Taxes and Risk in Financial Markets: A Separation Result

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Abstract

Two important factors in determining how investors select portfolios and how asset prices and returns are determined are taxes and risk. Absent risk, the effect of taxes has been captured by models that assume the existence of tax clienteles and characterize implicit taxes contained in the equilibrium returns of securities as a consequence of arbitrage by marginal clienteles. Absent taxes, the effect of risk has been captured by models in which arbitrage by risk averse investors leads to risk premia linearly related to economy-wide risk factors. When both risk and taxes are present, they interact in complex ways that eliminate the clean results obtainable when each characteristic is considered in isolation. The purpose of this paper is to demonstrate that the effects of taxes and risk on asset prices and returns can be linearly separated under the realistic assumption that there exists a parsimonious set of index futures contracts that spans the non-diversifiable risk factors in asset prices. The key implication of this separation is that under relatively weak assumptions, researchers can validly consider either taxes or risk in isolation, as if the other does not exist.

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I. Introduction

Considerable attention in finance has been given to assessing the effect of taxes and risk on investors' portfolio choice and asset pricing. These two characteristics, taxes and risk, have typically been considered separately to facilitate model tractability. While much progress has been achieved by considering taxes and risk independently, less progress has been made in models that consider them simultaneously as the two can interact in complex ways. This paper will develop a model with both taxes and risk, and demonstrate that under realistic assumptions, the effects of taxes and risk on portfolio choice and asset pricing can be linearly separated, and that the equilibrium effects of each characteristic (taxes or risk) are the same as they would be if the other characteristic is ignored (i.e. tax effects are the same with risk-aversion as they would be with risk-neutrality, and risk premia on assets are the same with capital income taxes as they would be under lump sum taxation). This implies that it is legitimate to model taxes and risk separately as has been commonly done in past research.

Absent consideration of risk, much progress can be (and has been) made in addressing the issue of how taxes affect portfolio choice and capital structure. Consider the most basic portfolio selection problem involving two securities, debt and equity. The income from debt ownership is fully taxed at the investor's marginal tax rate, while the income from equity ownership (largely in the form of capital gains) is effectively taxed at a lower rate (due to a lower statutory rate, benefit of deferral, selective realization of gains and losses, and step-up of basis at death). Hence, equity is a more attractive investment than debt from a tax perspective. However, some investors, particularly long-term investors in high tax brackets, have a relatively strong tax preference for equity, while tax-exempt investors, such as pension funds, face no tax

differential from interest and capital gains income. Ignoring risk considerations, these variations in tax attributes lead to the formation of tax clienteles, with investors who have a relatively large tax preference for equity investing in equity while investors with a relatively low (or zero) tax preference for equity invest in bonds. Equilibrium with this clientele formation requires that stocks provide a lower pre-tax return than bonds, known as an implicit tax, because stock investors incur this cost (reduction in pre-tax return) to avoid the explicit tax penalty associated with bonds. The exact implicit tax will equal the difference in tax rates on debt and equity for a marginal clientele that is indifferent between the two securities and that arbitrages them.

This type of tax clientele / implicit tax model has considerable utility. The model can be generalized to consider a variety of different assets and clienteles, such as municipal bonds, preferred stock, and common stock with different dividend yields, as well as more complex variations in taxpayer characteristics, such as corporations with low tax rates on dividend income.[1] The key in such models is the identification of the marginal clientele between any two securities that determines the magnitude of the implicit tax.[2] The model can also be used to explore other issues in finance, such as optimal capital structure. Miller (1977) develops a general equilibrium model based on the basic tax clientele model in which corporations adjust the supplies of stocks and bonds to change the marginal clientele and so force the implicit tax to equal the corporate tax rate (and equalize the cost of capital for issuers). With the cost of capital equalized, each firm is indifferent regarding its capital structure, and thus the Modigliani-Miller (1958) leverage irrelevance result holds even in the presence of taxes.

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[1] Williams (2001) considers clientele formation and implicit taxes with all these variations.

[2] The notion of a marginal clientele is commonly invoked in tax research in financial markets. For example, in studies of municipal bonds, the primary focus has been on identifying the marginal clientele and the alternative asset (taxable debt or equity) that the marginal clientele arbitrages against municipal bonds. Williams (2001), Mankiw and Poterba (1996), Fortune (1988), and Trczinka (1982) posit four different views on who is marginal in the municipal debt market and with respect to what alternative asset.

When assets provide risky returns, it is not generally true that a tax-optimal portfolio involves optimal risk management (in terms of minimizing idiosyncratic risk and balancing systematic risk against available risk premia). Thus, a conflict arises between tax and risk incentives that generally results in a compromise in which neither dimension is optimized completely. The joint tax / risk optimization program is far more complex than the simple tax optimization program that underpins the standard tax clientele model. The solution to the joint tax / risk problem typically involves a degree of diversification across assets[3] and implicit taxes that are combinations of multiple clienteles' tax and risk characteristics. There is no marginal clientele in such a model and no clean prediction of implicit taxes between assets.[4] Further, leverage irrelevance and any other result based on clientele formation fails to hold.

The primary purpose of this paper is to demonstrate that the problems of optimizing taxes and risk in portfolio choice can be separated under realistic assumptions. In particular, separation requires the existence of a parsimonious set of index futures contracts, where the number of contracts is at least as great as the number of non-diversifiable risk factors in asset prices. Since futures on nine stock indices are currently available on the Chicago Mercantile Exchange alone with average daily volume of approximately $50 billion (as well as several stock index futures on other U.S. exchanges), this does not appear to be a strong assumption.

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[3] For example, if bonds have low firm performance (default) risk and high interest rate risk, while stocks have low interest rate risk and high firm performance risk, then (absent tax considerations) most investors will prefer a combination of the two types of securities to minimize total risk. Taking taxes into account, investors will diversify to some extent, but overconcentrate their portfolios in tax-efficient assets.

[4] In fact, in such a model, it is impossible to separate risk premia from implicit taxes, which creates a severe problem for empirical research. The reason risk premia cannot be separated from implicit taxes is that different investors will not agree on the appropriate risk premium for any asset, since their portfolios are not fully optimized risk-wise and so at the margin their risk tolerances are not equal (whereas they would be absent tax effects). One way to think about this is that even if state prices and an equivalent martingale measure exist for differently taxed assets, they are not necessarily equal (Ross, 1987, demonstrates that even in the presence of taxes, a state price vector and equivalent martingale measure exist for each type of taxable income if a no arbitrage equilibrium exists, but these state price vectors need not bear any particular relationship across income types). Allen, Bernardo, and Welch (2000) and Gordon and Bradford (1980) develop models of this nature involving tax-risk tradeoffs in portfolio choice, mixed marginal clienteles, and commingled tax/risk premia on assets.

