have been assigned. Conversely if the assignment of items is as given above, then the assignment is FIFO.

Proof of Lemma 2.3: The proof proceeds by the use of induction in several parts.

Let $k = 1$.

Now we have arbitrarily assigned

$$
\begin{array}{ll}
S_n & \text{to } M_1 \\
S_{n-1} & \text{to } M_2 \\
\vdots & \\
S_{n-\nu+1} & \text{to } M_\nu
\end{array}
$$

and since $L(\cdot)$ is nonincreasing

$$L(S_n) \leq L(S_{n-1}) \leq \cdots \leq L(S_{n-\nu+1}) \tag{2.2.9}$$

hence the next assignment is $S_{n-\nu}$ to M_1 since $L(S_n)$ is the smallest field life. Now asume the lemma is true for assigning $S_{n-\nu-i+1}$ to M_i for $i \in \{1, 2, \ldots, \nu - 1\}$ we must prove

$$S_{n-\nu-i} \text{ is assigned to } M_{i+1} \,. \tag{2.2.10}$$

To prove (2.2.10) it is useful to show for all j such that $2 \leq j \leq i$ that

$$L(S_{n-\nu-j+2} + L(S_{n-j+2})) + L(S_{n-j+2}) \leq L(S_{n-\nu-j+1} + L(S_{n-j+1})) + L(S_{n-j+1}). \tag{2.2.11}$$

Inequality (2.2.11) states that the field life obtained from the two items assigned to M_{j-1} is less than the field life obtained from the two items assigned to M_j . For simplicity let $x \equiv L(S_{n-j+2})$ and $y \equiv L(S_{n-j+1})$ and since $L(\cdot)$ is nonincreasing $x \leq y$.

(i) If $x = y$ then (2.2.11) is obviously true.

(ii) If $x < y$, then since $L(\cdot)$ is continuous, since $L^{-}(S) \geq -1$ for $S \leq S_o$ and since by lemma 2.1 we have $S_{n-\nu-j+2} + y < S_o$ then

$$\frac{L(S_{n-\nu-j+2} + x) - L(S_{n-\nu-j+2} + y)}{x - y} \geq -1$$

hence

$$L(S_{n-\nu-j+2} + x) + x \leq L(S_{n-\nu-j+2} + y) + y$$

$$\leq L(S_{n-\nu-j+1} + y) + y$$

since

$$L(S_{n-\nu-j+2} + y) \leq L(S_{n-\nu-j+1} + y)$$

and (2.2.11) is satisfied, for all $j = 2, \ldots, i$.

Now we can telescope (2.2.11) into the inequality

$$L(S_{n-\nu} + L(S_n)) + L(S_n) \leq L(S_{n-\nu-j+1} + L(S_{n-j+1})) + L(S_{n-j+1})$$

for all $j = 1, \ldots, i$

and we will show

$$L(S_{n-i}) \leq L(S_{n-\nu} + L(S_n)) + L(S_n) \quad . \qquad (2.2.12)$$

Then, if (2.2.12) holds we have by (2.2.12) and (2.2.9) that demand source M_{i+1} (who received S_{n-i}) is the next source to demand an item from the stockpile which by FIFO and the inductive assumption is item $S_{n-\nu-i}$. We now prove (2.2.12). By lemma 2.1, $S_{n-i} + L(S_n) < S_o$ and since $L(\cdot)$ is continuous and $L^-(S) \geq -1$ for $S \leq S_o$ we have

$$\frac{L(S_{n-i} + L(S_n)) - L(S_{n-i})}{L(S_n)} \geq -1$$

$$L(S_{n-i}) \leq L(S_n) + L(S_{n-i} + L(S_n))$$

$$\leq L(S_n) + L(S_{n-\nu} + L(S_n))$$

since $$L(S_{n-i} + L(S_n)) \leq L(S_{n-\nu} + L(S_n)) \ .$$

Hence (2.2.12) holds and $S_{n-\nu-i}$ is assigned to M_{i+1} . Therefore by induction the lemma is true for $k = 1$.

Assume the lemma is true for $k = t$ and it will be proved true for $k = t + 1$. For all $j = 1, \ldots, \nu$ let x_j be the total field life of all the items assigned to M_j up through cycle $t - 1$, i.e., of items $S_{n-j+1}, S_{n-\nu-j+1}, \ldots, S_{n-(t-1)\nu-j+1}$. Then the inductive assumption states

$$x_1 \leq x_2 \leq \cdots \leq x_j \leq \cdots \leq x_\nu \ . \tag{2.2.13}$$

In order to assert that M_1 receives item $S_{n-(t+1)\nu}$, it is necessary to prove

$$L(S_{n-t\nu} + x_1) + x_1 \leq L(S_{n-t\nu-j} + x_{j+1}) + x_{j+1} \tag{2.2.14}$$

for all $j = 1, \ldots, \nu - 1$.

If $x_{j+1} = x_1$, then (2.2.14) obviously holds since $L(\cdot)$ is nonincreasing.

If $x_{j+1} > x_1$, then since $L(\cdot)$ is continuous and nonincreasing and by lemma 2.1, $S_{n-t\nu} + x_{j+1} < S_o$, thus

$$\frac{L(S_{n-t\nu} + x_{j+1}) - L(S_{n-t\nu} + x_1)}{x_{j+1} - x_1} \geq -1$$

implies

$$L(S_{n-t\nu} + x_1) + x_1 \leq L(S_{n-t\nu} + x_{j+1}) + x_{j+1}$$

$$\leq L(S_{n-t\nu-j} + x_{j+1}) + x_{j+1} \; .$$

And (2.2.14) holds for all $j = 1, \ldots , \nu - 1$ since j was arbitrary. Therefore item $S_{n-(t+1)\nu}$ is issued to M_1 and the lemma has been proved for the first assignment in the $t + 1^{st}$ cycle.

Now assume the lemma is true for the j^{th} assignment in the $t + 1^{st}$ cycle and it will be proved true for the $j + 1^{st}$ assignment in the $t + 1^{st}$ cycle $(j + 1 \leq \nu)$.

Let

y_1 be the total field life of items issued to M_1 up through cycle t

y_2 be the total field life of items issued to M_2 up through cycle t

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y_j be the total field life of items issued to M_j up through cycle t .

Then by the inductive assumption on t and on j

$$x_{j+1} \leq x_{j+2} \leq \cdots \leq x_\nu \leq y_1 \leq \cdots \leq y_j \qquad (2.2.15)$$

where $y_i = x_i + L(S_{n-t\nu-i+1} + x_i)$ for $i = 1, \ldots , j$.

It must be shown that

$$x_{j+1} + L(S_{n-t\nu-j} + x_{j+1}) \leq x_k + L(S_{n-t\nu-k+1} + x_k)$$

$$\leq y_i + L(S_{n-(t+1)\nu-i+1} + y_i) \qquad (2.2.16)$$

for all $k = j + 2, \ldots, \nu$ and $\nu = 1, \ldots, j$.

But (2.2.16) follows immediately by the same reasoning as used above. If $x_{j+1} = x_k$, then the first inequality in (2.2.16) holds since $L(\cdot)$ is nonincreasing. If $x_{j+1} < x_k$ then since $S_{n-t\nu-j} + x_k < S_o$ by lemma 2.1 then

$$\frac{L(S_{n-t\nu-j} + x_k) - L(S_{n-t\nu-j} + x_{j+1})}{x_k - x_{j+1}} \geq -1$$

implies

$$L(S_{n-t\nu-j} + x_{j+1}) + x_{j+1} \leq L(S_{n-t\nu-j} + x_k) + x_k$$

$$\leq L(S_{n-t\nu-k+1} + x_k) + x_k ,$$

for $k = j + 2, \ldots, \nu$ since k was arbitrary. Similarly if $x_k = y_i$ then the second inequality in (2.2.16) is obviously satisfied, and if $x_k < y_i$ then since $S_{n-t\nu-k+1} + y_i < S_o$ by lemma 2.1, then

$$L(S_{n-t\nu-k+1} + x_k) + x_k \leq L(S_{n-t\nu-k+1} + y_i) + y_i$$

$$\leq L(S_{n-(t+1)\nu-i+1} + y_i) + y_i$$

for all $i = 1, \ldots, j$ since i was arbitrary. Thus both inequalities in (2.2.16) hold.

