Ascending Price Vickrey Auctions

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Abstract

We show that an ascending price auction for multiple units of a homogeneous object proposed by Ausubel (i) raises prices for packages until they reach those non-linear and non-anonymous market clearing prices at which bidders get their marginal products and (ii) the auction is a primal-dual algorithm applied to an appropriate linear programming formulation in which the dual solution yields those same market clearing prices. We emphasize the similarities with efficient incentive compatible ascending price auctions to implement Vickrey payments when there is a single object or when objects are heterogeneous but each buyer does not desire more than one unit. A potential benefit of these common threads is that it helps to establish the principles upon which Vickrey payments may be implemented through decentralized, incentive compatible procedures.

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1 Introduction

In an ascending price (i.e., English) auction for a single indivisible item, prices start at a level at which there is excess demand. Prices rise until only one buyer remains. An ascending price auction therefore implements the smallest price at which demand equals supply. Under private values, buyers get their marginal product — called the Vickrey payoff point, below — and therefore truth-telling is a dominant strategy.

A similar result obtains in the standard assignment model. Although there are several heterogeneous indivisible objects, the *unit demand* assumption is satisfied, i.e., when offered a choice of several objects, each buyer wants to purchase at most one. Walrasian equilibrium and a smallest Walrasian price vector exist in the standard assignment model. The Hungarian method developed by Kuhn (1955) finds that smallest price vector; once again, demand equals supply and buyers get their marginal product at these market clearing prices. This has been exploited by Crawford and Knoer (1981) and Demange, Gale, and Sotomayor (1986) to construct incentive compatible ascending price auctions in which prices for objects with excess demand rise until markets clear at the smallest Walrasian prices.

This approach of implementing the Vickrey payoffs through market clearing prices can be extended to yield incentive compatible ascending price auctions for multiple units of a homogeneous object when each buyer's marginal utility for more than one object may be strictly positive, but non-increasing. As in the standard assignment model, we restrict attention to private values, i.e., the valuation of objects by one participant is not influenced by others' information; moreover, preferences are quasi-linear.

In Bikhchandani and Ostroy (2001) it is shown that efficiency and incentive compatibility are satisfied only when there exists an integer-valued solution to a linear programming formulation (of the underlying exchange economy) that implements the Vickrey payoff point. Such a solution must allow for prices that are nonlinear (i.e., the price of a bundle need not equal the sum of prices of objects in the bundle) and non-anonymous (i.e., the prices faced by different buyers may be different). Otherwise, if prices are required to be linear and anonymous, efficiency and incentive compatibility are inconsistent.

A necessary and sufficient condition for the Vickrey payoff point to be implementable through (nonlinear and non-anonymous) prices is that buyers are substitutes in the sense of Shapley (1962). When the objects for sale are identical, a sufficient condition for the buyers are substitutes property is that buyers have diminishing marginal utility. In this setting, Ausubel (1997) has constructed an ascending price version of the sealed bid Vickrey auction. We show that this ascending price auction is a primal-dual algorithm applied to the linear programming problem described above.

We demonstrate that the auction stops when prices increase to that pricing equilibrium in which buyers receive their marginal products, called the *marginal product pricing equilibrium*.

To highlight the role of the primal-dual algorithm, observe that if the simplex algorithm were used to solve the linear programming problem, the implementor would need to have complete knowledge of the data of the problem (tastes and technology). In contrast, the primal-dual algorithm mimics the emphasis on decentralized decision-making found in *tâtonnement*. It (1) starts with a dual feasible solution (prices); then (2) a tentative solution to the primal is found predicated on the hypothesis that the dual is optimal (individual maximization *as if* prices were market-clearing); (3) a check is made to see if the tentative solution to the primal is feasible (market-clearing). If so, the dual feasible solution and the primal are optimal. If not, discrepancies between the tentative solution and feasibility are used to select another dual feasible solution (adjust prices to excess demands) and steps (2) and (3) are repeated.

Informational economies are not, however, the only consideration. In a decentralized scheme, each individual must also have the incentive to act sincerely rather than adopt other behavior that would lead to a more favorable outcome. For the primal-dual algorithm to achieve incentive compatibility there should be no incentive for anyone to deviate from sincere decentralized decision-making along its path. Hence, unlike the traditional version of *tâtonnement* that relies on linear prices, this version of the primal-dual algorithm requires a richer set of prices to implement the Vickrey payoff point.

Thus, ascending price auctions for the standard assignment model and for multiple homogeneous objects are similar in two important respects. First, they are practical implementations of the primal-dual algorithm on a linear programming formulation of the underlying exchange economy. Second, by finding the prices that give them their marginal products, bidders are provided with the incentive to bid sincerely along the way. A potential benefit of these common threads is that it helps to establish the principles upon which Vickrey payments may be implemented through decentralized, incentive compatible procedures.

In Section 2, we describe Ausubel's implementation of the Vickrey auction for homogeneous objects. In Section 3, we present a linear programming formulation of an exchange economy with several buyers and one seller endowed. That Ausubel's ascending price auction is a primal-dual algorithm on this linear programming formulation is established in Section 4. Concluding remarks and prospects for extending these results to settings with heterogeneous objects are discussed in Section 5.

2 An example

We illustrate Ausubel's ascending price implementation of the Vickrey auction as an application of the primal-dual algorithm through an example.

Example 1: Four identical units are for sale to three buyers, $b = \mathrm{I, II, III}$. The buyers marginal utilities are in the table below. The seller's cost is zero.

Marginal utilities

z	1	2	3	4
$\Delta u_{\mathrm{I}}(z)$	**7**	2	1	0
$\Delta u_{\mathrm{II}}(z)$	**8**	**5**	2	0
$\Delta u_{\mathrm{III}}(z)$	**4**	4	2	0

The marginal utilities of units allocated at the efficient assignment are in bold face.

Steps in the auction

	p	$\underline{z}_{\mathrm{I}}$	$\underline{z}_{\mathrm{II}}$	$\underline{z}_{\mathrm{III}}$	OD	a_{I}, $p_{\mathrm{I}}(a_{\mathrm{I}})$	a_{II}, $p_{\mathrm{II}}(a_{\mathrm{II}})$	a_{III}, $p_{\mathrm{III}}(a_{\mathrm{III}})$
–	\$ 0	3	3	3	4	–	–	–
$0 < \theta < 1$	\$ θ	3	3	3	4	–	–	–
–	\$ 1	2	3	3	4	–	–	–
$0 < \theta < 1$	\$ $1 + \theta$	2	3	3	4	–	–	–
–	\$ 2	1	2	2	2	–	1, \$2	1, \$2
$0 < \theta < 2$	\$ $2 + \theta$	1	2	2	2	–	1, \$2	1, \$2
–	\$ 4	1	2	1	0	1, \$4	2, \$2+\$4	1, \$2

Notes: p is the price of each overdemanded (i.e., unallocated) unit.
$\underline{z}_b$is the smallest element in b's demand set, $b = \mathrm{I, II, III}$.
a_b is the amount allocated to b for a payment of $p_b(a_b)$, $b = \mathrm{I, II, III}$.
OD is the number of overdemanded units.

At any stage in the auction, units are allocated to (or "clinched" by) a buyer if the total demand from all other buyers is less than the total supply. Initially, as the price for each of the four units is raised from \$0 to \$1, buyers I, II, and III demand 3 units each. Since the sum of demands of buyers II and III, say, exceeds the total supply, nothing is allocated to buyer I. Similarly, nothing is allocated to buyers II and III. When the price reaches \$1, I reduces demand to 2 units; still, nothing can be allocated to buyers II or III. Hence, the price is raised further until it reaches \$2 for each of the 4 units. At this point, buyers I, II, and III demand 1 unit, 2 units, and 2 units respectively. Consider buyer II. If one were to satisfy the total demand of 3 units from buyers I and III, 1 of the 4 units would still remain. Therefore, if one assumes that I and III will not increase their demand, 1 unit may be allocated to II. At this stage in the auction, 1 unit is allocated to II at the current price of \$2. Similarly, III gets a unit at price \$2, and I is not allocated anything. The overdemanded set consists of 2 units, i.e., 2 of the 4 units are unallocated.

The prices of the 2 overdemanded units (but not of units allocated to I and III respectively) are raised from the current level of \$2 until at least one buyer changes demand. This happens when the price reaches \$4, at which point III reduces demand to 1. Now, one each of the 2 remaining units is allocated to buyers I and II at the price of \$4.[1]

The auction achieves the efficient allocation of 1 unit to I, 2 units to II, and 1 unit to III. As in the sealed bid Vickrey auction, each buyer pays his social opportunity cost. That is, each buyer pays an amount equal to the reduction in utility of other buyers due to this buyer's participation in the auction.

To clearly specify the auction one must define the set of overdemanded units, the rules for allocating units to buyers, and the payment rules. We provide formal definitions later, in Section 4. The steps in the auction are as follows:[2]

Step 0: Start with price zero.

Step 1: Buyers report their smallest demand at current prices. Compute for each buyer if he may be allocated additional units; allocate these units at the current price. Compute for each buyer if some of the allocations previously made have to be unmade;[3] these units revert to the set of overdemanded units at the current price. If the demand of all buyers is satisfied, go to step 3.

Step 2: Raise price of overdemanded units until a buyer changes demand. Go to step 1.

Step 3: All units are allocated. The payment of each buyer is the sum of the prices at which units were allocated to him. The auction ends.[4]

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[1] Actual transfers of objects and money payments take place after the auction is over.

[2] There are two minor differences between this auction and the one in Ausubel (1997): (i) a buyer may increase his demand as price increases and (ii) a buyer may decrease demand below the number of units he has been allocated.

[3] This possibility arises if some bidder increases demand with price. Under Assumption 1, buyers will not increase their demands along the equilibrium path.

[4] It is possible that the total demand is less than the total supply at the final price. In order to exhaust the available supply in this event, the total allocation to buyers who reduced their demand at the final price is an amount between their final demand and their demand prior to this last reduction.

If buyers have decreasing marginal utility, then sincere bidding is an ex post equilibrium in this auction. At this point, we leave unspecified what buyers know about others' demands, allocations, and their own allocations as the auction proceeds. As bidders' knowledge increases, equilibria involving non-sincere bidding may also exist.

For purposes of comparison, we draw attention to an ascending price implementation of the Vickrey auction for the standard assignment model. This auction is due to Demange, Gale, and Sotomayor (1986) and Crawford and Knoer (1981). Demange, Gale, and Sotomayor point out that this auction is a variant of the Hungarian method of Kuhn (1955).

In the standard assignment model, buyers' demands consist of single items. Hence, a set of objects is overdemanded if the number of items in the set is less than the number of buyers whose demands are in that set. As in the Ausubel auction for homogenous objects, in the auction presented by Demange et. al. prices are raised on a minimal overdemanded set. However, the notion of overdemanded sets seems to be different in the two settings. To reconcile the concept of overdemanded sets in the two models, imagine that every buyer has L agents (one for each unit sold), each of whom wishes to buy one unit. The reservation value of buyer b's zth agent is equal to b's marginal utility for the zth unit. If, at some stage, buyer b is allocated say z units, then these units and the first z agents of buyer b leave the economy. Each of the remaining agents of buyer b demands one unit from the set of unallocated units, provided that the price is less than the agent's reservation value (= the marginal utility) of that unit.

In Section 4, we elaborate further on this notion of overdemanded sets and show that the auction is a primal-dual algorithm on a linear programming formulation of the underlying exchange economy.

3 Linear programming and Vickrey payments

Before presenting the algorithm we must first describe the linear programming formulation upon which it is based. A key feature of this linear program is that the dual allows for sufficient nonlinear and non-anonymous pricing to implement Vickrey payments. This section builds on results from Bikhchandani and Ostroy (2001).

Let $b = 1, 2, ..., B$ be the buyers and let s denote the seller. The set of agents in the economy is denoted $N = \{1, 2, ..., B, s\}$. The seller has L identical indivisible units of an object (or equivalently has the capacity to make L units of the object).