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The separation of tax and risk optimization means that investors can achieve perfect tax optimization through clientele formation and portfolio separation (with no diversification across asset types) while simultaneously achieving optimal risk exposure through futures trading (which does not alter tax circumstances, as will be discussed later). Expected rates of return on assets are linear functions of their covariances with the systematic risk factors (as in the standard APT model) and their tax attributes (as in standard clientele theory). The key implication of this separation is that under relatively weal assumptions, researchers can ignore risk in developing models of how taxes influence portfolio choice and clientele behavior and can ignore taxes in developing models of how risk affects asset prices and expected returns. A model with an assumption of riskless assets (or equivalently risk neutrality) remains valid even if assets are risky and investors risk-averse so long as the random variables in the model are regarded as certainty equivalents,[5] and a model with an assumption of no taxes remains valid in terms of its predictions for risk premia, which are independent of the tax system.

Section II reviews the prior literature and motivates the current study. Section III derives the tax / risk separation theorem. Section IV discusses some extensions and applications of the theorem. Section V concludes the paper. Appendix A includes proves of the proposition and corollaries in the paper, and Appendix B outlines the nomenclature used in the paper.

II. Prior Research

The effects of taxes and risk on portfolio choice and asset pricing have been examined in numerous studies considering each of the effects separately. A seminal paper that considers the

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[5] Unlike the case in the absence of futures (discussed in note 4), risk premia are well-defined and universally agreed upon, since everyone now optimizes risk exposure. Thus, state prices, an equivalent martingale measure, and the certainty equivalent of risky streams are well-defined constructs, with only state prices varying for differently taxed

effect of taxes is Miller (1977), which also analyzes capital structure choice. Miller shows that if

investors are risk-neutral, they will separate into two clienteles based on their tax attributes.

Investors with a relatively strong preference for capital gains income will only own stocks while

investors with a relatively weak preference for capital gains income will only own bonds. An

implicit tax arises between debt and equity returns to offset the explicit tax difference between

them for the marginal tax clientele that is indifferent between the two securities.[6]

As noted above, Miller (1977) assumes risk neutrality. Extending Miller's equilibrium to

a risk-averse setting has required stringent restrictions in prior papers. DeAngelo and Masulis

(1980) derive a portfolio separation, leverage irrelevance equilibrium under the assumption that

every firm can issue a complete set of Arrow-Debreu state-contingent debt and equity claims.

DeAngelo and Masulis' approach allows agents to optimize portfolio and capital structure choice

on a state-by-state basis. Needless to say, the assumption of complete Arrow-Debreu debt and

equity markets is very restrictive.

Auerbach and King (1983) derive a Miller-type equilibrium by assuming mean-variance

utility for investors and allowing investors unlimited short selling ability for individual stocks so

long as the short sales are collateralized by other stocks (e.g. if you own $1000 of one stock, you

can short $1000 value of another stock, but if instead you own a $1000 bond, you cannot short

any stock at all). These assumptions, particularly the unconventional short sale constraint, are

highly restrictive. They are designed to allow investors to hold two portfolios, the market

portfolio and a riskless stock portfolio. Bond clientele members (those with a relatively low tax

preference for capital gains) do not eschew stocks entirely, as in Miller, but instead hold a long

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assets (and varying in a proportional manner), while the equivalent martingale measure is completely independent of
the tax system.

position in the market portfolio and a short position in the riskless equity so that their net equity position is zero.

In addition to considering the effect of taxes on financial markets, numerous studies consider the effect of risk on asset pricing and portfolio choice. The vast majority of such models exclude taxes from the analysis (e.g. the CAPM in Sharpe, 1964, and the APT in Ross, 1976). In doing so, these papers implicitly assume that either taxes do not matter (e.g. due to the marginal clientele being tax-exempt) or that taxes induce easily separable effects. For example, a multifactor model of stock returns is valid in the presence of taxes if taxes merely shift the intercept of the returns function. However, that is not generally the case.

Papers that have considered how taxes affect risk-based asset pricing models include Brennan (1970), which modifies the CAPM under the assumption of homogeneous (across investors) but differential taxation of dividends and capital gains. In this model, the intercept in the CAPM is a linear function of dividend yield. Elton and Gruber (1978) extend this analysis to consider tax-wise heterogeneous investors and find that if investors believe that asset prices follow Brennan's tax-augmented CAPM, then they will choose a composite portfolio consisting of a fraction of the market portfolio and a fraction of a dividend-weighted portfolio. However, Long (1977) demonstrates that mean-variance efficiency is not preserved under taxation, casting doubt on the CAPM approach in the presence of taxes.

Another stream of research involves investors choosing among a set of non-redundant differently-taxed assets (due to different dividend yields) without borrowing / short sale constraints. For example, Gordon and Bradford (1980) model portfolio selection as a choice among a set of risky stocks with different dividend yields (and thus different levels of taxation).

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[6] Miller further shows that the total debt and equity issued by firms in the economy will be adjusted to assure that the marginal clientele has a relative tax preference for equity equal to the corporate income tax, and as a result

Their model allows investors to weight shares of different companies differently. Given this choice, investors split their portfolios across the assets in a manner designed to diversify risk while overweighting relatively tax-favored assets (either high-yield or low-yield stocks depending on an investor's tax characteristics). Market equilibrium involves risk premiums for stocks that are mixtures of risk premiums based on the holdings of different investors. Gordon and Bradford need to impose several simplifying assumptions to reduce this to a linear relationship that can be empirically estimated. A key implicit assumption in this model is that the available securities are non-redundant. Otherwise, given the lack of borrowing / short sale constraints in the model, unbounded tax arbitrage between, for example, tax-exempt institutions and taxable individuals would occur and no equilibrium would exist.[7] Thus, this model can be characterized as a model of a "small" number of stocks. A similar modeling approach is taken by Allen, Bernardo, and Welch (2000) in which the choice of dividend policy by firms is endogenous (and motivated by signaling). As in Gordon and Bradford, expected returns for the two firms differ by an amount that depends on both tax rates and risk aversion parameters. Also, as in Gordon and Bradford, their model excludes the prospect of redundant securities, which would lead to unbounded tax arbitrage.

Ross (1987) examines no arbitrage equilibrium in the presence of taxes. He shows that if a no arbitrage equilibrium exists, then there exist state prices and equivalent martingale measures for each type of taxable income (e.g. ordinary income and capital gains), although the relationship among these state price vectors between them and the state prices that would exist absent taxation is not generally known and could be complex. The existence of different state

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individual firms are indifferent regarding their leverage.
[7] Dammon and Green (1987) discuss this situation, in which a borrowing / short-sale constraint is necessary for the existence of an equilibrium with finite asset demands. Redundancy in this context only requires that there exists a

prices for different types of income is not very useful, absent a characterization of the relationship between those state price vectors. In particular, if the state prices are not proportional, and the corresponding equivalent martingale measures differ, then the concept of certainty equivalence depends on the tax system, implying that risk-neutral tax models are not generally valid in risk-averse settings and no-tax models of risk premia are not generally valid in setting where taxes are levied.

This paper will develop a model in which, under realistic assumptions, state prices for different types of income are proportional, and there is a single equivalent martingale measure for the economy that is independent of the tax system. This is made possible by introducing sufficient redundancy into the set of assets available to investors so that they can achieve any risk exposure desired while maintaining tax efficiency in their portfolios. Thus, redundancy itself is the instrument that allows separation of tax and risk effects in asset prices and returns. Unlike the models of Gordon and Bradford (1980) and Allen, Bernardo, and Welch (2000) in which redundant assets would lead to unbounded tax arbitrage and no equilibrium, this paper will achieve redundancy through the introduction of stock index futures contracts that are by themselves useless for pure tax arbitrage purposes, but completely effective at providing access to any desired risk exposure while maintaining a tax-efficient portfolio.

