OPTIMAL ISSUING POLICIES IN INVENTORY MANAGEMENT

BY
WILLIAM P. PIERSKALLA

TECHNICAL REPORT NO. 7
August 28, 1965

PREPARED UNDER THE AUSPICES
OF
NATIONAL SCIENCE FOUNDATION GRANT GP-3739

PROGRAM IN OPERATIONS RESEARCH
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

OPTIMAL ISSUING POLICIES IN INVENTORY MANAGEMENT

by

William P. Pierskalla

TECHNICAL REPORT NO. 7

August 28, 1965

PREPARED UNDER THE AUSPICES

OF

NATIONAL SCIENCE FOUNDATION GRANT GP-3739

PROGRAM IN OPERATIONS RESEARCH

STANFORD UNIVERSITY

STANFORD, CALIFORNIA

Table of Contents

Chapter I

Introduction

1.1 Introduction and Characterization of the Model

The general inventory depletion problem can be described as the problem of finding an issue policy which maximizes or minimizes a prescribed function when the inventory itself is changing in quality over time. The change in quality may be either an appreciation or a deterioration of the useful life, the field life, of each item in the inventory as long as the item remains in the stockpile. An issue policy is a selected order of issue of the items in the stockpile when demands for the items are made from the field.

In 1958 Derman and Klein [2] and Lieberman [9] presented some analytic results concerning a more specific formulation of the general model. They obtained optimal policies of the form LIFO, last in first out, or FIFO, first in first out. The advantage of LIFO or FIFO policies is twofold. In practice, they are the most easily understood and most easily implemented policies. Second, they require only a knowledge of the relative ages of the items in the inventory and not their exact ages. However their model contains several restrictive assumptions which limit the application of the results to most real situations.

It is the primary purpose of this work to eliminate and/or modify some of the assumptions of the model and still, whenever possible, obtain LIFO or FIFO as optimal policies.

In order to be more specific as to which assumptions will be changed or removed, it will be advantageous to characterize the model explicitly.

The Model

A. Assumptions

(1) At the beginning of the process, a stockpile has n indivisible identical items of varying ages $S_1 < S_2 < \cdots < S_n$ where $S_1 \geq 0$. The ages S_i are called the initial ages of the items.

(2) Each item has a field life $L(S)$ which is a known non-negative function of the age S of the item upon being issued.

(3) Items are issued successively until either the entire stockpile is depleted or the remaining items in the stockpile have no further useful life, i.e. $L(S) = 0$ for the remaining items.

(4) No penalty or installation costs are associated with the issuance of an item from the stockpile.

(5) New items are never added to the stockpile after the process starts.

(6) An item is issued from the stockpile only when the entire life of the preceding item issued is ended.

(7) At the beginning of the process each item has positive field life, i.e., $L(S_i) > 0$ for all $i = 1, 2, \ldots, n$.

B. Objective

The objective is to find the issue policy which maximizes the total field life of the stockpile. An issue policy which achieves this maximum is called an optimal policy.

Since assumption (2) requires that $L(S)$ be known, this model is called deterministic. Moreover for all of the results in the following chapters, it is assumed that $L(S)$ is a deteriorating field life function. The appreciating field life function has not been studied since the mathematical techniques used in the case of deteriorating $L(S)$ generally carry over to the appreciating case with minor modifications. In addition, appreciating functions do not have the range of applications in the inventory context as do depreciating functions.

Many of the earlier results based on the above model are quite interesting and several of them are used in the subsequent chapters of this work. Some of these earlier results will now be presented.

Derman and Klein [2] stated the first major theorem giving sufficient conditions for LIFO optimality. However as Zehna [11] pointed out the statement was not quite correct. Zehna gave the corrected statement: "If $L(S)$ is a convex monotone function and LIFO is optimal for $n = 2$, then LIFO is optimal for all $n > 2$." Shortly after the Derman and Klein paper, Lieberman [9] presented three theorems which have been of benefit in further development of inventory depletion theory. In particular, his Theorem 3 is used many times in later chapters. The three theorems are:

Theorem 1: (i) If $\frac{dL}{dS} = L'(S) \geq -1$ and (ii) LIFO is an optimal policy when $n = 2$, then LIFO is an optimal policy when $n = 3, 4, \ldots$.

Theorem 2: (i) If $L(S)$ is a convex monotone function or if $L'(S) \geq -1$ and (ii) FIFO is an optimal policy when $n = 2$, then FIFO is an optimal policy for $n = 3, 4, \ldots$.

Theorem 3: (i) If $L'(S) \geq -1$ and (ii) $L(S)$ is a nonincreasing or nondecreasing concave function, then FIFO is an optimal policy.

