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Abstract

In our paper about optimal reverse pricing mechanisms (Spann, Zeithammer and Häubl 2010, hereafter labeled ORPM), some of the mathematical derivations implicitly assume that the name-your-own-price seller interprets the outside-market posted price $p$ differently than the buyers. This note shows that all of the qualitative results in ORPM continue to hold under the more natural assumption of common knowledge that $p$ is the upper bound of wholesale cost. Interestingly, the proofs and algebraic expressions are often simpler than those in ORPM.

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Introduction: the two assumptions about the meaning of $p$

Before going through the affected sections of the original paper, we expand on the nature of the original assumption implicit in the math of ORPM. In describing the model, ORPM assumes that the outside market price $p$ is common knowledge in the beginning of the game, and $p$ is an informative upper bound of the wholesale cost $w$. ORPM correctly shows how the buyers use $p$ in optimizing their bids. However, in deriving the seller profit, the algebra in ORPM is set up as if the seller believed his wholesale cost $w$ to be distributed Uniform[0,1] at the time of setting his strategy. Given the common knowledge of $p$, a more internally consistent assumption is that the seller has the same information as the buyers, and also believes his wholesale cost $w$ is distributed Uniform[0, $p$] when he sets his fee and minimum markup.

This note makes the latter assumption, and re-proves all results in ORPM that are affected by this change. On several occasions, the change requires a modification of a proposition or claim. The modifications needed are only in the exact algebraic and numerical expressions, not in the qualitative insights proposed by ORPM.

We start by noting that the analysis of buyer behavior is unaffected. Therefore, section 2.2 of the ORPM paper is unaffected. Section 2.3 changes as follows:

2.3 Optimal selling without commitment to a minimum markup ($m=0$)

The change to the profit function (equation 2 in the original paper) is highlighted in **bold red** (only **bold** in printed version) in the equations that follow.

$$\Pi(f) = \Pr(v < v)f + \mathbb{E}_{w,v \geq \frac{1}{2}}[1(b > w)(b - w)] =$$

$$= \left[1 - H\left(\frac{v}{2}\right)\right]f + \int_0^p \int_0^\frac{v}{2} \left(\frac{1}{2}w\right)\left(\frac{1}{p}\right)dwH\left(\frac{v}{2}\right) + \left[1 - H\left(\frac{v}{2}\right)\right]f + \int_0^p \pi(v)dH\left(\frac{v}{2}\right) + \left[1 - H\left(\frac{v}{2}\right)\right]\pi\left(\frac{v}{2}\right)$$

(2')

$$= \left[1 - H\left(2\sqrt{pf}\right)\right]f + \int_0^p \pi(v)dH\left(\frac{v}{2}\right) + \left[1 - H\left(\frac{v}{2}\right)\right]\pi\left(\frac{v}{2}\right)$$

where $\pi(v) = \int_0^v \frac{v}{2} \left(\frac{1}{p}\right)dw = \frac{v^2}{8p}$ is the profit from a bidder of valuation $v$.

A modified Proposition 2 implied by equation (2') is:
**Modified Proposition 2**: When \( \frac{3}{4}x - \frac{1 - H(x)}{h(x)} \) is an increasing function of \( x \), the optimal consumer entry threshold of a non-commitment seller satisfies either \( \frac{3}{4}v = \frac{1 - H(v)}{h(v)} \) or \( v = p \), whichever is lower. When \( H(x) = x \) on \([0,p]\), the implied optimal fee given \( m=0 \) is

\[
\sqrt{f^*(p)} = \min \left( \frac{2}{\sqrt{7p}}, \frac{\sqrt{p}}{2} \right) = \begin{cases} \frac{\sqrt{p}}{2} & \text{for } p < \frac{4}{7} \approx 0.57 \\ \frac{2}{\sqrt{7p}} & \text{for } p \geq \frac{4}{7} \approx 0.57 \end{cases}
\]

**Proof**: Please see the Online Appendix for all proofs in this note.

Interestingly, and unlike in ORPM, all \( p \geq \frac{4}{7} \) screen entering bidders at the same minimum valuation of \( v = \frac{4}{7} \).

