Chapter 8

From the Assignment Model to Combinatorial Auctions

Sushil Bikhchandani and Joseph M. Ostroy

1 Introduction

The goal of this chapter is to describe efficient auctions for multiple, indivisible objects in terms of the duality theory of linear programming. Because of its well-known incentive properties, we shall focus on Vickrey auctions.[1] These are efficient auctions in which buyers pay the social opportunity cost of their purchases and consequently are rewarded with their (social) marginal product.[2] We use the assignment model to frame our analysis.

The exposition will be divided into static and dynamic parts. By "static," we shall mean the characterization of a combinatorial sealed-bid auction as a *pricing equilibrium*. Pricing equilibria correspond to optimal solutions to the primal and dual of a certain linear programming (LP) problem. We shall give necessary and sufficient conditions, called *buyers are substitutes* (formulated by Shapley 1962), for the existence of a pricing equilibrium that yields the outcome of a Vickrey auction.

Algorithms for solving LP problems imply that the static characterization has a "dynamic" counterpart. But the simplex algorithm, for example, has the drawback that buyers' preferences must be available to the implementor, i.e., information must be centralized. Among the methods for solving LP problems, the primal dual algorithm is noteworthy because its informational requirements can be decentralized. The implementor can act as an auctioneer who calls out prices — otherwise known as feasible solutions to the dual — that requires no information about buyers' preferences. Using only knowledge of one's own preferences, each buyer makes utility maximizing bids based on those prices; and the auctioneer/implementor uses that information to adjust prices. By starting the algorithm at zero prices, the primal dual method can be interpreted as an ascending price auction.

The results reported here can be regarded as the grafting of two related ideas to form a useful hybrid. In linear programming, the primal dual algorithm was originally developed for the assignment model (see Section 2). In economics, the algorithm is the Law of Supply and Demand, i.e., the story of how market prices adjust to establish equilibrium.

In the auction problem, the metaphorical Auctioneer of the market is replaced by a literal entity, as well as a detailed procedure adjusting prices to bids. A complicating factor in the auction problem ignored in the Law of Supply and Demand is that bidders should have an incentive to bid truthfully. These incentive requirements conflict with the linearity and anonymity of pricing with which the Law is normally identified.[3] Nevertheless, the logic of prices adjusting to excess demands can be extended beyond its traditional boundaries to meet the added challenge of incentive compatible pricing.

Ausubel (1997) formulated an ingenious ascending price auction for homogeneous goods with diminishing marginal utility to implement a Vickrey scheme, which can be described as non-linear version of the Law of Supply and Demand. Bikhchandani and Ostroy (2002b) show that the indivisible commodity version of Ausubel's auction is a primal dual algorithm.

Heterogeneous commodities present new challenges. These are addressed in the LP formulation of Bikhchandani and Ostroy (2002a). The optimal solutions to the dual linear program correspond to pricing equilibria with possibly non-linear and non-anonymous prices. A decentralized procedure, such as the primal dual algorithm for solving the LP, that approaches the pricing equilibrium corresponding to the social opportunity cost point (from below) would yield an incentive compatible decentralized auction.

Parkes and Ungar (2000a) and Ausubel and Milgrom (2002) independently developed an ascending price auction for heterogeneous goods. See also Parkes (Chapter 2) and Ausubel and Milgrom (Chapter 3). Parkes and Ungar show that it converges to an efficient outcome and Ausubel and Milgrom prove that under a condition called buyers are submodular, a slightly stronger version of the buyers are substitutes condition, the auction is incentive compatible and converges to social opportunity cost point. de Vries et al. (2003) construct an incentive compatible primal dual auction which builds upon ideas in Demange, Gale, and Sotomayor (1986) and the LP formulation in Bikhchandani and Ostroy (2002a).

The basic results for the assignment model are described in Section 2 and for the combinatorial auction model in Section 3-6. The combinatorial auction model and pricing equilibria are defined in Section 3. Section 4 provides a necessary and sufficient condition for the equivalence between pricing equilibrium and the sealed-bid Vickrey auction. An LP formulation of the auction model is illustrated in Section 5. Section 6 describes an ascending price auction as a primal dual algorithm for solving the LP. Section 7 concludes with a perspective on the work of others, viewed through this LP framework.

2 The assignment model

The basic building block for our analysis is the assignment model. In common with combinatorial auctions, the assignment model concerns allocations with indivisibilities. Even though an optimal assignment is an integer programming problem, it is amenable to analysis as an LP problem because it has an *integral* optimal solution. A variant of the assignment model was the origin of one of the earliest application of LP (Kantorovich (1942)). The assignment model was also the first (modern) application of the core of a game to economics (Shapley 1955), and the core itself is an integral solution to an LP problem. Use of the primal dual algorithm is hardly coincidental since that algorithm was first developed to solve the assignment model (Kuhn, 1955). Demange, Gale, and Sotomayor (1986) formulated a particular application of the algorithm to implement a version of the ascending price auction in which buyers receive their Vickrey payoffs.

Normally, the assignment model is presented by taking the payoffs of pairwise matches as given. For economic applications, however, it is essential to provide a commodity representation of those payoffs.

Let $I = B \cup S$, where B, the set of buyers, has n elements and S, the set of sellers, has m elements. Buyers are indexed by $b \in B$, sellers are indexed $s \in S$, and agents (buyers or sellers) are indexed $i \in I$. Let $C = \{1, \ldots, m\}$ be the set of commodities having the same cardinality as S. Denote by $\mathbf{1}_c \in \mathbb{R}^m$, $c = 1, \ldots, m$, the vector with zeroes in all components except the c^{th} which has value 1.

The utility of buyer $b \in B$ for consuming commodity bundle $z_b \in \mathbb{R}^m$ is $v_b(z_b) \geq 0$. For each b, the set of possible z_b will be restricted to

$$Z^1 = \{\mathbf{1}_c : c = 1, \ldots, m\} \cup \{0\}.$$

This commodity representation is distinguished by the restriction that each buyer wishes to purchase at most one object, called the *unit demand* assumption. There is also a unit supply restriction for sellers. Let $v_s(z_s) \leq 0$ be the utility to seller $s \in S$ of supplying the bundle $z_s \in \mathbb{R}^m$. For each s, the set of possible z_s is

$$Z^s = \{\mathbf{1}_s, 0\},$$

i.e., seller s is the only supplier of commodity $c = s$. As a normalization, assume $v_b(0) = v_s(0) = 0$. Consequently, if b is matched with s, they could jointly obtain

$$v_{bs} = \max_{z \in Z^s}\{v_b(z) + v_s(z)\} \geq 0.$$

Throughout it is assumed that utility is *quasi-linear* in the commodity z_i and the amount of the money commodity, τ_i, that i pays or receives, which is to say that $u_i(z_i, \tau_i) = v_i(z_i) + \tau_i$. Thus, the utility of no trade is zero: $u_i(0, 0) = 0$. Further, buyers can supply any (negative) money quantity required.

Let $p \in \mathbb{R}^m$ denote prices for each of the m commodities. The utility maximizing choice by b at prices p is $v_b^*(p) = \max_{z_b \in Z^1}\{v_b(z_b) - p \cdot z_b\}$.[4] The utility maximizing choice by s at p is $v_s^*(p) = \max_{z_s \in Z^s}\{v_s(z_s) + p \cdot z_s\}$.