III. Portfolio Choice Model

Consider a model consisting of N firms, M investors, and a government. In the first period, exchange of securities occurs. In the second and final period, firms realize cash flow and distribute it to owners of their securities, including both stocks and bonds, based on

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linear combination of securities with a particular tax treatment that has identical returns to a linear combination of securities with a different tax treatment.

predetermined allocation rules.[8] The government levies personal taxes on individual interest

income on bonds and capital gains on stock and futures returns.[9]

The index futures contract is a third type of security; there are H index futures available

for trade in financial markets. Each index futures contract corresponds to an index of stocks (and

possibly bonds). In the first period, when parties contract, no money changes hands; however, a

futures price for the index is specified. In the second period, the buyer of the contract receives

the difference between the actual value of the index and the futures price. She pays the

difference to the seller if the futures price is higher. In equilibrium, this contract must have zero

net demand by investors.[10]

Let $r_{bn} + v_n$ be the return on bonds and $r_{en} + \varepsilon_n$ be the return on stock for firm n (with r_{bn}

and r_{en} their expectations, respectively) where each return is the ratio of the second period

change in value to the current price. Also $r_{fh} + \eta_h$ is the profit on a long position in a futures

contract representing one dollar of index h (with r_{fh} its expectation). That profit is the difference

between the second period price of the index and the futures price. If index h consists of weights

I_{hn} for the stock in each firm n (such that $\sum_{n=1}^{N} I_{hn} = 1$), then

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[8] If standard priority rules are strictly observed, this means that the firm will distribute all of its cash flow to bondholders up to the amount of principal plus accrued interest, with any remainder distributed entirely to shareholders (after corporate income taxes). Note that the exact distribution scheme is unimportant in this model.

[9] Stock returns could also be partially subject to dividend taxes, but that would merely change the tax rate and not alter the nature of the model. Likewise, the effective capital gains rate could be substantially lower than the statutory rate on capital gains due to deferral, selective realization, and possible step-up at death.

[10] In a two period model, as in this paper, this is a description of a forward contract as well as a futures contract. The difference in a multi-period model is that futures contracts involve periodic payments from one participant to the other based on price changes in the underlying stock to keep the price of the futures equal to 0. This "mark to market" maintains the zero basis characteristic of the futures contract over time. As will be demonstrated, the zero basis feature is what allows futures to effectively perform their role in the model and produce the tax-risk separation result. Thus, while the two period model would suggest that forward contracts would be equally effective, in a multi-period setting, only futures contracts will produce the results in this paper.

$$\eta_h = \mathbf{I}_h{}'\boldsymbol{\varepsilon}.$$[11]

Investors engage in costless trade of securities to maximize expected utility. Utility is a function of an investor's final wealth: W_{1m}, which is the product of his initial wealth, W_{0m}, and one plus the after-tax return on his portfolio. Only two restrictions are imposed on the utility function U_m: it must be concave and increasing in wealth.

Each investor takes all prices as given. Agents have identical information about future corporate cash flow and asset prices. Investor m is subjected to a tax rate t_m on interest income and z_m on equity income. Define the wealth-weighted distribution of tax ratios as

$$Y(a) = \frac{\displaystyle\sum_{m:(1-t_m)/(1-z_m)\le a} W_{0m}}{\displaystyle\sum_m W_{0m}}.$$

To insure a unique solution, assume that Y is strictly increasing in the interval $[\underline{a}, \overline{a}]$ where $\underline{a}$ and $\overline{a}$ are the minimum and maximum values of the tax ratio in the economy.[12]

For simplicity, the government is assumed to tax capital gains on futures at the ordinary tax rate (as is the case under current U.S. tax law). In fact, the actual tax rate on futures is irrelevant to the investor and inconsequential to the model. Since a futures position requires no initial investment, investor m facing a proportional tax of rate q_m on futures profits may undo the tax by increasing his position by the proportion $1/(1 - q_m)$ (Cox, Ingersoll, and Ross, 1981). This is, in fact, the attribute of futures contracts that makes them effective in this model. The zero

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[11] It is a simple extension to allow for the index to have positive weight on bonds as well as stocks.

[12] Strict monotonicity is not required for the existence of equilibrium, only uniqueness.

basis makes them effectively tax-free instruments. This is why futures are more effective than options in redistributing risk in the economy (in this model). While options could potentially be used by investors instead of futures to achieve the necessary transfer of risk, the inherent tax effects of the options, due to their non-zero basis, would cause problems.

For legal or institutional reasons, no investor may short any type of investment security (a security that involves an up-front investment, unlike a futures contract) beyond a fixed limit.[13] This means that an investor is constrained in the amount she can borrow or short stocks. Without loss of generally, the short sale bounds are assumed to be zero for every investor.

Formally, the constrained maximization problem for investor m is as follows (with B_{nm}, E_{nm}, and F_{hm} defined as m's investment in firm n's bonds, investment in firm n's stock, and futures position for index h, respectively):

max $EU_m (W_{1m})$ subject to

$$W_{1m} = \sum_{n=1}^{N}[1 + (1-t_m)(r_{bn} + v_n)]B_{nm} + [1 + (1-z_m)(r_{en} + \varepsilon_n)]E_{nm} +$$
$$\sum_{h=1}^{H}[1 + (1-t_m)(r_{fh} + \eta_h)]F_{hm}. \tag{1}$$

$$\sum_{n=1}^{N}(B_{nm} + E_{nm}) = W_{0m} \tag{2}$$

$$B_{nm}, E_{nm} \geq 0 \; \forall \; n. \tag{3}$$

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[13] Short sale constraints are commonly employed in models to prevent unbounded positions due to tax arbitrage (Dammon and Green, 1987, state conditions under which such short sale constraints are necessary for equilibrium; this model satisfies those conditions since taxes are linear and there exist assets that are linearly dependant on the set of differently taxed assets). A limit on futures positions, both short and long, can also be imposed without affecting the model, so long as the futures limits are not too strict.

Equation (1) sets terminal wealth equal to the sum of the after-tax returns on all of m's securities. (2) is the budget constraint, while (3) represents the short sale limits. Market clearing conditions (with B_n and E_n defined as the value of firm n's bonds and stocks, respectively) are:

$$B_n = \sum_{m=1}^{M} B_{nm} \ \forall \ n.$$
$$E_n = \sum_{m=1}^{M} E_{nm} \ \forall \ n. \tag{4}$$
$$0 = \sum_{m=1}^{M} F_{hm} \ \forall \ h.$$

The first two equations set the supply of securities by firms equal to the demand for securities by investors, while the third equation sets net futures demand equal to 0 (since the futures have no supply). (4) implies $\sum_{n=1}^{N} V_n = \sum_{m=1}^{M} W_{0m}$, where V_n is defined as the total market value of firm n ($B_n + E_n$). This constraint holds if each investor has an endowment of some quantity of securities, instead of a dollar amount, where each available security is initially held by at least one of the investors in the economy.

