The Futility of Soft Floors in Auctions

Robert Zeithammer

UCLA

July 6, 2017

Abstract: Several of the auction-driven exchanges that facilitate programmatic buying of internet display advertising have recently introduced “soft floors” in addition to standard reserve prices (called “hard floors” in the industry). A soft floor is a bid level below which a winning bidder in the auction pays his own bid instead of paying the second-highest bid as in a standard second-price auction the ad exchanges used to employ. This paper shows analytically that soft floors cannot increase the auctioneer’s revenue, and they can decrease it. When bidders are symmetric, soft floors have no effect on revenue, because there always exists a symmetric equilibrium in strictly monotonic bidding strategies, and standard revenue-equivalence arguments thus apply. The industry often motivates soft floors as tools for extracting additional expected revenue from an occasional asymmetrically high bidder, for example a bidder re-targeting the consumer making the impression. When such an asymmetrically high bidder is added to a set of symmetric regular bidders, soft floors decrease revenue because bid-shading by the regular bidders more than erodes the benefits of the additional pricing pressure on the high bidder. I find only one possible scenario in which soft floors may enhance revenue in the asymmetric situation: when the level of the soft floor is so high that nobody bids above it, and the level of the reserve price is simultaneously sub-optimally low.

Contact: UCLA Anderson School of Management, Los Angeles, CA 90095. Email: rzeitham@ucla.edu
1. Introduction

Since the world’s first banner ad in 1994 (Singel 2010), advertising dollars have followed the shift of consumer attention to digital media, reaching over a third of total U.S. advertising spending by 2016. Despite starting with display banner ads, the lion’s share of digital advertising dollars was initially spent on search ads because they offered an unparalleled level of targeting (Goldfarb 2014). But for the first time in the short history of digital advertising, spending on display ads surpassed spending on search ads in 2016 (emarketer 2016). An improved targeting ability is one of the key forces behind the return of banners: unlike the banners from the 1990s, today’s banner ads are targeted to the individual viewer one impression at a time by computer algorithms – a practice called “programmatic buying.” A dominant method of allocating and pricing the display advertising space sold programmatically is real-time bidding (RTB) whereby each available impression is sold to interested advertisers by a sealed-bid auction that lasts a fraction of a second. Experts estimate that over $20 billion in advertising is sold by RTB per year in the United States (emarketer 2016) in over 30 trillion unique transactions (Friedman 2015).

What are the rules of these trillions of auctions? The vast majority of the “ad exchange” auctioneers employ second-price sealed-bid “Vickrey” auctions - a dramatic shift from the obscurity of the Vickrey pricing rule in past auction-driven marketplaces (Rothkopf, Teisberg, and Kahn 1990). However, several important players in the RTB industry have recently partially reversed this shift by introducing “soft floors” – bid levels below which the auction’s pricing rule switches from second-price to first-price.\(^1\) The “soft” part of “soft floor” contrasts with a “hard floor” – a bid level below which the auctioneer will not sell the impression, also known as “reserve price” in the auction literature (Myerson 1981). This paper provides the first theoretical treatment of soft floors, and shows their use is misguided: when soft floors are low enough

---

\(^1\) The use of soft floors is widespread but not universal. There is no standard data about their prevalence, but at least AdX, OpenX and AppNexus exchanges have used them in the U.S. market. When they are employed, they tend to affect the pricing in many transactions: Shuai et al (2013) analyze a bidding platform, on which more than half of transactions involve a price equal to the winner’s bid, i.e. an active soft floor.
to kick in, they can only weakly reduce the auctioneer’s expected revenue. Moreover, soft floors reduce expected revenue strictly precisely in the situations for which they were initially designed. The next few paragraphs introduce the modeling assumptions and the results they imply.

As long as the bidders are symmetric, I show that soft floors have no impact on auction revenue, because they do not change the allocation rule of the underlying mechanism. In other words, when the different advertisers’ valuations of each impression are drawn from the same distribution, soft-floor auctions are revenue-equivalent with standard auctions that have the same hard floor. The revenue equivalence result is not a trivial extension of the well-known known equivalence between 1st and 2nd price auction: just because 1st and 2nd price auctions yield the same expected revenue (under bidder symmetry), it does not immediately follow that their hybrid arising from the presence of a soft floor will also be revenue-equivalent with the simple 2nd price auction: strategic bidders may react to the introduction of a soft floor by playing mixed strategies or by pooling, thus changing the relationship between valuations and the chance of winning. The first major contribution of this paper is a general proof that while bidders indeed react to the introduction of a soft floor by adjusting their bids, the resulting equilibrium is in pure monotonic strategies. The monotonicity of the bidding equilibrium in the soft-floor auction guarantees that the soft floor auction does not change any bidder’s chance of winning relative the 2nd price auction with the same hard floor, which in turn keeps the expected revenue of the auctioneer unaffected according to the revenue equivalence result of Myerson (1981). Analogous arguments can then be used to also show that revenue equivalence continues to hold in the symmetric model even when the soft floor is hidden from the bidders, as it tends to be on some exchanges.

Given the robust revenue equivalence in the symmetric model, this paper explores the obvious possibility that a rationalization of the soft-floor industry practice can arise from asymmetries among bidders. Inspired by the industry analysts who originally motivated the use of soft floors (e.g. Weatherman 2013), I consider the possibility that one high-value bidder may occasionally enter the auction. For example,
a “re-targeting advertiser” (whose website the customer has just visited before arriving to the publisher auctioning off the impression) usually values the impression much more than other advertisers who bid only on demographics. If such a high-value advertiser were always present, there would be little benefit to auctioning – the seller could and should just set a take-it-or-leave-it price (Riley and Zeckhauser 1983). But such a high-value advertiser may not participate in every RTB auction, so a high soft floor might seem to be a clever adaptive mechanism that automatically activates itself only when he does appear (Weatherman 2013). In contrast to this industry intuition, I show that adding a randomly appearing asymmetrically high bidder to the pool of bidders breaks down the revenue equivalence of the symmetric model, and makes soft floors sub-optimal for the auctioneer. Two cases exist depending on whether the soft floor is set low enough that it kicks in (i.e., the high bidder actually bids above it) or whether the soft floor is so high that the auction effectively becomes a first-price sealed-bid auction. I now discuss each case in turn, please see Figure 1 for a tree representation of all the modeling variants covered in this paper, and the relevant sections.

**Figure 1: Summary of all results in this paper, and assumptions they rely on**

**Effect of a soft floor on expected revenue of the seller:**

- $N$ symmetric bidders: (no effect, even if soft floor hidden) (Section 3)
- $N$ symmetric bidders + one high bidder randomly present (Section 4)

<table>
<thead>
<tr>
<th>Case</th>
<th>Assumption about distribution of valuations:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Soft floor kicks in for high bidder: (Section 4.1)</td>
<td>General</td>
</tr>
<tr>
<td>Soft floor never kicks in (Section 4.2)</td>
<td>Uniform</td>
</tr>
</tbody>
</table>

- **Hard floor $\geq$ optimal**
- **Hard floor $<<$ optimal**

When soft floors are set low enough that the asymmetrically high bidder actually bids above the soft floor, the sub-optimality of soft floors is strict and fully general: I show that the strategic bids-shading
by low-valuation bidders always more-than-offsets the additional pricing pressure on the high bidder generated by the soft floor. In the second major contribution of this paper, I derive the following simple expression of the lost expected revenue: the lost expected revenue is the same as if the pricing format remained a Vickrey auction, but one of the regular bidders disappeared whenever the soft floor “kicked in”, i.e. whenever the high bidder happened to be present and all the other bidders happened to have valuations low-enough that none of them would want to bid above the soft floor in a first-price sealed-bid auction.