A *feasible assignment* of units (of the object) from the seller to the buyer is

$$Z = (z_1, z_2, ..., z_b, ..., z_B, z_s),$$

where each z_b is a non-negative integer, $\sum_{b=1}^{B} z_b \leq L$, and $z_s = L - \sum_{b=1}^{B} z_b$. The allocation to buyer b is z_b and the seller is left with z_s. Let $\mathcal{Z}$ be the set of feasible assignments.

Buyers have utility over non-negative integer-valued quantities z and (divisible) money, $m \in \Re$. Buyer b's utility function is

$$U_b(z, m) \ = \ u_b(z) + m,$$

where $u_b(z)$ is buyer b's reservation value for z. It is convenient to assume that the buyer's initial endowment of the money commodity is normalized to be zero and that the buyer can supply any (negative) quantity required. Letting $u_b(0) = 0$, the utility of no trade for the buyer is $U_b(0, 0) = u_b(0) + 0 = 0$.

The seller's utility function over the bundle (z, m), $0 \leq z \leq L$, $m \in \Re$, is

$$u_s(z, m) \ = \ m,$$

where z is the number of the units sold, m is the money received in exchange.[5] The seller has no endowment of the money commodity and therefore the utility of no trade for the seller is $U_s(\phi, 0) = 0$, where $\phi = (0, \ldots, 0, L)$ is the null assignment.

Denote this economy $\mathcal{E}$.

An *efficient assignment*, Z^*, is a feasible assignment which maximizes the sum of buyers' reservation values:

$$Z^* \ \equiv \ \arg \max_{Z=(z_b)\in\mathcal{Z}} \left\{ \sum_{b=1}^{B} u_b(z_b) \right\}. \tag{1}$$

A *pricing function* is $\langle p_b(w) \rangle$, $\forall w \leq L$, $\forall b$. The price paid by buyer b for z units is $p_b(z)$. The price paid may be nonlinear in the units in the package and may depend on the identity of the buyer. The revenue received by the seller for assignment $Z = (z_b)$ is:

$$P(Z) \equiv \sum_{b=1}^{B} p_b(z_b).$$

Definition: A *pricing equilibrium* for $\mathcal{E}$ is a $[\, Z^* = (z_b^*),\ \langle p_b^*(\cdot) \rangle \,]$ such that $p_b^*(\cdot) \geq 0$, $p_b^*(0) = 0$, and

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[5] All the results in this paper can be proved under the assumption that the seller has a non-decreasing cost $c(z)$ of making z units and $u_s(z, m) \ = \ m - c(z)$.

- Z^* is a feasible assignment,
- buyers maximize utilities: for all b,

$$u_b(z_b^*) - p_b^*(z_b^*) \geq u_b(z) - p_b^*(z), \qquad \forall z = 0, 1, 2, ..., L$$

- seller maximizes profits: for all feasible assignments Z'

$$P^*(Z^*) \geq P^*(Z').$$

Note: If the pricing functions $\langle\, p_b^*(\cdot)\,\rangle$ were of the form $p_b^*(z_b) = p \cdot z_b$, this would be the usual definition of Walrasian equilibrium.

3.1 An LP characterization of the multi-unit model

There is a linear programming characterization of the set of efficient assignments and of pricing equilibrium in this economy. Regard $x_b(z) \in [0, 1]$ as the fraction of package z received by buyer b and $x_s(Z) \in [0, 1]$ as the fraction of feasible assignment Z delivered by the seller. Consider the following linear program,

LP:

$$\max \sum_{b=1}^{B} \sum_{z=0}^{L} x_b(z) u_b(z)$$

s.t.

$$\begin{aligned} \sum_{z=0}^{L} x_b(z) &= 1, \qquad b = 1, 2, ..., B && (2) \\ \sum_{Z \in \mathcal{Z}} x_s(Z) &= 1 && (3) \\ x_b(z) - \sum_{Z \in G^b(z)} x_s(Z) &= 0, \qquad z = 0, 1, 2, ..., L,\ b = 1, 2, ..., B && (4) \end{aligned}$$

$$x_b(\,\cdot\,),\, x_s(\,\cdot\,) \;\geq\; 0,$$

where $G^b(z) \subset \mathcal{Z}$ is the set of assignments in which buyer b gets z units. Constraint (2) ensures that the sum of fractions of packages bought is equal to one for each buyer. (As the sum in this constraint includes $z = 0$, the possibility that buyer b is allocated nothing is allowed.) Similarly, (3) requires that the sum of fractions of assignments sold is equal to one. Finally, demand equal to supply is guaranteed by (4).

For any feasible assignment $Z' = (z_b')$, define $x_b'(z) = 1$ if $z = z_b'$ and $x_b'(z) = 0$ if $z \neq z_b'$. Similarly, let $x_s'(Z) = 1$ if $Z = Z'$ and $x_s'(Z) = 0$ if $Z \neq Z'$. It is readily verified that $x_b'(\,\cdot\,)$, $x_s'(\,\cdot\,)$ satisfies (2), (3), and (4). Thus, all feasible assignments are (integer) feasible solutions of LP.

If one replaces the non-negativity constraints, $x_b(\,\cdot\,)$, $x_s(\,\cdot\,) \geq 0$, in LP by integer constraints, $x_b(\,\cdot\,)$, $x_s(\,\cdot\,) \in \{0,1\}$, then the optimal solution set of the resulting integer program is the set of efficient assignments. However, this restriction to integer constraints is unnecessary.[6]

Proposition 1 *The set of efficient assignments are (integer) optimal solutions to LP.*

The dual of LP is

DLP:

$$\min_{\pi_s,\pi_b} \; \pi_s + \sum_{b=1}^{B} \pi_b$$

s.t.

$$\begin{aligned} \pi_b - [u_b(z) - p_b(z)] &\geq 0, \qquad z = 0,1,2,...,L,\; b = 1,2,...,B && (5) \\ \pi_s - P(Z) &\geq 0, \qquad \forall Z = (z_b). && (6) \end{aligned}$$

where $P(Z) = \sum_b p_b(z_b)$. Interpret π_b as buyer b's surplus and π_s as the seller's surplus (profit). Inequalities (5) imply that buyers maximize utility at prices $p_b(\cdot)$, and inequalities (6) imply that the seller maximizes profits at prices $P(\cdot)$. The dual minimizes the sum of buyers' and the seller's surpluses subject to the constraint that surplus is at least equal to what can be obtained from a pricing function.

For any pricing function $\langle p_b(\cdot)\rangle$, define

$$\pi_b^*(p_b) \;\equiv\; \max_z [u_b(z) - p_b(z)], \quad \forall b, \qquad \pi_s^*(P) \;\equiv\; \max_Z [P(Z)].$$

It is readily seen that (a) *any* such $(\,\pi_s^*(P),\, (\pi_b^*(p_b)),\, \langle p_b(\cdot)\rangle\,)$ is a feasible solution to DLP; (b) any optimal solution will be of this form. In fact, the optimal solutions to LP/DLP characterize pricing equilibria.

Proposition 2 $[\,Z^* = (z_b^*),\, \langle p_b^*(\cdot)\rangle\,]$ *is a pricing equilibrium for* $\mathcal{E}$ *if and only if* Z^* *is an optimal solution to LP and* $(\,\pi_s^*(P^*),\, (\pi_b^*(p_b^*)),\, \langle p_b^*(\cdot)\rangle\,)$ *is an optimal solution to DLP.*

Vickrey payments as buyers' marginal products

Consider a solution to LP in which buyer b is assigned z_b. The social opportunity cost of z_b is the maximum amount that the *other* buyers would have been willing to pay for z_b. If b pays the social opportunity cost of z_b, then b receives all of the extra gains his participation confers on the economy, called b's marginal product. It

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[6] Propositions 1, 2, 5, and 6 are proved in Bikhchandani and Ostroy (2001). Hence, their proofs are omitted.

is well-known that the Vickrey payment in a private value auction gives buyers their marginal products. Hence, such payments provide buyers with the incentive to bid sincerely. Here, we formalize these properties with respect to LP and DLP.

Let $F(R)$ be the optimal value of LP as function of the right-hand side values of the constraints. In the above formulation, $R = (\mathbf{1}, \mathbf{0})$, where $\mathbf{1}$ is a vector of ones in $\Re^{B+1}$ ($B+1$ is the number of elements in N) and $\mathbf{0}$ is the zero element of $\Re^{(L+1)\times B}$. (Note: The number of constraints in (2) and (3) equals the number of elements in $\mathbf{1}$ and the number of constraints in (4) equals the dimension of $\mathbf{0}$.) $F(\mathbf{1}, \mathbf{0})$ is therefore the maximum gains from trade in the economy $\mathcal{E}$. Let $\mathbf{1_{-b}}$ be the vector in $\Re^{B+1}$ in which the b^{th} element of $\mathbf{1}$ is replaced with a zero denoting $\mathcal{E}_{-b}$, the economy without buyer b; and let $\mathbf{1}_{zb}$ be the element of $\Re^{(L+1)\times B}$ with a one in the $(z, b)^{th}$ place and zeroes elsewhere. The quantity $F(\mathbf{1}_{-b}, -\mathbf{1}_{zb})$ represents the maximum gains in $\mathcal{E}_{-b}$ when its members are required to give up one package of z units of the commodity to buyer b. Define the social opportunity cost of that transfer as

$$\phi_{-b}(z) = F(\mathbf{1}_{-b}, \mathbf{0}) - F(\mathbf{1}_{-b}, -\mathbf{1}_{zb}),$$

the difference between the maximum total gains to the economy $\mathcal{E}_{-b}$ when they keep z and the gains when they must give up z. The nonlinearity and non-anonymity of prices in a Vickrey auction is a direct result of the fact that $\phi_{-b}(\cdot)$ is typically nonlinear and $\phi_{-b'}$ need not equal ϕ_{-b} when $b \neq b'$.

Buyer b's *marginal product*, MP_b, is the increase in the gains from trade due to b, i.e.,

$$\mathrm{MP}_b \equiv F(\mathbf{1}, \mathbf{0}) - F(\mathbf{1}_{-b}, \mathbf{0}).$$

It is readily verified that if $Z^* = (z_1^*, z_2^*, ..., z_B^*)$ is an optimal solution to LP, then

$$\begin{aligned} F(\mathbf{1}, \mathbf{0}) &= \max_{0\leq z\leq L} \{u_b(z) + F(\mathbf{1}_{-b}, -\mathbf{1}_{zb})\} \\ &= u_b(z_b^*) + F(\mathbf{1}_{-b}, -\mathbf{1}_{z_b^* b}). \end{aligned}$$

Combining these results yields

$$\mathrm{MP}_b = u_b(z_b^*) - \phi_{-b}(z_b^*).$$

From Proposition 2, a pricing equilibrium $(Z^*, \langle p_b^*(\cdot)\rangle)$ in $\mathcal{E}$ can be identified with an optimal solution to LP. The duality theory of linear programming implies that

$$\sum_b \pi_b^*(p_b^*) + \pi_s^*(P^*) = F(\mathbf{1}, \mathbf{0}). \tag{7}$$

Because a feasible solution to the dual of the LP problem associated with $\mathcal{E}$ is also a feasible solution to the dual of the problem associated with $\mathcal{E}_{-b}$, linear programming theory implies

$$\sum_{b'\neq b} \pi_{b'}^*(p_{b'}^*) + \pi_s^*(P^*) \geq F(\mathbf{1}_{-b}, \mathbf{0}). \tag{8}$$

It follows immediately that:

Proposition 3 *For any pricing equilibrium* $(Z^*, \langle p_b^*(\cdot)\rangle)$ *in* $\mathcal{E}$*,*

$$MP_b \;\geq\; \pi_b^*(p_b^*), \qquad \forall b,$$

where $\pi_b^*(p_b^*) = u_b(z_b^*) - p_b^*(z_b^*)$*. Hence,* $p_b^*(z_b^*) \geq\; \phi_{-b}(z_b^*)$*.*

Proposition 3 says that the utility received by a buyer in any pricing equilibrium does not exceed his marginal product. This occurs because the buyer's payment is never less than the social opportunity cost of his purchase. We use these inequalities to define $\langle \underline{p}_b(\cdot)\rangle$ as a *marginal product pricing equilibrium* if $\pi^*(\underline{p}_b) = \mathrm{MP}_b$, $\forall b$. Equivalently, an mp pricing equilibrium satisfies $\underline{p}_b(z_b^*) = \phi_{-b}(z_b^*)$, $\forall b$. [Note: There may be many pricing functions yielding an mp pricing equilibrium, but they are all utility equivalent.]