Definition: A *Walrasian equilibrium* for the commodity representation of the assignment model is a $(p, (z_b), (z_s))$ such that

- $v_b(z_b) - p \cdot z_b = v_b^*(p)$ for all b

- $v_s(z_s) + p \cdot z_s = v_s^*(p)$ for all s

- $\sum_{b \in B} z_b = \sum_{s \in S} z_s$

The first two conditions stipulate utility maximization subject to prices and a budget constraint $\tau_b = -p \cdot z_b$ representing the transfer from the buyer and $\tau_s = p \cdot z_s$ being the transfer to the seller. The third condition is "market-clearance," the objects demanded equal the objects supplied.

The LP version of the commodity representation of the assignment model is:

$$\vartheta(\mathbf{1}_I, 0) = \max \sum_{b \in B} \sum_{z_b \in Z^1} v_b(z_b) x_b(z_b) + \sum_{s \in S} \sum_{z_s \in Z^s} v_s(z_s) x_s(z_s),$$

subject to:

$$\begin{aligned} \sum_{z_b \in Z^1} x_b(z_b) &= 1, \qquad \forall b \\ \sum_{z_s \in Z^s} x_s(z_s) &= 1, \qquad \forall s \\ \sum_b \Big(\sum_{z_b \in Z^1} z_b x_b(z_b) \Big) - \sum_s \Big(\sum_{z_s \in Z^s} z_s x_s(z_s) \Big) &= 0 \\ x_b(z_b), x_s(z_s) &\geq 0, \quad \forall b,\ \forall z_b \in Z^1,\ \forall s,\ \forall z_s \in Z^s \end{aligned}$$

The quantity $x_b(z_b)$ is the amount of z_b consumed by b and the quantity $x_s(z_s)$ is the amount of z_s supplied by s.

The expression $\vartheta(\mathbf{1}_I, 0)$ is the optimal value as a function of the right-hand side constraints: $\mathbf{1}_I$ is the restriction that there is one each of the individuals in I and $0 \in \mathbb{R}^m$ stands for the requirement that the demands of buyers for each of the m commodities must be exactly offset by the supplies of sellers.

An integral solution satisfies the above constraints *and* $x_b(z_b), x_s(z_s) \in \mathbb{Z}_+$. Such a requirement implies that for each b, $x_b(z_b) = 1$ for exactly one z_b, and for each s $x_s(z_s) = 1$ for exactly one z_s. If $z_b = \mathbf{1}_s$ then $z_b - z_s = 0$, i.e., b and s are matched. Because 0 is a possible choice by a buyer or seller, integral solutions do not require that all buyers and sellers be matched. Indeed, if $n \neq m$, they cannot be.

Let v_I be the maximum to the above problem when integral constraints are imposed. Therefore, $\vartheta(\mathbf{1}_I, 0) \geq v_I$. A key feature of the assignment model is that

$$\vartheta(\mathbf{1}_I, 0) = v_I,$$

i.e., there always exists integer optimal solutions to the LP problem.

The dual of this LP problem is:

$$\min \left(\sum_{b \in B} y_b + \sum_{s \in S} y_s \right),$$

subject to:

$$\begin{aligned} y_b + p \cdot z_b &\geq v_b(z_b), && \forall b, z_b \in Z^1 \\ y_s - p \cdot z_s &\geq v_s(z_s), && \forall s, z_s \in Z^s \end{aligned}$$

The dual constraints of the LP problem mirror utility maximization in Walrasian equilibrium. For each b, $y_b \geq v_b(z_b) - p \cdot z_b$, for all z_b. Hence, $y_b \geq v_b^*(p)$; and since y_b is to be minimized, y_b can be equated with $v_b^*(p)$. Similarly, for y_s and $v_s^*(p)$.

The assignment model also defines a game in characteristic function form, and from that its core is obtained. Because the only groupings of productive individuals are matches between buyers and sellers, we can bypass those formalities and concentrate on the equivalent notion of a "stable matching."

If the number of buyers is not equal to the number of sellers, add dummies to the smaller set to make them equal. For concreteness, define a dummy as having 0 as the only feasible trade. Let π be a matching of buyers to sellers, i.e., $\pi : B \to S$ is one-to-one and onto. There is an evident equivalence between a π and an integral solution to the LP problem.

The core of the assignment model is a stable matching defined by a $(\pi, (y_b), (y_s))$ such that $\pi(b) = s$ implies $y_b + y_s = v_{bs}$ and $\pi(b) \neq s'$ implies $y_b + y_{s'} \geq v_{bs'}$.

The various notions described above coincide:

Proposition 1 (Assignment model equivalence). *The following are equivalent:*

a. $(p, (z_b), (z_s))$ *is a Walrasian equilibrium where* $y_b = v_b^*(p)$ *and* $y_s = v_s^*(p)$

b. $(z_b), (z_s)$ *is an integral optimal solution to the primal and* $(y_b), (y_s)$ *and* p *is an optimal solution to the dual*

c. $(\pi, (y_b), (y_s))$ *belongs to the core of the assignment game*

The equivalence of (a) and (b) was established by Koopmans and Beckman (1957), using a result in von Neumann (1953) which showed that an LP version of the assignment model has integer solutions. The equivalence of (a) and (c) is due to Shapley and Shubik (1972).

The assignment model has another remarkable property that it shares with Vickrey auctions. Recall that $v_I (= \vartheta(\mathbf{1}_I, 0))$ is the maximum total gains for the individuals in I. Let v_{-b} be the maximum total gains without buyer b. This can be obtained as $v_{-b} = \vartheta(\mathbf{1}_I - \mathbf{1}_b, 0))$, or equivalently by forcing $x_b(\cdot) \equiv 0$. Define the *marginal product* of b as

$$\text{MP}_b = v_I - v_{-b}.$$

The aggregate resources in the assignment model are $\mathbf{1}_C$. If b were to obtain $z = \mathbf{1}_c$, the resources available to the others would be $\mathbf{1}_C - \mathbf{1}_c$. Let

$$V_{-b}(\mathbf{1}_C - \mathbf{1}_c) = \max\{\textstyle\sum_{i \neq b} v_i(z_i) : \sum_{i \neq b} z_i = \mathbf{1}_C - \mathbf{1}_c\},$$

be the maximum total gains in the model without b when their aggregate resources are $\mathbf{1}_C - \mathbf{1}_c$. Note that $v_{-b} = V_{-b}(\mathbf{1}_C)$ and

$$\begin{aligned} v_I &= \max_{z_b \in Z^1} \{v_b(z_b) + V_{-b}(\mathbf{1}_C - z_b)\} \\ &= \max\{\max_c \{v_b(1_c) + V_{-b}(\mathbf{1}_C - \mathbf{1}_c)\}, V_{-b}(\mathbf{1}_C)\} \end{aligned}$$

Define the *social opportunity cost* of giving one unit of c to b as the minimum amount of money the remaining individuals would be willing to accept to give one unit of c to b, i.e.,

$$\Delta V_{-b}(\mathbf{1}_c) = V_{-b}(\mathbf{1}_C) - V_{-b}(\mathbf{1}_C - \mathbf{1}_c).$$

If they were to receive $\Delta V_{-b}(\mathbf{1}_c)$, the remaining individuals would be exactly as well off as if they had kept those resources, namely $v_{-b} = V_{-b}(\mathbf{1}_C)$.