When the soft floor is so high that the auction effectively becomes a first-price sealed-bid auction, the equilibrium analysis becomes difficult because the asymmetrically high bidder plays a mixed strategy with continuous support (Vickrey 1961). It is well known that first-price auctions can dominate second-price auctions when there is no reserve (Vickrey 1961, Maskin and Riley 2000), but the understanding of the revenue ordering when the optimal reserve is present is less clear. This paper contributes to the analysis of asymmetric first-price sealed-bid auctions with a single high bidder by extending the model of Martínez-Pardina (2006) to the case when the hard floor may be binding as in Kaplan and Zamir (2012) and when the high bidder enters the auction only with some probability as assumed herein to capture the institutional details. The equilibrium is not in closed form, and numerical simulations are necessary to assess the revenue implications of using a high soft floor to effectively switch the pricing rule from second-price to first-price. I analyze the example of several uniformly distributed regular bidders in addition to the one asymmetrically high bidder, and find the first price rule can yield higher revenue, but only when the hard floor is suboptimally low. Thus, the well-known result that first-price auctions can dominate second-price auctions when there is one asymmetrically high bidder does seem to depend on the lack of a reserve price. In all other situations, the second-price rule dominates. Therefore, at least within the scope of a model with one asymmetrically high bidder, rationalizing the soft floor auction is difficult from the perspective of an auctioneer who sets his hard floor optimally. The only available rationalization is that soft-floor auctions effectively implement first-price sealed-bid auctions, and the auctioneer cannot optimize the hard floor.
The proposed model is generic, and thus applies to any auction market that is considering soft floors. Soft floors are an emerging industry practice in the market for online display advertising impressions, so this paper contributes to the growing literature on online display advertising. This literature (for thorough literature reviews, see Hoban and Bucklin, 2015, or Johnson, Lewis and Nubbemeyer, 2017) contains very few papers about soft floors: Yuan, Wang, and Zhao (2013) examine data from a large ad exchange that uses soft floors, and estimate that more than half of the exchange’s revenue is transacted using the first-price rule instead of the second-price rule. They conclude soft floors are an economically important phenomenon in the RTB marketplace. In contrast to the predictions of this paper, Försch, Heise, and Skiera (2016) analyze the profitability of soft floors using a large-scale field experiment, and conclude they can dramatically increase the auctioneer’s revenue. One possible explanation of the divergence between this paper’s predictions and the field experiment’s results is that the bidders did not have enough time to adjust their strategies to the novel mechanism: unlike the rational bidders assumed herein, the bidders in Försch et al. (2016) do not reduce their bids when soft floors are introduced. This paper predicts that any short-run benefits of soft floors will disappear in the long run as the bidders learn to play their correct best response to them. In the industry literature, other arguments against soft floors have emerged, for example the argument that an auctioneer who uses soft floors will lose bidders to a competitor who does not (Rubicon Project 2014).

More broadly, this paper also contributes to the Marketing literature on mechanism design, which concerns itself with the design of trading institutions (e.g., Wernerfelt 1994, Zeithammer 2015). Incentive compatibility plays a crucial role in that literature, ensuring buyers play their best response to the seller’s strategy. One way to frame the contribution of this paper is highlighting the importance of incentive compatibility in designing more profitable auctions for selling online advertising space: soft floors seem to have emerged on the assumption that bidders will not change their bidding strategy in response to the seller attempting to grab a piece of their surplus.
2. Soft-floor auction definition

One object (e.g., an ad impression in the RTB advertising context) is for sale. The auctioneer values the object at zero, and sets two reserves: a hard floor $h \geq 0$ and a soft floor $s > h$. The soft-floor sealed-bid auction collects bids, sorts them such that $b_{(1)} \geq b_{(2)} \geq b_{(3)} \ldots$, and determines the auction winner and the price paid as follows:

1) When $b_{(1)} > s$, the bidder who submitted $b_{(1)}$ wins and pays $\max\{s, b_{(2)}\}$.

2) When $s \geq b_{(1)} \geq h$, the bidder who submitted $b_{(1)}$ wins and pays $b_{(1)}$.

3) When $h > b_{(1)}$, the auctioneer keeps the object.

In words, the soft floor functions as a reserve price in a second-price sealed-bid auction (“2PSB”) as long as at least one bid exceeds it (case 1). When no bids exceed $s$, the auction reverts to a first-price sealed-bid auction (“1PSB”) with a reserve price equal to $h$ (cases 2 & 3).

Throughout this paper, I assume $h$ is common knowledge; that is, the auctioneer announces the reserve price before the auction. Regarding the bidders’ information about $s$, I first assume the auctioneer also pre-announces $s$ (or that, equivalently, the bidders figure out both values through experimentation), and then address the possibility of keeping $s$ hidden from bidders whenever tractable.

An analysis of the revenue implications of soft floors requires a model of bidders. This paper considers independent private-valuation (IPV) bidders – a standard assumption in auction theory (see Krishna 2002 for a discussion). IPV is also a reasonable model of bidders in the online display-ad market that motivates this paper: “valuation” of an impression is the increase in an advertiser’s profit from that advertiser winning the impression, “private” means no advertiser would want to update his estimate of his own valuation of the impression after learning another advertiser’s valuation, and “independent” means the values are statistically independent of each other in the population of bidders. Given the IPV assumption, a population distribution of valuations completes the model. This paper makes two nested assumptions about the distribution. First, section 3 analyzes the “symmetric” case when the valuations are drawn iid from some
continuous distribution $F$. Second, section 4 analyzes the “asymmetrically high bidder” case when a single high-valuation bidder is added to a group of symmetric bidders. I turn to the symmetric case next.

3. Symmetric bidders: soft floors have no impact on the auctioneer’s expected revenue

Suppose $N$ bidders have private valuations drawn independently from some distribution $F$ on $[0, M]$. Following Krishna’s (2002) notation, let $G$ be the distribution of the maximum from $N-1$ iid draws from $F$, itself denoted $Y_1$: $G(Y_1) = F^{N-1}(Y_1)$, and let $X_1$ be the highest of $N$ iid draws from $F$, distributed $F^N(X_1)$.

This section demonstrates that when the bidders are symmetric, soft floors have no impact on the auctioneer’s expected revenue. The proof proceeds in two steps. First, for any $s > h \geq 0$, I construct a monotonically increasing equilibrium bidding strategy $\beta(v)$ that best responds to $s$. Second, the fact that the bidding strategy is monotonic means the soft-floor auction allocates the object to the same bidder as a standard auction with a hard floor of $h$, and so the Revenue Equivalence Theorem of Myerson (1981) implies the soft-floor auction also produces the same expected revenue to the auctioneer. The exact form of the bidding equilibrium depends on the bidders’ information about the soft floor. The following subsection 3.1 analyzes the case of the soft floor being common knowledge among the bidders. Subsection 3.2 then takes up the case of bidders uncertain about the soft floor at the time of bidding.

3.1. Bidding equilibrium when the soft floor is common knowledge

Let $\beta_i(v)$ denote the standard symmetric bidding equilibrium in a 1PSB with a public reserve $h$ described in textbooks (for a detailed derivation, see Krishna 2002):

$$
\beta_i(v) = h \frac{G(h)}{G(v)} + \frac{1}{G(v)} \int_0^v xg(x)dx = E[\max \{Y_i, h\}|Y_i < v]
$$

(1)
where the roman-numeral subscript on $\beta$ indicates the pricing rule. Suppose the soft floor is common knowledge among the bidders at the start of the auction. Then the monotonically increasing bidding equilibrium $\beta(v)$ in the soft-floor auction can be characterized in terms of $\beta_{i}(v)$ as follows:

**Proposition 1:** When $s < \beta_{i}(M)$, the following in a unique symmetric monotonic pure-strategy equilibrium of the soft-floor auction: $\beta(v)=\left\{\begin{array}{ll}
v \leq \beta_{i}^{-1}(s) : \beta_{i}(M) \\
v > \beta_{i}^{-1}(s) : v
de$.

When $s \geq \beta_{i}(M)$, the soft-floor auction becomes a first-price sealed-bid auction, and $\beta(v) = \beta_{i}(M)$.