From (7) and (8), it is easy to see that $\langle \underline{p}_b(\cdot)\rangle$ is an mp pricing equilibrium if and only if

$$\sum_{b' \neq b} \pi_{b'}^*(\underline{p}_{b'}) + \pi_s^*(\underline{P}) = F(\mathbf{1}_{-b}, \mathbf{0}).$$

Therefore, since $(\pi^*(\underline{p}_b), \pi_s^*(\underline{P}), \langle \underline{p}_b(\cdot)\rangle)$ is feasible for the LP problem associated with $\mathcal{E}_{-b}$, it follows that it must also be optimal. Converting this result on optimal solutions for linear programming problems to its corresponding implications for pricing equilibria, we have:

Proposition 4 *If an mp pricing equilibrium* $\langle \underline{p}_b(\cdot)\rangle$ *exists for* $\mathcal{E}$*, then for all* b*,* $\langle \underline{p}_{b'}(\cdot)\rangle_{b' \neq b}$ *is a pricing equilibrium for* $\mathcal{E}_{-b}$*. Conversely, any pricing equilibrium* $\langle p_b(\cdot)\rangle$ *for* $\mathcal{E}$ *such that for all* b*,* $\langle p_{b'}(\cdot)\rangle_{b' \neq b}$ *is a pricing equilibrium for* $\mathcal{E}_{-b}$ *is an mp pricing equilibrium for* $\mathcal{E}$*.*

An individual may be defined as a perfect competitor if prices need not change when that individual is withdrawn from the economy. Hence, Proposition 4 says that the existence of an mp equilibrium is equivalent to the condition that buyers, but not necessarily the seller, are perfect competitors. Makowski and Ostroy (1987) describe the connections between perfect competition and mp pricing in a quasi-linear model when all participants are treated symmetrically.

Let

$$\underline{\mathrm{R}}_s \equiv F(\mathbf{1}, \mathbf{0}) - \sum_b \mathrm{MP}_b.$$

It may be verified that $\underline{\mathrm{R}}_s \geq 0$. (See Bikhchandani and Ostroy 2001.) Call the point $(\mathrm{MP}_1, \mathrm{MP}_2, ..., \mathrm{MP}_b, ..., \mathrm{MP}_B, \underline{\mathrm{R}}_s)$ the *Vickrey payoff point* (VPP).

When it exists, an mp pricing equilibrium combines the computational advantages of linear programming with the incentive properties of Vickrey auctions. To guarantee existence, define the marginal product of a subset of buyers T as

$$\mathrm{MP}_T \;\equiv\; F(\mathbf{1},\mathbf{0}) - F(\mathbf{1}_{-T},\mathbf{0}), \qquad T \subseteq \{1,2,...,b,...,B\},$$

where $\mathbf{1}_{-T}$ denotes the economy $\mathcal{E}_{-T}$, i.e., without the buyers in T. If for all $T \subseteq \{1,2,...,b,...,B\}$

$$\mathrm{MP}_T \;\geq\; \sum_{b\in T} \mathrm{MP}_b. \tag{9}$$

then we say, following Shapley (1962), that *buyers are substitutes* in $\mathcal{E}$.

The next proposition states that the substitutes condition is exactly what is required to permit the set of pricing equilibria to admit a VPP.

Proposition 5 *An mp pricing equilibrium exists if and only if buyers are substitutes.*

Thus, Figures 1a and 1b are illustrative of pricing equilibrium payoffs when the substitutes condition does and does not hold respectively.

Insert Figures 1a and 1b about here.

To give sufficient conditions for the substitutes hypothesis to be satisfied in the multi-unit auction, define the marginal utility for buyers in the usual manner:

$$\Delta u_b(z) \;\equiv\; u_b(z) - u_b(z-1), \qquad z = 1,2,...,L, \;\forall b = 1,2,...,B$$

Assumption 1: *Buyers have decreasing marginal utility. The seller has increasing marginal costs.*

Proposition 6 *Under assumption 1 buyers are substitutes.*

To summarize, when buyers have decreasing marginal utility and the seller has increasing marginal cost, the opportunity cost functions $\langle \phi_b(\cdot) \rangle$ constitute an mp pricing equilibrium.[7]

Example 1 (continued): The gains from trade of $F(\mathbf{1},\mathbf{0}) = 24$ are attained at the efficient assignment $(1,2,1,0)$. The Vickrey payoff point is ($\mathrm{MP}_{\mathrm{I}} = 3$, $\mathrm{MP}_{\mathrm{II}} = 7$, $\mathrm{MP}_{\mathrm{III}} = 2$, $\underline{\mathrm{R}}_s = 12$) and the opportunity costs of buyers' allocations (i.e., the

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[7] Propositions 1–5 generalize to models with heterogeneous objects. Further, if buyers' utility functions satisfy the gross substitutes assumption of Kelso and Crawford (1982) then buyers are substitutes.

Vickrey payments) are ($\mathrm{OC}_{\mathrm{I}} = 4$, $\mathrm{OC}_{\mathrm{II}} = 6$, $\mathrm{OC}_{\mathrm{III}} = 2$). The smallest Walrasian price (i.e., a pricing equilibrium in linear and anonymous prices) is $p = 4$; at this price bidders II and III pay more than their opportunity costs. An mp pricing equilibrium in nonlinear and non-anonymous prices exists and is given in the table below. The prices of buyer-packages in the efficient assignment are in bold-face; observe that these are equal to the Vickrey payments.

mp pricing equilibria

z	1	2	3	4
$p_{\mathrm{I}}(z)$	**4**	6	7–8	7–12
$p_{\mathrm{II}}(z)$	1–2	**6**	8	8–12
$p_{\mathrm{III}}z)$	**2**	6	8	8–12

4 An incentive compatible primal-dual algorithm

The previous section described how Vickrey payments in the homogeneous object auction could be characterized as the solution to DLP when buyers have diminishing marginal utility. In this section we show that an ascending price auction can be used to implement those payments via a primal-dual algorithm. The main result of this paper is:

Theorem 1 *If buyers have diminishing marginal utility then Ausubel's ascending price auction for multiple units is (1) a dynamic mechanism for discovering a pricing equilibrium yielding each bidder his marginal product, thereby ensuring that it is an ex post equilibrium to bid truthfully; moreover, (2) it is a primal-dual algorithm on the linear programs LP and DLP of Section 3.1. The auction implements the sealed bid Vickrey auction outcome.*

The proof is developed in the remainder of this section.

Every optimal solution to LP yields an efficient allocation, but not every optimal solution to DLP, equivalently every pricing equilibrium, encourages sincere bidding. To implement a Vickrey outcome via a primal-dual algorithm, it must converge to an mp pricing equilibrium.

Let $(\pi_s, (\pi_b), \langle p_b(\,\cdot\,)\rangle)$ be a DLP feasible solution with $p_b(\,\cdot\,) \geq 0$. Define:

$$\begin{aligned} D_b(p_b) &\equiv \{\, z \leq L \,|\, \pi_b + p_b(z) = u_b(z) \,\} \;=\; \{\, z \,|\, u_b(z) - p_b(z) \geq u_b(z') - p_b(z'),\ \forall z' \,\} \\ S(P) &\equiv \{\, Z \,|\, \pi_s - P(Z) = 0 \,\} \;=\; \{\, Z \,|\, P(Z) \geq P(Z'),\ \forall Z' \,\} \end{aligned}$$

where $P(Z) = \sum_b p_b(z_b)$, $Z = (z_b)$. Thus, $D_b(p_b)$ is the set of utility maximizing bundles (i.e., the demand set) for buyer b and $S(P)$ is the set of profit maximizing assignments (i.e., the supply set) for the seller at prices $p_b(\cdot)$, $P(\cdot) = \sum_b p_b(\cdot)$. As $z = 0$ and $Z = (0)$ are permissible packages and assignments respectively, $D_b(p_b)$ and $S(P)$ are non-empty for any dual feasible solution.

Associated with this feasible solution to DLP, we have a "solution" to LP, $(\,x_b(\,\cdot\,),\, x_s(\,\cdot\,)\,) \;\geq\; 0$, which satisfies complementary slackness:

$$\text{If } z \notin D_b(p_b) \qquad \text{then } x_b(z) = 0. \tag{10}$$

$$\text{If } Z \notin S(P) \qquad \text{then } x_s(Z) = 0. \tag{11}$$

The complementary slackness conditions which are used in the algorithm have a familiar interpretation. Condition (10) states that at every stage of the auction, each buyer b is notionally allocated a package z or a fraction of it only if z is utility maximizing at current prices. Condition (11) states that at every stage of the auction, the auctioneer notionally sells an assignment $Z = (z_1, z_2, ..., z_b, ..., z_B)$, or a fraction of it, only if this assignment maximizes her profit at current prices. At each stage of the auction, prices are increased in a manner described below. The auction ends when a feasible assignment which simultaneously satisfies (a) buyer maximization and (b) seller maximization is found.

Let $\underline{z}_b(p_b)$ and $\overline{z}_b(p_b)$ be the smallest and largest elements in $D_b(p_b)$, respectively.[8] A consequence of decreasing marginal utility is that $D_b(p_b)$ includes all integers between $\underline{z}_b$ and $\overline{z}_b$. Recall that $\Delta u_b(z) = u_b(z) - u_b(z-1)$, $z = 1, 2, ..., L$. Define $\Delta u_b(0) \equiv \infty$ and $p_b(0) \equiv p_b(-1) \equiv 0$. Then $D_b(p_b) = \{\underline{z}_b, \underline{z}_b + 1, ..., \overline{z}_b\}$ implies that

$$\begin{array}{rcll}
\Delta u_b(z) & > & p_b(z) - p_b(z-1), & \forall z = 0, 1, 2, ..., \underline{z}_b - 1 \\
\Delta u_b(\underline{z}_b) & \geq & p_b(\underline{z}_b) - p_b(\underline{z}_b - 1) & \\
\Delta u_b(z) & = & p_b(z) - p_b(z-1), & \forall z = \underline{z}_b + 1, \underline{z}_b + 2, ..., \overline{z}_b \\
\Delta u_b(z) & < & p_b(z) - p_b(z-1), & \forall z = \overline{z}_b + 1, \overline{z}_b + 2, ..., L.
\end{array}$$

Having fixed a DLP feasible solution $(\pi_s,\, (\pi_b),\, \langle p_b(\,\cdot\,)\rangle)$ with $p_b(\,\cdot\,) \geq 0$, the set of associated LP "solutions" satisfying (10) and (11) w.r.t. this DLP feasible solution is the feasible set of the *restricted primal*, hereafter RP:

RP:

$$\max \sum_b \sum_{z \in D_b(p_b)} \delta_b(z) w_b(z)$$

s.t.

$$\sum_{z \in D_b(p_b)} x_b(z) \;=\; 1, \quad \forall b \tag{12}$$

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[8] For notational simplicity, the dependence of $\underline{z}_b$ and $\overline{z}_b$ on prices $p_b(\,\cdot\,)$ is suppressed hereafter, except when this dependence is not clear from the context.

$$\sum_{Z \in S(P)} x_s(Z) \;=\; 1 \tag{13}$$

$$x_b(z) - \sum_{Z \in G^b(z) \cap S(P)} x_s(Z) - w_b(z) \;=\; 0, \quad \forall z \in D_b(p_b),\ \forall b \tag{14}$$

$$- \sum_{Z \in G^b(z) \cap S(P)} x_s(Z) - w_b(z) \;=\; 0, \quad \forall z \notin D_b(p_b),\ \forall b \tag{15}$$

$$x_b(\,\cdot\,),\, x_s(\,\cdot\,) \;\geq\; 0.$$

Observe that the constraints in RP are obtained from the constraints in LP by setting $x_b(z) = 0$ if $z \notin D_b(p_b)$, and setting $x_s(Z) = 0$ if $Z \notin S(P)$. However, unless we fortuitously start with an optimal DLP solution to obtain RP, the resulting constraints will be infeasible because there is excess demand at prices $\langle p_b(\cdot)\rangle$.[9] Therefore, we add new variables, $w_b(z)$, to the constraints (14) and (15) which track the "amount" of infeasibility in the associated LP "solution". As $D_b(p_b) \neq \emptyset$ and $S(P) \neq \emptyset$ for any DLP feasible solution, it is always possible to satisfy constraints (12) and (13) (without having to add infeasibility variables to these constraints).

