A FEEDBACK INVENTORY MODEL OF

A HOSPITAL BLOOD BANK

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ABSTRACT

This paper considers the problem of determining optimal stocking levels in a dynamic inventory system in which a portion of stock issued each period returns to the inventory λ periods later. We assume that demands in successive periods are known with certainty and that a known fraction of the issued inventory which is not used outdates at the end of each period. The fraction of unused stock issued each period is assumed to be a random variable with a known CDF. Costs are charged against shortage and outdating only. This model is suggested by the operation of a hospital blood bank.

I. INTRODUCTION

The problem of efficient management of blood inventories is becoming more important as the demand for health care increases. Since only a small fraction of the population in any region choose to act as regular donors, only a limited supply will be available to meet demands. Hence, one important aspect of the health care delivery system must be the efficient management of blood inventories.

That mathematical inventory models might successfully be applied to inventory management of whole blood appears to have first been recognized by Millard[7]. Most of the relevant work that followed used either computer simulation or heuristic methods to deal with the inventory management problem ([3,5,6,10]).

Recently attempts have been made to develop mathematical models which deal specifically with the characteristics of whole blood. One of the most important characteristics is that blood has a legal lifetime of twenty-one days. Classical inventory models assumed that once acquired, inventory could be held indefinitely. Recent work by Nahmias and Pierskalla[8], Nahmias[9], Fries[4] and Cohen[1] considers the fixed lifetime case and deals with the structure of both optimal and approximate stocking policies.

However, even these models will not be directly applicable at the hospital level. An important feature of the hospital blood bank is that it is likely that once units have been issued to meet demand (once they enter the assigned inventory), a portion of those units will return to the hospital bank some $\lambda \geq 1$ periods later. This results from the fact that blood needs during surgery cannot be estimated exactly and in general more blood is assigned than is actually used. In addition, blood units are tested for compatibility with the patient's blood prior to transfusion. If there is no evidence of an adverse reaction, then the units are said to be crossmatched to the patient and are effectively assigned to the patient in advance. Since aging occurs during the feedback process, this feature tends to contribute significantly to outdating and cannot be ignored. Jennings[6] was among the first to recognize that the feedback process was important, while later work of Cohen and Pierskalla[2] analyzed the relationship between outdates, shortages, the order of issue, ordering policy and the number of periods between issue and return.

The purpose of the current paper is to develop a theoretical inventory model which deals specifically with the feedback mechanism. In order to do so we approximate fixed life perishability by assuming exponential decay for the product over time.

II. NOTATION AND ASSUMPTIONS

As is customary with dynamic programming models, we assume that periods are numbered backwards and that there are a total of $N > 1$ periods in the planning horizon. Hence, period k implies that there are k periods remaining. We adopt the following notation:

x_k = total inventory on hand starting period k

y_k = amount ordered in period k

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β_k = known fraction of the amount returned in period k that does not outdate

I_k = quantity of stock issued in period k

d_k = known demand in period k

α_k = return random variable in period k ($\alpha_k \in [0, 1]$ and represents the fraction of stock issued in period $k+\lambda$ which is returned to inventory)

λ = positive integer indicating the number of elapsed periods from issuing until return

p = per unit shortage (penalty) cost > 0

θ = per unit outdate cost > 0.

We make the following assumptions:

(i) There are only two costs considered: shortage and outdating at p and θ per unit, respectively.
(ii) Demands in successive periods are known with certainty.
(iii) If in period $k+\lambda$, $I_{k+\lambda}$ is issued to meet demand, then $\alpha_k I_{k+\lambda}$ returns to stock in period k (λ periods later).
(iv) A fraction, $(1-\beta_k)$, of the quantity $\alpha_k I_{k+\lambda}$ outdates at the start of period k.
(v) The proportion of stock which re-enters inventory in period k, α_k, is a random variable which assumes values on [0, 1] and has a known C.D.F. F and density f.
(vi) $p > \theta$.
(vii) Excess demand is backlogged.
(viii) All orders are received instantly (zero lead time).

The process dynamics can best be represented by the diagram in FIGURE I below. In period k, the inventory on the shelf, x_k, is determined. Based on both the values of x_k and $I_{k+\lambda}$ (the inventory issued to meet demand λ periods before), an optimal order quantity $y_k(x_k, I_{k+\lambda})$ (denoted simply by y_k) is determined. After the optimal order quantity has been computed and the order placed, the value of the return random variable, α_k, is realized. At this point $\beta_k\alpha_k I_{k+\lambda}$ enters the inventory along with the order y_k so that $x_k + y_k + \beta_k\alpha_k I_{k+\lambda}$ is available to meet the demand, d_k. The amount of stock issued in period k, I_k, will either be all available stock or the demand (including backlog) whichever is smaller.