When $s < \beta_{i}(M)$, the equilibrium $\beta$ involves a jump discontinuity: bidders with $v \in [h, \beta_{i}^{-1}(s)]$ shade their bids as if they were in a first-price sealed-bid auction with a reserve of $h$. Higher-valuation bidders with $v \in [\beta_{i}^{-1}(s), M]$ bid their valuations. The proof of Proposition 1 is useful for understanding the bidding incentives in a soft-floor auction, so I include it in the main body of the paper:

**Proof of Proposition 1:** Given the soft-floor auction rules defined in section 2, every bidder $v$ can either bid below $s$ and face 1PSB pricing with the hard floor $h$ as a reserve price, or bid above $s$ and face 2PSB pricing with the soft floor as the reserve price. Suppose a symmetric monotonically increasing bidding equilibrium $\beta(v)$ exists such that $\beta(M) > s$. I first show that the equilibrium must have a jump discontinuity at the valuation level $v^*$ such that $\beta(v^*) = s$. By construction, the $v^*$ bidder pays $s$ and receives a positive surplus $\Pr(\text{win})(v^*-s) > 0$, so $v^* > s$. Now consider bidders with $v > v^*$, who also bid above $s$ by monotonicity of $\beta(v)$, and hence face 2PSB pricing with a reserve of $s$. By the dominant-strategy properties of 2PSB, these bidders thus bid $\beta(v) = v$, and so $\lim_{v \to v^*+} \beta(v) = v^* > s = \beta(v^*)$. 

9
To finish the proof, it remains to be shown that $v^*$ is unique and satisfies $v^* = \beta^{-1}_I(s)$. Fix a bidder with some valuation $v$, and consider bidding below $s$, which triggers 1PSB pricing. In a symmetric monotonic equilibrium, all bidders with valuations below $v$ also bid below $s$, and hence also face 1PSB incentives. Bidders who bid above $s$ and face different pricing do not affect the payoff of our focal bidder locally (as long as he continues bidding below $s$), because they win the auction for sure. Therefore, the standard 1PSB bidding strategy $\beta_I(v)$ is the only candidate for an equilibrium strategy among bidders who bid below $s$. The expected equilibrium surplus from bidding below $s$ is also the same as in 1PSB:

$$\pi_I(v) = G(v) \left( v - \beta_I(v) \right)$$  \hspace{1cm} (2)

Now consider the same bidder $v$, and suppose she bids above $s$, thus switching the auction format to 2PSB in the process. It is immediate from the properties of 2PSB that her best bid above $s$ is her valuation $v$, and we are tacitly assuming $v > s$; otherwise, a deviation above $s$ could not make sense. Denote her payoff from bidding her valuation to be $\pi^{II}(v)$. A rational bidder $v$ will bid below $s$ whenever $\pi^I(v) > \pi^{II}(v)$, and vice versa.

$\pi^{II}(v)$ is subtle to characterize in general, but it is simple for the bidder $v^*$ who is indifferent between bidding strictly above and weakly below $s$. The nature of the proposed equilibrium with a jump discontinuity at $v^*$ implies the indifferent bidder must be the lowest-valuation bidder who bids her valuation, and so she can only win the auction when all her competitors have lower valuations than $v^*$. But their valuations below $v^*$ mean the competitors are bidding according to the 1PSB bidding strategy $\beta_I(v) \leq s$, and so the indifferent bidder is guaranteed to pay $s$ should she win the auction after bidding her valuation. In other words, the soft floor must always “kick in” as the reserve price for the indifferent bidder. Therefore, the expected surplus of the indifferent bidder when she bids her valuation must be

$$\pi^{II}(v^*) = G(v^*) \left( v^* - s \right)$$  \hspace{1cm} (3)
Comparing equation 3 with equation 2, note the indifferent bidder is the bidder who would bid exactly $s$ in the 1PSB: $\beta_s(v^*) = s$. As long as $\beta^{-1}_s(s)$ is in the support of $F$, the resulting valuation is the unique solution to $\pi_s(v) = \pi_d(v)$ because $\pi_d(v)$ grows faster of the two expected surpluses above the lowest of potential solutions to the indifference equation. This completes the proof of Proposition 1.

Figure 2: Equilibrium bidding strategy in a soft-floor auction

Note to figure: $F =$ uniform $[0,1]$, $s=0.6$, and $h=0.5$ (the $h$ is optimal given the $F$). The dashed line indicates the 45-degree line; the dotted vertical lines indicate the jump discontinuities at $v^*$ for the given numbers of bidders indicated by the numbers next to the lines.

Example: $F=$Uniform$[0,1]$

It is useful to illustrate Proposition 1 on a concrete distributional example. A uniform distribution of valuations implies $G(x) = x^{N-1}$, so the 1PSB bidding strategy is

$$\beta(v) = \frac{N-1}{N} v + \frac{h^N}{Nv^{N-1}}$$  \hspace{1cm} (4)
The indifference equation \( \beta_i(v^*) = s \) becomes

\[
(N - 1)(v^*)^N - sN(v^*)^{N-1} + h^N = 0
\]

which does not have a closed-form solution for a general \( N \), but does for \( N=2 \):

\[
v^*(s) = s + \sqrt{s^2 - h^2} < 1 \iff s < \frac{1 + h^2}{2}
\]

Figure 2 illustrates the bidding function for \( N=2,3,4, \) and 10. Having established the existence of a symmetric monotonic pure-strategy equilibrium when the soft floor is common knowledge, I now turn to the possibility that the bidders are uncertain about the soft floor at the time of bidding.

### 3.2. Bidding equilibrium when the soft floor is hidden at the time of bidding

Suppose the bidders are uncertain about the soft floor at the time of bidding, and they summarize their beliefs about it by some distribution \( \Omega \) on \([h, M] \). Optimal bidding must now average over the possibility that the soft floor happens to be low (and second-price rules will thus apply) and the possibility that the soft floor happens to be high (and the price paid will be equal to the winning bid). Unlike in the previous subsection 3.1, characterizing the equilibrium in closed form is not possible even in the uniform example. However, the following proposition provides weak sufficient conditions for a symmetric monotonic equilibrium to exist and bounds the resulting bidding function below with \( \beta_i(v) \):

**Proposition 2:** When \( f(v) F^{N-2}(v) \) and \( \Omega(v) \) are continuous on \([h, M] \), the sealed-bid auction with a hard floor \( h \) and a hidden soft floor drawn from \( \Omega \) on \([h, M] \) has a symmetric pure-strategy equilibrium characterized by an increasing bidding function \( \beta(v) > \beta_i(v) \) that satisfies

\[
\beta'(v) = \frac{g(v)(v - \beta(v))}{G(v)[1 - \Omega(\beta(v))]} 
\]
The proof of Proposition 2 is relegated to the Appendix, and relies on the Peano Existence Theorem to assure us equation 7 has a solution. Compared to the 1PSB differential equation that gives rise to $\beta_i(v)$, equation 7 adds the term in the square bracket. Because $\beta(v)$ is thus steeper everywhere, the relative ranking of the two bidding functions follows. Intuitively, relative to 1PSB, random soft floors partially mitigate the increased payment associated with higher bids, by switching the pricing rule to second-price. The resulting “random discount” gives the bidders an incentive to raise bid levels, and so the $\beta(v)$ exceeds equilibrium bidding in a 1PSB with the same number of bidders everywhere above $h$. Illustrating the equilibrium bidding in a concrete example is again useful; see Figure 3.

**Figure 3: Bidding strategy when soft floor is hidden**

![Figure 3: Bidding strategy when soft floor is hidden](image)

**Note to figure:** $F = \text{uniform } [0,1]$, and $h=0.5$ (the $h$ is optimal given the $F$), and $s=\text{Uniform}[h,1]$. The dashed line indicates the 45-degree line. The dotted lines indicate 1PSB bidding strategies without a soft floor, and the solid lines indicate bidding strategies with a hidden soft floor, both for several levels of the number of bidders indicated by the numbers next to the lines.
3.3. Revenue equivalence under bidder symmetry

Both subsections 3.1 and 3.2 find a monotonically increasing symmetric pure-strategy equilibrium of the soft-floor auction game. Under both assumptions regarding the bidders’ information about the soft floor, the introduction of a soft floor therefore does not affect each \(v \geq h\) bidder’s probability of winning, namely, \(G(v)\). Under either informational assumption, the introduction of a soft floor also does not affect the payoff of the bidder with the lowest trading valuation \(v=h\), who makes zero surplus both with and without the soft floor. Therefore, the revenue-equivalence result of Myerson (1981) implies the soft floor does not affect the auctioneer’s profit and the bidders’ surpluses – a fact I summarize in the next Proposition:

**Proposition 3:** As long as bidders are symmetric in their valuations, and their beliefs about the soft floor are either exactly correct or captured by a continuous distribution on \([h,M]\), then for every hard floor \(h\), the introduction of a soft floor \(s>h\) has no effect on the auctioneer’s revenue or any bidder’s expected surplus.