To understand the role of $w_b(\cdot)$, let $(\,(w_b(z)),\, (x_b(z),\, z \in D_b(p_b)),\, (x_s(Z),\, Z \in S(P))\,)$, be any (integer) feasible solution to RP which is not feasible in LP. There is an imbalance between demand and supply. If buyer b's demand in this solution is $\tilde{z}_b$ (i.e, if $x_b(\tilde{z}_b) = 1$) but this is not met in the feasible assignment Z supplied by the seller, then set $w_b(\tilde{z}_b) = 1$ to satisfy the corresponding constraint (14). Similarly, if the seller's assignment is $\hat{Z} = (\hat{z}_b)$ but a buyer b' does not demand $\hat{z}_{b'}$ (i.e., $x_{b'}(\hat{z}_{b'}) = 0$) then set $w_{b'}(\hat{z}_{b'}) = -1$ to satisfy the appropriate constraint (15).

The penalties/costs associated with $w_b(\cdot)$, the infeasibility variables, are $(\,\delta_b(\cdot)\,) \leq 0$. The objective function of RP minimizes the total cost of using these infeasibility variables. The $\delta_b(\cdot)$ are defined later. They are selected so that buyers' payments are obtained from an mp pricing equilibrium (i.e., buyers pay their opportunity costs). It is shown in Lemma 3(ii) below that the absolute value of the optimal objective function value of RP is equal to the excess demand at the prices in the associated DLP feasible solution.

Let $RPS^* = (\,(w_b(z)^*),\, (x_b^*(z),\, z \in D_b(p_b)),\, (x_s^*(Z),\, Z \in S(P))\,)$ be an optimal solution to RP. Then $PS^* = ((x_b^*(z),\, z \in D_b(p_b)),\, (x_b^*(z) = 0,\, z \notin D_b(p_b)),\, (x_s^*(Z),\, Z \in S(P)),\, (x_s^*(Z) = 0,\, Z \notin S(P))\,)$ is an LP "solution" which satisfies complementary slackness conditions (10) and (11). Moreover, if $w_b(z)^* \equiv 0$, then PS^* is LP feasible, and because it satisfies complementary slackness with a DLP feasible solution, PS^* is also LP optimal; further, the feasible solution to DLP (with which RP is associated) is DLP optimal.

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[9] The primal-dual algorithm we describe starts at a low price at which there is excess demand. One could start the primal-dual algorithm at a high price at which there would be excess supply, but it is unlikely that this would converge to an mp pricing equilibrium.

The dual of RP or the *restricted dual*, hereafter RD, yields the rates at which prices are raised in the primal-dual algorithm.

$$\min_{\mu_s,\mu_b} \ \mu_s + \sum_{b=1}^{B} \mu_b$$

s.t.

$$\begin{array}{rcll}
\mu_b + \nu_{bz} & \geq & 0, & \forall z \in D_b(p_b),\ \forall b \\
\mu_s - \sum_b \nu_{bz_b} & \geq & 0, & \forall (z_b) = Z \in S(P) \\
-\nu_{bz} & = & \delta_b(z), & \forall z,\ \forall b
\end{array} \qquad (16)$$

The variables in the restricted dual have the following interpretation. μ_b is the rate of change of π_b, buyer b's surplus, and μ_s is the rate of change of π_s, the seller's surplus or profit. The rate of increase of buyer-package price $p_b(z)$ is equal to ν_{bz}. Constraints (16) state that ν_{bz} is set equal to $-\delta_b(z)$, the absolute values of penalties associated with the excess demand variables in RP. Hence, ν_{bz} may be eliminated and RD is re-written as:

RD:

$$\min_{\mu_s,\mu_b} \ \mu_s + \sum_{b=1}^{B} \mu_b$$

s.t.

$$\begin{array}{rcll}
\mu_b - \delta_b(z) & \geq & 0, & \forall z \in D_b(p_b),\ \forall b \qquad (17) \\
\mu_s + \sum_b \delta_{bz_b} & \geq & 0, & \forall (z_b) = Z \in S(P) \qquad (18)
\end{array}$$

As $\delta_b(z) \leq 0$, we have $\mu_b \leq 0$ and $\mu_s \geq 0$ at any optimal solution of RD. Thus, as the primal-dual algorithm proceeds, each buyer's surplus will decrease ($\mu_b \leq 0$, $\sum_b \mu_b < 0$) and the seller's surplus will increase ($\mu_s > 0$). Further, it will be shown that during the algorithm $\mu_s + \sum_b \mu_b < 0$. Thus, the sum of buyers' and seller's surpluses decreases at each iteration and the absolute value of the rate of change of this sum is maximized at each iteration. The algorithm terminates when the optimal solution to RD satisfies $\mu_s + \sum_b \mu_b = 0$.

If the objective were to find *any* optimal solution to DLP then there is considerable freedom in picking $(\ \delta_b(z)\ )$ -- for instance, setting $\delta_b(z) \equiv -1$, $\forall b$ is one of many options (see Luenberger 1973, pp. 81–86).[10] But, our objective is to find the optimal solution to DLP yielding an mp pricing equilibrium. Further, we want to ensure that

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[10] If, instead, we select $\delta_b(z) \equiv -z$, the algorithm converges to the smallest Walrasian price (i.e., the smallest pricing equilibrium in linear anonymous prices).

the rule for raising prices should be “simple.” We interpret this last requirement to mean that prices should be raised linearly and anonymously whenever possible. Recall that the rate of increase of price $p_b(z)$ is $-\delta_b(z)$. Therefore, we make $\delta_b(z)$ piecewise linear in order to get a simple auction, one in which the prices of overdemanded units (i.e., unallocated units) increase linearly and anonymously. Moreover, to ensure that buyers pay opportunity costs, prices of already allocated units should not increase. With this in mind, we define how buyers are allocated units.

For any buyer b, let

$$a_b \equiv \max\{0, L - \sum_{b' \neq b} \underline{z}_{b'}\} \tag{19}$$

$$A_{-b} \equiv \sum_{b' \neq b}^{B} a_{b'} \qquad \text{and} \qquad A \equiv \sum_{b'=1}^{B} a_{b'}.$$

The amount allocated to buyer b is a_b, A_{-b} is the total amount allocated to buyers other than b, and A is the total amount allocated at current prices. (The dependence of a_b and A on current prices $\langle p_b(\cdot)\rangle$ is dropped for simplicity.)

Let

$$ED \equiv \sum_{b=1}^{B} \underline{z}_b - L$$

be the excess demand at prices p_b. Observe that

$$a_b = \max\{\, 0, \underline{z}_b - ED \,\}. \tag{20}$$

Let

$$OD \equiv L - \sum_{b=1}^{B} a_b = L - A$$

be the set of overdemanded units, i.e., the number of units still in contention. The definitions imply that:

$$OD + ED = \sum_{b=1}^{B} (\underline{z}_b - a_b). \tag{21}$$

Thus, the aggregate demand, $\sum_{b=1}^{B} \underline{z}_b$, is equal to the sum of the excess demand, the overdemanded set, and the total amount allocated.

To elaborate on the extension of overdemanded sets mentioned in Section 2, suppose each buyer has L agents each of whom demands one unit. By decreasing marginal utility, the reservation value of the agent corresponding to a buyer’s zth unit decreases with z. If, at some stage, buyer b is allocated a_b units then these units and the first a_b agents of buyer b leave the economy. Each of the remaining $L - a_b$ agents of buyer b demands one unit from the set of unallocated units, provided that the price is less than the agent’s reservation value. Clearly, as long as $ED > 0$, the number of unallocated units, $L - A$, is less than the total number of agents who demand one unit at the current price. By analogy with the standard assignment model auction, $OD = L - A$ is the number of overdemanded units.

The following results will be useful.[11]

Lemma 1

(i) If $ED = 0$ *then* $OD = 0$*; if* $ED > 0$ *then* $OD > 0$.

(ii) If $ED > 0$ *then for each buyer* b *either* $\underline{z}_b > a_b \geq 0$ *or* $\underline{z}_b = a_b = 0$*; moreover,* $\underline{z}_b > a_b \geq 0$ *for at least two buyers.*

Thus, as long as there is excess demand, no buyer is allocated more units than his minimum demand and at least two buyers are allocated strictly less. Further, the total amount allocated is less than the total supply of units.

A selection of $\delta_b(z)$ which yields an mp pricing equilibrium as the solution to primal-dual auction is the following. If $ED \leq 0$ then define

$$\delta_b(z) \;\equiv\; 0. \tag{22}$$

If, instead, $ED > 0$ then define

$$\delta_b(z) \;\equiv\; \begin{cases} 0, & \text{if } z \leq a_b \\ -(z - a_b), & \text{if } a_b \leq z \leq \underline{z}_b \\ -(\underline{z}_b - a_b), & \text{if } \underline{z}_b \leq z \leq \underline{z}_b + A_{-b} \\ -(z - A), & \text{if } \underline{z}_b + A_{-b} \leq z \leq L, \end{cases} \tag{23}$$

where a_b, A, and A_{-b} are computed at current prices (i.e., at the current dual feasible solution). From Lemma 1 we know that $\underline{z}_b \geq a_b$ and therefore $\delta_b(z)$ is non-positive. Note that when $A = 0$ there are no units allocated and prices of all bundles increase linearly and at the same rate for all buyers.

Insert Figure 2

Once again regard each buyer as sending L agents, each with one unit demand, to the auction. The reservation price of the zth agent of buyer b is $\Delta u_b(z)$. The price for agent z of buyer b is equal to $p_b(z) - p_b(z-1)$. Thus, $p_b(z_b)$, the price of package z_b to buyer b is the sum of prices for b's first z_b agents. Each agent stays in the auction until he either clinches a unit or until the price becomes too high $[\, p_b(z) - p_b(z-1) \geq \Delta u_b(z)]$. As the rate of increase of $p_b(z_b)$ is $-\delta_b(z_b)$, from Figure 2 it follows that the rate of price increase for agent z of buyer b is 1 or 0 depending on which of the four cases in (23) the agent falls into. First, if agent (z, b)

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[11] The proof of Lemma 1 and all subsequent results are in an appendix.

is still in the auction [$a_b \leq z \leq \underline{z}_b$], then his rate of price increase is 1; thus all agents still in the auction face the same price. Second, if an agent has exited the auction but upon his exit no agent (of another buyer) clinched a unit [$\underline{z}_b + A_{-b} \leq z \leq L$], then his rate of price increase is 1. Third, if agent (z, b) has clinched a unit [$z \leq a_b$], then the rate of price increase of this agent is 0. Finally, the rate of price increase for agent (z, b) is 0 if he has exited the auction and his exit resulted in a unit being clinched by some agent of another buyer [$\underline{z}_b \leq z \leq \underline{z}_b + A_{-b}$]. In this fourth case $p_b(z) - p_b(z-1) = \Delta u_b(z)$ and this agent remains indifferent between buying or not buying at the end of the auction. This last fact will be useful in proving incentive compatibility of the primal-dual algorithm.