The lessons of the Vickrey-Clarke-Groves approach to incentive compatibility are: When a buyer is required to pay the social opportunity cost of his trade (equivalently, receive a payoff equal to his marginal product), the buyer has the incentive to report his characteristics truthfully. And, opportunities for profitable misrepresentations are created when a buyer pays more than the social opportunity cost.

It is readily established that

Proposition 2 (Marginal product pricing inequality). *For every*

Walrasian equilibrium price p,

$$p \cdot \mathbf{1}_c = p_c \geq \Delta V_{-b}(\mathbf{1}_c), \quad \forall b,\ \forall c.$$

Consequently,

$$MP_b \geq v_b^*(p), \quad \forall b.$$

When the inequality is strict, prices are manipulable. However, for each buyer there is a most favorable Walrasian equilibrium in which that buyer reaches the upper bound on his payoff beyond which he cannot manipulate.

Proposition 3 (One at a time property; Makowski and Ostroy 1987). *For any $b \in B$ there exists a Walrasian equilibrium price p^b such that*

$$MP_b = v_b^*(p^b).$$

Prices p^b are the best for b, i.e., $v_b^*(p^b) = \mathrm{MP}_b$, but another buyer $\hat{b}$ might think otherwise, i.e., $v_{\hat{b}}^*(p^b) < \mathrm{MP}_{\hat{b}}$. Let $\hat{p}^{\hat{b}}$ be the best Walrasian equilibrium for $\hat{b}$. For any two price vectors p and $\hat{p}$, define $\min\{p, \hat{p}\}$ as the vector consisting of the element by element $\min\{p_c, \hat{p}_c\}$. The assignment model guarantees that as far as the buyers are concerned, there need not be any conflict because of the following:

Proposition 4 (Lattice property; Shapley and Shubik 1972.) *If p and $\hat{p}$ are Walrasian equilibrium prices so is* $\min\{p, \hat{p}\}$.

Combining Propositions 3 and 4,

Proposition 5 (Buyers get their MP's; Gretsky, Ostroy, and Zame 1999). *There is an equilibrium price $\underline{p}$ such that for all b, if buyer b receives commodity c,*

$$\underline{p}_c = \Delta V_{-b}(\mathbf{1}_c);$$

and therefore

$$MP_b = v_b^*(\underline{p}).$$

Similar conclusions can be obtained for the sellers. Of course, it is typically not possible for both buyers and sellers to get their marginal products. This fact is implicitly recognized in the auction model, below, where the seller is treated asymmetrically as less than fully strategic in his actions and hence not requiring the same kind of incentive payments as the buyers.

Crawford and Knoer (1981) and Demange, Gale, and Sotomayor (1986) provide ascending price auctions which implement the smallest Walriasian price. As each buyer gets his marginal product the auction is incentive compatible. One of the auctions in Demange, Gale, and Sotomayor (1986) is a primal dual algorithm on an LP formulation of the assignment model.

Fix a price vector $p = (p_1, p_2, ..., p_c, ..., p_C)$ and let $D_b(p)$ be the demand set of buyer b at these prices. Let $M \subseteq C$. Let $R(M)$ be the set of individuals such that $\emptyset \neq D_b(p) \subseteq M$ for all $b \in R(M)$. A subset of objects M is *overdemanded* if $|R(M)| > |M|$. A necessary and sufficient condition for the existence of a feasible assignment in which every buyer is assigned an element of his demand set is that there is no overdemanded set of objects. This follows from Hall's theorem (see Gale 1960, p. 144). The Demange, Gale, and Sotomayor ascending-price auction described below raises prices of a minimal (w.r.t. set inclusion) overdemanded set:

Step 0: Start with price zero for each object.

Step 1: Buyers report their demand sets at current prices. If there is no overdemanded set, go to Step 3; otherwise go to Step 2.

Step 2: Choose a minimal overdemanded set. Raise prices of all objects in this set until some buyer changes his demand set. Go to Step 1.

Step 3: Assign each buyer an object in his demand set at current prices. Stop.

In the following sections the above properties of the assignment model serve as important guideposts for extending efficient and incentive compatible auctions beyond the unit demand assumption. To do this, an obvious hurdle needs to be cleared.

The Walrasian equilibria and associated LP problem underlying the assignment model yield linear and anonymous prices for commodities. That such prices suffice for sealed-bid and ascending price implementations of Vickrey auctions is a consequence of the unit demand assumption. In contrast, Vickrey payments are known to be typically discriminatory, i.e., the average amount an individual pays may depend on the quantity purchased, and two buyers purchasing the same objects may pay different amounts. To deal with multiple units of homogeneous objects and, *a fortiori* with multiple units of heterogeneous objects, the notion of Walrasian equilibrium and its corresponding LP problem must be reformulated to allow a greater variety of pricing than is needed in the assignment model, or that is normally associated with price-taking behavior and Walrasian equilibrium.[5]

3 The combinatorial auction model

In this and the remaining sections we shall construct the parallels between the assignment and auction models. The participants in the auction are $I = \{s\} \cup B$, where s is the seller and $B = \{1, \ldots, n\}$ are the buyers. As in the previous section, denote by $i \in I$ a buyer or seller, and by b a member of B.

Let $\mathbb{Z}_+$ be the non-negative integers and $\mathbb{Z}_+^m$ its m-fold Cartesian product. A commodity bundle assigned to b is denoted $z_b \in \mathbb{Z}_+^m$. An allocation of indivisible commodities to buyers is $Z = (z_b)$. The aggregate endowment of indivisible quantities of the m objects initially held by the seller is $\omega \in \mathbb{Z}_+^m$. The set of feasible assignments from the seller to the buyers is therefore

$$\mathcal{Z} = \{Z = (z_1, z_2, ..., z_b, ..., z_n) : \sum_{b \in B} z_b \leq \omega\}.$$

The utility to b of z_b is $v_b(z_b)$. Assume that $v_b(\cdot)$ is weakly monotonic on $\mathbb{Z}_+^m$ and, as a normalization, $v_b(0) = 0$.

The cost to the seller of supplying Z is $v_s(Z)$, where $v_s(Z) = v_s(Z')$ if $\sum_{b \in B} z_b = \sum_{b \in B} z'_b$, i.e., $v_s(Z) = v_s(\sum z_b)$.

It will be convenient to write the total utility from any $Z \in \mathcal{Z}$ as

$$(\textstyle\sum_{i \in I} v_i)(Z) = (\textstyle\sum_{b \in B} v_b)(Z) + v_s(Z),$$

rather than as $\sum_b v_b(z_b) + v_s(\sum_{b \in B} z_b)$.

The auction model is therefore described by $(\mathcal{Z}, \{v_i\})$.

3.1 Special Cases

Unit demands: As described in Section 2, $z_b \in Z^1$. There are several sellers in the assignment model and each is assumed to have one unit of their own commodity so that the aggregate endowment of commodities $\omega = \mathbf{1}_C$. In the auction version of the assignment model, ω is controlled by a single seller.

Homogeneous objects: Assume $m = 1$ while ω is an arbitrary element of $\mathbb{Z}_+$.

Multiple units of heterogeneous objects: The auction model above allows for the seller to have multiple units of several commodities and for buyers utilities to be similarly unrestricted. Hence, v_b is defined for $z_b \in \mathbb{Z}_+^m$, where m can be greater than 1 and z_b need not be less than or equal to $\mathbf{1}_C$. Unless otherwise indicated, that is the setting described below.