Zehna proved that Theorem 3 can be generalized to "Suppose $L(S)$ is a concave differentiable function. Then FIFO is an optimal policy for $n \geq 2$ if and only if $L'(S) \geq -1$."

With the Derman-Klein-Zehna theorem above, and the first two theorems of Lieberman, the search for an optimal policy reduced to a search for sufficient conditions when $n = 2$. Bomberger [1], Eilon [7], and Zehna presented many results establishing such sufficient conditions. In addition Zehna presented two theorems for $L'(S) < -1$ which are often quoted in later chapters. His Theorems 2.4 and 2.6 are combined in "If $L(S)$ is a convex or a concave differentiable function and $L'(S) < -1$ for all $S \geq 0$, then LIFO is optimal for $n \geq 2$."

But with the exception of Zehna's Chapters 4 and 5 and Eilon [4], none of the papers have considered removing any of the restrictive assumptions of the model. Looking at the assumptions of the model, it is apparent that there is a broad area of inventory depletion problems which are not covered by the model.

Assumption (6) implicitly assumes that there is only one demand source withdrawing items from the stockpile. Zehna and Eilon [4] independently approached this problem and both proved the result that if $L(S) = aS + b$ for $b > 0 > a > -1$, then for a stockpile of n items FIFO is optimal for one or more demand sources. Zehna also proved (Theorem 4.2) if $L(S)$ is either a convex or a concave differentiable function with $L'(S) < -1$, then LIFO is optimal for two demand sources and (Theorem 4.3) if FIFO (LIFO) is optimal for one and for two demand sources, then FIFO (LIFO) is optimal for more than two demand sources. Moreover Zehna demonstrated that for general $L(S)$ concave and differentiable with slope ≥ -1, FIFO is not always an optimal policy. In Chapter 2, however, we are able to show that when $L(S)$ is any concave nonincreasing function with slope ≥ -1, the FIFO issuing policy and the optimal issuing policy have the same upper and lower bounds for more than one demand source. Since not all policies are included in these bounds, then FIFO can be called a suboptimal policy in the sense that it differs from the optimal policy by not more than their common upper and lower bounds. Furthermore, it is shown that if the number of demand sources is greater than or equal to one half of the number of items in the stockpile, then FIFO is optimal.

All of the previous papers have assumed that there are no penalty costs, such as installation or work stoppage costs, when an item is issued from the stockpile. However, in most real situations, penalty costs are incurred, and if they are significant relative to the useful lives of some of the items in the stockpile, the optimal policy may change when these costs are considered. For example, if the policy which is optimal without penalty costs says to issue many items which have very little useful field life, then the penalty cost could more than outweigh the usefulness of the items. In Chapter 3 assumption (4) is removed and replaced by the assumption that there is a constant penalty cost, p, incurred every time an item is issued from the inventory. Theorems are then proved which reduce the search for the optimal policy from a search of $n!$ policies to the search of n policies. When $L(S)$ is linear an optimal policy can be specified exactly.

The most common assumption about $L(S)$ in the literature and in Chapters 2 and 3 is that $L(S)$ is a convex or concave function. It may be the case, however, that in a real problem $L(S)$ may be a concave function for some period of time and then a convex function after that time. Curves of this type will be called S-shaped curves and in Chapter 4 a special case of the S-shaped curve is examined and optimal policies obtained. The special case is $L(S)$ is concave nonincreasing and strictly positive for all $S \epsilon [0, t]$, $L(S) = L(t) = c$ for all $S \epsilon [t, \infty)$, and the left hand derivative of $L(S)$, $L^-(S)$, has $L^-(S) \geq -1$ for all $S \epsilon (0, t]$. We then show that for some $i = 1, \ldots, n$ the optimal policy is to issue the first i items by FIFO and the remaining $n - i$ items by LIFO. When the concave part of $L(S) = aS + b$ then sufficient conditions are given which locate the specific $i = 1, \ldots, n$ which yields the optimal policy.

Probably the most restrictive assumption of the model is assumption (5). Assumption (5) states that new items are never added to the stockpile after the process starts. This assumption makes the model completely static and not the dynamic representation of a "going concern" which is what most real inventory problems are. In Chapter 5 this assumption is removed and the dynamic inventory depletion model is presented. Here again we consider $L(S)$ concave nonincreasing and $L^{-}(S) \geq -1$, then if FIFO is optimal in the static model, FIFO is optimal when N items are added to the inventory at different times in the future. When $L(S)$ is concave or convex and $L^{-}(S) < -1$, then a policy called generalized-modified-LIFO (GML) is optimal. GML is the policy where LIFO is used until a new item arrives then the new item is immediately issued to the demand source which has the least life remaining on its item currently in consumption.