### 2.4 Optimal selling with commitment to a minimum markup \((m \geq 0)\)

The main result (dominance of fees over minimum markups in Proposition 3) survives intact, exactly as stated:

**Proposition 3**: Suppose \( H(x) = x \) on \([0,p]\). Even if the seller can credibly commit to a positive minimum markup, the optimal selling strategy uses zero minimum markup and the positive fee \( f^*(p) \) derived in Proposition 2.

Interestingly, the proof of Proposition 3 is substantially simplified by the more consistent assumption relative to the ORPM proof.

The illustrative example has to be re-calculated under the new assumption:

**Welfare and Profit Calculation of Illustrative Example in Section 2.4**

Let \( v=1 \) correspond to $1500, and let \( p = \frac{2}{3} \) (which thus corresponds to $1000), and assume consumer valuations are distributed uniformly on \([0,1500]\). The optimal bidding fee to charge is \( f^*(\frac{2}{3}) = \frac{6}{49} \approx $183 \), which screens at level \( v = \frac{4}{7} \approx $857 \) (the same level would hold for
all other \( p > 4/7 \) Thus, most low consumers do not enter. Those low consumers who do enter bid \( v/2 \), resulting in bids between $427 and $500. In addition, all high consumers enter, and all bid $500. The expected social welfare \( W \) realized through the reverse-pricing seller is the difference \( v - w \) when there is a trade, that is, when valuation exceeds \( v \) and the bid exceeds \( w \):

\[
W(f^*(3/7)) = \left(\frac{1}{p}\right) \int_0^{\frac{1}{p}} \int_{\frac{1}{p}}^{3/7} (v - w) \, dv \, dw + \left(\frac{1}{p}\right) \int_0^{\frac{1}{p}} \int_{\frac{1}{p}}^{3/7} (v - w) \, dv \, dw = \left(\frac{3}{2}\right) \frac{271}{3087} \approx $197
\]

with an overall probability of trading of about 14 percent. The seller’s profit is

\[
\Pi(f^*(3/7)) = \left(1 - \frac{4}{7}\right) \frac{6}{49} + \left(\frac{3}{2}\right) \int_0^{\frac{1}{p}} \int_{\frac{1}{p}}^{3/7} v^2 \, dv \, dw + \left(\frac{3}{2}\right) \int_0^{\frac{1}{p}} \int_{\frac{1}{p}}^{3/7} (v - w) \, dv \, dw = \frac{461}{5292} \approx $130 .
\]

Now consider the optimal minimum markup to charge contingent on bidding fees being zero.

To derive the optimal markup, let \( f = 0 \) in \( \Pi(m, f) \) in the Proof of Proposition 3:

\[
\Pi(m \mid f = 0) = \int_m^p \pi(m \mid v) \, dv + (1 - p) \pi(m \mid p) \Rightarrow FOC_m : m^* = \frac{3 - \sqrt{9 - 10p + 5p^2}}{5} \approx $260 .
\]

This is unchanged from before because the \( (1/p) \) term factors out of the profit. However, the welfare and profits are affected:

\[
W(m^*(3/7)) = \left(\frac{1}{p}\right) \left[ \int_{m^*}^{\frac{p - m^*}{2}} (v - w) \, dv \, dw + \int_{\frac{p - m^*}{2}}^{\frac{p}{2}} (v - w) \, dv \, dw \right] = \left(\frac{3}{2}\right) \frac{189 + 59\sqrt{41}}{6750} \approx $187 .
\]

Plugging the \( m^* \) into the profit equation yields

\[
\Pi(m^*(3/7) \mid f = 0) = \left(\frac{3}{2}\right) \frac{61 + 41\sqrt{41}}{8000} \approx $90 .
\]

Now consider the optimal bidding fee contingent on setting the minimum markup to

\[
m^* = \frac{9 - \sqrt{41}}{15} \approx $260 .
\]

For \( p = 2/3 \), the proof of Proposition 3 suggests a fee of

\[
f^*(m \mid p = \frac{2}{3}) = \frac{6(1 - 2m)^2}{49} \Rightarrow f^*(9 - \frac{\sqrt{41}}{15} \mid p = \frac{2}{3}) \approx $78 .
\]