Recall that the optimal solution to RD yields the rates at which the current feasible solution of DLP is changed; the amount by which the feasible solution of DLP is changed is determined by the requirement that the new solution be DLP feasible. This is described below.

Let $DFS = (\,\pi_s,\ (\pi_b),\ \langle p_b(\cdot)\rangle\,)$ be a feasible solution to DLP, and let $RPS = (\,w_b(z),\, x_b(\,\cdot\,), x_s(\,\cdot\,)\,)$ and $RDS = (\,\mu_s,\ \mu_b\,)$ be optimal solutions to the corresponding RP and RD respectively. For each b, let

$$\begin{array}{c} \hat{p}_b(z) \quad \equiv \quad p_b(z) - \hat{\theta}_b \delta_b(z), \qquad \forall z, \\ \hat{\pi}_b \;\equiv\; \pi_b + \hat{\theta}_b \mu_b \quad \text{and} \quad \hat{\pi}_s \;\equiv\; \pi_s + \hat{\theta}_b \mu_s, \end{array} \tag{24}$$

where $\hat{\theta}_b > 0$. Thus, the new DLP feasible solution has higher prices of packages and consequently, lower buyers' surpluses, and higher seller's surplus. Prices are increased in the direction defined by the $\delta_b(z)$'s until at least one buyer decreases his demand (i.e., the smallest element in his demand set decreases).

If $\pi_b + p_b(z) > u_b(z)$, then for small $\hat{\theta}_b$ we have $\hat{\pi}_b + \hat{p}_b(z) > u_b(z)$. Thus, for small $\hat{\theta}_b$, $D_b(\hat{p_b}) \subseteq D_b(p_b)$. Using (24), we see that

$$\begin{array}{lrcll} \text{if} & \pi_b + \hat{\theta}_b\mu_b + p_b(z) - \hat{\theta}_b\delta_b(z) & > & u_b(z), & \forall z \notin D_b(p_b) \\ \text{then} & \pi_b + p_b(z) - u_b(z) & > & -\hat{\theta}_b(\mu_b - \delta_b(z)), & \forall z \notin D_b(p_b). \end{array}$$

If $\mu_b - \delta_b(z) \geq 0$ then the above inequality is always satisfied as $\hat{\theta}_b > 0$.[12] Hence define:

$$\begin{aligned} \Theta_b(DFS, RPS) \;&\equiv\; \begin{cases} \min_{\{z \,|\, \mu_b - \delta_b(z) < 0\}} \frac{\pi_b + p_b(z) - u_b(z)}{-[\mu_b - \delta_b(z)]}, & \text{if } \{z \,|\, \mu_b - \delta_b(z) < 0\} \neq \emptyset, \\ \infty, & \text{otherwise.} \end{cases} \\ \Theta(DFS, RPS) \;&\equiv\; \min_b \Theta_b(DFS, RPS). \end{aligned} \tag{25}$$

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[12] From (17) and (23) we know that $\mu_b - \delta_b(z) \geq 0$, $\forall z \geq \underline{z}_b$. Thus, as prices are raised according to (24), any element $z \in D_b(\hat{p_b}) \backslash D_b(p_b)$ satisfies $z < \underline{z}_b$.

For all $\hat{\theta}_b < \Theta_b(DFS, RPS)$, we have $D_b(\hat{p}_b) \subseteq D_b(p_b)$ and for $\hat{\theta}_b \geq \Theta_b(DFS, RPS)$, $D_b(\hat{p}_b) \nsubseteq D_b(p_b)$. It may be verified that LP feasibility implies $\Theta(DFS, RPS) < \infty$. If $\hat{\theta}_b > \Theta(DFS, RPS)$, then $(\,\hat{\pi}_s,\ (\hat{\pi}_b),\ \langle \hat{p}_b(\,\cdot\,)\rangle\,)$ is DLP infeasible. In particular, (5) is violated for some b and $z < \underline{z}_b$.

The initial DLP feasible solution (also the starting point of the auction based on the primal-dual method) is:

$$\begin{aligned} \pi_s^0 &= 0, \qquad \pi_b^0 = u_b(L), \quad \forall b \\ P^0(Z) &\equiv 0, \qquad p_b^0(z) \equiv 0. \end{aligned} \tag{26}$$

An LP (infeasible) solution which satisfies complementary slackness w.r.t. the above DLP feasible solution is:[13]

$$\begin{aligned} x_b^0(z) &\equiv \begin{cases} 1, & \text{if } z = L \\ 0, & \text{otherwise} \end{cases} \\ x_s^0(Z) &\equiv \begin{cases} 1, & \text{if } Z = (L, 0, 0, ..., 0) \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{27}$$

We are now ready to describe the auction as primal-dual algorithm. Observe that the steps in the algorithm correspond to the steps in the auction in Section 2.

The primal-dual algorithm

Step 0: Let $n = 0$ and let the initial DLP feasible solution, $DFS^0 = (\,\pi_s^0,\ (\pi_b^0),\ \langle p_b^0(\,\cdot\,)\rangle\,)$, be as in (26). Let PS^0 be as in (27). Thus, PS^0 is the LP (infeasible) solution which satisfies complementary slackness w.r.t. the current DLP feasible solution DFS^0. Let $D_b^0 \equiv D_b(p_b^0) = \{L\}$, and $S^0 \equiv S(P^0)$ be any feasible assignment.

Step 1: Using (22), (23), compute $\delta_b(z)^n$, $\forall b,\ z$.[14] Let RP^n be the restricted primal corresponding to DFS^n, and let RD^n be the dual of RP^n. Compute $RPS^n = (\,w_b(z)^n,\ (x_b^n(z),\ z \in D_b^n),\ (x_s^n(Z),\ Z \in S^n\,)$, an optimal solution of RP^n. If the optimal objective function value of RP^n is 0 then go to Step 3; otherwise compute $RDS^n = (\,\mu_s^n,\ \mu_b^n\,)$, an optimal solution to RD^n.

Step 2: Let $\theta^n = \Theta(DFS^n, RPS^n)$ be as defined in (25). Define

$$\begin{aligned} \pi_s^{n+1} &\equiv \pi_s^n + \theta^n \mu_s^n \\ \pi_b^{n+1} &\equiv \pi_b^n + \theta^n \mu_b^n, \qquad \forall b \\ p_b^{n+1}(z) &\equiv p_b^n(z) - \theta^n \delta_b(z)^n, \qquad \forall z,\ \forall b \end{aligned}$$

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[13] For simplicity we assume that each buyer's marginal utility is strictly positive. Therefore, at price 0 every buyer demands exactly L.

[14] If $A^n = 0$ at stage n, we have $\delta_b(z)^n \equiv z$, $\forall b,\ z$; i.e., prices for all agents of each buyer increase at the rate 1. If some buyers have been allocated units in Step 2 of stages $\ell < n$, i.e., $A^n > 0$, then $\delta_b(z)^n$ is of the form in Figure 2.

Let $DFS^{n+1} = (\,\pi_s^{n+1},\ (\pi_b^{n+1}),\ \langle p_b^{n+1}(\,\cdot\,)\rangle\,)$. Compute $D_b^{n+1} = D_b(p_b^{n+1})$, $S^{n+1} = S(P^{n+1})$. Increment n by one. Go to Step 1.

Step 3: The auction/algorithm is over. Let $PS^n = ((x_b^n(z_b),\ z_b \in D_b^n),\ (x_b^n(z_b) = 0,\ z_b \notin D_b^n),\ (x_s^n(Z),\ (z_b) = Z \in S^n),\ (x_s^n(Z) = 0,\ Z \notin S^n)\,)$. PS^n is an efficient assignment at prices DFS^n.

The computations of demand sets and the price increment, θ^n, in Step 2 of the algorithm require centralized knowledge of buyers' utilities. On the other hand, knowledge of buyers' utilities is decentralized in the auction implementation, with each buyer knowing only his own utility function. As prices are increased in the direction obtained from the restricted dual, the first buyer to lower his demand in the auction determines the value of θ^n. Computation of demand sets is unnecessary; it is an ex post equilibrium for buyers to bid sincerely and thus report their demand sets (or the smallest bundle in their demand sets).

To establish that this algorithm is an incentive compatible auction, we need to show that at each iteration DFS^n is DLP feasible, and that the algorithm terminates in a finite number of iterations at an mp pricing equilibrium.[15]

Let $\langle p_b(\,\cdot\,)\rangle$ be a set of prices. Let $B(Z)$ be the set of buyers who are assigned an element of their demand sets in an assignment $Z = (z_b)$. That is, $z_b \in D_b(p_b)$, $\forall b \in B(Z)$.[16] Let $B^c(Z) \equiv \{1, 2, ..., B\} \backslash B(Z)$. Define $Z = (z_b)$ to be a *maximal demand assignment* at prices $\langle p_b(\,\cdot\,)\rangle$ if

a. For all b, $z_b \leq \underline{z}_b$.
b. $\sum_b z_b = L$.
c. For all $b' \in B^c(Z)$, $L - \sum_{b \in B(Z)} z_b < \underline{z}_{b'}$.

The definition of $B(Z)$, together with (a) above, implies that if Z is a maximal demand assignment then $z_b = \underline{z}_b$, $\forall b \in B(Z)$. As the largest element in $D_b(p_b)$ is less than or equal to L, (c) implies that at any maximal demand assignment Z, $B(Z) \neq \emptyset$.

A maximal demand assignment, Z, for a set of prices at which there is excess demand may be constructed as follows. Elicit buyer demands at these prices. Order the buyers in some sequence $b_1, b_2, ..., b_B$. First, assign $\underline{z}_{b_1}$ units to b_1. Next, assign $\underline{z}_{b_2}$ to b_2 if this is feasible, i.e., if $\underline{z}_{b_2} \leq L - \underline{z}_{b_1}$; otherwise move on to the next buyer in the sequence. Proceed in this manner until the last buyer is reached. The buyers who have been assigned the smallest element in their demand sets constitute the set $B(Z)$. As there is excess demand, $B^c(Z) \neq \emptyset$. Assign any remaining units to buyers

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[15] We cannot appeal to standard results for primal-dual algorithms to establish this for two reasons. First, some of the $\delta_b(\cdot)$ are zero. Second, we need to establish convergence to an mp pricing equilibrium rather than any pricing equilibrium.

[16] The dependence of $B(\,\cdot\,)$ on prices $\langle p_b(\,\cdot\,)\rangle$ is suppressed in the notation.

in $B^c(Z)$ in an arbitrary manner. Thus, a maximal demand assignment exists for any set of prices at which there is excess demand.[17]

As shown in Lemma 3(i) below, at each stage n maximal demand assignments are in S^n, the set of profit maximizing assignments for the seller. Thus, if price controls were imposed on the seller, the seller would prefer a maximal demand assignment to any assignment that partially satisfied every buyer's demand.

The next lemma shows that if Z is a maximal demand assignment then no other assignment satisfies the demands of a larger subset of buyers. Hence, the qualifier "maximal demand" attached to assignments satisfying $a - c$ above.

Lemma 2 *If* $Z = (z_b)$ *is a maximal demand assignment, then:*
(i) There does not exist an assignment Z' *such that* $B(Z) \subset B(Z')$, $B(Z) \neq B(Z')$.
(ii) If $|B^c(Z)| = 1$ *with* $B^c(Z) = \{b'\}$, *then* $z_{b'} = a_{b'}$.
(iii) If $|B^c(Z)| \geq 2$ *then* $a_b = 0$ *for all* $b \in B^c(Z)$.

The links between the overdemanded units, the excess demand, maximal demand assignments, and optimal solutions to the restricted primal and dual are investigated in the following lemma.