The “magic” of revenue equivalence stems from the fact that we only need to consider the allocation probability (the chance of winning) for every bidder type – the revenues are then implied by incentive compatibility. Please see Myerson (1981) and Krishna (2002) for a more detail exposition. In the Appendix, I verify the auctioneer’s revenue is indeed unaffected under the assumptions of subsection 3.1.

Looking at both Figures 2 and 3, the empirical prediction about the impact of soft floors on bidding is clear: keeping the hard floor constant, adding a soft floor should lead to bid-shading by low-valuation bidders, and so the distribution of the observed bids should become more skewed to the right. If the soft floor is common knowledge, the distribution of observed bids should also have a hole just above the soft floor. The data collected by Försch et al. (2016) does not have either of these features, suggesting that the bidders in the experiment did not rationally adjust their bidding strategies to the presence of the soft floor.
4. Asymmetric bidders: soft floors reduce auctioneer’s revenue as long as hard floors are set optimally

In a prominent explanation of soft floors, Kevin Weatherman of the MoPub platform used a stylized example of a seller who sets a hard floor of $1 and faces an occasional single high bid of $2 in addition to regular bidding activity in the $0.75-$0.90 range (Weatherman 2013). Weatherman’s argument for why such a seller would benefit from soft floors is that the seller can lower his hard floor to below $0.75 while introducing a soft floor of $1: such an arrangement seems to preserve the price pressure on the high bidder whenever he is present while also collecting more revenue from low bidders. Or, as Weatherman puts it: “the goal is to ‘harvest’ higher bids while not compromising on lower bid opportunities” (Weatherman 2013). Diksha Sahni of AppLift eloquently makes the same argument by pointing out that “when the gap between a bid and the second bid is significant, it may create a gap between potential revenues and actual revenues” (Sahni 2016). Motivated by these industry experts, this section analyzes the possibility of bidder asymmetry somehow making soft floors profitable for the auctioneer.2

Assume that, in addition to N “regular” bidders drawn from the same $\mathcal{F}$ on $[0,M]$ as in the previous section, a “high” bidder with valuation $M$ also exists, and he only appears with some probability $\alpha<1$. In a 2PSB with just a hard floor $h$, the auctioneer makes

$$
\Pi_{2PSB}^A(h) = \alpha E\left(\max\{X, h\}\right) + \left(1-\alpha\right)\Pi(h)
$$

where the $A$ subscript indicates the “asymmetric” situation with an additional high bidder, and $\Pi(h)$ is just the expected revenue in a standard auction with a reserve price of $h$ (e.g., equation 23 in the Appendix).

2 Note that this is just one of many possible asymmetries one could analyze: more generally, each bidder could be drawn from a completely different distribution (for examples, see Maskin and Riley 2000 and Kaplan and Zamir 2012). This paper does not attempt to characterize the impact of soft floors on all possible asymmetries, but instead focuses on a particular asymmetry used in the industry to justify the soft-floor practice.
To set the profit baseline for the soft-floor auction, note how the high bidder’s presence changes the optimization of the hard floor. Recall that without the high bidder ($\alpha = 0$), the optimal reserve does not depend on the number of bidders, and satisfies the classic relationship $\psi(h^*) = 0$, where

$$
\psi(v) = v - \frac{1 - F(v)}{f(v)}
$$

is the so-called “virtual value” function of Myerson (1981)\(^3\) assumed here for simplicity to be increasing ($F$ is assumed to be “regular” in Myerson’s terminology).

The demand increases with the addition of the randomly present high bidder, and so, intuitively, the optimal reserve should rise above $\psi^{-1}(0)$. This is indeed the case: a reserve price of $M$ fetches the profit of $\alpha M$, which is obviously the best the seller can achieve as $\alpha$ approaches 1.\(^4\) So one would expect the optimal reserve $h^*$ to grow as $\alpha$ increases, eventually reaching $M$. The first-order condition (FOC) of equation 8 is (please see the Appendix for a detailed derivation):

$$
\psi(h^*) = \frac{\alpha F(h^*)}{(1-\alpha)Nf(h^*)} > 0
$$

(9)

Because the LHS of the FOC in equation 9 is increasing, the FOC confirms the optimal reserve exceeds that without the high bidder ($\alpha = 0$) and rises as $\alpha$ increases. Moreover, because the RHS of the FOC rises without bound as $\alpha$ increases while the LHS is bounded above by $M$, an $\alpha < 1$ always exists such that for all $\alpha \geq \alpha$, the optimal reserve is a corner solution at $h^* = M$ as hypothesized above. For example, when $F$ is Uniform[0,1] and $N=2$, then $h^* = \frac{2(1-\alpha)}{5\alpha - 4}$ for $\alpha \leq \frac{2}{3}$ and $h^* = 1$ otherwise.

What is the soft-floor auction profit here, and how does it compare to $\Pi_{PSB}^1(h^*)$? Two distinct

\(^3\) The virtual value of a bidder with valuation $v$ is the marginal revenue a seller can extract from the bidder in a direct-revelation mechanism (Myerson 1981). In the $\psi(h^*) = 0$ equation, the zero on the RHS is the implicitly assumed opportunity cost of the seller considered here.

\(^4\) Because the seller knows the high bidder’s valuation, the selling problem becomes trivial when the asymmetrically high bidder is always present. $\alpha < 1$ keeps the problem interesting while also capturing the essence of Weatherman’s (2013) stylized example.
cases emerge, depending on the soft-floor magnitude relative to bids: First, the soft floor can be low enough to have a direct effect on bidding, i.e. to induce at least the high-valuation bidders to bid above it. Second, the soft floor can be so high as to only allow an indirect effect by making all bidders bid below \( s \) and thus switching the auction pricing rule to 1PSB for all bidders. I analyze these two cases in turn.

**4.1 Soft floor low enough that some bidders bid above it**

Conveniently for analysis of this case, the soft-floor auction has the same bidding equilibrium (outlined in Proposition 1) as without the high bidder whenever \( s \) is small enough that \( s < \beta_i(M) \):

**Lemma 1:** When \( s \) is small enough that \( s < \beta_i(M) \), randomly (with probability \( \alpha < 1 \)) adding one high bidder with \( v = M \) to a soft-floor auction with \( N \) symmetric bidders with \( v \sim F[0, M] \) does not change the bidding equilibrium described in Proposition 1.

The proof of Lemma 1 is important for understanding the nature of asymmetry arising from the high bidder’s presence, so I present it in the main text:

**Proof of Lemma 1:** First, if the high bidder indeed bids his valuation \( M \), the regular bidders assume they can only win when the high bidder is not present, and so they behave the same as in an auction without him. Second, the high bidder bids his valuation \( M \) because he does not prefer to mimic the \( v^* \) bidder and switch pricing to 1PSB. He is assured of victory with a bid of \( M \) and 2PSB pricing (with a reserve of \( s \)). The only non-trivial deviation I need to analyze is bidding \( s \) or below to trigger 1PSB pricing, resulting in winning a lot less often but also paying less. Given the high bidder’s high valuation, bidding exactly \( s \) is the best such deviation from all possible bids weakly below \( s \). Mathematically, bidding \( M \) yields a surplus:
\[ M - F^N(v^*(s))s + (1 - F^N(v^*(s)))E(X_i|X_i > v^*(s)) = \]
\[
= F^N(v^*(s))(M - s) + (1 - F^N(v^*(s)))\left[ M - E(X_i|X_i > v^*(s)) \right] \]

which strictly exceeds the \( F^N(v^*)(M - s) \) = the surplus from bidding \( s \). In words, bidding \( M \) yields an additional expected surplus compared to bidding (and paying) \( s \). The additional expected surplus arises from winning more often, namely, when the highest regular competitor is above \( v^* \). \( QED \) Lemma 1.