In the second and final period, firms distribute payouts to bondholders and shareholders, based on their (random) cash flow and their allocation rules (based on the bond contract terms). Define corporate cash flow as X_n; this is the amount that is split between stock- and bondholders of the firm. Assume that X_n is generated by the following (K-1)-factor model:

$$X_n = \overline{X}_n + \sum_{k=1}^{K-1} \mu_{nk} f_k + \phi_n. \tag{5}$$

where $\overline{X}_n$ is expected level of cash flow, $\mathbf{f}$ is the vector of systematic risk factors (there are K-1 of them in the economy, normalized to be orthogonal to each other and have expected value

of 0) and ϕ_n is idiosyncratic noise that is uncorrelated with the risk factors. It is assumed that a sufficient number of firms exists, such that ϕ_n can be diversified. That is to say that there exists a set of portfolio weights such that the idiosyncratic noise vanishes within the portfolio. Let **L** be a vector of weights for a portfolio of corporate cash flows that is fully diversified in this sense. Thus,

$$\sum_{n=1}^{N} L_n = 1 \quad \text{and} \quad \mathbf{L'}\boldsymbol{\phi} = 0. \tag{6}$$

Stock and bond returns depend on the payoffs the firm makes on each security. This in turn involves an allocation of the firm's cash flow that depends on bond contract terms. For example, the firm might give the total cash flow up to some maximum amount to bondholders, with the remainder (if any) distributed to shareholders. For purposes of this model, it does not matter precisely how the allocation occurs. It is important that the sum of the payouts to all investors equals the firm's total cash flow. Thus,

$$(1 + r_{bn} + v_n)B_n + (1 + r_{en} + \varepsilon_n)E_n = X_n. \tag{7}$$

(7) ignores the effect of the corporate income tax (and associated debt-tax shield), but inclusion of such a tax in the model would have no material effect. The corporate tax and debt-tax shield would be relevant to firms' choice of capital structure, but in this model capital structure is modeled as being exogenous (see Section IV for a discussion of how the model can be generalized to include endogenous capital structure choice).

The division of cash flow between debt and equity is non-linear, and therefore, it is unlikely that the K-1 risk factors in corporate cash flow are sufficient to characterize all systematic risks in stocks and bonds. Regress stock and bond returns on the K-1 risk factors:

$$(1 + r_{bn} + v_n)B_n = \bar{B}_n + \sum_{k=1}^{K-1} \beta_{nk} f_k + \zeta_n \tag{8a}$$

$$(1 + r_{en} + \varepsilon_n)E_n = \bar{E}_n + \sum_{k=1}^{K-1} a_{nk} f_k + \psi_n \tag{8b}$$

It is not necessarily the case that ζ_n is diversifiable in any portfolio consisting exclusively of bonds. Likewise, ψ_n might not be diversifiable in any stock portfolio. However, it is possible to construct an additional risk factor, f_K, that overcomes the non-linearization of corporate cash flow caused by the stock-bond allocation rule. In particular, construct a portfolio of stocks weighted by **L**. From (8b), its payoff is

$$\sum_{n=1}^{N} (1 + r_{en} + \varepsilon_n)E_n L_n = \sum_{n=1}^{N} \bar{E}_n L_n + \sum_{k=1}^{K-1} \left(\sum_{n=1}^{N} a_{nk} L_n \right) f_k + \sum_{n=1}^{N} \psi_n L_n. \tag{9}$$

Define f_K as the last term in (9), that is

$$f_K \equiv \sum_{n=1}^{N} \psi_n L_n. \tag{10}$$

Clearly, f_K has expectation zero and is orthogonal to the risk factors in corporate cash flow, by construction. Let **f** be a vector of K risk factors that includes the risk factors in corporate cash flow and f_K, defined in (10). Regress stock and bond returns on the full set of risk factors

$$(1 + r_{bn} + v_n)B_n = \overline{B}_n + \sum_{k=1}^{K} \beta_{nk} f_k + \zeta_n^*. \tag{11a}$$

$$(1 + r_{en} + \varepsilon_n)E_n = \overline{E}_n + \sum_{k=1}^{K} a_{nk} f_k + \psi_n^*. \tag{11b}$$

All the αs and βs are unchanged from (8a) and (8b), except for α_{nK} and β_{nK}, which are new (this is due to the orthogonality of the risk factors). Moreover, it is clear, given the choice of f_K that

$$\mathbf{L'\psi}^* = 0.$$

Given (6) and (7),

$$\phi = \psi^* + \zeta^*.$$
$$\mathbf{L'\zeta}^* = \mathbf{L'}(\phi - \psi^*) = 0.$$

Thus, given the set of systematic risk factors, $\mathbf{f}$, it is possible to diversify all non-systematic risk using a portfolio consisting entirely of stocks or of bonds. This is critical for the derivation of Proposition 1.

Finally, a portfolio of securities is defined as well-diversified if it exhibits no idiosyncratic risk. That is, its return can be expressed as a linear function of the K risk factors exclusively. Note that the stock and bond portfolios with weights $\mathbf{L}$ are well-diversified, but presumably so are many others, including the portfolios of all stocks and of all bonds.

Given the model developed above, the following proposition specifies sufficient (but not necessary) conditions under which a portfolio choice equilibrium exists with clientele formation and tax-risk separation in asset returns.[14]

Proposition 1

Assume that there are at least K futures contracts available for exchange, on well-diversified indices, such that the vector of returns on these indices spans the risk factors. Then,

a) With the available futures contracts sorted so that the first K are well-diversified and span the risk factors, expected returns on bonds and stocks are

$$r_{bn} = r_b + \sum_{h=1}^{K} \gamma_{nh} r_{fh}. \tag{12a}$$

$$r_{en} = r_z + \sum_{h=1}^{K} \lambda_{nh} r_{fh}. \tag{12b}$$

where r_b and r_z are the risk-free equivalent rates of return on debt and equity respectively, and γ_{hn} and λ_{hn} are the coefficients for regressions of bond and stock returns on the futures returns, respectively.

b) Each investor m will invest his entire wealth in a well-diversified portfolio of bonds (any such portfolio is optimal) if the following condition holds:

$$\frac{r_z}{r_b} < \frac{1 - t_m}{1 - z_m}. \tag{13}$$

If condition (13) fails to hold, m will invest entirely in a well-diversified portfolio of stocks.

    c) Equilibrium returns are characterized by

$$Y\left(\frac{r_z}{r_b}\right) = \frac{\sum_{n=1}^{N} E_n}{\sum_{n=1}^{N} V_n}. \tag{14}$$

Proof: Appendix A.

Since the futures contracts span the systematic risk factors and their taxation is irrelevant, they offer a perfect means for optimizing risk. Investors can choose any well-diversified portfolio of stocks or bonds and use futures positions to achieve any risk exposure desired. Thus, investors can choose tax efficient portfolios without sacrificing any risk optimality. They select assets that yield the highest after-tax riskfree-equivalent return and form pure clienteles based solely on their tax characteristics. Likewise, investors choose risk exposures based only on the shapes of their utility functions and tax-adjusted wealth.[15] Taxes and risk can be separated from each other in both decision making and pricing, as the following corollary states.