Since the combination \( (m^*, f^*) \) is now on the locus of optimal fees given markups, we can simply plug the \( m^* \) into the profit equation the proof of Proposition 3 to find that the profit is \( \approx $107 . \) As expected, this profit is
more than $\Pi(m^*(\frac{2\epsilon}{3}) | f = 0)$, but less than $\Pi(m = 0, f^*(\frac{2\epsilon}{3}))$. Interestingly, the resulting screening level is similar to that with the optimal fee and no markup, namely, $\approx$ $820. The welfare is thus decidedly lower because of bid-shading:

$$W(m^*, f^*(m^*)) = \left(\frac{1}{p}\right)^{\frac{2}{3}} \int_0^{\frac{v-m^*}{2}} \int_0^{\frac{p-m^*}{2}} (v-w)dw \, dv + \left(\frac{1}{p}\right)^{\frac{2}{3}} \int_0^{\frac{v-m^*}{2}} \int_0^{\frac{p-m^*}{2}} (v-w)dw \, dv = \frac{3}{2} \cdot \frac{18 + 277 \sqrt{41}}{27000} \approx$ $150.$

Modified Table 1 displays the results for (1) the optimal bidding fee given the minimum markup set to zero, (2) the case of an optimal minimum markup given no bidding fee, and (3) for the case of the optimal fee to use given the minimum markup suggested in (2).

**Modified Table 1: Illustrative example of a plane ticket from New York to London that costs $1000 on Expedia and that consumers value uniformly between $0 an $1500.**

<table>
<thead>
<tr>
<th></th>
<th>optimal bidding fee, no minimum markup</th>
<th>optimal minimum markup, no bidding fee</th>
<th>optimal minimum markup and its associated optimal bidding fee</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Bidding fee (f)</strong></td>
<td>$183</td>
<td>$0</td>
<td>$78</td>
</tr>
<tr>
<td><strong>Minimum markup (m)</strong></td>
<td>$0</td>
<td>$260</td>
<td>$260</td>
</tr>
<tr>
<td><strong>Screening level (v)</strong></td>
<td>$857</td>
<td>$260</td>
<td>$820</td>
</tr>
<tr>
<td><strong>Social welfare (gains from trade)</strong></td>
<td>$197</td>
<td>$187</td>
<td>$150</td>
</tr>
<tr>
<td><strong>Seller profit</strong></td>
<td>$130</td>
<td>$90</td>
<td>$107</td>
</tr>
</tbody>
</table>
2.5 Should the seller who charges a bidding fee facilitate or hinder consumer learning about the current bid-acceptance threshold?

Claim 1: The claim that when potential consumers learn the marginal cost before their entry decision, the optimal bidding fee is a solution to \( 2f_i = E[\min(p, v) | v > f_i] \) holds exactly because the \((1/p)\) correction factors out of the profits (see proof for details).

While the FOC is unaffected, the profit of the seller facing informed consumers is \((1/p)\)-times higher than in the original paper. The profits of the seller facing uncertain consumers is different as well (as discussed previously), so the numerical details of Proposition 4 change:

**Modified Proposition 4:** A unique outside price \( \frac{4}{\sqrt{p}} < \hat{p} < 1 \) exists such that facilitating consumer learning about the seller’s current bid-acceptance threshold is profitable for the seller when \( p > \hat{p} \), and vice versa.

The modified Figure 4 illustrates Proposition 4.
Figure 4: Seller profit under two consumer-information scenarios

Note to Figure 4: This figure illustrates Proposition 4. The two curves represent seller profits as a function of the price on the outside posted-price market. The solid curve involves consumers informed about seller cost before making their entry decision. The dashed curve involves consumers uncertain about seller cost at the time of their entry decision.