3.2 Pricing equilibria in the auction model

Let $P_b(z_b)$ be the money payment by b for the allocation z_b. Thus, b receives the total (or net) utility $v_b(z_b) - P_b(z_b)$ from the allocation z_b.

Define

$$v_b^*(P_b) = \max_{z_b}\{v_b(z_b) - P_b(z_b)\},$$

as the indirect utility of b facing pricing schedule $P_b(\cdot)$.

Let $(\sum_{b\in B} P_b)(Z) = \sum_{b\in B}(z_b)$ where $Z = (z_b)$. Given the pricing schedules P_b for each b, the utility maximizing choice of assignments by the seller is

$$v_s^*(\textstyle\sum_{b\in B} P_b) = \max_{Z\in\mathcal{Z}}\{v_s(Z) + (\sum_{b\in B} P_b)(Z)\}.$$

While each b is concerned only with the benefits and costs of the b^{th} component of Z, the seller is interested in the entire assignment. The term $(\sum_{b \in B} P_b)(Z)$ is reminiscent of the supplier of a public good who offers the bundle Z to buyers and collects $P_b(z_b)$ from each one. The similarity arises from the possibility of price discrimination. Just as the supplier of a public good can charge different prices to different buyers, so can the seller in a combinatorial auction. For example, the seller can receive different payments for the same quantities delivered to different buyers and the seller's average revenue from any single buyer can vary with the quantities purchased. The difference, of course, is that unlike the supplier of a public good, the seller who gives the bundle z_b to buyer b cannot also supply those same objects to any other buyer.

Definition: $(Z = (z_b), \{P_b\})$ is a *pricing equilibrium* for the auction model $(\mathcal{Z}, \{v_i\})$ if

- $v_b(z_b) - P_b(z_b) = v_b^*(P_b)$
- $v_s(Z) + (\sum_{b \in B} P_b)(Z) = v_s^*(\sum_{b \in B} P_b)$
- $Z \in \mathcal{Z}$.

Pricing equilibrium is a variation on Walrasian equilibrium. In fact, we have

DEFINITION: $(Z, \{P_b\})$ is a *Walrasian equilibrium* for the auction model $(\mathcal{Z}, \{v_i\})$ if it is a *pricing equilibrium* and there exists a $p \in \mathbb{R}^m$ such that for all b and z,

$$P_b(z) = p \cdot z$$

Pricing equilibrium is therefore an extension of Walrasian equilibrium in which each buyer chooses a z_b to maximize utility subject to a budget constraint in which prices depend on both b and z, while the seller maximizes utility subject to a revenue function that permits discriminatory pricing.

The concept of pricing equilibrium *could be* used to describe "first degree price discrimination" in which the seller seeks to extract the maximum possible gains from the buyers, as if the seller knew the buyers' utility functions and the seller could prevent resale. In fact, it will be used to illustrate the very opposite in the sense that the buyers will receive the maximum — not the minimum — surplus consistent with pricing equilibrium and consequently have an incentive to reveal their utilities in a sealed-bid auction or bid straightforwardly in an ascending price auction. In the remainder of this section, we describe some of its relevant features.

An allocation Z is *efficient* for the auction model $(\mathcal{Z}, \{v_i\})$ if it achieves the maximum total gains

$$v_I = \max_{Z \in \mathcal{Z}} (\textstyle\sum_{i \in I} v_i)(Z).$$

It is readily seen that for any $\{P_b\}$, $v_b^*(P_b) \geq v_b(z_b) - P_b(z_b)$ and $v_s^*(\sum_{b \in B} P_b) \geq v_s(Z) + (\sum_{b \in B} P_b)(Z)$. These inequalities lead directly to the conclusion that, as with standard Walrasian equilibrium,

Proposition 6 (Bikhchandani and Ostroy 2002a)**.** *All pricing equilibria are efficient.*

Let

$$v_{-b} = \max_{Z \in \mathcal{Z}} (\textstyle\sum_{i \neq b} v_i)(Z),$$

be the maximum total gains in the auction model without b. Again, the marginal product of b is

$$\mathrm{MP}_b = v_I - v_{-b}.$$

We shall want to compare the utility each b receives in a pricing equilibrium with that buyer's marginal product.

If b were to obtain z, the resources available to the others would be $\omega - z$. Let

$$V_{-b}(\omega - z) = \max\{(\textstyle\sum_{i \neq b} v_i)(Z) : \sum_{b' \neq b} z_{b'} = \omega - z\},$$

be the maximum total gains in the model without b when their aggregate resources are $\omega - z$. The difference between this and the assignment model is that here ω is not necessarily $\mathbf{1}_C$ and z is not necessarily $\mathbf{1}_c$. Nevertheless, again $v_{-b} = V_{-b}(\omega)$ and

$$v_I = \max_{z_b \in \mathbb{Z}_+^m} \{v_b(z_b) + V_{-b}(\omega - z_b)\}.$$

Again the *social opportunity cost* of giving z to b is

$$\Delta V_{-b}(z) = V_{-b}(\omega) - V_{-b}(\omega - z),$$

If the individuals other than b were to receive $\Delta V_{-b}(z)$, they would be exactly as well off as if they had kept those resources, namely v_{-b}.

The analog of Proposition 2 for the auction model is:

Proposition 7 (MP PRICING EQUILIBRIUM INEQUALITY; Bikhchandani and Ostroy 2002a). *If* $(Z, \{P_b\})$ *is a pricing equilibrium, then*

$$P_b(z_b) \geq \Delta V_{-b}(z_b).$$

Consequently,

$$MP_b \geq v_b^*(P_b).$$

Proposition 7 says that in any pricing equilibrium buyers pay at least as much as the social opportunity cost of their purchases, and possibly more. If the buyer's payment does equal the social opportunity cost of his purchase, it must be that among all pricing equilibria the buyer is paying the minimum, i.e.,

$$\min_{P_b} P_b(z_b) = \Delta V_{-b}(z_b),$$

among all equilibrium $P_b(\cdot)$.

DEFINITION: An *MP pricing equilibrium* (for buyers) is a pricing equilibrium $(Z, \{\underline{P}_b\})$ with the added property that for all b,

$$\underline{P}_b(z_b) = \Delta V_{-b}(z_b).$$

Equivalently,

$$v_b(z_b) - P_b(z_b) = MP_b.$$

4 Sealed bid Vickrey auctions as pricing equilibria

The characterizing property of a Vickrey auction is that buyers pay the social opportunity cost of their purchases. As in the assignment model (Proposition 3), for any one buyer there is always a pricing equilibrium in which that buyer makes his Vickrey payment.

Proposition 8 (ONE AT A TIME PROPERTY; Parkes and Ungar 2000b). *For any $b^0 \in B$, there exists a pricing equilibrium $(Z, \{P_b\})$ such that*

$$P_{b^0}(Z) = \Delta V_{-b^0}(z_{b^0}),$$

and consequently,

$$v^*_{b^0}(P_{b^0}) = MP_{b^0}.$$

Unlike the assignment model, however, a pricing equilibrium for which all buyers pay their social opportunity cost need not exist (see Example 5.1 in Bikhchandani and Ostroy 2002a). The following qualification gives necessary and sufficient conditions for a pricing equilibrium to satisfy that condition.