Lemma 3 *At any stage* n *of the primal-dual algorithm:*
(i) All maximal demand assignments are profit maximizing for the seller.
(ii) The optimal solution to RD^n *is* $\mu_s^n = OD^n$, $\mu_b^n = -(\underline{z}_b^n - a_b^n)$, $\forall b$. *The optimal value of the objective function of* RD^n *(and hence also of* RP^n*) is* $-ED^n$.
(iii) In an optimal solution to RP^n, *each buyer* b *demands and receives* $\underline{z}_b^n$ *and the seller supplies any maximal demand assignment.*
(iv) $DFS^{n+1} = (\pi_s^{n+1}, (\pi_b^{n+1}), \langle p_b^{n+1}(\cdot)\rangle)$ *is DLP feasible.*

Lemma 3(iv) implies that at each iteration of the algorithm, the new set of prices is DLP feasible. Thus, the algorithm maintains DLP feasibility and complementary slackness between the LP and DLP solutions. As the optimal objective function value of RD is the negative of the excess demand (Lemma 3ii), the algorithm stops when the excess demand becomes zero; this happens in a finite number of iterations since the excess demand decreases by at least one unit per iteration. At this point, it is possible to select an LP feasible assigment from buyers' demand sets. This assignment is efficient as it is also an optimal solution to the LP. Moreover, as this assignment is a maximal demand assignment (Lemma 3iii), it maximizes the seller's surplus (Lemma 3i) at the final prices.

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[17] At high enough prices, all buyers will demand the empty set. Therefore, at these prices no assignment satisfies (a) and (b) of the definition, and a maximal demand assignment will not exist.

Lemma 4 *The primal-dual algorithm described above converges to an mp pricing equilibrium in a finite number of iterations.*

4.1 Ex post equilibrium

To complete the proof of Theorem 1, it remains to define ex post equilibrium and show that sincere bidding has that property for the algorithm described above.

Let U be the set of all (utility) functions over L units of the object while $D \subset U$ are those functions displaying diminishing marginal utility, i.e., $u(z) - u(z-1) \geq u(z+1) - u(z)$, $z = 1, \ldots, L-1$. For $u_b \in D$ if z_b is a utility maximizing choice at prices $p_b(\cdot)$, z'_b is a utility maximizing choice at prices $p'_b(\cdot)$, and $p_b(z) - p_b(z-1) \leq p'_b(z) - p'_b(z-1)$, $\forall z$ then $z'_b \leq z_b$. That is, increasing the price for each "agent" of b never leads to an increase in the total number of units demanded by b. If a bidder does not increase demand as prices for his agents increase, then we say that his behavior is consistent with diminishing marginal utility. If bidders' demands were monitored to enforce such behavior (i.e., once an agent of a buyer exits an auction he is not allowed to re-enter), the result below implies that there would be no gain to bidding insincerely regardless of others' bidding strategies: sincere bidding would be a dominant strategy. We do not assume that bidders' behavior is monitored.

A strategy for buyer b with utility $u_b \in D$ in an ascending price auction is defined as the choice of a $\tilde{u}_b \in U$ such that b bids as if his utility function is $\tilde{u}_b$. Let $u_{-b} = (u_1, u_2, ..., u_{b-1}, u_{b+1}, ..., u_B) \in D_{-b}$. Define $\pi_b(\tilde{u}_b, u_{-b} \,|\, u_b)$ as buyer b's payoff when he bids according to $\tilde{u}_b$ although his actual utility is u_b and the other buyers bid according to their utility functions in u_{-b}. So, $\pi_b(u_b, u_{-b} \,|\, u_b)$ is the payoff from sincere bidding. When buyers have diminishing marginal utility, bidding sincerely in the primal-dual algorithm/auction is an *ex post equilibrium* if for all b,

$$\pi_b(u_b, u_{-b} \,|\, u_b) \geq \pi_b(\tilde{u}_b, u_{-b} \,|\, u_b), \qquad \forall \tilde{u}_b \in U, \forall u_b \in D, \forall u_{-b} \in D_{-b}. \tag{28}$$

Lemma 5 *When buyers have decreasing marginal utility, it is an ex post equilibrium for all buyers to bid sincerely in the primal-dual algorithm/auction.*

The requirements of ex post equilibrium are that as long as every other bidder is following a strategy of never increasing the quantity demanded as prices rise — whether or not they are basing their demands on their actual utilities, there is no gain to the remaining buyer to bid insincerely. If, however, one bidder were to increase demand during the auction, the buyers are substitutes condition cannot be invoked and pricing equilibrium payoffs may be as in Figure 1b; sincere bidding no longer guarantees any of the other bidders his marginal product and a deviation may now be profitable.

If buyers are not required to bid in a manner consistent with diminishing marginal utility, it is easy to construct examples having a small number of buyers, each possessing substantial knowledge about others' preferences so as to know when to strategically increase demand after a price increase, such that there are insincere ex post equilibria. We conjecture that in a Bayesian setting, if the domain from which bidders valuations are drawn is "large" ex post equilibria would involve sincere bidding.

5 Concluding remarks

The primal-dual algorithm is a practical implementation of the market heuristic that prices adjust to excess demands. This algorithm adjusts feasible solutions to the dual (i.e., prices) to decrease the infeasibility of the primal "solution" (i.e., the quantities that buyers' would demand at these prices) in such a manner that when the primal does become feasible, it will be optimal. The same principles underlie the adjustment scheme known as *tâtonnement* where prices adjust to reduce excess demands until markets clear, i.e., the primal solution is feasible. In an economic setting, the primal-dual method has advantages not enjoyed by other linear programming algorithms. Without knowing the data of the problem, the auctioneer adjusting prices can find a solution by announcing prices and letting individuals, who know their own (but not others') data, submit the quantities they would desire. Knowledge of excess demands, rather than the data of the problem, suffices to adjust prices. Thus, the primal-dual algorithm in the LP problem described above decentralizes the task of finding an optimal solution.

Decentralization brings with it the possibility that individuals might bid insincerely to manipulate the process in their favor with potentially adverse consequences for the optimality of the solution. In fact, such possibilities would typically exist in the homogeneous units case if we confined our attention to optimal solutions implemented by prices that were linear and anonymous; a restriction that is normally part of *tâtonnement*. Nevertheless, when the set of primal and dual solutions contains an mp pricing equilibrium, its implementation implies that no matter what the characteristics of the buyers (assuming decreasing marginal utility), if others submit their responses sincerely the remaining buyer cannot do better by responding otherwise. Remarkably, the primal-dual algorithm can be applied even though prices are nonlinear and non-anonymous. Moreover, the extension to nonlinear and non-anonymous prices is essential for *tâtonnement* with incentive compatibility.

We have shown that the notion of an ascending price Vickrey auction corresponds to the implementation of an mp pricing equilibrium via the primal-dual algorithm. The correspondence applies to auctions for the assignment model, including the special case of a single object, and for the homogeneous units model. In each, the auction focuses on an overdemanded set on which prices are raised until the set is reduced. The process is repeated until the overdemanded set is empty. In one case, this process leads to linear anonymous prices, and in the other case not. The reason is that for incentive compatibility buyer prices should not change when any one buyer is removed (Propositions 4 and 5); this is possible with linear anonymous prices only when buyers have unit demands, as in the assignment model.

One motivation for drawing out similarities between these auctions is that it points the way to discovering ascending-price Vickrey auctions in other settings. Propositions 1–5 of Section 3 extend to heterogeneous objects. Therefore, provided buyers are substitutes, the framework developed in Sections 3 and 4 may be applied to heterogeneous objects. Several papers, including Parkes and Ungar (2000), Ausubel (2000), Ausubel-Milgrom (2001), Bikhchandani, de Vries, Schummer, and Vohra (2001) and de Vries, Schummer, and Vohra (2001) present ascending-price auctions in a heterogeneous objects model in which the buyers are substitutes assumption is satisfied. Of these, the latter two papers are closest to the approach developed here; they use matroid formulations whose dual admits a mp pricing equilibrium to obtain primal-dual auctions. In particular, de Vries et. al. present an incentive compatible primal-dual algorithm/auction for heterogeneous objects under the assumption that buyers' utility function satisfies the gross substitutes assumption.

6 Appendix

Proof of Lemma 1: (i) If $ED = 0$ then $a_b = \underline{z}_b$, $\forall b$, and $\sum_b a_b = \sum_b \underline{z}_b = L$. Hence $OD = 0$.

Assume that $ED > 0$. First, suppose that $a_b = 0$, $\forall b$. Clearly $0 = \sum_{b=1}^{B} a_b = A < L$. Next, suppose that there exists $\hat{b}$ such that $a_{\hat{b}} > 0$. Let $H = \{b \,|\, a_b > 0\}$. Observe that

$$\begin{aligned}
\sum_{b=1}^{B} a_b \;=\; \sum_{b\in H} a_b \;&=\; \sum_{b\in H}\left[\underline{z}_b - ED\right] \\
&\leq\; \underline{z}_{\hat{b}} - ED + \sum_{b\in H\setminus\{\hat{b}\}} \underline{z}_b \\
&\leq\; \underline{z}_{\hat{b}} - ED + \sum_{b\in H\setminus\{\hat{b}\}} \underline{z}_b + \sum_{b\notin H} \underline{z}_b \\
&=\; \sum_{b=1}^{B} \underline{z}_b - ED \;=\; L,
\end{aligned}$$

where the first inequality follows from $ED > 0$ and the fact that $\hat{b} \in H$. Finally, observe that if $|H| \geq 2$ then the first inequality is strict and if, instead, $|H| = 1$ then the second inequality is strict. Thus, $OD > 0$.

(ii) Suppose that $ED > 0$. If $\underline{z}_b > 0$ then, from (20), $\underline{z}_b > a_b \geq 0$. If $\underline{z}_b = 0$ then, again from (20), $a_b = 0$. As $\underline{z}_b \leq L$, $\forall b$, $ED > 0$ implies that at least two buyers have strictly positive smallest elements in their demand set. ▮

Proof of Lemma 2: Let $Z = (z_b)$ be a maximal demand assignment.

(i) Part (a) of the definition of a maximal demand assignment implies that $z_b = \underline{z}_b$, $\forall b \in B(Z)$. Hence, (c) of the definition implies that there does not exist a Z' such that $B(Z) \subset B(Z')$, $B(Z')\setminus B(Z) \neq \emptyset$.

(ii) As $z_b = \underline{z}_b$, $\forall b \in B(Z) = \{b = 1, 2, ...B\}\setminus\{b'\}$, $z_{b'} = L - sum_{b\neq b'}\underline{z}_b$. Hence, (19) implies that $z_{b'} = a_{b'}$.

(iii) From $z_b = \underline{z}_b$, $\forall b \in B(Z)$ and (c) of the definition we see that

$$L - \sum_{b\in B(Z)} \underline{z}_b \;=\; L - \sum_{b\in B(Z)} z_b < \; \underline{z}_{\hat{b}}, \qquad \forall \hat{b} \in B^c(Z).$$

Hence, for any $b' \in B^c(Z)$, $L - \sum_{b\in B(Z)} \underline{z}_b - \sum_{\hat{b}\in B^c(Z)\setminus\{b'\}} \underline{z}_{\hat{b}} \;<\; 0$. Therefore, (19) implies that $a_{b'} = 0$. ▮

Proof of Lemma 3: First, we establish that during Step 2, $ED^n > 0$ and $OD^n > 0$. Suppose that in Step 1, $ED^n \leq 0$. From (22) we know that $\delta_b^n(z) \equiv 0$. Thus, (17) and (18) reduce to $\mu_b^n \geq 0$, $\forall b$ and $\mu_s^n \geq 0$ respectively. Hence, the optimal solution to RD^n is $\mu_b^n \equiv 0$, $\forall b$, $\mu_s^n = 0$, and the optimal objective function values of RD^n and RP^n are both zero. Therefore, at the end of Step 1, the algorithm goes to Step 3 and terminates. Hence, at any time in Step 2 we must have $ED^n > 0$. Moreover, Lemma 1(i) implies that $OD^n > 0$.

(i) For any $Z = (z_b)$, let $\Delta^n(Z) \equiv -\sum_b \delta_b^n(z_b)$. As $p_b^n(z_b)$ increases at the rate $-\delta_b^n(z_b)$, the total sales revenue from assignment Z increases at rate $\Delta^n(Z)$. First, we show that $\Delta^n(Z) \leq OD^n$, $\forall Z$.