Online appendix

Proof of Modified Proposition 2: The FOC of profit in equation (2') are:

\( (3') \quad FOC_{f} : \frac{3\sqrt{pf}}{2} h \left( 2\sqrt{pf} \right) = 1 - H \left( 2\sqrt{pf} \right) \)

Rewriting in terms of \( v = 2\sqrt{pf} \) yields \( FOC_{v} : \frac{3v}{4} h \left( v \right) = 1 - H \left( v \right) \). Setting \( H(x) = x \) yields the rest. QED

Modified proof of Proposition 3: In stage 1, the reverse-pricing seller maximizes his expected profit \( \Pi(m, f) \) by selecting the optimal combination of bidding fee \( f \) and minimum markup \( m \). As long as \( m + 2\sqrt{pf} \leq p \) (i.e., as long as consumer \( v = p \) enters), the seller obtains the following total expected profit:

\[
\Pi(m, f) = Pr(v > \frac{v}{2}) f + E_{w,v>\frac{v}{2}} \left[ 1(b(v|m) > w+m) b(v|m)(b-w) \right] = \\
= \left[ 1 - H \left( m + 2\sqrt{pf} \right) \right] f + \int_{m + 2\sqrt{pf}}^{p} \pi(m|v) dH(v) + \left[ 1 - H(p) \right] \pi(m|p)
\]

(4')

where \( \pi(m|v) = \int_{0}^{\frac{v-m}{2}} \left( \frac{1}{p} \right) dw = \left( \frac{1}{p} \right) \left( \frac{v-m}{2} \right) \left( \frac{v+3m}{4} \right) \).

To maximize \( \Pi(m, f) \), the seller first finds the optimal bidding fee for every possible level of minimum markup \( m \). This contingent optimal fee satisfies the following first-order condition that equates the marginal cost of raising \( f \) slightly with its marginal benefit:

\[
f^*(p) : \left( \frac{2m + 3\sqrt{pf}}{2} \right) h \left( m + 2\sqrt{pf} \right) = 1 - H \left( m + 2\sqrt{pf} \right)
\]

(5')

As in the case of \( m = 0 \), the entry threshold must be below \( p \), so whenever \( m + 2\sqrt{pf^*}(p) > p \), the optimal fee is the lower value that satisfies \( m + 2\sqrt{pf} = p \). When the \( FOC_f \) suggest a
negative $f^*(p)$, the minimum markup $m$ is already too large, and the optimal fee is zero.

The second-order condition is:

$$SOC_f : 7h(m + 2\sqrt{pf}) + 2(2m + 3\sqrt{pf})h'(m + 2\sqrt{pf}) > 0$$

The interesting part of the FOC and SOC is that $f$ always occurs inside the $\sqrt{pf}$ term, so one can characterize the solution by:

$$FOC_f : \left(\frac{2m + 3z}{2}\right)h(m + 2z) = 1 - H(m + 2z) \text{ and } f^* = \frac{z^2}{p}$$

$$SOC_f : h'(m + 2z) < \frac{7h(m + 2z)}{2(2m + 3z)}$$

It is clear that the SOC holds for all non-increasing pdfs.

Let $H(x) = x$ on $[0,p]$. In other words, assume $p$ low consumers are distributed uniformly on $[0,p]$ and an additional mass $(1-p)$ of arbitrarily distributed high consumers are above $p$. For this uniform distribution of low consumers, the optimal fee contingent on $m$ implied by $FOC_f$ becomes

$$\sqrt{f^*(p | m)} = \frac{2(1-2m)}{7\sqrt{p}} \Rightarrow 2\sqrt{pf^*(p | m)} = \frac{4(1-2m)}{7} \text{ and } v = \frac{4-m}{7} \quad (6')$$

Note that the screening level does not depend on $p$.

The full solution is thus:

$$\sqrt{f^*(p | m)} = \begin{cases} 
0 & \text{when } m \geq \frac{1}{2} \\
\frac{2(1-2m)}{7\sqrt{p}} & \text{when } m < \frac{1}{2} \text{ and } p \leq \frac{4-m}{7} \\
\frac{p-m}{2\sqrt{p}} & \text{when } m < \frac{1}{2} \text{ and } p > \frac{4-m}{7}
\end{cases}$$

Plugging the optimal fee into the profit function yields:
\[ \Pi(m, f^*(m)) = \frac{32 + 49(3 - 2p)p^2 - 3m(64 - 49(2 - p)p) - 57m^2 - 11m^3}{1176p} \]  

(7')

\[ \Rightarrow \frac{d\Pi(m, f^*(m))}{dm} \propto -\left[64 + m(38 + 11m) - 49(2 - p)p\right] < 0 \]

where the last inequality follows because \((2-p)p\) is maximized at \(p=1\), at which point the inequality holds. Since profit \(\Pi(m, f^*(m))\) is thus decreasing in \(m\) along the locus of possible solutions, the globally optimal solution is captured by the \(FOC_f\) given \(m = 0\). \(QED\) P3