Since the equilibrium bidding is the same as described in Proposition 1, the auctioneer’s profit from a soft-floor auction thus becomes:

\[
\Pi^s(h,s) = \alpha \left[ F^N(v^*(s))s + (1 - F^N(v^*(s)))E(X_i|X_i > v^*(s)) \right] + \frac{(1 - \alpha)\Pi(h)}{\text{high bidder not present} \rightarrow \text{high bidder wins and pays max}(X_i,s)} \]

where \( \Pi(h) \) is just the soft-floor auction profit with \( N \) regular bidders shown in Proposition 3 not to depend on \( s \). Since equations 8 and 11 have similar structures, comparing \( \Pi_{PSB}^d(h) \) with \( \Pi^s(h,s) \) is easy. First notice that when the high bidder is not present, the two formats deliver the same profits (Proposition 3 + Lemma 1). Second, recall that \( s \) can be expressed as a conditional expectation of an order statistic:

\[ s = \beta_i(v^*) = E\left[ \max\{Y, h\}|Y < v^* \right] \]

Substituting for \( s \) in equation 11 and suppressing the dependence of \( v^* \) on \( s \) for clarity yields

\[
\frac{\Pi_{PSB}^d(h) - \Pi^s(h,s)}{\alpha} = F^N(v^*)E\left( \max\{X_i, h\}|X_i < v^* \right) + (1 - F^N(v^*))E\left( \max\{X_i, h\}|X_i > v^* \right)
\]

\[
- F^N(v^*)E\left( \max\{Y_i, h\}|Y_i < v^* \right) - (1 - F^N(v^*))E(X_i|X_i > v^*) = \]

\[
= F^N(v^*)\left[ E\left( \max\{X_i, h\}|X_i < v^* \right) - E\left( \max\{Y_i, h\}|Y_i < v^* \right) \right] \]

(12)
where the second equality follows from noticing the conditional expectation above \( v^* \) simplifies for all \( v^* > h \): 
\[
E\left( \max \{ X_i, h \} \mid X_i > v^* \right) = E\left( X_i \mid X_i > v^* \right).
\]
In words, the lost expected revenue due to a soft floor is the same as if the pricing format remained a Vickrey auction, but one of the regular bidders disappeared whenever the soft floor would “kick in”, i.e. whenever the high bidder happened to be present and all the other bidders happened to have valuations low-enough that none of them would want to bid above the soft floor in a first-price sealed-bid auction. The maximum of \( N \) draws (\( X_i \)) always exceeds the maximum of \( N-1 \) draws from the same distribution (\( Y_i \)), so the 2PSB auction delivers a higher profit.

Formally, I have shown the following:

**Proposition 4**: Assume \( N \) bidders drawn independently from a continuous \( F[0,M] \) participate in the auction, and one high bidder with \( v=M \) participates with probability \( \alpha \leq 1 \). Then, for every hard floor \( h \), adding a soft floor \( s > h \) low enough that at least some bidders bid above \( s \) reduces the expected auction profit by 
\[
\alpha F^N(v^*(s)) [E(\max \{ X_i, h \} \mid X_i < v^*(s)) - E(\max \{ Y_i, h \} \mid Y_i < v^*(s))] ,
\]
which increases in the magnitude of the soft floor \( s \).

In summary, note that soft floors actually reduce expected auctioneer revenue precisely when they “kick in” to put pricing pressure on the high bidder, that is, precisely in the situation discussed by soft-floor advocates (e.g., Weatherman 2013, Sahni 2016). The industry analysts are correct in noting that the soft floor adds pricing pressure on the high bidder whenever he is present. But Proposition 4 shows that the co-incident bid-shading by low-valuation bidders more than erodes the benefits of the added pricing pressure.

The following example illustrates Proposition 4:

**Example: \( F=\text{Uniform}[0,1] \)** Under the uniform assumption,
\[
F^N(z) E\left( \max \{ X_i, h \} \mid X_i < z \right) = h^{N+1} + N \int_{h}^{z} x^{N} \, dx = \frac{h^{N+1} + N z^{N+1}}{N+1}
\]  
(13)
Therefore, equation 12 becomes:

\[
\frac{\Pi^A_{2PSB}(h) - \Pi^A(h,s)}{\alpha} = \frac{N(v^*(s) - h)}{N(N+1)} \sum_{k=0}^{\infty} \left( h^{k+1} \left( v^*(s) \right)^{N-k+1} - h^N \right) > 0 
\]  

(14)

which is obviously increasing in \( v^* \), and so it is also increasing in \( s \). In words, the lost expected profit due to the introduction of a soft floor about \( h \) increases as the soft floor increases, ceteris paribus.

**4.2 Soft floor so high the soft-floor auction becomes a first-price auction**

In the previous subsection, I have shown that randomly adding an asymmetrically high bidder makes the soft-floor auction perform strictly worse than a standard 2PSB auction with the same hard floor, as long as the soft floor is low enough that some bidders actually bid above it. But what happens to profits when the soft floor is high enough that the auction reverts to a 1PSB? In the symmetric case of the previous section, we could rely on revenue equivalence to argue profits remain unaffected because 1PSB is revenue equivalent to 2PSB under bidder symmetry. But it is well known that bidder asymmetry breaks the revenue equivalence between 1PSB and 2PSB (Vickrey 1961, Maskin and Riley 2000). Therefore, soft-floor auctions might increase the auctioneer’s revenue by effectively implementing the 1PSB within a bidding environment initially set up as a 2PSB. In this subsection, I compare the 1PSB profits with 2PSB profits under the asymmetric model to explore whether effectively implementing the 1PSB can rationalize the use of soft floors.

Suppose \( s \) is so high that all bidders bid below \( s \), effectively engaging in a 1PSB. The equilibrium bidding strategies are not known for general \( \alpha, F, \) and \( N \). Kaplan and Zamir (2012) extend the analysis of Vickrey (1961) without a binding reserve to the case of a binding reserve and \( \alpha=1, F=\text{Uniform}, \) and \( N=1 \). Martínez-Pardina (2006) extends the analysis of Vickrey (1961) to the case of \( \alpha=1 \), general \( F \), and \( N>1 \), but does not consider binding reserves. I extend the analyses of Kaplan and Zamir (2012) and Martínez-Pardina (2006) to \( \alpha < 1, N > 1 \), and general \( F \), covering both binding and non-binding reserves. My analysis
has two parts: Lemma 2 provides necessary properties of an equilibrium, and an example numerically solves for the equilibrium for $F$ uniform on $[0,1]$, $N=2$, and a range of $a$.

**Lemma 2**: Suppose ties at $h$ are broken in favor of the regular bidders, and let $b$ satisfy $M = b + \frac{F(b)}{Nf(b)}$.

If an equilibrium exists in which the high bidder uses a mixed strategy $H(b)$ on a continuous interval, the equilibrium must have the following properties:

- **All the regular bidders use a symmetric strategy** $\beta(v)$ with partial pooling at $\max(h, b)$ such that
  $$\beta(v) = \max(h, b) \text{ on } v \in \left[\max(h, b), v_h\right], \text{ and } \beta(v) \text{ is increasing on } v \in (v_h, M].$$

- **The support of $H(b)$ is** $\left[\max(h, b), M - F^N(v_h)(M - \max(h, b))\right]$.

In words, the hard floor $h$ is only binding when it exceeds $b$. The pooling at the lowest bid level arises from the possibility that the high bidder may not be present. Please see the Appendix for a detailed proof and intuition.