Corollary 1: The Tax-Risk Separation Theorem

    a) An asset's return is linearly separable into two pieces, one which corresponds exclusively to the tax characteristics of the asset (r_b and r_z) and one which corresponds exclusively to the risk characteristics of the asset ($\gamma_n' r_f$ and $\lambda_n' r_f$).

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[15] Tax-adjusted wealth is initial wealth times one plus the riskfree-equivalent after-tax rate of return on either stocks or bonds, whichever is the optimal asset.

b) An alternative characterization of the tax-risk separation in asset pricing is in terms of state prices. Specifically, if $\mathbf{q}_b$ is the vector of state prices for valuing bonds, then the following is the vector of state prices for valuing stocks:

$$\mathbf{q}_e = \frac{1 + r_b}{1 + r_z} \mathbf{q}_b.$$

Moreover, the equivalent martingale measure is identical for both types of securities.

c) Investors can optimize taxes without regard to risk characteristics of assets by maximizing the after-tax value of the tax component of returns. Similarly, investors can optimize risk without regard to tax characteristics of assets by using effectively tax-free futures contracts.

Proof: Appendix A.

The tax-risk separation theorem validates prior research that considered either the effect of taxes or risk on portfolio choice and asset pricing without explicitly accounting for the other (e.g. Miller, 1977, and Ross, 1976). The approach of considering each objective in isolation is not only simpler, but also correct if a sufficient number of index futures contracts exists. In fact, with these few index futures in the economy, an even stronger result obtains. Taxes do not affect resource allocation or welfare in any manner, except through wealth redistribution, as the following corollary states.[16]

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[16] This result does depend on the assumption that saving is fixed. If the choice between current consumption and saving is endogenous (as is likely), then taxes on interest and capital gains will have distortionary effects on the

Corollary 2: Tax Neutrality

Assume that the tax system is replaced by distributionally equivalent lump sum taxation. Then, the Proposition 1 portfolios remain optimal, investor expected utility is unaffected by the tax reform, and for some r*, asset returns obey

$$r_{bn} = r* + \gamma'_n \mathbf{r}_f, \text{ and } r_{en} = r* + \lambda'_n \mathbf{r}_f. \tag{15}$$

The equivalent martingale measure in an economy with taxes on capital gains and interest is also the equivalent martingale measure in an otherwise identical economy with lump sum taxes. Thus, for any distributionally equivalent linear tax system, utility of investors and risk premia are identical.

Proof: Appendix A.

This corollary indicates that if you adjust endowments for the wealth effects of taxes, you do not need to know anything else about the tax system to characterize the influence of risk in asset pricing. You could construct certainty equivalents to all risky streams without reference to the distribution of tax rates in the economy. In turn, to analyze the effect of taxes on portfolio choice and asset prices, you could convert all risky streams to their certainty equivalents (which do not depend on taxes) and model investors as if they are risk-neutral. Numerous papers have taken these approaches (analyzed risk in a tax-free environment and analyzed taxes in a risk-free environment); these corollaries validate the generality of such models.

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equilibrium interest rate. The point of the corollary is to suggest that the consumption-investment margin could be

Returning to Proposition 1, it can be seen that if equity is tax-advantaged relative to debt (i.e. $z_m < t_m$ for all investors), then $r_z < r_b$. The risk-adjusted rate of return on equity is less than the risk-adjusted return on debt, a phenomenon known as implicit tax. This implicit tax is necessary to compensate bondholders for paying higher taxes than stockholders. The exact level of the implicit tax depends on the identity of the marginal clientele. The marginal clientele is the clientele whose relative tax rates make it just indifferent between holding debt and equity. If the marginal clientele has a strong tax preference for equity, the implicit tax will be high; if the marginal clientele is a tax-exempt entity, the implicit tax will be zero. The identity of the marginal clientele depends on both the distribution of tax rates among investors and the aggregate debt / equity ratio.

IV. Extensions and Applications of the Model

The tax-risk separation theorem and tax-neutrality corollaries can be extended to consider corporate level decisions, such as capital structure, dividend policy, and real investment. With the inclusion of corporate income taxes (with interest deductibility), the model can easily be extended in a manner consistent with the leverage irrelevance model of Miller (1977). In that case, notwithstanding the effects of capital structure on the risk characteristics of a firm's debt and equity,[17] it is optimal to issue debt or equity exclusively depending on whether the combination of corporate income and capital gain taxes exceeds the ordinary tax rate of the marginal clientele. Consequently, an interior general equilibrium (with positive quantities of both debt and equity in the economy) requires that the marginal clientele have tax rates that

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the only margin on which capital income taxes have a distortionary effect.

[17] Grossman and Stiglitz (1977), as well as several other papers, point out that in incomplete markets, changes in the risk characteristics of a firm's securities in response to corporate decisions, particularly capital structure, could affect shareholder investment opportunities (and thus utility) and should be a relevant factor in making these decisions.

exactly equalize the total tax burden of debt and equity; and in that case, each firm is indifferent regarding its capital structure. This result is identical to Miller (1977) except that it does not require the risk-neutrality assumption in Miller, and is less restrictive than prior models of leverage irrelevance that allow risk-aversion.[18] So long as the requirements of the tax-risk separation theorem hold (in particular, that a set of spanning futures contracts exists), the complications introduced by risk drop away leaving a model that substantively behaves as if agents are risk-neutral.

This approach can be extended to establish dividend irrelevance[19] and to demonstrate that corporate investment is invariant to capital income taxes (including corporate income taxes).[20] This second result is a broad extension of Corollary 2 (tax-neutrality) and builds on Stiglitz (1973). Whereas Stiglitz develops a model in which portfolio and capital structure choice are influenced by taxes but firm-level production and investment are not, the tax-risk separation theorem allows for a model in which none of these decisions are affected by taxes (at least not in any efficiency reducing manner; taxes could induce shifts along the Pareto frontier). The only margin along which capital income taxes could distort the economy and generate deadweight loss is the savings-consumption decision (since capital income taxes penalize saving).

V. Concluding Remarks

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[18] As mentioned in section II, prior studies extended Miller (1977) to allow for risk aversion only under stringent conditions, such as all firms being capable of issuing complete Arrow-Debreu state-contingent debt and equity claims (DeAngelo and Masulis, 1980).

[19] Kemsley and Williams (2001) develop a model of dividend irrelevance under the assumption that tax-risk separation occurs.

[20] This follows from the fact that corporate value is proportional to what value would be absent taxation (the factor of proportionality is one minus the ordinary income tax rate of the marginal clientele). Thus, any firm value maximizing production or investment decision absent taxes will maximize value in the presence of taxes.

This paper has shown that when investors face differential linear taxes on interest and capital gains, the availability of a limited set of stock index futures contracts[21] guarantees that:

i)   investors will separate into tax clienteles investing in either debt or equity exclusively (Proposition 1),

ii)   asset returns will be linearly separable into pure tax-based and pure risk-based components (Corollary 1),

iii)   equivalently, state price vectors for valuing stocks and bonds will be proportional, so that the equivalent martingale measure of each type of security is identical (Corollary 1),

iv)   taxes do not distort portfolio choice in a way that creates deadweight loss (Corollary 2),

v)   and consequentially, the equivalent martingale measure for the economy is independent of the tax system (Corollary 2).