For $R \subseteq B$, let

$$v_{-R} = \max_{Z \in \mathcal{Z}} (\textstyle\sum_{i \in B \setminus R} v_i)(Z),$$

be the maximum gains without the buyers in R. Define the marginal product of R as

$$\mathrm{MP}_R = v_I - v_{-R}.$$

In other words, v_{-R} and MP_R are extensions of v_{-b} and MP_b to include subsets of buyers other than singletons.

Following Shapley (1962) who established this property for the assignment model,

Definition: *Buyers are substitutes* if for every $R \subseteq B$,

$$\mathrm{MP}_R \geq \sum_{b \in R} \mathrm{MP}_b.$$

Proposition 9 (Vickrey auctions as pricing equilibrium;

Bikhchandani and Ostroy 2002a). *An MP pricing equilibrium exists if and only if buyers are substitutes.*

Let $\hat{B} \subset B$ and for $b \in \hat{B}$, let $MP_b(\hat{B}) \equiv \max_{Z \in \mathcal{Z}} (\sum_{i \in \hat{B}} v_i)(Z) - \max_{Z \in \mathcal{Z}} (\sum_{i \in \hat{B} \setminus b} v_i)(Z)$ be the marginal product of b in $\hat{B} \cup s$. Similarly, for $R \subset \hat{B}$, $MP_R(\hat{B})$ is the marginal product of R in $\hat{B} \cup s$.

DEFINITION: *Buyers are strong substitutes* if for every $R \subseteq \hat{B} \subseteq B$,

$$\mathrm{MP}_R(\hat{B}) \geq \sum_{b \in R} \mathrm{MP}_b(\hat{B}).$$

Ausubel and Milgrom (2002) (see also Chapter 3) use the term "buyer submodularity" instead of "buyers are strong substitutes."

4.1 MP Pricing equilibria in various auction models

SINGLE OBJECT AUCTION: This is the simplest and best known example. Assuming the seller's valuation is zero and buyers valuation are such that $v_1 \geq v_2 \geq \cdots v_n \geq 0$, a pricing equilibrium is described by single p and $z_1 \neq 0$ such that $v_1 \geq p \geq v_2$. When $p = v_2$, buyer 1 pays the social opportunity cost of the object.

UNIT DEMANDS: Assume buyers' demands are in Z^1. The unit demand assumption implies that without loss of generality pricing equilibrium can be limited to those $P_b(\cdot)$ for which there exists $p \in \mathbb{R}^m$ such that for all b and z_b, $P_b(z_b) = p \cdot z_b$, i.e., to Walrasian equilibria. Proposition 5 implies that the pricing equilibrium $(Z, \underline{p})$ satisfies the requirement of Proposition 9, confirming the fact that buyers are always substitutes in the assignment model.

Homogeneous units: With multiple units of a homogeneous object, let

$$\Delta v_b(z_b) = v_b(z_b + 1) - v_b(z_b),$$

be the increase in utility from adding one more unit to the purchases of buyer b. Buyer b exhibits *diminishing marginal utility* if $z' > z$, implies $\Delta v_b(z') \leq \Delta v_b(z)$. *The buyers are substitutes condition is satisfied if each* v_b *exhibits diminishing marginal utility* (Bikhchandani and Ostroy 2002a). Conversely, when there is a buyer who does not exhibit diminishing marginal utility, examples in which the buyers are substitutes condition is not satisfied are readily constructed.

Unlike the single object and unit demand models in which linear anonymous prices characterize Vickrey payments, when there are multiple units of a homogeneous commodity a single price will not do. The social opportunity cost function $\Delta V_{-b}(z)$ varies non-linearly with z and also varies with b. It therefore requires the full level of generality of pricing equilibrium, along with the hypothesis of diminishing marginal utility, to capture Vickrey payments.

MULTIPLE UNITS OF HETEROGENEOUS COMMODITIES: Pricing equilibria always exist for auction models and for any one b there is always a smallest pricing equilibrium $\underline{P}_b$ such that the buyer's price coincides with the Vickrey payment. However, unless the buyers are substitutes condition is satisfied, there will be a conflict between choosing the smallest pricing equilibrium for one buyer and the smallest for another. A condition proposed by Kelso and Crawford suffices to eliminate that conflict.

Kelso and Crawford considered the following many-to-one variant of the assignment model. On one side of the market, each seller can offer one unit of his own good (his labor). On the other side, buyers can employ more than one worker. Interpret $v_b(z_b)$ as the output of employer b hiring workers z_b. Walrasian equilibrium in this many-to-one model combines the price-taking behavior of the sellers in the assignment model with the price-taking behavior of buyers in the heterogenous object auction model. To ensure the existence of such an equilibrium, they assumed that v_b satisfies *gross substitutes*: If the prices of some but not all commodities increase, there is always a utility maxmizing demand in which the goods that do not rise in price continue to be demanded. More formally, if $v_b(z) - p \cdot z = v_b^*(p)$ and $p' \geq p$, then there exists a z' such that

$v_b(z') - p' \cdot z' = v_b^*(p')$ and $z'_c \geq z_c$ for those c such that $p_c = p'_c$.

The gross substitutes condition guarantees the existence of Walrasian equilibrium, a particular form of pricing equilibrium. By Proposition 7, the payments that buyers (employers) make will be at least as large as their social opportunity costs. Because prices are restricted to be linear, however, there may be no Walrasian equilibrium for which *any* buyer makes his Vickrey payments. Nevertheless, gross substitutes is relevant to our analysis:

Proposition 10 (Bikhchandani, de Vries, Schummer and Vohra 2002, Ausubel and Milgrom 2002) *If each v_b satisfies the gross substitutes condition, then buyers are strong substitutes.*

Therefore, even though Walrasian equilibrium is not generally compatible with buyers paying the social opportunity costs of their purchases, conditions guaranteeing the existence of Walrasian equilibrium imply the existence of a pricing equilibrium that does.

5 LP formulation of the auction model

The set of pricing equilibria for the auction model corresponds to the set of *integral* optimal solutions to the primal and the set of optimal solutions to the dual of an LP problem. The LP connection implies that LP algorithms can be employed to find pricing equilibria. This section is therefore the bridge between the sealed-bid or static approach and the ascending price or dynamic approach to pricing equilibria and Vickrey auctions.

To construct the LP problem defined by the auction model $(\mathcal{Z}, \{v_i\})$, regard (b, Z) as an "activity" the unit value of which is written as $v_b(Z) \equiv v_b(z_b)$. Similarly, (s, Z) is an activity the unit value of which is $v_s(Z)(= 0)$. The LP problem is:

$$\vartheta(\mathbf{1}_I, 0_L) = \max_{Z \in \mathcal{Z}} [v_s(Z)x(s, Z) + \sum_{b \in B} v_b(Z)x(b, Z)]$$

subject to

$$\begin{aligned}
\textstyle\sum_{Z \in \mathcal{Z}} x(b, Z) &= 1, \qquad \forall b \\
\textstyle\sum_{Z \in \mathcal{Z}} x(s, Z) &= 1 \\
x(b, Z) - x(s, Z) &= 0, \qquad \forall (b, Z) \\
x(b, Z), x(s, Z) &\geq 0
\end{aligned}$$

The quantity $x(b, Z)$ represents the level of operation of the "activity" (b, Z), and similarly for $x(s, Z)$. For each buyer, there is a constraint stipulating that his total participation among all Z must be 1 and there is a similar constraint for the seller, for a total of $n + 1$ constraints for which the RHS is 1. The third line of constraints says that for each (b, Z), the level of participation by b in Z, $x(b, Z)$, must be the same as the level of participation by s in Z, $x(s, Z)$. Letting d be the number of elements Z in $\mathcal{Z}$, there are $L = nd$ third line constraints. The expression $\vartheta(\mathbf{1}_I, 0_L)$ is the optimal value function of the LP problem as a function of its RHS constants. This LP problem has $\mathbf{1}_I$ in common with the LP problem defining the assignment model, but its vector of zero constraints is of a different dimension. (Note: I is used in both models to describe the participants, even though the number of sellers in the two models differs.)