Take any assignment $Z = (z_b)$. Suppose that $a_b^n \leq z_b \leq \underline{z}_b^n + A_b^n$ for all b. Then, from (23), $-\delta_b^n(z_b) \leq z_b - a_b^n$, $\forall b$. Clearly, $\Delta^n(Z) = -\sum_b \delta_b^n(z_b) \leq \sum_b(z_b - a_b^n) \leq L - A^n = OD^n$.

Suppose, instead, that there exists b' such that $z_{b'} > \underline{z}_{b'}^n + A_{b'}^n$. From (23) we know that $-\delta_{b'}^n(z_{b'}) = z_{b'} - A^n$ and $-\delta_b(z) \leq z$, $\forall z$, $\forall b \neq b'$. Thus, $\Delta^n(Z) \leq z_{b'} - A^n + \sum_{b \neq b'} z_b = \sum_{b=1}^{B} z_b - A^n \leq L - A^n = OD^n$.

Finally, suppose that $Z = (z_b)$ is such that $z_b \leq \underline{z}_b^n + A_b^n$ for all b, and that there exists b' such that $z_{b'} < a_{b'}^n$. Therefore, $a_{b'} > 0$. Define

$$z'_b \equiv \begin{cases} \underline{z}_b^n, & \text{if } b \neq b', \\ a_{b'}^n, & \text{if } b = b'. \end{cases}$$

As $a_{b'}^n > 0$, (19) implies that $\sum_b z'_b = L$ and therefore $Z' = (z'_b)$ is a feasible assignment. Observe that $z'_b - a_b^n = -\delta_b^n(z'_b) \geq -\delta_b^n(z_b)$ for all b. Therefore,

$$\Delta^n(Z) = -\sum_b \delta_b^n(z_b) \leq -\sum_b \delta_b^n(z'_b) = \sum_b (z'_b - a_b^n) = L - A^n = OD^n.$$

Thus, $\Delta^n(Z) \leq OD^n$, $\forall Z$.

Next, let $\hat{Z} = (\hat{z}_b)$ be a maximal demand assignment at stage n. Suppose that $|B^c(\hat{Z})| = 1$ with $B^c(\hat{Z}) = \{b'\}$. As $\hat{z}_{b'} = a_{b'}^n$ from Lemma 2(ii), we have $\delta_{b'}^n(\hat{z}_{b'}) = \hat{z}_{b'} - a_{b'}^n = 0$. Further, for all $b \neq b'$, $\hat{z}_b = \underline{z}_b^n$ and therefore $\delta_b^n(\hat{z}_b) = \hat{z}_b - a_b^n$. Thus, $\Delta^n(\hat{Z}) = \sum_b(\hat{z}_b - a_b^n) = L - A^n = OD^n$.

On the other hand, if $|B^c(\hat{Z})| \geq 2$ then Lemma 2(iii) implies that $a_{b'} = 0$ for all $b' \in B^c(\hat{Z})$. Thus, $A^n = \sum_{b \in B(Z)} a_b^n$. Hence, we have

$$\begin{aligned} \Delta^n(\hat{Z}) &= -\sum_{b \in B^c(\hat{Z})} \delta_b^n(\hat{z}_b) - \sum_{b \in B(\hat{Z})} \delta_b^n(\hat{z}_b) \\ &= \sum_{b' \in B^c(\hat{Z})} \hat{z}_{b'} + \sum_{b \in B(\hat{Z})} (\hat{z}_b^n - a_b^n) \\ &= L - A^n = OD^n. \end{aligned}$$

Thus, in any stage n, the rate of increase of revenue from a maximal demand assignment is at least as large as that from any other assignment. Observe that at the beginning of stage 0, all assignments (including maximal demand ones) yield zero revenues and therefore are revenue maximizing. This establishes (i).

(ii) Observe that (17), (18), and (23) imply that for any RD^n feasible $(\,(\mu_b),\,\mu_s\,)$, we have

$$\begin{aligned}\mu_b &\geq \max_{z\in D_b^n}\delta_b^n(z) \;=\; \delta_b^n(\underline{z}_b^n) \;=\; -(\underline{z}_b^n - a_b^n)\\ \mu_s &\geq \max_{(z_b)=Z\in S(P)} -\sum_b \delta_{bz_b}^n \;=\; OD^n.\end{aligned}$$

Thus, $\mu_s^n = OD^n$, $\mu_b^n = -(\underline{z}_b^n - a_b^n)$, $\forall b$ is the unique optimal solution of RD^n.

Inserting the values of the optimal $(\,(\mu_b^n),\,\mu_s^n\,)$ in the objective function of RD^n, we have $\mu_s^n + \sum_b \mu_b^n = OD^n - \sum_b(\underline{z}_b^n - a_b^n) = -ED^n$, from (21).

(iii) We need to show that the following is an optimal solution to RP^n. The seller supplies $\hat{Z}$, a maximal demand assignment. Each buyer b receives/demands $\underline{z}_b^n$. To take care of demand supply imbalances, define $w_b(z)$ to be 1 if b demands z but z is not supplied to b by the seller; $w_b(z) = -1$ if z is supplied by the seller to b but z is not demanded by b; $w_b(z) = 0$ for all other (b, z) combinations. This solution to RPS^n is represented by:

$$\begin{aligned}x_b^n(z) &\equiv \begin{cases}1, & \text{if } z = \underline{z}_b^n,\ \forall b\\ 0, & \text{otherwise.}\end{cases}\\ x_s^n(Z) &\equiv \begin{cases}1, & \text{if } Z = \hat{Z}\\ 0, & \text{otherwise.}\end{cases}\\ w_b^n(z) &\equiv \begin{cases}1, & \text{if } z = \underline{z}_b^n,\ b\in B^c(\hat{Z}),\\ -1, & \text{if } z = \hat{z}_b,\ b\in B^c(\hat{Z}),\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

It is easily verified that this is a feasible solution to RP^n. The value of the objective function at this solution is

$$\begin{aligned}-\sum_{b\in B^c(\hat{Z})}(\underline{z}_b^n - a_b) + \sum_{b\in B^c(\hat{Z})}(\hat{z}_b - a_b) &= -\sum_{b\in B^c(\hat{Z})}\underline{z}_b^n + \sum_{b\in B^c(\hat{Z})}\hat{z}_b\\ &= -\sum_{b\in B^c(\hat{Z})}\underline{z}_b^n + (L - \sum_{b\in B(\hat{Z})} z_b^n)\\ &= L - \sum_b \underline{z}_b^n\\ &= -ED^n\end{aligned}$$

where the second and third inequalities follow from parts (b) and (a), respectively, of the definition of a maximal demand assignment. From (ii) we know that this is equal to the optimal objective objective function value of RD^n. Hence, RPS^n is an optimal solution to RP^n.

(iv) Suppose that $DFS^n = (\,\pi_s^n,\, (\pi_b^n),\, \langle\, p_b^n(\,\cdot\,)\,\rangle)$ is DLP feasible. This is true for $n = 0$. The definition of θ^n implies that (5) is satisfied by DFS^{n+1}. Recall that

$$\begin{aligned} \pi_s^{n+1} &\equiv \pi_s^n + \theta^n \mu_s^n \\ p_b^{n+1}(z) &\equiv p_b^n(z) - \theta^n \delta_b^n(z), \qquad \forall z,\, \forall b \end{aligned}$$

Hence, for any assignment $Z = (z_b)$,

$$\begin{aligned} \pi_s^{n+1} - \sum_b p_b^{n+1}(z_b) &\equiv \pi_s^n + \theta^n \mu_s^n - \sum_b p_b^n(z_b) + \theta^n \sum_b \delta_b^n(z_b) \\ &\geq \pi_s^n + \theta^n(L - A^n) - \sum_b p_b^n(z_b) - \theta^n(L - A^n) \\ &= \pi_s^n - \sum_b p_b^n(z_b) \\ &\geq C(Z), \end{aligned}$$

where the first inequality follows from the fact that $\mu_s^n = L - A^n$ and $\sum_b \delta_b^n(z_b) = -\Delta^n(Z) \geq -(L - A^n)$, and the last inequality from DLP feasibility of DFS^n. Hence, (6) is also satisfied by DFS^{n+1}. ▮

Proof of Lemma 4: In view of the paragraph preceding this lemma, we know that the final assignment is efficient and that buyers and the seller maximize at the final prices. Hence, the algorithm stops at a pricing equilibrium in a finite number of iterations. We need to verify that it stops at a mp pricing equilibrium.

Let $(z_1^*, z_2^*, ..., z_B^*)$ be the assignment and $\langle p_b^* \rangle$ the prices at the end of the primal-dual algorithm. Consider any buyer b'. If $z_{b'}^* = 0$, then $\langle p_b^* \rangle$ remains a pricing equilibrium in $\mathcal{E}_{-b'}$. Suppose, instead, $z_{b'}^* > 0$. If buyer b' did not participate in the auction, then his $z_{b'}^*$ units would be allocated efficiently amongst the remaining buyers. Let $(z_b^* + \eta_b)_{b \neq b'}$, $\eta_b \geq 0$, $\sum_{b \neq b'} \eta_b = z_{b'}^*$, be an efficient allocation of L units to buyers other than b'.[18] For each $b \neq b'$, agents $z = z_b^* + 1, z_b^* + 1, ..., z_b^* + \eta_b$ of buyer b are precisely those whose exit from the auction coincided with the clinching of a unit by an agent of b'; thus, recalling the discussion immediately after (23), these (z, b) agents, $z = z_b^* + 1, z_b^* + 2, ..., z_b^* + \eta_b$, are indifferent between buying or not buying at the end of the auction. That is, $p_b^*(z) - p_b^*(z-1) = \Delta u_b(z)$, $\forall z = z_b^* + 1, z_b^* + 1, ..., z_b^* + \eta_b$, and $z_b^* + \eta_b \in D_b(p_b^*)$, $\forall b \neq b'$. In other words, $\langle p_b^* \rangle$ remains a pricing equilibrium in $\mathcal{E}_{-b'}$. Hence, by Proposition 4, $\langle p_b^* \rangle$ is a mp pricing equilibrium. ▮

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[18] Decreasing marginal utility implies that there exists an efficient allocation in $\mathcal{E}_{-b'}$ such that $\eta_b \geq 0$.