One key result in this paper is *iv*; capital income taxes have only wealth effects, not substitution effects on resource allocation (holding the level of saving fixed). As discussed in Section III, this result can be extended to corporate-level finance, production, and investment decisions. While taxes could alter these decisions, they simply shift them along the Pareto frontier, generating no sub-optimal resource allocation or deadweight cost. Apart from a potential effect on the saving-consumption decision, capital income taxes do not cause any inefficiency.

The main result in this study is the separability of tax and risk incentives, represented by findings *ii*, *iii*, and *v*. The prospect of tax-risk separation is particularly important in justifying

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[21] As stated in Section III, the only requirements for these futures are that they represent well-diversified indices and that they collectively span the systematic risk factors in asset prices.

the common research strategy of modeling taxes and risk independently, as it eliminates the concern that they could confound each other. As v states, the appropriate measure of certainty equivalence does not depend on the tax system. Therefore it is legitimate for a modeler to adjust all risky streams to their certainty equivalents and then analyze the effects of taxes in a world of essentially risk neutral agents. The assumption of risk neutrality has been assumed in several tax models,[22] and this paper demonstrates that the results of these models are generalizable to economies with risk-aversion. The converse of this situation is also true; models of how risk influences asset prices and returns, and how investors select risk exposures, that have been developed under the assumption of no taxes are generalizable to economies with taxes.

Extending this analysis to empirical research, it is clear that researchers cannot simply ignore one of the effects (taxes or risk) while focusing on the other. Asset prices and returns exhibit both tax and risk effects commingled. One cannot, for example, estimate risk premia or control for risk using a model that does not account for taxes, as is commonly done, due to mis-specification.[23] While the Tax-Risk Separation Theorem does not eliminate the need to untangle tax and risk effects in empirical studies, it does suggest that these effects have a simple additive form. Thus, valid empirical estimation can be conducted in a two step procedure in which the tax effect is estimated in one step (controlling for risk by subtracting risk premia from stock

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[22] Recent examples include Kemsley and Williams (2001), who develop a model of dividend policy in which such policy is irrelevant because the dividend tax is fully capitalized into stock prices regardless of timing (and is thus a sunk cost), and Williams (2001), who develops a three-clientele model (individuals, corporations, and tax-exempts) with stocks (with varying dividend yields), bonds, and municipal bonds. With effective risk neutrality, he derives a non-linear relationship between stock returns and dividend yields and derives municipal bond yields that are complex functions of individual and corporate tax rates for dividends and capital gains.

[23] For example, many papers test hypotheses regarding abnormal returns (including event studies and market efficiency studies). These papers use a risk model (such as the CAPM) to construct expected returns. Since the CAPM estimates each stock's expected return as the certainty equivalent return plus β times the excess of the stock market return over the certainty equivalent return, a mis-specification occurs if the researcher uses the certainty equivalent bond instead of certainty equivalent stock return (r_b instead of r_z in the nomenclature of this paper). Even if the purpose of the study is a cross-sectional comparison (in which case the intercept drops out), the mis-measurement of the slope on β can be problematic if β is correlated with the variable of interest in the study.

returns) and risk effects are estimated in the other step (controlling for taxes by adjusting the certainty equivalent return to account for any implicit tax in stock returns). Aboody and Williams (2001) do the first step and estimate implicit taxes on equity by comparing an estimated certainty equivalent return on multiple stock indices (based on using their corresponding futures to hedge any risk) to a certainty equivalent return on bonds. Aboody and Williams find significant implicit taxes for each index. Their findings can (and should) be used to adjust certainty equivalent returns used in any empirical test that estimates or controls for risk premia.

Finally, the paper demonstrates the potential importance of futures contracts (distinct from other derivatives such as options and forward contracts). Futures are the key to the tax-risk separation result, due to their zero-basis characteristic, a feature unique to futures among all standard financial securities. Thus, while other derivative securities allow investors to redistribute risk, only futures do so in a tax-neutral manner.

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Appendix A

Proofs of the Proposition and Corollaries

Proposition 1

a) Since the first K futures contracts represent well-diversified portfolios and $\mathbf{f}$ is normalized such that $E[\mathbf{f}] = \mathbf{0}$, the unexpected returns on the futures can be expressed (in scalar and matrix form) as

$$\eta_h = \sum_{k=1}^{K} \pi_{hk} f_k$$

$$\eta = \Pi'\mathbf{f}.$$

By the assumption that the first K futures span the risk factors, Π is invertible.

Consider the stock in firm n. Based on (11b), its unexpected return can be expressed as

$$\varepsilon_n = (\psi_n^* / E_n) + \sum_{k=1}^{K} (\alpha_{nk} / E_n) f_k$$

where (ψ_n / E_n) is idiosyncratic noise. The systematic risk in n's stock can be offset by acquiring the following vector of futures positions:

$$-\frac{1 - z_m}{1 - t_m} \lambda_n$$

where

$$\lambda_n = \frac{\Pi^{-1}\alpha_n}{E_n}.$$

Consider an investor m holding a well-diversified portfolio of stocks (we will subsequently demonstrate that such investors exist). If stock n is included in her portfolio, then she could sell some of her holdings of n (say ω dollars) and purchase stock s. If this trade is combined with the following futures trades,

$$\omega \frac{1-z_m}{1-t_m}(\lambda_n - \lambda_s),$$

then given the well-diversified nature of the investor's portfolio, this leaves her risk exposure unchanged. For both stocks n and s to have positive demand, this portfolio shift must also leave expected returns unchanged. Therefore,

$$\omega\left[(1-z_m)r_{es} - (1-t_m)\frac{1-z_m}{1-t_m}\lambda_s'\mathbf{r}_f - \left((1-z_m)r_{en} - (1-t_m)\frac{1-z_m}{1-t_m}\lambda_n'\mathbf{r}_f\right)\right] = 0.$$

(12b) follows directly. Every stock must have the same risk-adjusted return, r_z, to guarantee positive demands for all equities. (12a) follows from a similar analysis of the bond market.

b) Now consider portfolio choice by investors. It can be readily seen that any portfolio that is not well-diversified is dominated by any well-diversified portfolio with the same mix of stocks and bonds combined with suitable adjustments to futures positions to maintain identical systematic risk. Expected return will be the same by (12a) and (12b), systematic risk will also be

the same, but idiosyncratic risk will be less for the well-diversified portfolio, which is desirable since the utility function is concave.

An investor m with a well-diversified portfolio that includes bond n, could sell ω dollars of the bond and buy ω dollars of stock s. The systematic risk effect of this transaction could be offset with suitable futures trades as previously shown. Given the returns expressed in (12a) and (12b), the expected return of this arbitrage is

$$\omega\left[(1-z_m)r_{es} - (1-t_m)\frac{1-z_m}{1-t_m}\lambda_n'\mathbf{r}_f - \left((1-t_m)r_{bn} - (1-t_m)\gamma_n'\mathbf{r}_f\right)\right] =$$

$$\omega\left[(1-z_m)r_z - (1-t_m)r_b\right]$$

This expected return is negative if condition (13) holds, demonstrating that keeping the bond is desirable and, in fact, selling any stocks to buy more bonds improves expected return, while the level of risk can be maintained at a constant level so long as the investor continues to hold a well-diversified portfolio. If condition (13) holds, m will trade stocks for bonds until the short sale constraint (3) binds. At that point, m will only hold bonds, which is optimal. Of course, any well-diversified portfolio of bonds is as good as any other.