An integer feasible solution is an $\{x(b, Z), x(s, Z)\}$ satisfying the above constraints and $x(b, Z), x(s, Z) \in \mathbb{Z}_+$. For the constraints above, integrality implies there exists exactly one Z such that $x(b, Z) = 1$, $\forall b$ and $x(s, Z) = 1$. The value v_I is precisely the optimal value subject to the integer restriction. Because the LP problem includes the integer feasible solutions, $\vartheta(\mathbf{1}_I, 0_L) \geq v_I$. Inspection of the constraint set reveals that the integer solutions are in fact its extreme points, from which it follows that like the assignment model, there is always an *integral optimal solution*,

$$\vartheta(\mathbf{1}_I, 0_L) = v_I.$$

The dual is:

$$\min \sum_{b \in B} y_b + y_s,$$

subject to

$$y_b + P_b(Z) \geq v_b(Z), \qquad \forall (b, Z)$$

$$y_s - (\textstyle\sum_{b\in B} P_b)(Z) \geq v_s(Z), \qquad \forall Z$$

Rewriting the dual constraints, for each b, $y_b \geq v_b(Z) - P_b(Z),\ \forall Z$, and $y_s \geq v_s(Z) + (\sum_{b\in B} P_b)(Z),\ \forall Z$. Since y_b and y_s are to be minimized, we can set $y_b = v_b^*(P_b)$ and $y_s = v_S^*(\sum_{b\in B} P_b)$. Therefore, the dual can be rewritten as

$$\min_{\{P_b\}} \sum_{b\in B} v_b^*(P_b) + v_s^*(\textstyle\sum_{b\in B} P_b)$$

This reduction leads to:

Proposition 11 (LP equivalence**;** Bikhchandani and Ostroy 2002a)**.** *$(Z, \{P_b\})$ is a pricing equilibrium for the auction model $(\mathcal{Z}, \{v_i\})$ if and only if Z is an integral optimal solution to the primal and $\{P_b\}$ is an optimal solution to the dual, i.e., $y_b = v_b^*(P_b)$ for all b and $y_s = v_s^*(\sum_{b\in B} P_b)$.*

The integral optimality observed above along with the evident existence of optimal solutions to this LP problem implies the existence of pricing equilibria.

5.1 LP and the core

In the assignment model, the core corresponds to the set of optimal solutions to the dual. The same conclusion holds for the auction model.

For the auction model, recall that $I = \{s\} \cup B$, where $B = \{1, \ldots, n\}$. The game theoretic characteristic function for an auction model is described by the collection $\{v_T : T \subseteq I\}$, where $v_\emptyset = 0$, and for non-empty T,

$$v_T = \vartheta(\mathbf{1}_T, 0_L) = \begin{cases} \max_{Z \in \mathcal{Z}} \left(\sum_{i \in T} v_i\right)(Z), & \text{if } s \in T, \\ 0, & \text{otherwise.} \end{cases}$$

The core of $\{v_T\}$ is the set of those (y_i) such that (i) $\sum_{i \in T} y_i \geq v_T$ for all T and (ii) $\sum_{i \in I} y_i = v_I$. If $(Z, \{P_b\})$ is a pricing equilibrium, Proposition 11 implies that setting $y_b = v_b^*(P_b)$ for all b and $y_s^* = v_s^*(\sum_{b \in B} P_b)$, y will satisfy (i) and (ii). The converse also holds.

Proposition 12 (Core Equivalence; Bikhchandani and Ostroy 2002a). *For any (y_i) in the core of the characteristic function $\{v_T\}$ defined by the auction model $(\mathcal{Z}, \{v_i\})$, there exists a pricing equilibrium $(Z, \{P_b\})$ such that $y_b = v_b^*(P_b)$ for all b and $y_s^* = v_s^*(\sum_{b \in B} P_b)$.*

6 An ascending price auction

The logic of dual algorithms is familiar to economists as the idea that prices adjust to excess demand. But the application of that price adjustment rule is identified with linear, anonymous prices. What is remarkable here is that the principle can be applied more generally.

There are two types of dual algorithms which lend themselves to an auction implementation — the sub-gradient algorithm and the primal dual algorithm. It is easier to find a direction of (dual) price change in a sub-gradient algorithm, whereas a restricted primal and restricted dual linear program must be solved to obtain a price change in a primal dual algorithm. The payoff to this additional computation in the primal dual algorithm is that at each iteration price changes can be large. In a sub-gradient algorithm, on the other hand, assurance of close proximity to an optimal solution requires that prices changes be small. Thus, a subgradient algorithm has weaker convergence properties. The ascending price auction in Parkes and Ungar (2000a) and Ausubel and Milgrom (2002) is a sub-gradient algorithm. The reader is referred to Parkes (Chapter 2) and Ausubel and Milgrom (Chapter 3) for a description and properties of this auction. We describe the primal dual auction of de Vries et al. (2003), which builds upon ideas in Demange, Gale, and Sotomayor (1986) and the LP formulation in Bikhchandani and Ostroy (2002a).

Note that any $\{P_b\}$ defines a feasible solution to the dual. If buyers and sellers faced those prices, they could announce utility maximizing demands and supplies. In particular, buyer b is only concerned with the z_b component of Z whereas all the z_b would be relevant for the seller. If prices are such that the amounts buyers wish to purchase exceeds the amount the seller wishes to supply, the primal dual algorithm proposes an adjustment in $\{P_b\}$ to reduce excess demands. When prices are such that each of the buyers together with the seller choose the same Z, that would only be possible if the Z were a feasible solution to the primal. In that case, the primal dual algorithm would stop. And it is readily verified that in such a case the value of the primal would equal the value of the dual and hence an optimal solution would be achieved.

A primal dual algorithm applied to the auction model could begin with any dual feasible solution and converge to some optimal solution to the primal and dual. As far as overall efficiency is concerned, there would be no reason to choose one version of the algorithm over another since they would differ only with respect to the distribution of the gains from trade, not its total. For example, starting from prices that are too high a primal dual algorithm could be constructed mimicking a descending price auction leading to the highest prices at which the commodities could be sold. This would require the willing cooperation of the buyers bidding straightforwardly even though that information would be used against them. But, just as it is commonly presumed that without sufficient information about buyers, a seller cannot enforce "first degree" or perfect price discrimination, here we presume that buyers would bid strategically to lower the prices they pay. To preclude such behavior, it is desirable to have an auction in which each buyer can rely on the fact that the information gleaned from his bidding is used in his favor both with respect to the seller and with respect to the information supplied by the other bidders. This will be the case if buyers understand that they will pay no more than the social opportunity cost of their purchases, i.e., buyers make their Vickrey payments.