But (2.2.16) implies that M_{j+1} is in need of an item before $M_{j+2}, \ldots, M_{\nu}, M_1, \ldots, M_j$ in that order. Therefore the next item to be assigned must be assigned to M_{j+1}. However the last item assigned (by the inductive assumption) was item $S_{n-(t+1)\nu-j+1}$ and since we are following FIFO then $S_{n-(t+1)\nu-j}$ is the next item and is assigned to M_{j+1}. But this last assignment is precisely what this lemma states it should be. Hence by induction on $k = t$ and on j the lemma is proved. The converse is obviously true since we assign the oldest item each time an assignment is made.

q.e.d.

There is an interesting corollary to this lemma which states exactly how many items each demand source receives under FIFO issuance when the field life function is as given in lemma 2.3.

Corollary 2.3.1: Let $L(S)$ be a continuous nonincreasing function with $L^{-}(S) \geq -1$ for all S such that $0 < S \leq S_0$. If FIFO is used to assign the n items to $\nu \geq 1$ demand sources, then demand source M_j receives exactly $\left[\frac{n-j}{\nu}\right] + 1$ items $(j = 1, \ldots, \nu)$ where $[x]$ denotes the greatest integer $\leq x$.

Proof of Corollary 2.3.1: By lemma 2.3 any demand source M_j receives items indexed by $n - k\nu - j + 1$ for $k = 0, 1, \ldots, t$ where t is the largest integer such that

$$n - t\nu - j + 1 \geq 1 .$$

Thus $n - t\nu - j \geq 0$ and $t \leq \frac{n-j}{\nu}$. But t is the largest integer satisfying this condition, hence $t = \left[\frac{n-j}{\nu}\right]$. Now since $k = 0, 1, \ldots, t$ items then M_j receives exactly

$$\sum_{k=0}^{[\frac{n-j}{\nu}]} 1 = 1 + \left[\frac{n-j}{\nu}\right] \text{ items.}$$

q.e.d.

For example, let $n = 11$, $\nu = 3$ then

M_1 receives $1 + \left[\frac{11-1}{3}\right] = 1 + \left[\frac{10}{3}\right] = 4$ items

M_2 receives $1 + \left[\frac{11-2}{3}\right] = 1 + \left[\frac{9}{3}\right] = 4$ items

M_3 receives $1 + \left[\frac{11-3}{3}\right] = 1 + \left[\frac{8}{3}\right] = 3$ items

total 11 items

The next two lemmas are very useful in the proofs of theorems in subsequent chapters. In the case $\nu = 1$ demand source, lemma 2.4 states that the FIFO issuance of a set of n items has a greater total field life than another set of n items also issued by FIFO whenever the initial ages of each member of the second set of items is at least as great as its corresponding member in the first set. This result holds under fairly general $L(S)$. Lemma 2.5 generalizes lemma 2.4 to the case of more than one demand source. We have stated the $\nu = 1$ and $\nu \geq 1$ cases separately since the proof of lemma 2.5 becomes quite simple once lemma 2.4 has been proved.

Lemma 2.4: Let $L(S)$ be a continuous nonincreasing function with $L^-(S) \geq -1$ for all S such that $0 < S \leq S_o$. Let $\nu = 1$. Consider two sets of n items which the following characteristics:

$$I = \{S_1, \ldots, S_n | S_i < S_{i+1} < S_o \text{ for all } i = 1, \ldots, n-1\}$$

$$II = \{\hat{S}_1, \ldots, \hat{S}_n | \hat{S}_i < \hat{S}_{i+1} < S_o \text{ for all } i = 1, \ldots, n-1\}$$

and $S_i \leq \hat{S}_i$ for all $i = 1, \ldots, n$. Denote by Q_F and $\hat{Q}_F$ the total field life by FIFO issuance of the items in sets I and II respectively. Then $Q_F \geq \hat{Q}_F$.

Proof of Lemma 2.4: The proof is by induction.

Let $n = 1$. Since $L(\cdot)$ is nonincreasing then

$$Q_F = L(S_1) \geq L(\hat{S}_1) = \hat{Q}_F .$$

Now assume the lemma is true for $n = k$ items and it will be proved true for $n = k + 1$ items. Let x and $\hat{x}$ denote the total field life obtained by the FIFO issuance of the first k items issued (i.e., the k oldest items) of sets I and II respectively. Thus by the inductive assumption $x \geq \hat{x}$. If $x = \hat{x}$, then

$$Q_F = L(S_1 + x) + x \geq L(\hat{S}_1 + \hat{x}) + \hat{x} = \hat{Q}_F$$

since $L(\cdot)$ is nonincreasing. If $x > \hat{x}$, then since $L(\cdot)$ is continuous nonincreasing and $L^-(S) \geq -1$ for $S \leq S_o$ and by lemma 2.1, $S_1 + x < S_o$ and $\hat{S}_1 + \hat{x} < S_o$ then

$$\frac{L(S_1 + x) - L(S_1 + \hat{x})}{x - \hat{x}} \geq -1$$

implies

$$Q_F = L(S_1 + x) + x \geq L(S_1 + \hat{x}) + \hat{x} \geq L(\hat{S}_1 + \hat{x}) + \hat{x} = \hat{Q}_F \ .$$

And by induction the lemma is proved.

q.e.d.

Lemma 2.5: Let $L(S)$ and sets I and II have the same properties as in lemma 2.4. Let $\nu \geq 1$. Denote by $Q_{F_{n,\nu}}$ and $\hat{Q}_{F_{n,\nu}}$ the total field life by FIFO issuance of the n items to the ν demand sources with the items from sets I and II respectively. Then

$$Q_{F_{n,\nu}} \geq \hat{Q}_{F_{n,\nu}} \ .$$

Proof of Lemma 2.5: Using lemma 2.3 the following table for the assignment of the n items to the ν demand sources can be constructed

Demand Source	Set I	Set II	Total field life contributed by M_i: Set I	Set II
M_1	$[S_n, \dots, S_{n-k\nu}]$	$[\hat{S}_n, \dots, \hat{S}_{n-k\nu}]$	x_1	$\hat{x}_1$
M_2	$[S_{n-1}, \dots, S_{n-k\nu-1}]$	$[\hat{S}_{n-1}, \dots, \hat{S}_{n-k\nu-1}]$	x_2	$\hat{x}_2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
M_ν	$[S_{n-\nu+1}, \dots, S_{n-(k+1)\nu+1}]$	$[\hat{S}_{n-\nu+1}, \dots, \hat{S}_{n-(k+1)\nu+1}]$	x_ν	$\hat{x}_\nu$

where the subscripts on the S's are such that $n - k\nu - i \geq 1$ for all $k = 0, 1, \dots$ and $i = 0, 1, \dots, \nu - 1$, i.e., the inventory is exhausted. Note that lemma 2.3 tells us that the subscripts on the items for a particular demand source are the same for both sets I and II. Hence the items assigned to M_j from sets I and II obey the conditions

(i) $S_{n-k\nu-j+1} \leq \hat{S}_{n-k\nu-j+1}$ for all $k = 0, 1, 2, \ldots$

(ii) there are the same number of items assigned to M_j from set I as there is from set II.

But these conditions hold for all M_j, $j = 1, \ldots, \nu$. Hence by lemma 2.4 $x_j \geq \hat{x}_j$ for all $j = 1, \ldots, \nu$ and

$$Q_{F_{n,\nu}} = \sum_{j=1}^{\nu} x_j \geq \sum_{j=1}^{\nu} \hat{x}_j = \hat{Q}_{F_{n,\nu}} .$$

q.e.d.

Zehna [11] stated that for $L(S)$ concave and differentiable with $0 \geq L'(S) \geq -1$ for $0 \leq S \leq S_o$ when there are $\nu = 2$ demand sources and either $n = 3$ or $n = 4$ items in the stockpile, then FIFO is the optimal issuing policy. He did not present the proof of this statement and since these results are essential to the proof of Theorem 2.6, the proof is presented here.

Lemma 2.6: Let $L(S)$ be a concave function with $L^{-}(S) \geq -1$ for $0 < S \leq S_o$. Let $\nu = 2$ and $n = 3$ or $n = 4$. Then FIFO is the optimal issue policy.