**Example: $F=$Uniform[0,1]**

When $F$ is uniform, the equilibrium proposed by Lemma 2 exists and has the following properties: the cutoff for binding hard floors is $b = \frac{N}{N+1}$. The bidding function of the regular bidders is

$$\beta(v) = \begin{cases} 
\max(h, b) \leq v \leq v_h : \max(h, b) \\
v > v_h : 1 - \left(1 - \max(h, b)\right)\left(\frac{v_h}{v}\right)^{v_h}
\end{cases}$$

(15)

The high bidder’s mixing function satisfies
where the $v_b$ point satisfies $(1-\alpha)v_b^{N-1} = \exp \left\{ - \int_{\min(h,b)}^{1-v_b^{N}(1-\max(h,b))} v_h^{N} \left( 1 - \frac{\max(h,b)}{1 - x} \right)^{-1} dx \right\}$.

Unfortunately, the key integral in the definition of $H$ is not tractable analytically. However, it can be easily computed numerically. Figure 4 shows the auctioneer’s revenue as a function of hard floor in the case of $N=2$. The dotted lines represent the (analytical) 2PSB benchmark derived in equation 8, and the solid lines represent the expected 1PSB revenue from the numerical simulation. With $N=2$, the bid level below which the hard floor is not binding in 1PSB is $b = \frac{2}{3}$, so the solid (1PSB) lines are constant for $h < \frac{2}{3}$. The optimal 2PSB hard floor defined by equation 9 whenever it is below $M=1$ is $h^* = \min \left( \frac{2(1-\alpha)}{4-5\alpha}, 1 \right)$, which is equal to 1 for $\alpha \geq \frac{2}{3}$. In other words, the optimal 2PSB reserve amounts to a take-it-or-leave-it offer at the high bidder’s valuation whenever the high bidder is more than 2/3 likely to be present, and so the dotted (2PSB) lines are increasing in $h$ for $\alpha \geq \frac{2}{3}$. Having explained the key features of each auction format’s revenue, I now turn to comparing the formats to each other.

Figure 4 shows 1PSB can indeed dominate the 2PSB in terms of expected revenue, but it can only do so when $\alpha$ is high and $h$ is sub-optimally low for the 2PSB (i.e., in the region above the locus of optimal 2PSB reserves shown by the dashed (red) line). The $\alpha=1$ case extends the standard finding in the literature that 1PSB revenues exceed 2PSB revenues under $h=0$ (Martinez-Pardina 2006) to all $h<0.75$. Not surprisingly, the revenue order persists also for $\alpha$ near 1 and low-enough $h$. A more novel finding is that the
ordering flips when $\alpha$ drops below approximately $\frac{1}{2}$: 2PSB with a high-enough reserve dominate 1PSB with the same reserve then. Finally, as long as the hard floor is set optimally for the 2PSB, it clearly outperforms the 1PSB not only for the same hard floor, but also for all possible hard floors. We summarize these observations in the following result:

**Figure 4: Revenue comparison of 2PSB and 1PSB as a function of hard floor ($N=2, F=\text{Uniform}$)**
Numerical result: Assume two bidders drawn independently from the uniform distribution on $[0, 1]$ always participate in the auction, and one high bidder with $v = 1$ participates with probability $\alpha \leq 1$. Thus,

1) Adding a soft floor $s > h$ high-enough that no bidders bid above $s$ increases the auctioneer’s revenue only when $h$ is set sub-optimally low for a standard 2PSB and $\alpha$ is high enough (approximately above $1/2$).

2) For every $\alpha$, the 2PSB with a hard floor set optimally generates more revenue than the 1PSB with any hard floor.

I conclude that, at least for uniformly distributed regular bidders, adding a randomly participating high bidder to the auction does not provide a strong reason to switch the pricing rule from second-price to first-price. The only possible reason would be that the hard floor is set too low for some exogenous reason, for example, because the worker running the auction faces negative consequences from his manager if too many auctions end without a sale. But such a worker is not optimizing his expected revenue, so how he would value the distribution of results produced by the 1PSB is unclear. Moreover, recall the example of Weatherman (2013), who suspects the hard floors in the existing 2PSB are set too high, not too low.

5. Discussion

Soft floors have emerged in the RTB digital display advertising industry as a potential tool for increasing publisher revenues. This paper shows soft floors are not likely to deliver on this promise, even if the auctioneer keeps them hidden from bidders, or when one of the bidders tends to have a very high valuation, e.g. arising from re-targeting considerations. The main reason soft floors cannot increase revenue is that low-valuation bidders respond to them by shading their bids down from their valuations, and the lowest-valuation bidder who does not shade her bid has a valuation well above the soft floor.
When the bidder valuations are all drawn from the same distribution (and the bidders are thus “symmetric” in the auction theory parlance), the shading by low-valuation bidders makes soft floors completely irrelevant to the auctioneer’s revenue because the resulting equilibrium bidding function remains monotonically increasing as in the benchmark second-price auction with the same hard floor. The monotonicity of equilibrium bidding continues to hold even when the auctioneer hides the exact level of the soft floor before bidding. Together with the bidders’ symmetry, the monotonicity of their bidding strategy implies the introduction of soft floors does not change any bidder’s chance of winning the auction, and the classic revenue-equivalence result of Myerson (1981) thus implies it does not change the auctioneer’s expected revenue.

Soft-floor advocates often point to a gap between the winning bid and the second-highest bid in RTB auctions, and argue the auctioneer can capture some of this gap as extra profit using a soft floor. The symmetric case discussed in the previous paragraph shows an occasional random realization of a large gap by a set of otherwise similar bidders is not a good argument for soft floors. But the possibility remains that the gap is systematic and can be explained by the presence of a bidder whose valuation is known to be high. For example, one of the bidders bidding on a particular impression may be a re-targeting advertiser whose website the customer has just visited. It may seem that a soft floor can put pricing pressure on such an “asymmetrically high” bidder while preserving revenues when he happens not to show up at the auction. I show this argument is also incomplete: the industry analysts are correct in noting that the soft floor adds pricing pressure on the high bidder whenever he is present, but the co-incident bid-shading by low-valuation bidders always more than erodes the benefits of the added pricing pressure. The reason is that even in a soft-floor auction (as opposed to a simple second-price auction), the pricing pressure on the asymmetrically high bidder ultimately stems from lower-valuation bidders, more of whom shade their bids down when the soft floor increases. As a result, soft floors actually reduce expected auctioneer revenue precisely in the asymmetric situation that seems to motivate their advocates.
Even if soft floors reduce revenues when they actually “kick in”, they might still increase revenue when they have an indirect effect, that is, when they are so high that all bidders bid below them. In other words, soft-floor auctions might just be a clunky implementation of a first-price sealed-bid auction in a bidding environment originally designed for second-price auctions. Unlike in the case of the categorical revenue reduction under the direct effect discussed in the previous paragraph, I find such an indirect benefit of soft floors is possible. Specifically, I provide an example of an asymmetric situation in which the first-price auction outperforms the second-price auction. However, the example involves systematic mis-pricing by the auctioneer: switching to first-price rules only increases the auctioneer’s revenue when the auctioneer sets his hard floor sub-optimally low to begin with. As long as the auctioneer sets his hard-floor reserve correctly, second-price auctions always outperform first-price auctions in my asymmetric bidder example.

Throughout the paper, I focused on a single auction attended by several independent private-value bidders. However, advertisers looking to purchase impressions on ad exchanges face a sequence of opportunities to show their ad, and they often view these opportunities as substitutes because they are budget constrained. Some advocates of soft floors correctly point out that bidding one’s full private valuation in a sequential auction for substitutes is not optimal (e.g., Nolet 2010, Strong 2012). Instead, one needs to bid the valuation of winning net of the opportunity cost of trying again, and the opportunity-cost calculation needs to take into account equilibrium considerations because the opportunity cost depends on the behavior and types of competing bidders (see, e.g., Milgrom and Weber 2000, Engelbrecht-Wiggans 1994). Based on the resultant observation that bidding one’s full private value of an impression is no longer a dominant strategy, some advocates of soft floors leap to the conclusion that first-price auctions would be more suitable for selling impressions (e.g., Nolet 2010, Strong 2012). Existing theory does not support this leap, but rather continues to find first-price and second-price auctions are revenue equivalent even in sequential settings under the symmetric model (e.g., Reiß and Schöndube 2010, Chattopadhyay and Chatterjee 2012). Given these revenue-equivalence results, one can conjecture that an analysis analogous
to section 3 of this paper would show soft floors have no effect on auction revenue even in a sequential-auction model under bidder symmetry. An interesting future direction of inquiry would examine soft floors in sequential auctions with asymmetric bidders.