If the buyers are substitutes condition is satisfied, there will exist a lowest pricing equilibrium $\{\underline{P}_b\}$ such that if Z is an efficient assignment, $\underline{P}_b(z_b)$ is the social opportunity cost of z_b for all b. Consequently, starting from prices that are too low and adjusting upwards, i.e., creating an ascending price auction, a primal dual algorithm can be constructed that converges to $\{\underline{P}_b\}$. If the buyers are substitutes condition is not satisfied, then in any pricing equilibrium there will be at least one buyer paying more than his social opportunity cost. This buyer would have the incentive to shade his bids.

In the remainder of this section, we outline the ascending-price auction for the heterogeneous object auction model, due to de Vries, Schummer, and Vohra (2003) that leads to the Vickrey payments when buyers are substitutes. This is followed by a simple illustrative example describing the step by step details.

6.1 The auction algorithm

Define utility maximizing demands for b at prices P_b as

$$D_b(P_b) = \{z_b : v_b(z_b) - P_b(z_b) = v_b^*(P_b)\}.$$

This information is used by the auctioneer as follows:

(a) For any $\{P_b\}$, the set of price consistent feasible assignments $\Phi(\{P_b\}) \subset \mathcal{Z}$ is defined as

$$Z \in \Phi(\{P_b\}) \text{ such that } z_b \neq 0 \text{ implies } z_b \in D_b(P_b).$$

(b) Among the price consistent feasible assignments, the set of revenue maximizing feasible assignments for the seller at $\{P_b\}$ is

$$\overline{\Phi}(\{P_b\}) = \operatorname{argmax}\{(\textstyle\sum_{b\in B} P_b)(Z) : Z \in \Phi(\{P_b\})\}.$$

(c) A set of buyers T is *unsatisfied* at prices $\{P_b\}$ if there does not exist a revenue maximizing feasible assignment which simultaneously satisfies the demands of all buyers in T. That is, if $\widehat{Z}$ is an assignment such that $\hat{z}_b \in D_b(\{P_b\})$ for all $b \in T$, then $\widehat{Z} \notin \overline{\Phi}(\{P_b\})$. There may be several unsatisfied sets of buyers at prices $\{P_b\}$. Accordingly, let $\underline{T}$ be a *minimally unsatisfied* set of buyers if no subset of $\underline{T}$ is unsatisfied and let $\underline{B}(\{P_b\})$ be the set of minimally unsatisfied sets of buyers at prices $\{P_b\}$. $\underline{B}(\{P_b\}) = \emptyset$ if and only if $(Z, \{P_b\})$ is a pricing equilibrium for all efficient assignments Z.

Assume that buyers' valuations are discrete, i.e., for all b, $v_b(z_b) \in k^{-1}\mathbb{Z}_+$ for some integer k. To simplify further, assume $k = 1$ so that valuations are integral.

The steps of the auction/algorithm are as follows:[6]

Step 0: Set $t = 1$. Start with prices $P_b^1(z_b) = 0$ for all b and z_b.

Step 1: Buyers report $D_b(P_b^t)$, from which the auctioneer constructs $\underline{B}(\{P_b^t\})$ following (a)–(c) above. If $\underline{B}(\{P_b^t\}) = \emptyset$, go to Step 3. If not, select a $\underline{T}^t \in \underline{B}(\{P_b^t\})$ go to Step 2.

Step 2: For each $b \in \underline{T}^t$ and $z_b \in D_b(P_b^t)$, set $P_b^{t+1}(z_b) = P_b^t(z_b) + 1$. Let $t \leftarrow t + 1$. Go to Step 1.

Step 3: Each b receives a $z_b \in D_b(P_b^t)$ and the auction ends.

This auction is a primal dual algorithm on the LP formulation of the auction model. It converges to a pricing equilibrium (whether or not buyers are substitutes). If buyers are strong substitutes then the auction is incentive compatible and it is an ascending price implementation of the Vickrey auction.

Proposition 13 (Primal dual auction**;** de Vries, Schummer, and Vohra 2003**).** *If buyers are strong substitutes then the above algorithm implements the Vickrey auction, i.e., it converges to a MP pricing equilibrium.*

6.2 An example

There are three buyers, 1, 2, and 3, and two indivisible objects a and b. Buyers' reservation utilities and Vickrey payments are in Table 8.1. At the unique efficient assignment, buyer 1 gets b and 3 gets a.

Insert Table 8.1 around here.

The auction starts with all prices equal to zero. Buyers' demand $\underline{\text{ab}}$ at these prices; in addition, $\underline{\text{a}}$ is also in buyer 3's demand set (see Table 8.2). The set $\overline{\Phi}(\{P_b^1\})$ consists of feasible assignments which give $\underline{\text{ab}}$ to one buyer and nothing to the other two and the feasible assignment which gives $\underline{\text{a}}$ to

Insert Table 8.2 around here.

buyer 3 and nothing to the other two. Any pair of buyers is minimally unsatisfied; we select the first two buyers and raise by one unit prices of all elements in their demand sets, i.e., raise prices of $\underline{\text{ab}}$ for buyers 1 and 2. Thus, at the start of round 2 we have:

Insert Table 8.3 here.

Now, $\underline{\text{b}}$ is added to buyer 1's demand set.[7] Observe that {3} is a minimally unsatisfied set of buyers because for any $Z \in \overline{\Phi}(\{P_b^2\})$, we have $z_3 = 0$, whereas $D_3(p_3^2) = \underline{\text{a}}, \underline{\text{ab}}$. Select the minimally unsatisfied set {1,2} and increment prices of objects in these two buyers' demand sets.

Insert Table 8.4 here.

From Table 8.4 we see that at the start of round 3, the minimally unsatisfied sets are the same as at start of round 2; select {1,2,} again.

Insert Table 8.5 here.

The demand sets at start of round 4 (Table 8.5) are the same as at the end of round 3; however, {3} is the only minimally unsatisfied set of buyers. Thus, in round 4, prices at which $\underline{\mathrm{a}}$ and $\underline{\mathrm{ab}}$ are available to buyer 3 are incremented by one unit.

Insert Table 8.6 here.

Table 8.6 indicates that $(\underline{b}, 0, \underline{a})$ is added to the set of revenue maximizing feasible assignments at the start of round 5. Now, the only minimally unsatisfied set in round 5 is $\{2, 3\}$. After incrementing prices for buyers 2 and 3, there are no unsatisfied buyers. The prices $\{P_b\}$ implement the Vickrey outcome (Table 8.7).

Insert Table 8.7 here.

7 Concluding remarks

Our goal has been to explore the properties of heterogeneous object auctions as extensions of the assignment model. The essential common thread between the assignment and auctions models is that both exhibit integer optimal solutions in their respective linear programming formulations. In both models,

- Optimal solutions to the primal and dual can be identified as pricing equilibria.

- The dual solutions can be identified with the core, and hence with pricing equilibria.

- The primal dual algorithm can be used to implement an ascending price auction.

One contrast between the two models is:

- Integral optimality is a built-in feature of the linear programming formulation of the assignment model, whereas in the combinatorial auction model the linear program must be specifically tailored to achieve it.

This first contrast leads to a second:

- Pricing equilibria in the assignment model are characterized by anonymous pricing, whereas in the combinatorial auction model pricing equilibria are non-linear and non-anonymous.