Proof of Lemma 2.6: The proof will proceed by the elimination of all non-FIFO policies. By Lieberman [9] Theorem 3, it is only necessary to consider allocations to each demand source, M_1 and M_2, which are FIFO within the allocation. Hence the possible policies are:

$n = 4$	$n = 3$
$1 = [S_4 \,;\, S_3, S_2, S_1]$	$A = [S_3 \,;\, S_2, S_1]$
$2 = [S_4, S_3 \,;\, S_2, S_1]$	$B = [S_2 \,;\, S_3, S_1]$
$3 = [S_4, S_2 \,;\, S_3, S_1]$	$C = [S_1 \,;\, S_3, S_2]$
$4 = [S_4, S_1 \,;\, S_3, S_2]$	
$5 = [S_3 \,;\, S_4, S_2, S_1]$	
$6 = [S_2 \,;\, S_4, S_3, S_1]$	
$7 = [S_1 \,;\, S_4, S_3, S_2]$	

Policies 1, 5, and A may be eliminated immediately since they are not feasible (they contradict assumption (6)'). A general result is now proved which eliminates policies 4, 7, and C. Consider the two policies $I = [A, S_2 \,;\, B, S_1]$ and $II = [A, S_1 \,;\, B, S_2]$ where A and B are any items in FIFO order. Let y and x denote the total field life of the items represented by A and B respectively. It will be shown that for $x \geq y \geq 0$

$$Q_I = L(S_2 + y) + y + L(S_1 + x) + x \geq L(S_1 + y) + y + L(S_2 + x) + x = Q_{II} . \qquad (2.2.17)$$

If $x = y$ then (2.2.17) holds with equality. If $x > y$ then since $S_2 + y < S_o$ by lemma 2.1 and since $L(\cdot)$ is concave for $S \leq S_o$ then

$$\frac{L(S_2 + y) - L(S_1 + y)}{S_2 - S_1} \geq \frac{L(S_2 + x) - L(S_1 + x)}{S_2 - S_1} .$$

With $S_2 > S_1$ we then have $L(S_2 + y) + L(S_1 + x) \geq L(S_2 + x) + L(S_1 + y)$ and (2.2.17) holds for all $x \geq y$.

Now in policy 4 $A = S_4$, $B = S_3$ and $L(S_4) \leq L(S_3)$

in policy 7 $A = \phi$, $B = S_4, S_3$ and $0 < L(S_4) + L(S_3 + L(S_4))$

in policy C $A = \phi$, $B = S_3$ and $0 < L(S_3)$.

We apply the above result for $Q_I \geq Q_{II}$ and see that policy 4 is dominated by policy 3, policy 7 is dominated by policy 6, and policy C is dominated by policy B.

Thus for $n = 3$, A and C have been eliminated, hence B is optimal but by lemma 2.3, policy B is FIFO.

For $n = 4$ by lemma 2.3, policy 3 is FIFO. It is necessary to show that policy 3 dominates policies 2 and 6. That is, show

$$L(S_4) + L(S_2 + L(S_4)) + L(S_3) + L(S_1 + L(S_3)) \geq L(S_4) + L(S_3 + L(S_4)) + L(S_2) + L(S_1 + L(S_2)) \tag{2.2.18}$$

and

$$L(S_4) + L(S_2 + L(S_4)) + L(S_3) + L(S_1 + L(S_3)) \geq L(S_4) + L(S_3 + L(S_4)) + L(S_1 + L(S_4) + L(S_3 + L(S_4))) + L(S_2). \tag{2.2.19}$$

For (2.2.18) since $S_3 + L(S_4) < S_o$ and $S_1 + L(S_2) < S_o$ by lemma 2.1 and since $L(\cdot)$ is concave for $S \le S_o$ then

$$\frac{L(S_3 + L(S_4)) - L(S_2 + L(S_4))}{S_3 - S_2} \le \frac{L(S_3) - L(S_2)}{S_3 - S_2}$$

implies

$$L(S_3 + L(S_4)) + L(S_2) \le L(S_2 + L(S_4)) + L(S_3) \ . \tag{2.2.20}$$

Furthermore since $L(\cdot)$ is nonincreasing

$$L(S_4) + L(S_1 + L(S_2)) \le L(S_4) + L(S_1 + L(S_3)) \ . \tag{2.2.21}$$

Combining (2.2.20) and (2.2.21) we obtain (2.2.18). For (2.2.19) since $L(\cdot)$ is nonincreasing and by lemma 2.2

$$L(S_3) \le L(S_4) + L(S_3 + L(S_4))$$

implies

$$L(S_1 + L(S_3)) \ge L(S_1 + L(S_4) + L(S_3 + L(S_4))) \ . \tag{2.2.22}$$

Combining (2.2.22) and (2.2.20) we obtain (2.2.19). Hence policy 3, which is FIFO, dominates policies 2 and 6. FIFO is optimal for $n = 4$.

q.e.d.

In the next section we will use some of the foregoing lemmas and corollaries to prove some interesting results on optimal inventory depletion policies when $\nu > 1$.

2.3 Bounds on the Optimal Policy

As Zehna [11] points out, the extension of the results for $\nu = 1$ to the case $\nu \geq 1$ when $L(S)$ is concave nonincreasing is not a simple matter. He gives a counterexample to show that such an extension is not possible in general. However, for the particular case $L(S) = aS + b$, $(b > 0 > a > -1)$, for $0 \leq S \leq S_o$, the results for $\nu = 1$ and $\nu \geq 1$ coincide, viz. FIFO is optimal in both cases.

Presented below are a set of theorems which provide upper and lower bounds on the optimal policy when $\nu > 1$ and $L(S)$ is concave non-increasing for $S \leq S_o$. These bounds for the optimal policy coincide with the bounds for the FIFO policy for the same n items and $\nu > 1$. And since not all policies are included in these bounds, the optimal policy and the FIFO policy are "close" in the sense that the difference between the optimal policy and the FIFO policy cannot exceed the difference between their common upper and lower bounds.

Since $L(S)$ takes the same form for all of the theorems and lemmas of this section we will say:

$L(S)$ has property Ω if $L(S)$ is a concave nonincreasing function for all S such that $0 \leq S \leq S_o$ and $L^-(S) \geq -1$ for $0 < S \leq S_o$.

Theorem 2.1: Let $L(S)$ have property Ω. Let $\nu \geq 1$. Denote by $Q^*_{n,\nu}$, the total field life obtained from the n items in the stockpile when the number of demand sources is ν and when an optimal issuing policy is followed. Then

$$Q^*_{n,\nu} \leq Q^*_{n,\nu+1} \quad \text{for any } \nu = 1, \ldots, n-1 .$$

Proof of Theorem 2.1: Let the optimal policy which achieves $Q^*_{n,\nu}$ be denoted by $A = [S_{i_{11}}, S_{i_{12}}, \ldots, S_{i_{1j_1}}; S_{i_{21}}, \ldots, S_{i_{2j_2}}; \ldots; S_{i_{\nu 1}}, \ldots, S_{i_{\nu j_\nu}}]$. Now since $n > \nu$ then at least one of the subscripts $j_1, j_2, \ldots, j_\nu$ is an integer strictly greater than 1 (i.e., at least one demand source must have two or more items assigned to it). Let us say $j_k > 1$. Then the total field life contributed by demand source M_k to the total field life $Q^*_{n,\nu}$ is given by

$$Q_{M_k} = L(S_{i_{k1}}) + L(S_{i_{k2}} + L(S_{i_{k1}})) + \cdots + L(S_{i_{kj_k}} + L(S_{i_{k1}}) + \cdots) . \tag{2.3.1}$$

Now consider the following issuing policy $B_{\nu+1}$ for the case of $\nu + 1$ demand sources:

> Issue the same items in the same order to all demand sources M_i for all $i \neq k$ as are issued to them when policy A is followed. Issue item $S_{i_{kj_k}}$ to demand source $M_{\nu+1}$ and issue the remaining $j_k - 1$ items to demand source M_k in the same order as under policy A.

Let $Q_{B_{n,\nu+1}}$ denote the total field life obtained from policy $B_{\nu+1}$. We will show $Q_{B_{n,\nu+1}} \geq Q^*_{n,\nu}$.