The managerial recommendations of the results presented in this paper are clear. First, soft floors should be eliminated from ad exchanges because they cannot increase publisher revenue as long as bidders are rational and strategic (i.e., in the long run). Second, instead of introducing soft floors, managers should focus on setting their hard-floor levels correctly given the demand they face. Finally, managers should not worry about the “revenue gap” between the top two bids in the second-price auction identified by Sahni (2016). As Rothkopf et al. (1990) pointed out, the temptation to somehow grab that gap as revenue is quite strong, and bidder suspicions of auctioneers breaking the auction’s rules in order to grab the gap make the second-price auction difficult to implement. RTB exchange managers should resist such temptation, and rest easy knowing the winner needs to capture the entire gap as his surplus in order to continue bidding truthfully in dominant strategies, that is, in order to preserve the clear bidding incentives that make the second-price rules desirable. One of the most powerful implications of Myerson’s (1981) revenue equivalence is that this strategic simplicity for bidders does not come at a cost to the auctioneer as long as the bidders are symmetric – the second-price auction with a correctly chosen reserve is at least as profitable as any other auction format the manager may wish to implement.
Appendix: Proofs not covered in the main text

Proof of Proposition 2: Consider one bidder with valuation $v$, and suppose all $N-1$ of his competitors bid according to some increasing bidding function $\beta(v)$. The focal bidder solves

$$\max_b G\left(\beta^{-1}(b)\right) \left[ \Omega(b) \left( v - E \left[ \max_{s, Y_i} \{ s, \beta(Y_i) \} | s < b \land Y_i < \beta^{-1}(b) \} \right] + (1 - \Omega(b))(v - b) \right]$$

The $E\left[ \max_{s, Y_i} \{ s, \beta(Y_i) \} | s < b \land Y_i < \beta^{-1}(b) \}$ term, which captures price paid whenever the bid exceeds the soft floor, seems rather complex at first, but simplifies to

$$E\left[ \max_{s, Y_i} \{ s, \beta(Y_i) \} | s < b \land Y_i < \beta^{-1}(b) \right] = b - \int_{b \beta^{-1}(s)}^b \frac{G\left(\beta^{-1}(x)\right)}{G\left(\beta^{-1}(b)\right)} \frac{\Omega(s)}{\Omega(b)} dx$$

I now prove the above simplification: Write the expected payment as a double integral:

$$E\left[ \max_{s, Y_i} \{ s, \beta(Y_i) \} | s < b \land Y_i < \beta^{-1}(b) \right] = \frac{1}{G\left(\beta^{-1}(b)\right)} \left[ \int_0^{\beta^{-1}(s)} sdG(Y_i) + \int_{\beta^{-1}(s)}^{\beta^{-1}(b)} \beta(Y_i) dG(Y_i) \right] \frac{\Omega(s)}{\Omega(b)}$$

The material in the square bracket simplifies as follows:

$$\frac{1}{G\left(\beta^{-1}(b)\right)} \left[ \int_0^{\beta^{-1}(s)} sdG(Y_i) + \int_{\beta^{-1}(s)}^{\beta^{-1}(b)} \beta(Y_i) g(Y_i) dY_i \right] = G\left(\beta^{-1}(s)\right)s + G\left(\beta^{-1}(b)\right)b - G\left(\beta^{-1}(s)\right)s - \int_s^{\beta^{-1}(b)} G(Y_i) \beta'(Y_i) dY_i = G\left(\beta^{-1}(b)\right)b - \int_s^{\beta^{-1}(b)} G\left(\beta^{-1}(x)\right) dx$$

where the second line follows from the first line using integration by parts, and the third line follows from the second line by $G\left(\beta^{-1}(s)\right)s$ cancelling out and a change of variables $x = \beta(Y_i)$.

Plugging the simplified material in the square bracket into equation (19) yields equation (18).

In words, equation (18) shows the expected payment of a winner who randomly faces a soft floor below his bid $b$ involves a discount below $b$ (the inside integral), which in turn depends on the realized $s$: when $s=b$, there is no discount. When $s$ is low, the discount arises from paying the second-highest price, and so
it is related to the integrated probability of winning. Given equation (18), the bidder’s problem in equation 17 simplifies to

\[
\max_b G\left(\beta^{-1}(b)\right)(v-b) + \int_{h}^{b} G\left(\beta^{-1}(x)\right)dx \Omega(s)
\]

(20)

The first-order condition of the bidding problem is

\[
g\left(\beta^{-1}(b)\right)\frac{1}{\beta'(\beta^{-1}(b))}(v-b) - G\left(\beta^{-1}(b)\right) + G\left(\beta^{-1}(b)\right)\Omega(b) = 0
\]

(21)

where the first two terms are the same as in a textbook solution of a 1PSB problem, and the third term arises from the hidden soft floor. In a symmetric equilibrium, it must be that \(b = \beta(v)\), and so the equilibrium bidding function must satisfy the differential equation in equation 7. The differential equation does not have a closed-form solution, but the Peano Existence Theorem implies a solution exists whenever the RHS of equation 7 is continuous in \((v, \beta)\), for which a sufficient condition is that \(g(y) = (N-1)f(y)F^{N-2}(y)\) and \(\Omega(v)\) are continuous. QED Proposition 2.

Verification of revenue equivalence when soft floor is common knowledge

To verify the auctioneer’s revenue does not depend on \(s\), note the auctioneer makes

\[
\Pi(h, s) = \Pr(h \leq X_1 < v^*) E_{X_1}[\beta_1(X_1) | h \leq X_1 < v^*] + \Pr(X_1 > v^*) E_{X_1} \left[ s \Pr(\beta_1(X_2) < s) + X_2 \Pr(\beta_1(X_2) > s) \right] X_1 > v^*
\]

(22)

where \(X_2\) is the second-highest \(N\) valuations, and its distribution conditional on \(X_1 = x\) is just \(\frac{G(x)}{G(x)}\) : the highest of \(N-1\) draws from \(F\) conditional on the draws being below \(x\). I can plug in the \(\beta_l\) bidding strategy from equation 1 to obtain
\[
\Pi(h,s) = \int_h^v \left[ \frac{h G(h)}{G(x)} + \frac{1}{G(x)} \int_h^x zg(z)dz \right] dF^N(x) + \int_s^v \left[ \frac{s G(v^*)}{G(x)} + \frac{1}{G(x)} \int_h^x zg(z)dz \right] dF^N(x) = \\
= \int_h^v \left[ \frac{h G(h)}{G(x)} + \frac{1}{G(x)} \int_h^x zg(z)dz \right] dF^N(x) = E[\beta_i(X_1)] = \Pi(h)
\] (23)

where the second line substitutes for \( G(v^*) \) at which \( \beta(v^*) = s \) using equation 1, which implies

\[
G(v^*) = h \frac{G(h)}{s} + \frac{1}{s} \int_h^v zg(z)dz.
\]

This substitution makes the expected revenue from a given level of \( X_1 \) the same for \( X_1 < s \) (delineated by the curly brackets) and \( X_1 > s \) (delineated by square brackets), and so \( s \) has no impact on \( \Pi \), because neither it nor \( v^* \) are present on the second line.