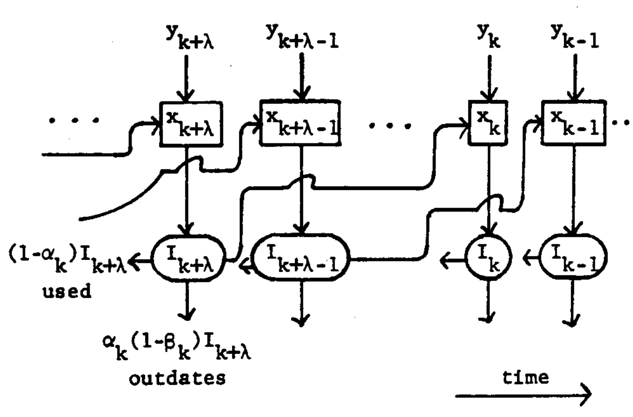

FIGURE I. THE PROCESS DYNAMICS

This leads to the following transfer equations:

$$x_k = x_{k+1} + y_{k+1} + \beta_{k+1}\alpha_{k+1} I_{k+\lambda+1} - d_{k+1}$$

$$I_k = \min[d_k + (-x_k)^+,\ y_k + x_k^+ + \beta_k\alpha_k I_{k+\lambda}]$$

$$\text{for } 1 \le k \le N$$

where $x_k^+ = \max(x_k, 0)$.

III. THE SINGLE PERIOD PROBLEM

The expected cost of outdating and shortage incurred in period k is given by

$$L(x_k, I_{k+\lambda}, y_k) = p\int_0^1 (d_k - x_k - y_k - \beta_k a I_{k+\lambda})^+ f(a)da + \theta\int_0^1 (1-\beta_k) a I_{k+\lambda} f(a)da$$

$$= p\int_0^1 (d_k - x_k - y_k - \beta_k a I_{k+\lambda})^+ f_k(a)da + \sigma_k I_{k+\lambda}$$

where $\sigma_k = \theta(1-\beta_k)E(\alpha_k)$.

The following result, which gives the optimal policy for the single period problem, can be established by standard arguments.

LEMMA 1. The function $L(x_k, I_{k+\lambda}, y_k)$ is jointly convex in the triple $(x_k, I_{k+\lambda}, y_k)$. For every pair of fixed values of $(x_k, I_{k+\lambda})$ the values of y_k which minimize $C(x_k, I_{k+1}, y_k)$ are

$$\{\bar{y}_k(x_k, I_{k+\lambda}) : \bar{y}_k(x_k, I_{k+\lambda}) \ge [d_k - x_k]^+\}.$$

That all values of $y \ge [d_k - x_k]^+$ are optimal is readily transparent from the defining equation for $L(x_k, I_{k+\lambda}, y_k)$, since the outdate cost is completely independent of y_k. All of the order quantities given in LEMMA 1 yield an expected shortage cost of zero.

IV. THE MULTI-PERIOD DYNAMIC PROBLEM

Due to the interaction of y_k and I_k through the transfer function, it is far less obvious that this same solution is optimal for the multi-period problem. When more than a single period remains in the planning horizon, both the optimal cost function and the optimal policy will depend on more than just the two variables y_k and $I_{k+\lambda}$. Denote by $\underline{I}_{k+\lambda} = (I_{k+\lambda}, I_{k+\lambda-1}, \ldots, I_{k+1})$, which represents the vector of stocks issued during the λ periods preceding period k. Assuming that future costs are discounted to the present and the discount factor is $0 < \gamma < 1$, define

$C_k(x_k, \underline{I}_{k+\lambda})$ = minimum expected discounted cost when the current stock level is x_k, $\underline{I}_{k+\lambda}$ has been issued during the past λ periods, and k periods of demand remain

Writing

$$C_k(x_k, \underline{I}_{k+\lambda}) = \min_{y_k \geq 0} \{B_k(x_k, \underline{I}_{k+\lambda}, y_k)\}$$

it follows that

$$B_k(x_k, \underline{I}_{k+\lambda}, y_k) = L(x_k, I_{k+\lambda}, y_k) + \gamma E_{\alpha_k}[C_{k-1}(x_{k-1}, \underline{I}_{k+\lambda-1})] ,$$

with the transferred values of the state variables, $(x_{k-1}, \underline{I}_{k+\lambda-1})$, given above. The notation E_{α_k} should be interpreted as taking the expectation with respect to probability density, f. These equations constitute a statement of the principle of optimality as it applies to this problem.