And this, in turn, leads to a difference with respect to our characterization of the outcome of a Vickrey auction where buyers pay the social opportunity cost of their purchases, called an MP pricing equilibrium.

- In the assignment model, MP pricing equilibria (for buyers) always exists because the buyers are substitutes condition is always satisfied, whereas in the auction model the buyers are substitutes condition must be assumed, either directly or derived from a prior condition such as gross substitutes on buyers' utilities.

The linear programming framework can be used to describe research on efficient, ascending price auctions in private value models. The rows in the Table 8.8 list several models and the papers analyzing them. In the first five rows, buyers are substitutes. Thus, an MP pricing equilibrium exists; the ascending-price auctions that are incentive compatible are exactly the ones that implement an MP pricing equilibrium. In the assignment model, the smallest Walrasian equilibrium is MP pricing. In a combinatorial auction model, Walrasian equilibrium is not typically incompatible with MP pricing. Therefore, the ascending-price auctions in Kelso and Crawford (1982), Gul and Stacchetti (2000), and Milgrom (2001) that implement Walrasian equilibria are not incentive compatible — typically buyers pay more than their social opportunity costs. The auctions in Parkes and Ungar (2000a), Ausubel and Milgrom (2002), and de Vries, Schummer, and Vohra (2003) in the last row implement pricing equilibria. Because buyers need not be substitutes, MP pricing equilibrium need not exist and therefore the Vickrey outcome need not be implemented.

Thus, a number of combinatorial ascending price auctions have important features in common: they implement pricing equilibria; the auction itself is often a primal dual algorithm on an LP formulation of the underlying exchange economy; and they are incentive compatible if and only they implement an MP pricing equilibrium. These features should serve as useful guidelines in comparing common value auctions and in addressing future challenges in auction theory.

Insert Table 8.8 around here.

Acknowledgments: We are grateful to Sven de Vries, David Parkes, Jim Schummer, Rakesh Vohra, and two anonymous referees for helpful comments. Bikhchandani gratefully acknowledges support from the National Science Foundation.

8 References

Ausubel, Lawrence M. (1997), "An Efficient Ascending-bid Auction for Multiple Objects," to appear in *American Economic Review*.

Ausubel, Lawrence M. (2000), "An efficient dynamic auction for heterogeneous commodities," working paper, University of Maryland.

Ausubel, Lawrence M. and Paul Milgrom (2002), "Ascending Auctions with Package Bidding," *Frontiers of Theoretical Economics*, 1, 1, article 1.

Bikhchandani, Sushil and John W. Mamer (1997), "Competitive Equilibrium in an Exchange Economy with Indivisibilities," *Journal of Economic Theory*, 74, 385-413.

Bikhchandani, Sushil and Joseph M. Ostroy (2002a), "The Package Assignment Model," *Journal of Economic Theory*, 107, 377-406.

Bikhchandani, Sushil and Joseph M. Ostroy (2002b), "Ascending Price Vickrey

Auctions," to appear in *Games and Economic Behavior*.

Bikhchandani, Sushil, Sven de Vries, James Schummer, and Rakesh Vohra, (2002), "Linear Programming and Vickrey Auctions," in *Mathematics of the Internet: E-Auction and Markets*, Brenda Dietrich and Rakesh Vohra (eds.), New York: Springer Verlag, pp. 75-115.

Crawford, Vincent P. and Elsie M. Knoer (1981), "Job Matching with Heterogeneous Firms and Workers," *Econometrica*, 49, 437-450.

Demange, Gabrielle, David Gale, and Marilda Sotomayor (1986), "Multi-item Auctions," *Journal of Political Economy*, 94, 863-872.

de Vries, Sven, James Schummer, and Rakesh Vohra (2003) "On Ascending Vickrey Auctions for Heterogeneous Objects," working paper, Northwestern University.

Gale, David (1960), *The Theory of Linear Economic Models*, Chicago: The University of Chicago Press.

Gretsky, Neil, Joseph M. Ostroy, and William Zame (1999) "Perfect Competition in the Continuous Assignment Model," *Journal of Economic Theory*, 88, 60–118.

Gul, Faruk and Ennio Stacchetti (1999), "Walrasian Equilibrium with Gross Substitutes," *Journal of Economic Theory*, 87, 96–124.

Gul, Faruk and Ennio Stacchetti (2000), "The English Auction with

Differentiated Commodities," *Journal of Economic Theory*, 92, 66-95.

Kantorovich, Leonid V. (1942), "On the Translocation of Masses," *Doklady Akad. Navk S.S.S.R.* 37, 199-201. Translated in *Management Science*, 5, October 1958, 1-4.

Kelso, Alexander and Vincent Crawford (1982), "Job Matching, Coalition Formation, and Gross Substitutes," *Econometrica*, 50, 1483-1504.

Koopmans, Tjallings C. and Martin Beckman, "Assignment Problems and the Location of Economic Activities," *Econometrica* **25** (1957), 53–76.

Kuhn, Harold (1955), "The Hungarian Method for the Assignment Problem," 3, 253-258, *Naval Research Logistics Quarterly*.

Makowski, Louis (1979), "Value Theory with Personalized Trading," *Journal of Economic Theory*, 20, 194-212.

Makowski, Louis and Joseph M. Ostroy (1987), "Vickrey-Clarke-Groves Mechanisms and Perfect Competition," *Journal of Economic Theory*, 42, 244-261.

Milgrom, Paul (2001), "Putting Auction Theory to Work: The Simultaneous Ascending Auction," *Journal of Political Economy*, 108, 245-272.

von Neumann, John (1953), "A Certain Zero-Sun Two-Person Game Equivalent to the Optimal Assignment Problem," in *Contributions to the Theory of* *games*, Harold Kuhn and Albert Tucker (eds.), Princeton University Press.

Parkes, David C. and Lyle H. Ungar (2000a), "Iterative Combinatorial Auctions: Theory and Practice," *Proc. 17th National Conference on AI*, 74-81.

Parkes, David C. and Lyle H. Ungar (2000b), "Preventing strategic manipulation in iterative auctions," *Proc. 17th National Conference on AI*, 82-89.

Shapley, Lloyd (1955), "Markets as Cooperative Games," RAND Corporation working paper P-629, Santa Monica, CA.

Shapley, Lloyd (1962), "Complements and Substitutes in the Optimal Assignment Problem," *Naval Research Logistics Quarterly* 9, 45-48.

Shapley, Lloyd and Martin Shubik (1972), "The Assignment Game I: The Core," *International Journal of Game Theory*, 1, 111-130.

Notes

[1] See also Ausubel and Milgrom (Chapter 1).

[2] Such a scheme mimics the reward principle in a perfectly competitive market. Makowski and Ostroy [1987].

[3] See Kelso and Crawford (1982), Bikhchandani and Mamer (1997), and Gul and Stacchetti (1999).

[4] If $z_b = 1_c$, then $p \cdot z_b = p_c$, i.e., $p \cdot z_b$ is the dot product of p and z_b.

[5] An exception is Makowski (1979) who gives a non-linear, non-anonymous version of Walrasian equilibria.

[6] In this auction, the packages in the demand sets of buyers in a minimal unsatisfied set play the role of a minimal overdemanded set of objects in the ascending price auction for the assignment model described in Section 2.

[7] Under the rules of the auction, the demand sets of buyers are non-decreasing; that is once in the demand set, a package never leaves it.