**Proof of Proposition 3:** Equation 12 derives the formula for the difference in profits. It is enough to show it is positive and increasing in \( v^* \). Omitting the asterisk for clarity, I can plug in the distributions of \( X_1 \) and \( Y_1 \) in terms of \( F \):

\[
\frac{\Pi^d_{PSSB}(h) - \Pi^d(h,s)}{\alpha} = F^N(v) \left[ E\left( \max\{X_1,h\} \mid X_1 < v\right) - E\left( \max\{Y_1,h\} \mid Y_1 < v\right) \right] = \\
= F^N(v) \left[ \frac{h}{F^N(v)} \int_h^v xdF^N(x) - h \frac{F^{N-1}(h)}{F^{N-1}(v)} \right] \\
= \int_h^v xdF^N(x) - F(v) \int_h^v xdF^N(x) - hF^{N-1}(h)(F(v) - F(h))
\]

The last expression is clearly zero when \( v = h \). To show that \( \frac{\Pi^d_{PSSB}(h) - \Pi^d(h,s)}{\alpha} \) is positive and increasing in \( v \) for \( v > h \), it is enough to show its derivative in \( v \) is positive for \( v > h \):
where the last line follows from the previous expression by adding and subtracting $f(v)F^{N-1}(h)$. QED.

**Proof of Lemma 2**: The construction of the equilibrium starts with the same two steps as the proof of Proposition 4 in Kaplan and Zamir (2012), which it generalizes to $\alpha < 1, N > 1$, and general $F$: First, the high bidder’s indifference fixes the inverse bidding function of the regular bidders. Second, the fact that each regular bidder is playing the best response to $H$ determines the shape $H$ must take. The third step in the equilibrium construction is new to this paper, and arises from the pooling by low-valuation regular bidders: $v_h$ needs to be set so that $H(\max(h, b)) = 0$.

**Preliminaries**: The $M = b + \frac{F(b)}{Nf(b)}$ is just equation 6 in Martinez-Padina (2006), which is not affected by $\alpha < 1$, because it is derived from the high bidder’s incentives, and thus conditional on the high bidder being present. I construct the equilibrium with the binding hard floor, that is, the case of $h > b$, in detail — the equilibrium with non-binding reserve is constructed analogously.

**Step 1**: High bidder’s indifference determines the regular bidders’ inverse bidding function

Let $v(b)$ be the inverse of the $\beta$ bidding function. The tie-breaking rule simplifies the high bidder’s chance of winning with a bid of $h$ to $F^N(v_h)$. The high bidder’s indifference on $[h, b]$ then characterizes the inverse bidding strategy $v(b) : [h, b] \to [v_h, M]$ as follows:
\[ F^N(v_h)(M-h) = F^N(v(b))(M-b) \]

\[ \Rightarrow F^N(v(b)) = \frac{F^N(v_h)(M-h)}{M-b} \Rightarrow v(b) = \text{inv} \left( F(v_h) \right)^N \frac{M-h}{M-b} \]

To determine the upper bound \( \bar{b} \) of the support of \( H \), note that bidding \( \bar{b} \) wins for sure, and so

\[ F^N(v_h)(M-h) = F^N(v(\bar{b}))(M-\bar{b}) \Rightarrow \bar{b} = M - F^N(v_h)(M-h) \]

**Step 2: Increasing part of the regular bidders’ bidding function determines the shape of \( H \)**

The regular bidders who do not pool at \( h \) play their best response by solving

\[
\max_{b > \bar{b}} F^{N-1}(v(b)) \left[ 1 - \alpha + \alpha H(b) \right] (v-b)
\]

The first-order condition is

\[
\left( (N-1)f(v(b))v'(b) + F(v(b)) \frac{\alpha H'(b)}{1 - \alpha + \alpha H(b)} \right) (v-b) = F(v(b))
\]

Given the symmetry of the equilibrium among regular bidders, I can substitute \( v(\bar{b}) \) for \( v \), and separate variables to obtain the following differential equation:

\[
\frac{d \log(1 - \alpha + \alpha H(b))}{db} = \frac{1}{v(b)-b} - \frac{(N-1)f(v(b))v'(b)}{F(v(b))} \Rightarrow \frac{1}{v(b)-b} = \frac{d \log(F^{N-1}(v(b)))}{db}
\]

Integration yields the solution up to a constant:

\[
\log(1 - \alpha + \alpha H(b)) = \log(C) + \int_{\bar{b}}^{b} \frac{1}{v(b)-b} dx - \log(F^{N-1}(v(b)))
\]
Finally, I can exponentiate both sides, and substitute $v(b)$ from step 1:

$$\exp\left(\int_{b}^{h} \left( invF\left(F\left(v_{h}\right)\sqrt[\alpha]{\frac{M-h}{M-x}} - x\right)^{-1}\right) dx\right)$$

$$1 - \alpha + \alpha H(b) = C \frac{F^{-1}\left(v_{h}\right)\left(\frac{M-h}{M-b}\right)^{\frac{N-1}{N}}}{F^{-1}\left(v_{h}\right)\left(\frac{M-h}{M-b}\right)^{\frac{N-1}{N}}}$$

where $invF$ is the inverse cdf of the distribution of the regular bidders’ valuations.

To finish the equilibrium construction, I need to determine two constants: $C$ and $v_{h}$. These constants are pinned down by the two extremes of $H$. At the upper bound $\tilde{b}$ of the support of $H$, the LHS equals 1, giving rise to the following expression for $C$: $-\log(C) = \int_{h}^{\tilde{b}} \left( invF\left(F\left(v_{h}\right)\sqrt[\alpha]{\frac{M-h}{M-x}} - x\right)^{-1}\right) dx$

which I can plug back into the main equation to get

$$\exp\left(- \int_{h}^{\tilde{b}} \left( invF\left(F\left(v_{h}\right)\sqrt[\alpha]{\frac{M-h}{M-x}} - x\right)^{-1}\right) dx\right)$$

$$1 - \alpha + \alpha H(h) = \frac{F^{-1}\left(v_{h}\right)\left(\frac{M-h}{M-\tilde{b}}\right)^{\frac{N-1}{N}}}{F^{-1}\left(v_{h}\right)\left(\frac{M-h}{M-\tilde{b}}\right)^{\frac{N-1}{N}}}$$

**Step 3: Upper bound of pooling balances the extent of the high bidder’s mixing strategy**

Everything so far assumed the high bidder uses a continuous mixing distribution on $[h, \tilde{b}]$. Therefore, it must be the case that $H(h) = 0$, resulting in the second equation:

$$(1 - \alpha) F^{-1}\left(v_{h}\right) = \exp\left(- \int_{h}^{\tilde{b}} \left( invF\left(F\left(v_{h}\right)\sqrt[\alpha]{\frac{M-h}{M-x}} - x\right)^{-1}\right) dx\right)$$
The construction without a binding hard floor is analogous except that $h$ is replaced with $b$ as the lower bound of the bidding distribution:

**Step 1: High bidder’s indifference determines the regular bidders’ inverse bidding function**

The inverse bidding function becomes $v(b) = \text{inv}F \left( F(v_h)^N \sqrt{\frac{M - b}{M - b}} \right)$

The upper bound $\bar{b}$ of the support of $H$ becomes $\bar{b} = M - F^N(v_h)(M - b)$

**Steps 2 and 3: The regular bidders’ bidding function determines shape of $H$**

The regular bidders play their best response by solving

$$\max_{b > \bar{b}} F^{-1}(v(b)) \left[ 1 - \alpha + \alpha H(b) \right] (v - b)$$

which is essentially the same as in the case of binding reserve with $H(b) = 0$

QED Lemma 3

**Details of optimal reserve with asymmetrically high bidder**

$$\Pi_{2PSB}^\alpha(h) = \alpha \left[ hF^N(h) + \int h x dF^N(x) \right]$$

$$+ (1 - \alpha) \left[ hN F^{N-1}(h)(1 - F(h)) + \int x N(N - 1) F^{N-2}(x)(1 - F(x)) dx \right]$$

$$\Rightarrow \frac{d\Pi_{2PSB}^\alpha(h)}{dh} = F^{N-1}(h) \left[ \alpha F(h) + (1 - \alpha)N(1 - F(h) - hf(h)) \right]$$

$$\Rightarrow h - \frac{1 - F(h)}{f(h)} = \frac{\alpha F(h)}{(1 - \alpha) N f(h)}$$
References


Friedman, Jay (2015) I study billions of online ad impressions to know the truth. on Quora.com.