Now the total field life contributed by demand sources M_i for all $i \neq k$ is the same for both policy A and policy $B_{\nu+1}$. Hence we only need to examine the field life contributed by M_k and $M_{\nu+1}$. Let

$$x = L(S_{i_{k1}}) + L(S_{i_{k2}} + L(S_{i_{k1}})) + \cdots + L(S_{i_{kj_k-1}} + L(S_{i_{k1}}) + \cdots) \tag{2.3.2}$$

then

$$Q_{M_k} = x + L(S_{i_k j_k} + x) \tag{2.3.3}$$

by using (2.3.1). We must show

$$x + L(S_{i_k j_k}) \geq Q_{M_k} \tag{2.3.4}$$

but $L(\cdot)$ is nonincreasing hence $L(S_{i_k j_k}) \geq L(S_{i_k j_k} + x)$ since $x > 0$. Therefore (2.3.4) holds. But x is the field life contributed by M_k and $L(S_{i_k j_k})$ is the field life contributed by $M_{\nu+1}$ under policy $B_{\nu+1}$.

Therefore

$$Q_{B_{n,\nu+1}} = \sum_{\substack{i=1 \\ i \neq k}}^{\nu} Q_{M_i} + x + L(S_{i_k j_k})$$

$$\geq \sum_{i=1}^{\nu} Q_{M_i} = Q^*_{n,\nu} .$$

Now $Q^*_{n,\nu+1}$ is the optimal policy for $\nu + 1$ demand sources, hence

$$Q^*_{n,\nu+1} \geq Q_{B_{n,\nu+1}} \geq Q^*_{n,\nu} .$$

q.e.d.

Theorem 2.2: Let $L(S)$ have property Ω . Let $\nu \geq 1$. Then when the FIFO issuing policy is followed

$$Q_{F_{n,\nu}} \leq Q_{F_{n,\nu+1}} \quad \text{for any} \quad \nu = 1, \ldots , n-1 .$$

Proof of Theorem 2.2: By lemma 2.1 (applied to each demand source separately) the FIFO issuance of the n items in the stockpile results in each item having positive field life on issuance under either $F_{n,\nu}$ or $F_{n,\nu+1}$. Furthermore in any FIFO ordering of the n items for any $\nu \leq n$ there are then exactly n terms $L(S_i + \cdots)$ for $i = 1, \ldots , n$. Hence there is a one - one correspondence between the terms in $Q_{F_{n,\nu}}$ and $Q_{F_{n,\nu+1}}$ where this correspondence is established on the basis of the index letter i for $L(S_i + \cdots)$ and $i = 1, \ldots , n$. Now using lemma 2.3

$$\begin{aligned} Q_{F_{n,\nu}} = {} & L(S_n) + L(S_{n-\nu} + L(S_n)) + \cdots & M_1 \\ & + L(S_{n-1}) + L(S_{n-\nu-1} + L(S_{n-1})) + \cdots & M_2 \\ & + \\ & \quad \ddots & \vdots \\ & \qquad + L(S_{n-\nu+1}) + L(S_{n-2\nu+1} + L(S_{n-\nu+1})) + \cdots & M_\nu \end{aligned} \tag{2.3.5}$$

$$
\begin{aligned}
Q_{F_{n,\nu+1}} = {} & L(S_n) + L(S_{n-\nu-1} + L(S_n)) + \cdots && M_1 \\
& + L(S_{n-1}) + L(S_{n-\nu-2} + L(S_n)) + \cdots && M_2 \\
& + \\
& \quad \ddots && \vdots \\
& \qquad + L(S_{n-\nu+1}) + L(S_{n-2\nu} + L(S_{n-\nu+1})) + \cdots && M_\nu \\
& \qquad\quad + L(S_{n-\nu}) + L(S_{n-2\nu-1} + L(S_{n-\nu})) + \cdots && M_{\nu+1}
\end{aligned}
\tag{2.3.6}
$$

Now choose any $L(S_i + x_i)$ for $i = 1, \ldots, n$ belonging to $Q_{F_{n,\nu}}$ and the corresponding $L(S_i + y_i)$ for $i = 1, \ldots, n$ belonging to $Q_{F_{n,\nu+1}}$. We will show

$$L(S_i + y_i) \geq L(S_i + x_i) \quad \text{for all} \quad i = 1, \ldots, n ;$$

but since $L(\cdot)$ is nonincreasing, it is only necessary to show $x_i \geq y_i$ for all $i = 1, \ldots, n$.

<u>Case 1</u>: $i \in \{n - \nu, n - \nu + 1, \ldots, n\}$

Then $y_i = 0$ and since $x_i \geq 0$ we have $x_i \geq y_i$ (2.3.7)

<u>Case 2</u>: $1 \leq i \leq n - \nu - 1$

Then

$$x_i = L(S_{i+t\nu}) + L(S_{i+(t-1)\nu} + L(S_{i+t\nu})) + \cdots + L(S_{i+\nu} + \cdots)$$

$$y_i = L(S_{i+s(\nu+1)}) + L(S_{i+(s-1)(\nu+1)} + L(S_{i+s(\nu+1)})) + \cdots + L(S_{i+\nu+1} + \cdots)$$

where these equations follow from lemma 2.3. Now $s \leq t$ since by lemma 2.3 every ν^{th} item is assigned to the j^{th} demand source (say M_j receives S_i) under $Q_{F_{n,\nu}}$ and every $(\nu + 1)^{st}$ item is assigned under $Q_{F_{n,\nu+1}}$. Hence when the $F_{n,\nu}$ policy is followed, the demand source which receives S_i will have already received more (or equal) items than the demand source which receives S_i under $F_{n,\nu+1}$. Hence x_i and y_i have the following policies

$$F_{x_i} = [S_{i+t\nu}, S_{i+(t-1)\nu}, \cdots , S_{i+\nu}]$$

$$F_{y_i} = [S_{i+s(\nu+1)}, S_{i+(s-1)(\nu+1)}, \cdots , S_{i+\nu+1}] .$$

But

$$\begin{aligned}
i + \nu &< i + \nu + 1 &&\Rightarrow S_{i+\nu} < S_{i+\nu+1} \\
i + 2\nu &< i + 2(\nu + 1) &&\Rightarrow S_{i+2\nu} < S_{i+2(\nu+1)} \\
&\vdots && \vdots \qquad \vdots \\
i + s\nu &< i + s(\nu + 1) &&\Rightarrow S_{i+s\nu} < S_{i+s(\nu+1)} .
\end{aligned} \qquad (2.3.8)$$

Now consider the FIFO policy of issuing the s items $S_{i+\nu}, \cdots , S_{i+s\nu}$ and denote this policy by A i.e.,

$$A = [S_{i+s\nu}, S_{i+(s-1)\nu}, \cdots , S_{i+\nu}] .$$

Now by (2.3.8) and lemma 2.4

$$Q_A \geq y_i$$

where Q_A is the field life from policy A. Furthermore, since $s \leq t$ then by lemma 2.2

$$Q_A \leq x_i$$

Thus $x_i \geq y_i$. And since the choice of $L(S_i + x_i)$ was arbitrary for $1 \leq i \leq n - \nu - 1$

$$x_i \geq y_i \quad \text{for all } i \text{ with } 1 \leq i \leq n - \nu - 1 . \tag{2.3.9}$$

Combining (2.3.7) and (2.3.9) we have

$$x_i \geq y_i \qquad \text{for all } i = 1, \ldots , n ,$$

therefore

$$L(S_i + x_i) \leq L(S_i + y_i) \qquad \text{for all } i = 1, \ldots , n$$

and

$$Q_{F_{n,\nu}} = \sum_{i=1}^{n} L(S_i + x_i) \leq \sum_{i=1}^{n} L(S_i + y_i) = Q_{F_{n,\nu+1}} .$$

q.e.d.

<u>Theorem 2.3</u>: Let $L(S)$ have property Ω . Let $\nu \geq 1$. If one item is added to the initial stockpile of n items prior to the issuance of any of the items, then

$$Q_{F_{n,\nu}} \leq Q_{F_{n+1,\nu}}$$

when the FIFO issuing policy is followed.

<u>Proof of Theorem 2.3</u>: Before beginning the proof it should be noted that for $\nu = 1$, this theorem reduces to lemma 2.2.