Proof of Lemma 5: Suppose that buyers' utility functions are $(u_1, u_2, ..., u_B)$ and all buyers other than b bid sincerely. If b bids sincerely, the primal-dual algorithm implements an efficient assignment (z_b^*) and b pays social opportunity cost $\phi_{-b}(z_b^*)$. If, instead, he bids as if his utility function is $\tilde{u}_b \in U$ the resulting assignment is $(\tilde{z}_b)$ and payment $\phi_{-b}(\tilde{z}_b)$. From the definition of $\phi_{-b}(\cdot)$ in Section 3.1, we have

$$\begin{aligned}
\pi_b(u_b, u_{-b} \,|\, u_b) &= u_b(z_b^*) - \phi_{-b}(z_b^*) \\
&= u_b(z_b^*) - F(\mathbf{1}_{-b}, \mathbf{0}) + F(\mathbf{1}_{-b}, -\mathbf{1}_{z_b^* b}) \\
&= \sum_{b'=1}^{B} u_{b'}(z_{b'}^*) - F(\mathbf{1}_{-b}, \mathbf{0}) \\
&\geq \sum_{b'=1}^{B} u_{b'}(\tilde{z}_{b'}) - F(\mathbf{1}_{-b}, \mathbf{0}) \\
&= u_b(\tilde{z}_b) - F(\mathbf{1}_{-b}, \mathbf{0}) + F(\mathbf{1}_{-b}, -\mathbf{1}_{\tilde{z}_b b}) \\
&= u_b(\tilde{z}_b) - \phi_{-b}(\tilde{z}_b) \\
&= \pi_b(\tilde{u}_b, u_{-b} \,|\, u_b),
\end{aligned}$$

where the inequality follows from efficiency. ∎

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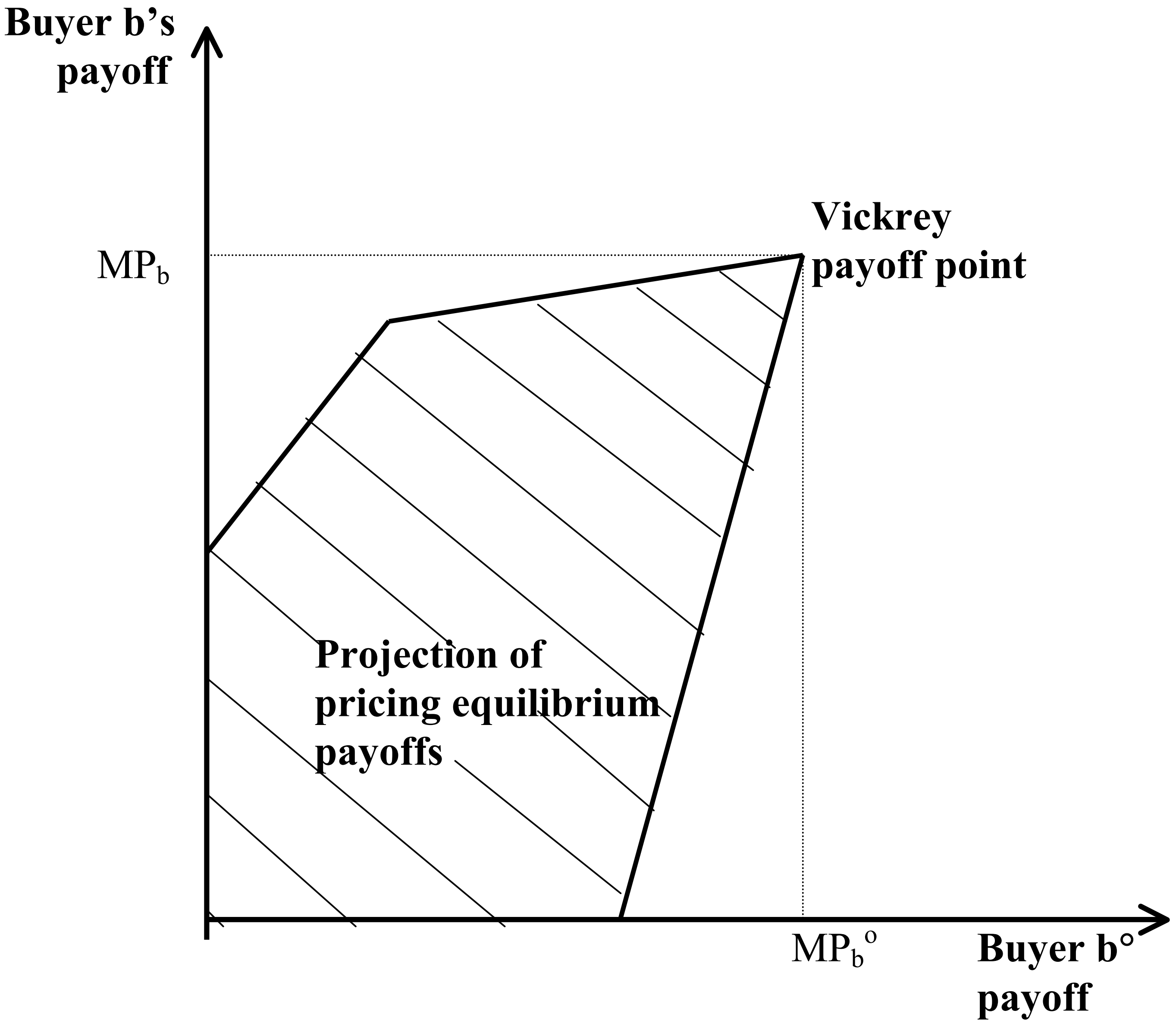

Figure 1a: Projection of pricing equilibrium payoffs into the space of payoffs of any two buyers, when (all) buyers are substitutes

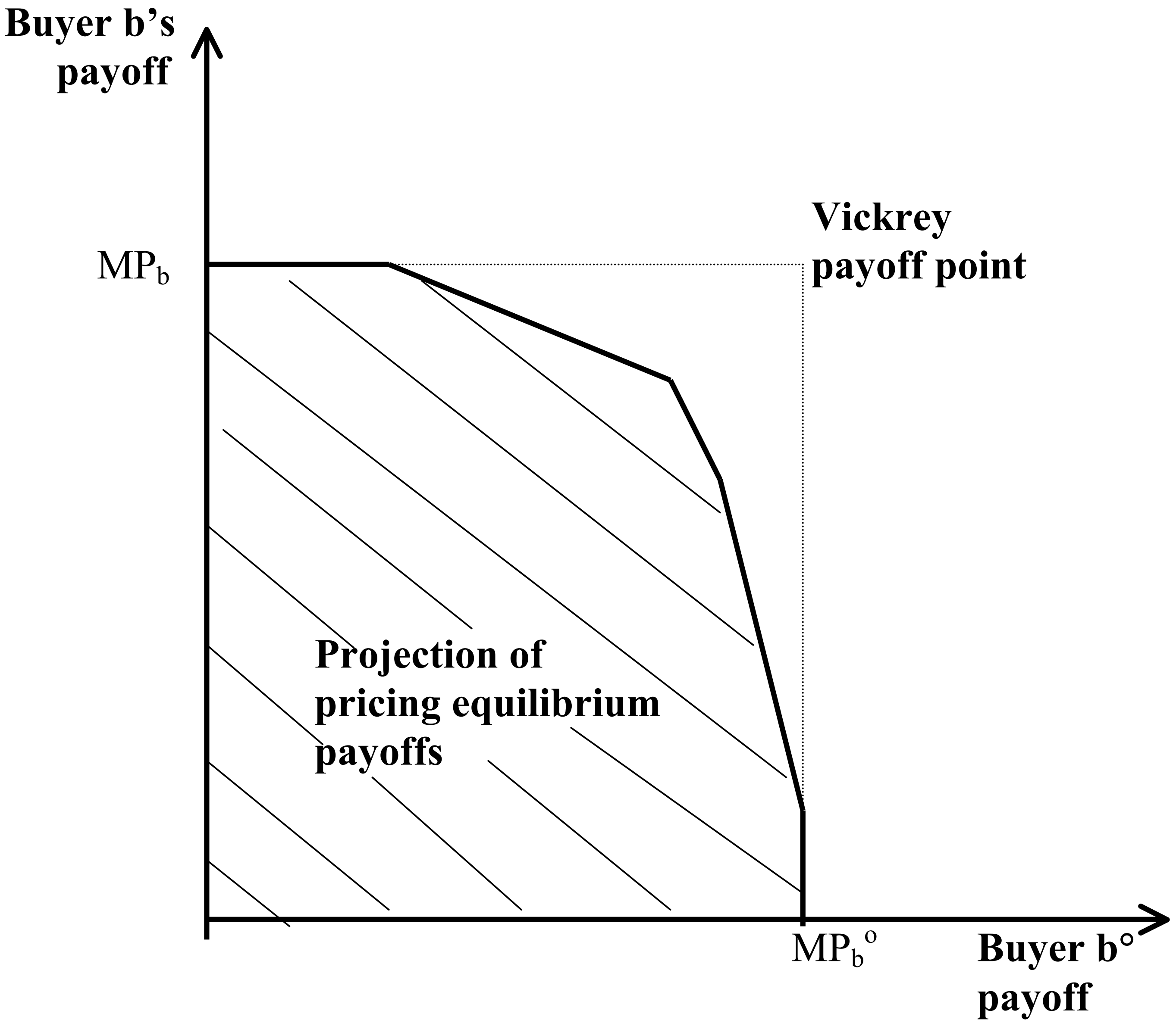

Figure 1b: Projection of pricing equilibrium payoffs into the space of payoffs of any two buyers, when buyers are not substitutes

$\delta_b(z)$: **rate of price increase of package z for buyer b**

$\delta_b(z)$: **penalty for using $w_b(z)$**

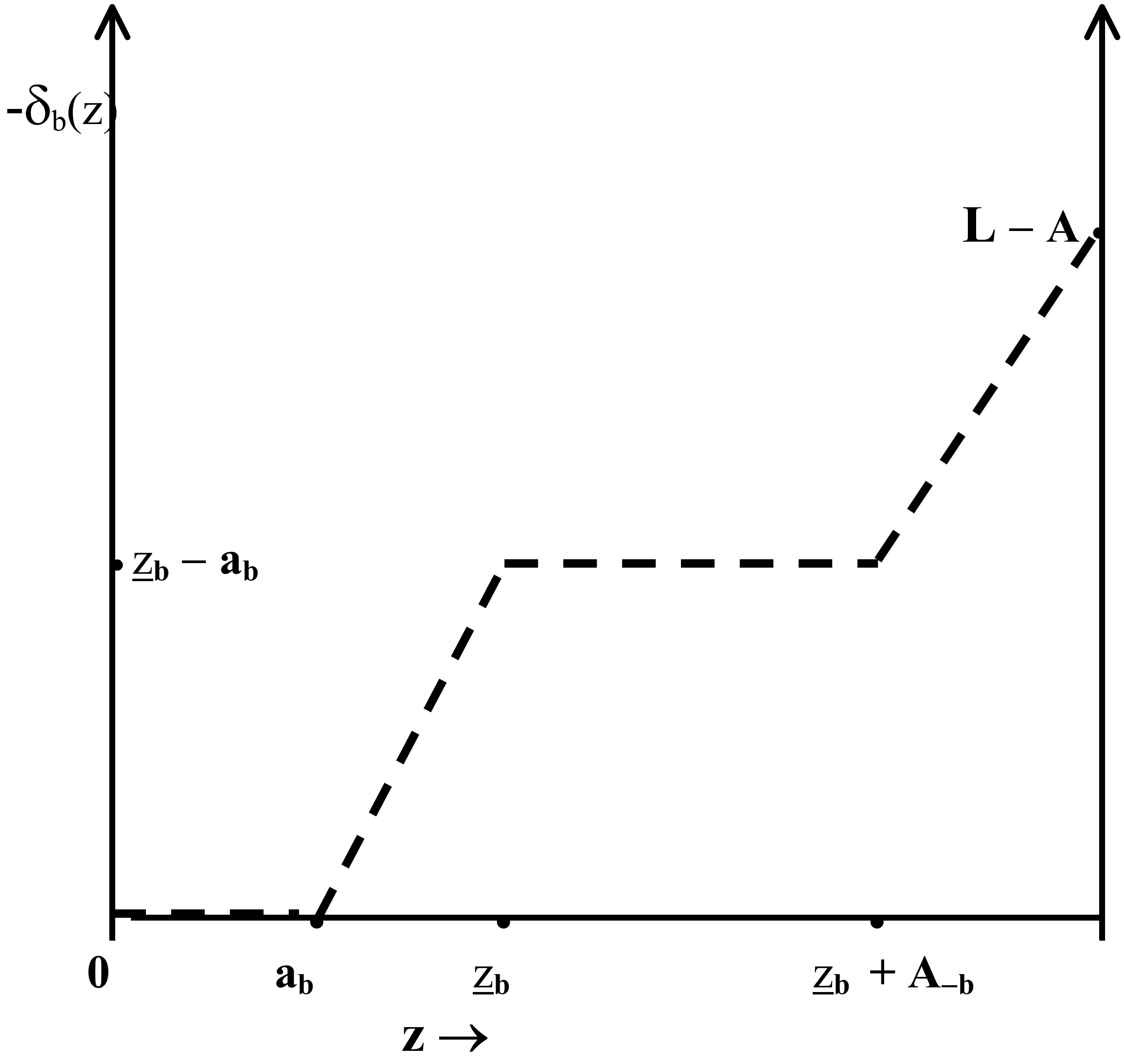

Figure 2