The main result of the paper is the following:

THEOREM. Given assumptions (i)-(viii) above we have that

(1) $B_k(x_k, \underline{I}_{k+\lambda}, y_k)$ is convex in y_k for all fixed values of the $(\lambda+1)$ dimensional vector $(x_k, \underline{I}_{k+\lambda})$,

(2) An optimal policy in each period is

$$y_k^*(x_k, \underline{I}_{k+\lambda}) = y_k^*(x_k) \geq [d_k - x_k]^+ ,$$

(3) $$C_k(x_k, \underline{I}_{k+\lambda}) = \sum_{i=0}^{\lambda-1} \gamma^i \sigma_{k-i} I_{k+\lambda-i} + \sum_{i=1}^{k-\lambda} \gamma^{k-i} \sigma_i d_{i+\lambda} + \gamma^\lambda \sigma_{k-\lambda}[-x_k]^+ ,$$

where we adopt the notational conventions that

$$\sum_{i=1}^{-t} (\cdot) \equiv 0 , \quad \sigma_{-t} \equiv 0 , \quad \text{for all } t \geq 0 ,$$

and the initial condition $C_0(x_0, \underline{I}_\lambda) = 0$.

A formal proof is given in the appendix. Essentially, the optimal policy is to order at least enough stock to meet the demand each period. This is different than the type of policy obtained by Nahmias and Pierskalla[8] where fixed life perishability was assumed and only shortage and outdate costs were present. The difference arises from the fact that here we are only approximating the outdating process by an exponential decay model in order to concentrate on the feedback mechanism.

The third result is also of some interest since it gives an explicit expression for the cost associated with an optimal policy. Notice that when $1 \leq k \leq \lambda-1$, $C_k(x_k, \underline{I}_{k+\lambda})$ depends only on x_k and $(I_{\lambda+k}, \ldots, I_{\lambda+1})$ and is independent of $(I_\lambda, \ldots, I_{k+1})$, which indicates that only k periods of the total amounts issued need to be considered in order to determine the value of the optimal return function. The second term shows the dependence between the amount issued each period and the total demand, d_k. It is perhaps most interesting to note that at an optimal solution, the expected discounted cost is completely independent of the shortage cost, p. Since demand is deterministic, it is possible to order in such a way as to guarantee that no shortage will occur. It will be optimal to do so as long as $p > \theta$ as we have assumed.

Preliminary investigation indicates that when proportional ordering and holding costs are included, these results no longer hold. For the more general case, it will not be optimal to order to meet the demand each period; the optimal policy and return functions become complex nonlinear functions of the state vector. This extension of the current model to include ordering and holding costs as well as the effect of uncertainty in the periodic demand will be the subject of future investigation.

APPENDIX. Proof of Theorem

The proof is by induction on k, the number of periods remaining in the horizon. When k = 1 the form of the optimal policy follows from the lemma. Since

$$C_1(x_1, \underline{I}_{\lambda+1}) = L\big(x_1, \underline{I}_{\lambda+1}, y^*(x_1)\big) = \sigma_1 I_{\lambda+1}$$

the theorem holds for k = 1.

Assume it holds for 1, 2,..., k - 1. By definition

$$\begin{aligned} B_k(x_k, \underline{I}_{k+\lambda}, y_k) &= p\int_0^1 [d_k - x_k - y_k - \beta_k I_{k+\lambda} a]^+ f(a)da \\ &\quad + \sigma_k I_{k+\lambda} \\ &\quad + \gamma \int_0^1 C_{k-1}(x_{k-1}, \underline{I}_{k+\lambda-1}) f(a)da \\ &= p\int_0^1 [d_k - x_k - y_k - \beta_k I_{k+\lambda} a]^+ f(a)da \\ &\quad + \sigma_k I_{k+\lambda} \\ &\quad + \gamma \int_0^1 \Big\{\sum_{i=0}^{\lambda-2} \gamma^i \sigma_{k-i-1} I_{k+\lambda-i-1} + \gamma^{\lambda-1}\sigma_{k-\lambda} I_k \\ &\quad + \sum_{k=1}^{k-\lambda-1} \gamma^{k-i-1} d_{i+\lambda} \\ &\quad + \gamma^\lambda \sigma_{k-\lambda-1}[-x_{k-1}]^+\Big\} f(a)da \end{aligned}$$

by the inductive assumption

$$\begin{aligned} &= p\int_0^1 [d_k - x_k - y_k - \beta_k I_{k+\lambda} a]^+ f(a)da \\ &\quad + \gamma^\lambda \sigma_{k-\lambda} \int_0^1 \min\Big(d_k + [-x_k]^+, y_k \\ &\quad + [x_k]^+ + \beta_k I_{k+\lambda} a\Big) f(a)da \end{aligned}$$

$$+\gamma^{\lambda+1}\sigma_{k-\lambda-1}\int_0^1 [d_k - x_k - y_k - \beta_k I_{k+\lambda} a]^+ f(a)da + K$$

where

$$K = \sigma_k I_{k+\lambda} + \sum_{i=0}^{\lambda-2} \gamma^{i+1}\sigma_{k-i-1} I_{k+\lambda-i-1} + \sum_{k=1}^{k-\lambda-1} \gamma^{k-i}\sigma_i d_{i+\lambda}$$

includes the terms independent of y_k. The second and third terms above result from substituting the definition of the transfer function.