For Chapters 3, 4, and 5 the results are given for one or more demand sources. Also, in Chapters 2, 3, 4, and 5, we have been concerned with the deterministic model, i.e., $L(S)$ is a known function. Zehna considered the case where the field life of an item is a nonnegative random variable, $X(S)$, dependent on the age, S, of the item upon being issued. In this case the objective function now becomes: maximize the total expected return (utility) of the stockpile. Zehna obtained two theorems for this stochastic model. The first theorem (5.1) holds only when the distribution of $X(S)$ has an increasing mean value function as S increases. The second theorem (5.2) holds when the mean value function is decreasing and since this work is concerned with deteriorating inventories we state his Theorem 5.2: "Suppose for each $S \geq 0$, $X(S)$ has density

$$\frac{1}{[L(S)]^{\alpha+1}\Gamma(\alpha+1)} x^{\alpha} e^{-\frac{x}{L(S)}} \qquad \text{for } x \geq 0$$

where $L(S) = e^{-ks}$, $k > 0$ and integer $\alpha > -1$. Then LIFO is optimal when $n = 2$." The stochastic model presented in Chapter 6 differs somewhat from Zehna's model. In Chapter 6 $X(S)$ can take on any one of a countable number of values, $L_i(S)$ with probability p_i where $i = 1, \ldots, M$, $\sum_{i=1}^{M} p_i = 1$, and $p_i \geq 0$ (M may be replaced by $+\infty$). If $L_i(S) = a_i S + b_i$ where $b_i > 0 > a_i > -1$, $L_i(S) < L_{i+1}(S)$ for all i and any $S < S_o$ (where S_o is defined in the next section), and $L_i(S_o) = L_{i+1}(S_o) = 0$ for all i, then FIFO is optimal for n items in the stockpile and one demand source. This result says that if we know that each item deteriorates according to some linear field life function then even though the specific function is unknown for any item, FIFO is a policy which maximizes the total expected return. If we change $L_i(S) = a_i S + b_i$ above to $L_i(S)$ is concave and differentiable with $0 \geq L_i'(S) > L_{i+1}'(S) \geq -1$ for all i and $S < S_o$, then FIFO is optimal for $n = 2$.

In Chapter 7 we consider the case when there are batches of items of the same age in the stockpile. Eilon [5] considered this problem in regard to the obsolescence of commodities which are subject to deterioration in the stockpile. However he did not consider the batch assumption's effect on the optimality of LIFO or FIFO. In Chapter 7 this latter consideration is made. A general result is proved: If $L(S)$ is continuous and if FIFO (LIFO) is optimal in the case of no batches, then FIFO (LIFO) is optimal when batches are permitted.

1.2 The Truncation Point S_o

Assumption (2) of the model requires that $L(S)$ be a nonnegative function of S. In the case where we assume that $L(S)$ is a concave decreasing function then there is a point, say S_o, such that $L(S) > 0$ for all $S \in [0, S_o)$ and $L(S) \leq 0$ for all $S \geq S_o$. Thus for all $S \geq S_o$, $L(S)$ must be redefined to be identically zero and S_o is a finite truncation point. In another case, e.g., $L(S) = \frac{1}{S}$ for all $S > 0$ then as $S \to +\infty$, $L(S) \to 0$ and $S_o = +\infty$ is called the truncation point.

In general, if $L(S)$ is a decreasing function of S and $L(0) > 0$, then $S_o \leq +\infty$ is a truncation point for $L(S)$ if and only if $S_o = \inf \{S \in [0, \infty) | L(S) \leq 0\}$ and then $L(S)$ is redefined to be

$$L(S) = \begin{cases} L(S) > 0 & \text{for all } S \in [0, S_o) \\ 0 & \text{for all } S \geq S_o \end{cases}$$

(Ref. Zehna [11].)

From a practical point of view it makes little sense to permit $L(S)$ to be arbitrarily large for some S. Hence we will assume that there is some number $k < \infty$ such that $L(S) < k$ for all S of interest. If $L(S) = \frac{1}{S}$, as shown in the example above, we will assume this $L(S)$

applies only to those $S > 0$ such that $L(S) = \frac{1}{S} < k$. Then if a finite number, n, of items are issued by any policy A, the total field life, Q_A, is bounded by $0 < Q_A < nk = K$ for all policies A and any n items $0 \leq S_1 < S_2 < \cdots < S_n$.