Let S_{n+1} denote the initial age of the new item. We consider three cases:

<u>Case 1</u> $S_{n+1} < S_1$ and say S_1 is assigned to M_j for some $j \in \{1, \ldots , \nu\}$. Then by lemma 2.3

$$\begin{aligned} Q_{F_{n,\nu}} = {} & L(S_n) + L(S_{n-\nu} + L(S_n)) + \cdots \\ & + L(S_{n-1}) + L(S_{n-\nu-1} + L(S_{n-1})) + \cdots \\ & + L(S_{n-2}) + L(S_{n-\nu-2} + L(S_{n-2})) + \cdots \\ & + \\ & \ddots \\ & + L(S_{n-j+1}) + \cdots \quad + L(S_1 + L(S_{n-j+1}) + \cdots) \\ & + \\ & \ddots \\ & + L(S_{n-\nu+1}) + \cdots \end{aligned} \tag{2.3.12}$$

$$
\begin{aligned}
Q_{F_{n+1,\nu}} = {} & L(S_n) + \cdots \\
& + L(S_{n-1}) + \cdots \\
& \quad + \\
& \quad \ddots \\
& \qquad + L(S_{n-j+1}) + \cdots \qquad \cdots + L(S_1 + L(S_{n-j+1}) + \cdots) \\
& \qquad + L(S_{n-j}) + \cdots \qquad \cdots + L(S_{n+1} + L(S_{n-j}) + \cdots) \\
& \qquad\quad + \\
& \qquad\quad \ddots \\
& \qquad\qquad + L(S_{n-\nu+1}) + \cdots \ .
\end{aligned}
\tag{2.3.13}
$$

and

$$
Q_{F_{n+1,\nu}} - Q_{F_{n,\nu}} = L(S_{n+1} + L(S_{n-j}) + \cdots) > 0
$$

by lemma 2.1. Therefore $Q_{F_{n+1,\nu}} > Q_{F_{n,\nu}}$ for this case.

Case 2 $\qquad S_n < S_{n+1} < S_o$

$Q_{F_{n,\nu}}$ is still given by (2.3.12); however $Q_{F_{n+1,\nu}}$ now becomes

$$
\begin{aligned}
Q_{F_{n+1,\nu}} = {} & L(S_{n+1}) + L(S_{n-\nu+1} + L(S_{n+1})) + \cdots \\
& + L(S_n) + L(S_{n-\nu} + L(S_n)) + \cdots \\
& \quad + L(S_{n-1}) + L(S_{n-\nu-1} + L(S_{n-1})) + \cdots \\
& \qquad + \\
& \qquad \ddots + L(S_{n-j+1}) + L(S_{n-\nu-j+1} + L(S_{n-j+1})) + \cdots \\
& \qquad\quad + \\
& \qquad\quad \ddots \\
& \qquad\qquad + L(S_{n-\nu+2}) + L(S_{n-2\nu+2} + L(S_{n-\nu+2})) + \cdots \ .
\end{aligned}
$$

And

$$Q_{F_{n+1,\nu}} - Q_{F_{n,\nu}} = L(S_{n+1}) + L(S_{n-\nu+1} + L(S_{n+1})) + \cdots$$

$$- [L(S_{n-\nu+1}) + L(S_{n-2\nu+1} + L(S_{n-\nu+1})) + \cdots] \geq 0$$

by lemma 2.2 since $Q_{F_{n+1,\nu}} - Q_{F_{n,\nu}}$ represents the difference of the two policies

$$A = [S_{n+1}, S_{n-\nu+1}, S_{n-2\nu+1}, \cdots , S_{n-k\nu+1}, \cdots]$$

$$B = [\qquad S_{n-\nu+1}, S_{n-2\nu+1}, \cdots , S_{n-k\nu+1}, \cdots]$$

where B has the same items as A after A has issued its first item S_{n+1} .

Therefore

$$Q_{F_{n+1,\nu}} \geq Q_{F_{n,\nu}} .$$

<u>Case 3</u> $\qquad S_i < S_{n+1} < S_{i+1}$ for any $i = 1, \ldots , n - 1$

Then for items S_n down through S_{i+1} the two total field life functions are identical. Let x_j denote the total field life contributed by M_j $(j = 1, \ldots , \nu)$ down through item S_{i+1} . Without loss of generality let S_{n+1} be assigned to M_j . Now we can rewrite (2.3.12) in the following manner:

$$
\begin{aligned}
Q_{F_{n,\nu}} = x_1 &+ L(S_{n-t\nu} + x_1) + \cdots \\
&+ x_2 + L(S_{n-t\nu-1} + x_2) + \cdots \\
&+ \ddots \\
&+ x_j + L(S_{n-(t-1)\nu-j+1} + x_j) + \cdots \\
&+ x_{j+1} + L(S_{n-(t-1)\nu-j} + x_{j+1}) + \cdots \\
&+ \ddots \\
&+ x_\nu + L(S_{n-t\nu+1} + x_\nu) + \cdots .
\end{aligned}
$$

where $n - t\nu < i + 1$, and $S_{n-(t-1)\nu-j+1} \equiv S_i$ when numbering from above.

And

$$
\begin{aligned}
Q_{F_{n+1,\nu}} = x_1 &+ L(S_{n-t\nu+1} + x_1) + \cdots \\
&+ x_2 + L(S_{n-t\nu} + x_2) + \cdots \\
&+ \ddots \\
&+ x_j + L(S_{n+1} + x_j) + \cdots \\
&+ x_{j+1} + L(S_{n-(t-1)\nu-j+1} + x_{j+1}) + \cdots \\
&+ x_{j+2} + L(S_{n-(t-1)\nu-j} + x_{j+2}) + \cdots \\
&+ \ddots \\
&+ x_\nu + L(S_{n-(t-1)\nu-\nu+2} + x_\nu) + \cdots
\end{aligned}
$$

By induction we will prove $Q_{F_{n+1,\nu}} \geq Q_{F_{n,\nu}}$ for this case. Let $S_i = S_1$ then $S_{n-(t-1)\nu-j+1} \equiv S_1$ and

$$Q_{F_{n+1,\nu}} - Q_{F_{n,\nu}} = L(S_{n+1} + x_j) + L(S_1 + x_{j+1}) - L(S_1 + x_j) \ . \quad (2.3.14)$$

Now by the definition of the x_i's and since FIFO is being followed then by lemma 2.3

$$x_j \leq x_{j+1} \leq x_{j+2} \leq \cdots \leq x_\nu \leq x_1 \leq \cdots \leq x_{j-1} \ . \qquad (2.3.15)$$

If $x_j = x_{j+1}$ then (2.3.14) has

$$Q_{F_{n+1,\nu}} - Q_{F_{n,\nu}} = L(S_{n+1} + x_j) > 0 \quad \text{and} \quad Q_{F_{n+1,\nu}} > Q_{F_{n,\nu}} \ .$$

If $x_j < x_{j+1}$, since $S_{n+1} + x_{j+1} < S_o$ by lemma 2.1 and since $L(\cdot)$ is concave for $S \leq S_o$ then

$$\frac{L(S_{n+1} + x_{j+1}) - L(S_{n+1} + x_j)}{x_{j+1} - x_j} \leq \frac{L(S_1 + x_{j+1}) - L(S_1 + x_j)}{x_{j+1} - x_j}$$

implies

$$L(S_{n+1} + x_j) + L(S_1 + x_{j+1}) \geq L(S_{n+1} + x_{j+1}) + L(S_1 + x_j)$$
$$> L(S_1 + x_j) \ .$$

Therefore in (2.3.14) we have

$$Q_{F_{n+1,\nu}} > Q_{F_{n,\nu}} \ .$$

Now assume $Q_{F_{n+1,\nu}} - Q_{F_{n,\nu}} \geq 0$ for $S_i = S_k$ (i.e., $S_k < S_{n+1} < S_{k+1}$) and it will be proved true for $S_i = S_{k+1}$ (i.e., $S_{k+1} < S_{n+1} < S_{k+2}$) for $k = 1, 2, \ldots , n-2$.

Now for $S_i = S_{k+1}$ we have

$$Q_{F_{n+1,\nu}} - Q_{F_{n,\nu}} = L(S_{n+1} + x_j) + L(S_{k+1} + x_{j+1}) + L(S_k + x_{j+2}) + \cdots + L(S_1 + x_{m+1} + \cdots) - [L(S_{k+1} + x_j) + L(S_k + x_{j+1}) + \cdots + L(S_1 + x_m + \cdots)] \tag{2.3.16}$$

where we assume that S_1 is assigned to M_{m+1} under $F_{n+1,\nu}$.