A similar argument applies if condition (13) strictly fails. In that case, expected return can be increased by selling all bonds and buying stocks while taking appropriate futures positions to maintain constant risk. The assumption that Y is strictly monotonic guarantees that (13) fails weakly (i.e. the ratio of risk-free returns equals the ratio of tax rates) for a set of investors with measure zero. However, for those investors, there is no gain from exchanging any stock or bond for any other asset. In that case, any well-diversified portfolio is equally desirable.

c) The final equilibrium issue involves the market clearing conditions (4). The ratio of risk-free returns r_z/r_b will adjust to a level such that (13) holds for enough investors to buy all available bonds and (13) fails for enough investors to buy all available equity. The fraction of wealth held by investors with tax ratios less than a is $Y(a)$ by definition, so the quantity of stock demanded by investors is $Y(r_z/r_b)\sum_{n=1}^{N}V_n$ which must equal the supply of equity by firms, yielding (14). Futures market clearing occurs through adjustment of the factor risk premia.

Corollary 1

a) follows straightforwardly from Proposition 1, noting that the relative risk-free returns are determined by (14) as

$$\frac{r_z}{r_b} = Y^{-1}\left(\frac{\sum_{n=1}^{N}E_n}{\sum_{n=1}^{N}V_n}\right)$$

Recall that Y is the wealth-weighted distribution function of tax rate ratios. Hence the relationship between r_b and r_z is purely tax-driven. The risk premia are defined identically for debt and equity as the regression coefficients for the futures returns, so taxes do not influence them.

To prove b), consider the required restrictions on state prices for valuing equity. Given a vector of states, Θ, the price of all stocks must correspond to the state prices, $\mathbf{q}_e$, as

$$1 = \sum_{\theta \in \Theta} q_e(\theta)[1 + r_z + \boldsymbol{\alpha}'(\mathbf{p} + \mathbf{f}(\theta))] \quad \forall \boldsymbol{\alpha} \in R^K,$$
$$0 = \sum_{\theta \in \Theta} q_e(\theta)\psi(\theta) \quad \forall \psi \text{ s.t. } E[\psi\mathbf{f}] = \mathbf{0}$$

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where $\mathbf{p}$ is the vector of risk premia on the K factors ($\mathbf{p} = \mathbf{\Pi}^{-1\prime}\mathbf{r}f$). The first expression implies K + 1 linear restrictions on $\mathbf{q}_e$, which can be expressed as the same equation with $\boldsymbol{\alpha} = \mathbf{0}$ and with each $\boldsymbol{\alpha}$ in which $\alpha_k = 1 + r_z$ and $\alpha_{l \neq k} = 0$. These restrictions imply

$$\sum_{\theta \in \Theta} q_e(\theta) = \frac{1}{1 + r_z},$$

$$\sum_{\theta \in \Theta} q_e(\theta)[1 + p_k + f_k(\theta)] = \frac{1}{1 + r_z} \quad \forall\, k, \quad \text{and}$$

$$\sum_{\theta \in \Theta} q_e(\theta)\psi(\theta) = 0 \quad \forall\, \psi \text{ s.t. } \mathrm{E}[\psi\mathbf{f}] = \mathbf{0}.$$

A similar analysis performed for the state price vector for bonds yields these restrictions:

$$\sum_{\theta \in \Theta} q_b(\theta) = \frac{1}{1 + r_b},$$

$$\sum_{\theta \in \Theta} q_b(\theta)[1 + p_k + f_k(\theta)] = \frac{1}{1 + r_b} \quad \forall\, k, \quad \text{and}$$

$$\sum_{\theta \in \Theta} q_b(\theta)\psi(\theta) = 0 \quad \forall\, \psi \text{ s.t. } \mathrm{E}[\psi\mathbf{f}] = \mathbf{0}.$$

Clearly, for any vector $\mathbf{q}_b$ that solves these restrictions,

$$\mathbf{q}_e = \frac{1 + r_b}{1 + r_z}\mathbf{q}_b$$

solves the restrictions on $\mathbf{q}_e$. Since the ratio of the sum of state prices equals the ratio of state prices in every state, the martingale measures $\mathbf{q}_b / \sum q_b(\theta)$ and $\mathbf{q}_e / \sum q_e(\theta)$ are identical.

c) is a direct restatement of portfolio strategy in Proposition 1.

With taxes on returns, the portfolio choice problem can be characterized, given (1), (2),

and (12) as the selection of a well-diversified portfolio that maximizes the utility of

$$W_{1m} = W_{0m} + (1 - t_m)r_b B_m + (1 - z_m)r_z E_m + (1 - t_m)\left(\mathbf{F}_m + \gamma\mathbf{B}_m + \frac{1 - z_m}{1 - t_m}\lambda\mathbf{E}_m\right)(\mathbf{r}_f + \mathbf{\eta}_f)$$

where $B_m = \mathbf{B}_m'\mathbf{1}$ and $E_m = \mathbf{E}_m'\mathbf{1}$, subject to (2) and (3).

With lump sum taxes of $t_m r_b B_m + [r_b - (1 - z_m)r_z]E_m + (t_m\mathbf{F}_m + t_m\gamma\mathbf{B}_m + z_m\lambda\mathbf{E}_m)(\mathbf{r}_f + \mathbf{\eta}_f)$,

if asset returns are characterized by (15) with $r^* = r_b$ and futures return unchanged,[24] then the

portfolio selection problem becomes the choice of $\mathbf{F}_m^*$, $\mathbf{B}_m^*$, and $\mathbf{E}_m^*$ that maximizes expected

utility, subject to (2) and (3), for

$$\begin{aligned}W_{1m} = (1 + r_b)W_{0m} &+ \left(\mathbf{F}_m^* + \gamma\mathbf{B}_m^* + \lambda\mathbf{E}_m^*\right)(\mathbf{r}_f + \mathbf{\eta}_f) - \\ &\left\{t_m r_b B_m + [r_b - (1 - z_m)r_z]E_m + (t_m\mathbf{F}_m + t_m\gamma\mathbf{B}_m + z_m\lambda\mathbf{E}_m)(\mathbf{r}_f + \mathbf{\eta}_f)\right\}\end{aligned}$$

Note that $\mathbf{F}_m^* = \mathbf{F}_m$, $\mathbf{B}_m^* = \mathbf{B}_m$, and $\mathbf{E}_m^* = \mathbf{E}_m$ is a feasible portfolio that provides identical

expected utility to the same portfolio with investment taxes. Also note that given an arbitrary

portfolio $\mathbf{F}_m^*$, $\mathbf{B}_m^*$, and $\mathbf{E}_m^*$, there is an alternative portfolio $\mathbf{B}_m^{**} = \mathbf{B}_m$, $\mathbf{E}_m^{**} = \mathbf{E}_m$, and