Since

\[ \frac{d\Pi(m, f^*(m))}{dm} \bigg|_{m=0} \propto -\left[64 - 49(2 - p)p\right] < 0, \]

Corollary to Proposition 3 also continues to hold. \(QED\)

**Proof of Claim 1:** Consumers enter and bid \(\min(w, p)\) when \(\min(v, p) - f - w > 0\). The seller collects all of his revenue through the fee \(f\), resulting in the following profit function:

\[ \Pi(f) = f \Pr(entry | f) = f \left[ \int_f^p \Pr(w < v - f) dH(v) + \int_p^1 \Pr(w < p - f) dH(v) \right] = \]

\[ = f \left[ \int_f^p \frac{(v_f - f)}{p} dH(v) + \frac{(p - f)}{p} \left(1 - H(p)\right) \right] \]

where the first term in \(\Pr(entry | f)\) averages the probability that \(w < v - f\) over the valuations of the low consumers. The second term adds the probability that \(w < p - f\) times the probability of a high consumer occurring. The first-order conditions are

\[ 0 = \Pr(entry | f) + f \frac{d\Pr(entry | f)}{df} = \int_f^p (v_f - f) dH(v) + (p - f)(1 - H(p)) - f[1-H(f)] \]

\[ \Rightarrow 2f = \frac{p[1-H(p)] + \int_f^p v dH(v)}{1-H(f)} = E[\min(p, v) | v > f] \]

\(QED\)
Proof of Modified Proposition 4.

We start by analyzing the seller profits under the two consumer-information scenarios. In the uniform case of $H(x)=x$ on $[0,p]$, the optimal fee is a solution to the equation

$$0 = 3f_i^2 - 4f_i + 2p - p^2,$$

which has a unique root below $p$ of $f_i^*(p) = \frac{2-\sqrt{4-6p+3p^2}}{3}$. The profit of the seller facing informed consumers is $\Pi_i(f_i^*(p)) = f_i^*(p - f_i^*)\left(\frac{2-p-f_i^*}{2p}\right)$.

The algebraic expression is complicated by the square root in the optimal fee $f_i^*$, but it is easy to evaluate $\Pi_i(f_i^*(1)) = \frac{2}{27}$. $\Pi_i(f_i^*(p))$ has a local maximum at $\Pi_i\left(f_i^*\left(\frac{1}{\sqrt{2}}\right)\right) = \frac{3}{2} - \sqrt{2}$. The profit can be expressed as $\Pi_i(f_i^*(p)) = \frac{1}{54p}(2-D)(4+D-3p)(D-2+3p)$ for $D = \sqrt{4-3p(2-p)} \leq 2$, where the inequality follows from the fact that $D$ is decreasing in $p$ (and hence $f_i^*(p) = \frac{2-D}{3}$ is positive and increasing in $p$ as would be expected).

Now turn to the seller facing uncertain consumers. When $p$ is low enough ($p < 4/7$), the seller facing uncertain consumers only serves the high consumers by charging $\sqrt{f} = \sqrt{p}/2$ (with $\nu = p$). His profit is $\Pi\left(f^*(p) \mid p < 4/7\right) = (1-p)\frac{p}{4} + (1-p)\int_0^{\sqrt{p}/2} \left(\frac{p}{2} - w\right)\left(\frac{1}{p}\right) dw = \frac{3p(1-p)}{8}$.

For $p \geq 4/7$, the optimal profit simplifies to $\Pi\left(f^*(p) \mid p \geq 4/7\right) = \frac{4}{147p} + \frac{p}{8} - \frac{p^2}{12}$.

As in the original paper, it can be shown that a unique point $4/7 < z < 1$ exists such that $\Pi_i(f_i^*(z)) = \Pi_i(f^*(z))$, and the seller facing informed consumers makes a higher profit for $p > z$ and vice versa. Finding $z$ algebraically would involve solving a high-order polynomial equation, but it can be easily numerically estimated as $z \approx 0.694$. QED