Now using (2.3.15) and since $S_{n+1} + x_{j+1} < S_o$ by lemma 2.1 and $L(\cdot)$ is concave for $S \leq S_o$ then if $x_{j+1} > x_j$, then

$$\frac{L(S_{n+1} + x_{j+1}) - L(S_{n+1} + x_j)}{x_{j+1} - x_j} \leq \frac{L(S_{k+1} + x_{j+1}) - L(S_{k+1} + x_j)}{x_{j+1} - x_j} .$$

This implies

$$L(S_{n+1} + x_j) + L(S_{k+1} + x_{j+1}) \geq L(S_{k+1} + x_j) + L(S_{n+1} + x_{j+1}) . \tag{2.3.17}$$

If $x_{j+1} = x_j$, then (2.3.17) holds with equality. Now adding the same quantities to both sides of (2.3.17) we obtain

$$L(S_{n+1} + x_j) + L(S_{k+1} + x_{j+1}) + L(S_k + x_{j+2}) + \cdots + L(S_1 + x_{m+1} + \cdots)$$

$$\geq L(S_{k+1} + x_j) + L(S_{n+1} + x_{j+1}) + L(S_k + x_{j+2}) + \cdots + L(S_1 + x_{m+1} + \cdots).$$

(2.3.18)

But by the inductive assumption

$$L(S_{n+1} + x_{j+1}) + L(S_k + x_{j+2}) + L(S_{k-1} + x_{j+3}) + \cdots + L(S_1 + x_{m+1} + \cdots)$$

$$\geq L(S_k + x_{j+1}) + L(S_{k-1} + x_{j+2}) + \cdots + L(S_1 + x_m + \cdots)$$

(2.3.19)

where the left side of (2.3.19) is just the right side of (2.3.18) after omitting $L(S_{k+1} + x_j)$. Hence we can write the new inequality using (2.3.18) and (2.3.19)

$$L(S_{n+1} + x_j) + L(S_{k+1} + x_{j+1}) + L(S_k + x_{j+2}) + \cdots + L(S_1 + x_{m+1} + \cdots)$$

$$\geq L(S_{k+1} + x_j) + L(S_k + x_{j+1}) + L(S_{k-1} + x_{j+2}) + \cdots + L(S_1 + x_m + \cdots).$$

(2.3.20)

However, (2.3.20) is precisely what we want to show for (2.3.16). Therefore $Q_{F_{n+1,\nu}} - Q_{F_{n,\nu}} \geq 0$. And by induction case 3 has been proved. Combining cases 1, 2, and 3 the theorem is proved.

q.e.d.

Theorem 2.4: Let $L(S)$ have property Ω . Let $\nu \geq 1$. If one item is added to the initial stockpile of n items prior to the issuance of any of the items, then

$$Q^*_{n,\nu} \leq Q^*_{n+1,\nu}$$

when an optimal issuing policy is followed.

Proof of Theorem 2.4: Let ψ_n denote the optimal policy which yields total field life $Q^*_{n,\nu}$. Let A be the policy where the original n items are issued according to ψ_n and the new item, S_{n+1} , is issued last, to the demand source which first finishes its consumption of its items under ψ_n. Then if the field life of item S_{n+1} at the time of issuance is denoted by x we have $x \geq 0$, and

$$Q^*_{n,\nu} \leq Q^*_{n,\nu} + x = Q_{A_{n+1,\nu}} \leq Q^*_{n+1,\nu} .$$

q.e.d.

Before presenting the next theorem, we should point out an interesting extension of Theorems 2.3 and 2.4.

Corollary 2.4.1: Let L(S) have property Ω . Let $\nu \geq 1$. If $M \geq 1$ items are added to the initial stockpile of n items prior to the issuance of any of the items, then

(i) $Q_{F_{n+M,\nu}} \geq Q_{F_{n,\nu}}$ if the FIFO issuing policy is followed

(ii) $Q^*_{n+M,\nu} \geq Q^*_{n,\nu}$ if an optimal issuing policy is followd.

Proof of Corollary 2.4.1: We will just prove (i) since the proof for (ii) follows *mutatis mutandis*.

In Theorem 2.3 we have already proved the corollary true for $M = 1$. Assume the corollary is true for $M > 1$ and consider adding $M + 1$ items to the stockpile. Ignoring item S_{M+1} temporarily, the total field life of the remaining items satisfies $Q_{F_{n+M,\nu}} \geq Q_{F_{n,\nu}}$ by the inductive hypothesis. Then adding S_{M+1} can only increase the total field life by $M = 1$ hence by Theorem 2.3

$$Q_{F_{n+M+1,\nu}} \geq Q_{F_{n+M,\nu}} \geq Q_{F_{n,\nu}} .$$

q.e.d.

<u>Theorem 2.5</u>: Let $L(S)$ have property Ω. Let $\nu \geq 1$. If $[\frac{1}{2}(n + 1)] \leq \nu \leq n$, then any feasible policy which assigns more than two items to any demand source has a lower total field life than some policy which assigns at most two items to each demand source.

Before beginning the proof of this theorem two things should be pointed out. First, although the theorem doesn't explicitly state the improved policy, the proof does state it. Second, if we call the set "G" the set of all policies which issue at most two items to each demand source, the theorem states that for any feasible policy for issuing the n items, there is a member of G which dominates it. The theorem does not state that this member of G is a feasible policy. Indeed, this may not be the case at all. At this point though, it should be noted that FIFO ϵ G and by lemma 2.3 FIFO is feasible. Theorem 2.6 will show that of all the policies in G, FIFO maximizes the total field life for the n items; hence FIFO is the optimal policy.

<u>Proof of Theorem 2.5</u>: Since ν is an integer which is greater than or equal to $\frac{1}{2}$ the number of items in the stockpile then if i demand sources have $k_i > 2$ items assigned to them, there are at least $\sum_i (k_i - 2)$ demand sources which have only one item assigned to them (since all demand sources must have at least one item by the initial assignment).

We only need to consider one demand source with $k_i > 2$ items and $k_i - 2$ demand sources with only one item each since the same procedure (following) applies to all other demand sources with $k_j > 2$ items assigned to them.

Let $i > 2$ items be assigned to M_k. In particular let these items be denoted by $S_{k_1} < S_{k_2} < \cdots < S_{k_i}$. Let M_j be a demand source with only one item assigned to it.

Let $\psi = [S_{t_i}, \ldots, S_{t_2}, S_{t_1}; S_{j_1}]$ be the part of any arbitrary feasible policy which assigns $S_{t_i}, \ldots, S_{t_2}, S_{t_1}$ to M_k and S_{j_1} to M_{j_1} where $S_{t_i}, \ldots, S_{t_2}, S_{t_1}$ is any permutation of the items $S_{k_1}, \ldots, S_{k_i}$.

We will now show $S_{j_1} < S_{k_2}$. Assume to the contrary that $S_{j_1} > S_{k_2}$. Then since $L(\cdot)$ is nonincreasing $L(S_{j_1}) \leq L(S_{k_2})$. We have two cases:

Case (i): $S_{k_2} \neq S_{t_1}$ then S_{k_2} is issued before item S_{t_1}. Let x be the total field life up to but not including the issuance of item S_{k_2}. If $x = 0$ then $L(S_{j_1}) \leq L(S_{k_2})$ above. If $x > 0$ then

$$\frac{L(S_{k_2} + x) - L(S_{k_2})}{x} \geq -1$$

implies

$$L(S_{j_1}) \leq L(S_{k_2}) \leq L(S_{k_2} + x) + x . \qquad (2.3.21)$$

But (2.3.21) says that policy ψ is infeasible since item S_{j_1} is consumed prior to S_{k_2} hence some S_{t_j} should be assigned to M_{j_1} rather than M_k. This result contradicts the hypothesis that ψ is feasible. Therefore in this case $S_{j_1} < S_{k_2}$.

Case (ii): $S_{k_2} = S_{t_1}$ then $S_{k_1} \neq S_{t_1}$ and item S_{k_1} is issued before item S_{t_1}. Since we are assuming $S_{j_1} > S_{k_2}$ then $S_{j_1} > S_{k_1}$ and $L(S_{j_1}) \leq L(S_{k_1})$. By the same argument as in case (i) above we obtain

$$L(S_{j_1}) \leq L(S_{k_1}) \leq L(S_{k_1} + x) + x$$

and we obtain the contradiction that some S_{t_j} should be issued to M_{j_1} rather than M_k. Hence in this case also $S_{j_1} < S_{k_1} < S_{k_2}$. Thus

$$S_{j_1} < S_{k_2} . \qquad (2.3.22)$$

Now from Lieberman [9] Theorem 3 we have that the FIFO issuance of $S_{k_i}, \ldots, S_{k_1}$ yields a greater total field life than any other permutation such as given by $S_{t_i}, \ldots, S_{t_1}$ in policy ψ. Therefore if we let policy A be

$$A = [S_{k_i}, \ldots, S_{k_2}, S_{k_1} ; S_{j_1}]$$

then

$$Q_A \geq Q_\psi .$$

Now policy A may not be a feasible policy; however since we only wish to show that there exists a policy which belongs to G which is better than ψ, in the sense of greater total field life, we do not need feasibility for A. It will be shown that the policy which belongs to G has field life Q and $Q \geq Q_A$. Thus $Q \geq Q_\psi$.