For convenience define $\delta_k = (d_k - x_k - y_k)/(\beta_k I_{k+\lambda})$. We will consider two cases:

Case 1: $\beta_k I_{k+\lambda} > 0$.

It is easy to verify that on the three open intervals the function $B_k(x_k, \underline{I}_{k+\lambda}, y_k)$ is partially differentiable with respect to y_k. Performing the differentiation one obtains:

$$\frac{\partial B_k(x_k, \underline{I}_{k+\lambda}, y_k)}{\partial y_k} = \begin{cases} -(p+\gamma^{\lambda+1}\sigma_{k-\lambda-1}) + \gamma^{\lambda}\sigma_{k-\lambda} & \text{for } y_k < d_k - x_k - \beta_k I_{k+\lambda} \\ -(p+\gamma^{\lambda+1}\sigma_{k-\lambda-1} - \gamma^{\lambda}\sigma_{k-\lambda}) \cdot F(\delta_k) & \text{for } d_k - x_k - \beta_k I_{k+\lambda} < y_k < d_k - x_k \\ 0 & \text{for } y_k > d_k - x_k. \end{cases}$$

Clearly the partial derivative is continuous at the interval endpoints. Furthermore, since $p > \theta$, it follows that $p > \sigma_{k-\lambda}$, and since $F(\delta_k) \le 1$, $\partial B_k/\partial y_k \le 0$ for $y_k \le d_k - x_k$.

Convexity follows since $F(\delta_k)$ is nonincreasing in y_k and the optimal solution where the partial vanishes, namely, for any $y_k \ge d_k - x_k$, $y_k \ge 0$. Hence choose $y_k^*(x_k) = [d_k - x_k]^+$.

Case 2: $\beta_k I_{k+\lambda} = 0$.

In this case one obtains:

$$\frac{\partial B_k(x_k, \underline{I}_{k+\lambda}, y_k)}{\partial y_k} = \begin{cases} -(p+\gamma^{k+1}\sigma_{k-\lambda-1}) + \gamma^{\lambda}\sigma_{k-\lambda} & \text{for } y_k < d_k - x_k \\ 0 & \text{for } y_k > d_k - x_k. \end{cases}$$

In this case the partial derivative is not continuous at the boundary point $d_k - x_k$, but since it is nonpositive on $[-\infty, d_k - x_k)$ and 0 on $[d_k - x_k, \infty)$ and $B_k(x_k, \underline{I}_{k+\lambda}, y_k)$ is continuous in y_k, it will also be convex in y_k. As in Case 1 all $y_k \ge d_k - x_k$ are optimal so choose $y_k^*(x_k) = [d_k - x_k]^+$.

In order to complete the proof of the Theorem, note that at an optimal solution

$$[-x_{k+1}^*]^+ = [-x_k - y_k^*(x_k) - \beta_k I_{k+\lambda} + d_k]^+ = 0$$

and

$$I_k^* = \min[d_k + [-x_k]^+,\ y^*(x_k) + [x_k]^+ + \beta_k I_{k+\lambda}] = d_k + [-x_k]^+.$$

Hence

$$\begin{aligned} C_k(x_k, \underline{I}_{k+\lambda}) &= B_k\left(x_k, \underline{I}_{k+\lambda}, y_k^*(x_k)\right) \\ &= \sigma_k I_{k+\lambda} + \gamma\int_0^1 C_{k-1}(x_{k-1}, \underline{I}_{k+\lambda-1}) f(a)da \\ &= \sigma_k I_{k+\lambda} + \gamma\int_0^1 \left\{\sum_{i=0}^{\lambda-1} \gamma^i \sigma_{k-i-1} I_{k+\lambda-i-1} + \sum_{i=1}^{k-\lambda-1} \gamma^{k-i-1}\sigma_i d_{i+\lambda}\right\} f(a)da \\ &= \sum_{i=0}^{\lambda-1} \gamma^i \sigma_{k-i} I_{k+\lambda-i} + \sum_{i=1}^{k-\lambda-1} \gamma^{k-i}\sigma_i d_{i+\lambda} + \gamma^{\lambda}\sigma_{k-\lambda} d_k + \gamma^{\lambda}\sigma_{k-\lambda}[-x_k]^+ \\ &= \sum_{i=0}^{\lambda-1} \gamma^i \sigma_{k-i} I_{k+\lambda-i} + \sum_{i=1}^{k-\lambda} \gamma^{k-i}\sigma_i d_{i+\lambda} + \gamma^{\lambda}\sigma_{k-\lambda}[-x_k]^+. \end{aligned}$$

Q.E.D.

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