$\mathbf{F}_m^{**} = \mathbf{F}_m^* + \gamma(\mathbf{B}_m^* - \mathbf{B}_m) + \lambda(\mathbf{E}_m^* - \mathbf{E}_m)$, that provides identical expected utility. This portfolio is

tax efficient if investment taxes are levied and provides the same expected utility under both tax

regimes. Thus, any expected utility obtainable with lump sum taxes is obtainable with

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[24] The choice of r^* is arbitrary. With another r^*, a different lump sum tax would be required, but the outcome
would be the same.

investment taxes (and vice versa if only tax efficient portfolios are considered), so the investment opportunity sets in the two economies are the same. Further, since $\mathbf{F}_m$, $\mathbf{B}_m$, and $\mathbf{E}_m$ is an optimal portfolio with investment taxes, it must be optimal with lump sum taxes (of course, it is only optimal in the weak sense; many other portfolios provide identical utility). Since markets clear in the investment tax case, and all investors have the incentive to choose identical portfolios in the lump sum tax case, markets clear for asset returns governed by (15).

That the equivalent martingale measure with investment taxes remains one with lump sum taxes, follows from the fact that factor risk premia are the same (since futures returns are the same) under both (and in fact, all compensated linear) tax regimes. Thus, the state price ($\mathbf{q}_l$) restrictions with lump sum taxes are

$$\sum_{\theta \in \Theta} q_l(\theta) = \frac{1}{1 + r^*},$$

$$\sum_{\theta \in \Theta} q_l(\theta)[1 + p_k + f_k(\theta)] = \frac{1}{1 + r^*} \quad \forall \, k, \quad \text{and}$$

$$\sum_{\theta \in \Theta} q_l(\theta)\psi(\theta) = 0 \quad \forall \, \psi \text{ s.t. } \mathrm{E}[\psi \mathbf{f}] = \mathbf{0}.$$

If $\mathbf{q}_b$ is the state price vector for bonds in the investment tax economy, then

$$\mathbf{q}_l = \frac{1 + r_b}{1 + r^*} \mathbf{q}_b$$

meets the above restrictions. Clearly, the equivalent martingale measures based on these state prices ($\mathbf{q}_b$ and $\mathbf{q}_l$) are the same.

Appendix B

Notation Used in the Paper

Scalars

a	Ratio of tax rates, $(1-t_m)/(1-z_m)$
$\underline{a}$	Minimum value of a in the economy
$\overline{a}$	Maximum value of a in the economy
B_m	Total bonds held by investor m
B_n	Total bonds issued by firm n
$\overline{B}_n$	Expected total payout to n's bondholders
B_{nm}	Bonds issued by firm n held by investor m
E_m	Total equity held by investor m
E_n	Total equity issued by firm n
$\overline{E}_n$	Expected total payout to n's shareholders
E_{nm}	Equity issued by firm n held by investor m
F_{hm}	Position in futures on index h held by investor m
f_k	Risk factor k
H	Number of different indices with futures contracts
h	An index with a futures contract
I_{hn}	Weight of stock n in index h
K	Number of risk systematic risk factors
k	A risk factor
L_n	Weight of firm n is a diversified portfolio of corporate cash flow
M	Number of investors
m	An investor
N	Number of firms
n	A firm
p_k	Risk premium on factor k
$q_b(\theta)$	State price for valuing bonds
$q_e(\theta)$	State price for valuing equity
$q_l(\theta)$	State price for valuing assets in a no-tax economy
q_m	Hypothetical tax rate on futures profits
r^*	Risk-free return on assets in a no-tax economy
r_b	Risk-free return on bonds
r_{bn}	Expected return on n's bonds
r_{en}	Expected return on n's stock
r_{fh}	Expected profit on futures contract h
r_z	Risk-free return on equity
s	Stock purchased in portfolio reallocation
t_m	Ordinary income tax rate of investor m
U_m	Utility function of investor m
V_n	Value of securities issued by firm n

W_{1m}	Ending wealth of investor m
W_{0m}	Beginning wealth of investor m
X_n	n's cash flow
$\overline{X}_n$	n's expected cash flow
$Y(a)$	Wealth-weighted distribution of tax ratios $(1-t_m)/(1-z_m)$
z_m	Capital gains tax rate of investor m
α_{nk}	Coefficient on factor k in regression of n's stock payout on the risk factors
β_{nk}	Coefficient on factor k in regression of n's bond payout on the risk factors
ε_n	Unexpected return on n's stock
ϕ_n	Idiosyncratic noise in n's cash flow
γ_{hn}	Coefficient on index h in regression of n's bond return on the first K futures
η_h	Unexpected profit on futures contract h
λ_{hn}	Coefficient on index h in regression of n's stock return on the first K futures
ν_n	Unexpected return on n's bonds
μ_{hk}	Coefficient on factor k in regression of n's cash flow on the risk factors
π_{hk}	Coefficient on factor k in linear combination of the factors that equals η_h
θ	A state of nature
ω	Dollar amount of incremental portfolio change
ψ_n	Residual noise in the total payout to n's stockholders using K-1 risk factors
ψ^*_n	Residual noise in the total payout to n's stockholders using K risk factors
ζ_n	Residual noise in the total payout to n's bondholders using K-1 risk factors
ζ^*_n	Residual noise in the total payout to n's bondholders using K risk factors

Vectors and Matrices

$\mathbf{B}_m$	Bondholdings of investor m
$\mathbf{B}_m^*$	Bondholdings of investor m in a no-tax economy
$\mathbf{B}_m^{**}$	Alternative bondholdings of investor m in a no-tax economy
$\mathbf{E}_m$	Stockholdings of investor m
$\mathbf{E}_m^*$	Stockholdings of investor m in a no-tax economy
$\mathbf{E}_m^{**}$	Alternative stockholdings of investor m in a no-tax economy
$\mathbf{f}$	Systematic risk factors
$\mathbf{F}_m$	Futures position of investor m
$\mathbf{F}_m^*$	Futures position of investor m in a no-tax economy
$\mathbf{F}_m^{**}$	Alternative futures position of investor m in a no-tax economy
$\mathbf{I}_h$	Weights of stocks in index h
$\mathbf{L}$	Weights of firms in a diversified portfolio of corporate cash flow
$\mathbf{p}$	Risk premia on the systematic risk factors
$\mathbf{q}_b$	State prices for valuing bonds
$\mathbf{q}_e$	State prices for valuing equity
$\mathbf{q}_l$	State prices for valuing assets in a no-tax economy
$\mathbf{r}_f$	Expected profit on the first K futures
α_n	Coefficients from a regression of n's stock payout on the risk factors
ε	Unexpected return on stocks
ϕ	Vector of residual noise in corporate cash flow
γ_n	Coefficients from a regression of n's bond return on the first K futures
η	Unexpected profit on futures
λ_n	Coefficients from a regression of n's stock return on the first K futures
Π	Solution to the linear equations system: $\eta = \Pi' \mathbf{f}$
Θ	States of nature
ψ^*	Vector of residual noise in stocks
ζ^*	Vector of residual noise in bonds