Now since $S_{j_1} < S_{k_2}$ and $S_{k_1} < S_{k_2}$ we consider two cases.

<u>Case 1</u> $S_{j_1} < S_{k_1}$

Then policy $B = [S_{k_i}, S_{k_{i-1}}, \cdots, S_{k_2}, S_{j_1} ; S_{k_1}]$ results in a greater total field life than policy A. The proof of this statement follows:

Let Q_A and Q_B be the total field life from policy A and policy B respectively.

Let $x = L(S_{k_i}) + \cdots + L(S_{k_2} + L(S_{k_i}) + \cdots)$ then

$$Q_A = x + L(S_{k_1} + x) + L(S_{j_1})$$

$$Q_B = x + L(S_{j_1} + x) + L(S_{k_1}) \; ;$$

we must show $Q_B \geq Q_A$. Now $x > 0$ by lemma 2.1 and $S_{k_1} - S_{j_1} > 0$. Furthermore by lemma 2.1 $x + S_{k_1} < S_o$ and $L(\cdot)$ is concave for $S \leq S_o$ by hypothesis, thus

$$\frac{L(S_{k_1} + x) - L(S_{j_1} + x)}{S_{k_1} - S_{j_1}} \leq \frac{L(S_{k_1}) - L(S_{j_1})}{S_{k_1} - S_{j_1}}$$

which implies

$$L(S_{k_1} + x) + L(S_{j_1}) \leq L(S_{j_1} + x) + L(S_{k_1})$$

hence $Q_B \geq Q_A$.

<u>Case 2</u> $S_{k_1} < S_{j_1}$

Then policy $C = [S_{k_i}, S_{k_{i-1}}, \dots, S_{k_3}, S_{j_1} ; S_{k_2}, S_{k_1}]$ results in a greater total field life than policy A.

Let $y = L(S_{k_i}) + \cdots + L(S_{k_3} + \cdots)$ then

$$Q_A = y + L(S_{k_2} + y) + L(S_{k_1} + y + L(S_{k_2} + y)) + L(S_{j_1})$$

$$Q_C = y + L(S_{j_1} + y) + L(S_{k_2}) + L(S_{k_1} + L(S_{k_2})) .$$

We must show $Q_C \geq Q_A$.

By lemma 2.2 since $y > 0$, $y + L(S_{k_2} + y) \geq L(S_{k_2})$. Now since $L(\cdot)$ is nonincreasing

$$L(S_{k_1} + y + L(S_{k_2} + y)) \leq L(S_{k_1} + L(S_{k_2})) . \qquad (2.3.23)$$

By lemma 2.1, $S_{k_2} + y < S_o$ and since $L(\cdot)$ is concave for $S \leq S_o$ and since $S_{k_2} - S_{j_1} > 0$ then

$$\frac{L(S_{k_2} + y) - L(S_{j_1} + y)}{S_{k_2} - S_{j_1}} \leq \frac{L(S_{k_2}) - L(S_{j_1})}{S_{k_2} - S_{j_1}}$$

which implies

$$L(S_{k_2} + y) + L(S_{j_1}) \leq L(S_{j_1} + y) + L(S_{k_2}) \ . \qquad (2.3.24)$$

Upon combining (2.3.23) and (2.3.24) we have proved

$$Q_C \geq Q_A \ .$$

Note that policy C has reduced the problem by assigning only $i - 1$ items to M_k and 2 items to M_{j_1}. We will now show that there exists a policy D which is better than policy B where D assigns $i - 1$ items to M_k and 2 items to M_{j_1}.

Let $D = [S_{k_i}, S_{k_{i-1}}, \ldots, S_{k_3}, S_{k_1}; S_{k_2}, S_{j_1}]$ where $S_{j_1} < S_{k_1}$ But the same argument as in case 2 above applies since policy B presents the same situation relative to policy D that policy A presents to policy C. Thus $Q_D \geq Q_B$. Hence in either case $S_{j_1} < S_{k_1}$ or $S_{k_1} < S_{j_1}$ we have a policy $Q_D \geq Q_A$ or $Q_C \geq Q_A$ which is better than policy A and which assigns $i - 1$ items to M_k and 2 items to M_{j_1}.

This reduction process continues. If $i - 1 > 2$, we consider demand source M_{j_2} with only one item, S_{j_2} then

$$A^* = [S_{k_i}, S_{k_{i-1}}, \ldots, S_{k_3}, S^*_{k_1}; S_{j_2}]$$

where $S^*_{k_1}$ is either S_{k_1} or S_{j_1} of the first iteration. In the same manner that it was shown that $S_{j_1} < S_{k_2}$, it is also the case that $S_{j_2} < S_{k_3}$ (actually $S_{j_2} < S_{k_2}$ also but this is not necessary at this point). Then policy A^* is dominated by

$$C^* = [S_{k_i}, S_{k_{i-1}}, \ldots, S_{k_4}, S_{j_2}; S_{k_3}, S^*_{k_1}] \quad \text{if} \quad S_{j_2} > S^*_{k_1}$$

or

$$D^* = [S_{k_i}, S_{k_{i-1}}, \ldots, S_{k_4}, S^*_{k_1}; S_{k_3}, S_{j_2}] \quad \text{if} \quad S_{j_2} < S^*_{k_1}$$

which reduces the problem to $i - 2$ items assigned to M_k and 2 items assigned to M_{j_2}.

We must now show that it is better to go from an $i = 3$ problem to an $i = 2$ problem and then by reduction the theorem has been proved.

Let $S_{k_1} < S_{k_2} < S_{k_3}$ be the $i = 3$ items assigned to M_k and let S_{j_1} be the single item assigned to M_j. Now $S_{j_1} < S_{k_2}$ by the same reasoning as given before.

<u>Case 1a</u>: $S_{j_1} > S_{k_1}$ and let $A = [S_{k_3}, S_{k_2}, S_{k_1}; S_{j_1}]$; $B = [S_{k_3}, S_{j_1}; S_{k_2}, S_{k_1}]$

$$Q_A = L(S_{k_3}) + L(S_{k_2} + L(S_{k_3})) + L(S_{k_1} + L(S_{k_3}) + L(S_{k_2} + L(S_{k_3}))) + L(S_{j_1})$$

$$Q_B = L(S_{k_3}) + L(S_{j_1} + L(S_{k_3})) + L(S_{k_2}) + L(S_{k_1} + L(S_{k_2})) .$$

We must show $Q_B \geq Q_A$. Now by lemma 2.2

$$L(S_{k_3}) + L(S_{k_2} + L(S_{k_3})) \geq L(S_{k_2}) ;$$

therefore since $L(\cdot)$ is nonincreasing

$$L(S_{k_1} + L(S_{k_3}) + L(S_{k_2} + L(S_{k_3}))) \leq L(S_{k_1} + L(S_{k_2})) . \qquad (2.3.25)$$

By lemma 2.1 $S_{k_2} + L(S_{k_3}) < S_o$ and since $L(\cdot)$ is concave for $S \leq S_o$

$$\frac{L(S_{k_2} + L(S_{k_3})) - L(S_{j_1} + L(S_{k_3}))}{S_{k_2} - S_{j_1}} \leq \frac{L(S_{k_2}) - L(S_{j_1})}{S_{k_2} - S_{j_1}} .$$

Thus

$$L(S_{k_2} + L(S_{k_3})) + L(S_{j_1}) \leq L(S_{j_1} + L(S_{k_3})) + L(S_{k_2}) \qquad (2.3.26)$$

and combining (2.3.25) and (2.3.26) we have $Q_B \geq Q_A$ as desired.

<u>Case 2a</u> $S_{j_1} < S_{k_1}$

Let $C = [S_{k_3}, S_{k_2}, S_{j_1} ; S_{k_1}]$ then

$$Q_C = L(S_{k_3}) + L(S_{k_2} + L(S_{k_3})) + L(S_{j_1} + L(S_{k_3}) + L(S_{k_2} + L(S_{k_3}))) + L(S_{k_1}).$$

We must show $Q_C \geq Q_A$. By lemma 2.1 $S_{k_1} + L(S_{k_3}) + L(S_{k_2} + L(S_{k_3})) < S_o$ and since $L(\cdot)$ is concave $S \leq S_o$

$$\frac{L(S_{k_1} + L(S_{k_3}) + L(S_{k_2} + L(S_{k_3}))) - L(S_{j_1} + L(S_{k_3}) + L(S_{k_2} + L(S_{k_3})))}{S_{k_1} - S_{j_1}}$$

$$\leq \frac{L(S_{k_1}) - L(S_{j_1})}{S_{k_1} - S_{j_1}}$$

and

$$L(S_{k_1} + L(S_{k_3}) + L(S_{k_2} + L(S_{k_3}))) + L(S_{j_1})$$

$$\leq L(S_{j_1} + L(S_{k_3}) + L(S_{k_2} + L(S_{k_3}))) + L(S_{k_1}) \; . \qquad (2.3.27)$$

Hence by (2.3.27), $Q_C \geq Q_A$. Now let

$$D = [S_{k_3}, S_{k_1} \; ; \; S_{k_2}, S_{j_1}]$$

the proof that $Q_D \geq Q_C$ is the same as given in case 1a only with the proper subscripts interchanged. Hence $Q_D \geq Q_A$, and we have shown a better policy exists where 2 items are assigned to M_k and 2 items are assigned to M_j.

By reduction, the theorem is proved.

q.e.d.

Theorem 2.6: Let $L(S)$ have property Ω. If $[\frac{1}{2}(n + 1)] \leq \nu \leq n$, then FIFO is the optimal issuing policy.

Proof of Theorem 2.6: Note that not only does FIFO ϵ G and FIFO is feasible but also that FIFO issues all n of the items, i.e., none of the items deteriorate to zero in the stockpile.

We will now show that an optimal policy for the conditions given in this theorem also must issue all of the items. This last statement is proved by contradiction. Assume that the optimal policy allows at least one item, say S_j, to expire in the stockpile. Then since $[\frac{1}{2}(n + 1)] \leq \nu \leq n$ there is at least one demand source which receives only one item, say S_i. In addition $S_i < S_j$ or else by lemma 2.1, S_j would have positive field life upon the consumption of S_i, i.e., $S_j + L(S_i) < S_o$, and S_j would then be issued. Thus assume $S_i < S_j$. Now by Lieberman [9] Theorem 3 we have

$$L(S_j) + L(S_i + L(S_j)) \geq L(S_i)$$

where equality holds only if $L'(S) = -1$ over the range of S_i and S_j, and strict inequality holds at all other times. Therefore letting S_j deteriorate to zero in the stockpile can't be optimal. And we obtain a contradiction to the assumption of optimality. But S_j was a general item which deteriorated in the stockpile, thus the contradiction obtained applies to all S_j, and the optimal policy must issue all n items.

Thus the optimal policy as well as the FIFO policy issues all items in the stockpile. Now in looking at all policies in G we can restrict our attention to looking at only those policies which issue all n items. Let $A \in G$ be one of these policies and consider any two demand sources M_i and M_j under policy A.

Case 1 M_i receives S_{i_1}, S_{i_2} with $S_{i_1} < S_{i_2}$

M_j receives S_{j_1}, S_{j_2} with $S_{j_1} < S_{j_2}$

Then if the four items $S_{i_1}, S_{i_2}, S_{j_1}, S_{j_2}$ are not assigned to M_i and M_j according to FIFO, then the total field life can be increased by a FIFO assignment since by lemma 2.6 FIFO is optimal for $n = 4$, $\nu = 2$.

Case 2 M_i receives S_{i_1}

M_j receives S_{j_1}, S_{j_2} with $S_{j_1} < S_{j_2}$

Again, if the three items $S_{i_1}, S_{j_1}, S_{j_2}$ are not assigned to M_i and M_j according to FIFO then the total field life can be increased by a FIFO assignment. By lemma 2.6 FIFO is optimal for $n = 3$, $\nu = 2$.

Case 3 M_i receives S_{i_1}; M_j receives S_{j_1}. Then FIFO is obviously optimal (there is only one policy).

Thus the total field life from all demand sources can be improved until every demand source has a FIFO ordering of its items relative to every other demand source. We will call such an ordering a pairwise-FIFO ordering. It should be noted that any other ordering results in a lower total field life hence pairwise-FIFO is optimal.

We must now show that pairwise-FIFO is the same as FIFO for the total assignment of the n items to the ν demand sources. Assume the items are in pairwise-FIFO order. Now relabel the demand sources such that

$$\begin{array}{llll} M_\nu & \text{has item} & S_n & \text{assigned to it} \\ M_{\nu-1} & \text{has item} & S_{n-1} & \text{assigned to it} \\ \vdots & & \vdots & \\ M_p & \text{has item} & S_{n-i+1} & \text{assigned to it} \\ M_1 & \text{has item} & S_{n-\nu+1} & \text{assigned to it .} \end{array} \tag{2.3.28}$$

This relabelling is possible since no two of the items $S_n, \ldots, S_{n-\nu+1}$ can be assigned to the same demand source under pairwise-FIFO. Now consider the demand source M_p which has the two items S_{n-i+1} and S_{i_1} assigned to it, for any $p = 1, \ldots, \nu$. We must show that $S_{i_1} = S_{n-\nu-i+1}$ then by lemma 2.3 we have a FIFO ordering for the total assignment (since p was arbitrary). The proof of $S_{i_1} = S_{n-\nu-i+1}$ is by contradiction. Assume $S_{i_1} \neq S_{n-\nu-i+1}$.

<u>Case 1</u> $\qquad S_{i_1} > S_{n-\nu-i+1}$

Now from above we know $S_{i_1} < S_{n-\nu+1}$ and since $S_{i_1} > S_{n-\nu-i+1}$ there are at most $(n - \nu) - (n - \nu - i + 1) - 1 = i - 2$ items, with initial life greater than S_{i_1}, which are available for assignment to demand sources $M_\nu, \ldots, M_{p+1}$. $M_\nu, \ldots, M_{p+1}$ are the first $i - 1$ demand sources to consume their initial items. Hence some item $S_{j_1} < S_{i_1}$ must be assigned to one of these $i - 1$ demand sources, say demand source M_{p+j} $(j \geq 1)$. Then the pairwise ordering for M_p and M_{p+j} is $[S_{n-i+1}, S_{i_1}; S_{n-i+1+j}, S_{j_1}]$; but $S_{n-i+1} < S_{n-i+1+j}$ and $S_{i_1} > S_{j_1}$ is not a FIFO ordering, hence we obtain a contradiction to the assumption of pairwise-FIFO. Therefore $S_{i_1} \not> S_{n-\nu-i+1}$.

<u>Case 2</u> $S_{i_1} < S_{n-\nu-i+1}$

As shown above (2.3.28) there are $i - 1$ demand sources with items whose initial ages are greater than S_{n-i+1} . And since $S_{i_1} < S_{n-\nu-i+1}$ there are at least $(n - \nu + 1) - (n - \nu - i + 1) = i$ items such that $S_{n-\nu-i+1+j} > S_{i_1}$ for $j = 0, 1, \ldots, i - 1$ and these items are available for issuance to the first $i - 1$ demand sources requesting items <u>viz</u>. $M_\nu, \ldots, M_{p+1}$. Hence there is at least one $S_{n-\nu-i+1+j}$ which must be issued to one of the M_t where $t = 1, 2, \ldots, i - 1$. But then the issue policy for M_p and M_t is $[S_{n-i+1}, S_{i_1}; S_{n-i+1-k}, S_{n-\nu-i+1+j}]$ where $k = p - t$. But $S_{n-i+1} > S_{n-i+1-k}$ and $S_{i_1} < S_{n-\nu-i+1+j}$ is not a FIFO ordering; hence we obtain a contradiction to pairwise-FIFO. Therefore $S_{i_1} \not< S_{n-\nu-i+1}$.

Combining cases 1 and 2 we have $S_{i_1} = S_{n-\nu-i+1}$ and by lemma 2.3 since p was arbitrary, FIFO is optimal.

q.e.d.