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THE VALUATION OF WARRANTS: IMPLEMENTING A NEW APPROACH*

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The option pricing model developed by Black and Scholes and extended by Merton gives rise to partial differential equations governing the value of an option. When the underlying stock pays no dividends – and in some very restrictive cases when it does – a closed form solution to the differential equation subject to the appropriate boundary conditions, has been obtained. But, in some relevant cases such as the one in which the stock pays discrete dividends, no closed form solution has been found. This paper shows how to solve these equations by numerical methods. In addition, the optimal strategy for exercising American options is derived. A numerical illustration of the procedure is also presented.

1. Introduction

In their papers on rational warrant pricing, Samuelson (1965) and McKean (1965) present a warrant valuation model which for the first time takes into account the non-negative extra value to the warrant-holder of the right to exercise a warrant at any time in the interval prior to its maturity. Following Samuelson's and McKean's studies, Chen (1969, 1970) derives a functional equation for the value of a warrant by applying a dynamic programming technique. The problem with these approaches as Chen recognizes in (1970), is that: 'in applying the warrant valuation model to compute the theoretical value of a warrant, the required rate of return on the warrant (i.e. the discount rate) and on the stock must be known.'

Black and Scholes (1973), in a seminal paper, present a market equilibrium option valuation model which has had fundamental implications for the valuation of corporate liabilities. This approach to option valuation does not require knowledge of investors' tastes (utility) nor their beliefs about the expected returns on the option or on the underlying common stock. The mathematical relationship between the option value and the value of the associated stock and

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time to maturity of the option is obtained by the arbitrage principle that in market equilibrium there are no riskless profits to be made with a zero net investment. A zero net investment portfolio is obtained by taking long and short positions on the stock, the option and the riskless asset. A partial differential equation governing the value of the option is the result of this analysis, which together with the boundary condition can be used under certain conditions to derive an analytical expression for the value of the option. The pricing formulae obtained depend for the most part, on observable variables.

To derive their option pricing formula Black and Scholes[1] assume 'ideal conditions' in the market for the stock and the option. These conditions are: (1) The short-term interest rate is known and constant through time. (2) The stock price follows a Geometric Brownian motion through time. Thus the distribution of possible stock prices at the end of any finite interval is log-normal. The variance rate of return on the stock is constant. (3) The stock pays no dividends or other distributions during the life of the option. (4) The option can only be exercised at maturity (European type). (5) 'Frictionless' markets exist, there are no transaction costs. The borrowing and the lending rates are equal. Borrowing and short selling are permitted without restrictions and with full use of the proceeds.

In a major extension of Black and Scholes' model, Merton (1973b) proves that their basic method of analysis also obtains under somewhat less restrictive assumptions. In particular he shows that in the case where the stock is assumed to pay continuous dividends, the hedging process described by Black and Scholes can also be applied and a different partial differential equation obtained. He points out, however, that, in general, this partial differential equation can not be solved by analytical methods (i.e., it does not have a closed form solution).

Merton (1973b) also shows that if a stock pays no dividends or the option is 'dividend payout protected', it will never pay to exercise an American option before maturity and, hence, the value of an American option is equal to the value of its European counterpart. But if the stock pays dividends and the option is not dividend protected, it may pay to exercise the American option before maturity because the option holder foregoes the dividend paid to the stockholder and, hence, its value may be greater than its European counterpart. Merton also shows that if a stock pays discrete dividends, it may pay to exercise an American option just before the stock goes ex-dividend, but never in between dividend payments dates.

Black (1974) proves that the value of an option on a stock that pays discrete dividends is also governed by the same partial differential equation derived by Black and Scholes (1973) for the no-dividend case. The boundary conditions, however, change at each dividend payment date to reflect the fact that it may

[1]See Black and Scholes (1973, p. 640). The term 'option' is used here to refer to a call option.

pay to exercise the American option at those points in time.[2] No closed form solution has been found in this case.

This paper develops a numerical procedure for valuing options[3] on dividend paying stocks. A general numerical solution to the partial differential equation governing the value of an option on a stock which pays discrete dividends is presented.[4] In addition, the optimal strategy for exercising American options is derived. For a sufficiently large value of the stock it may pay to exercise the American option at dividend payment dates. This paper shows how to determine the 'critical stock price' above which it will pay to exercise the option.

In section 2 the differential equation with its boundary conditions is derived and in section 3 the solution algorithm is given. Finally, section 4 presents a numerical example.

2. The model

Let

S = price of the common share at time t,
W = equilibrium value of a call option on the stock,
T = time to maturity of the option; note that $\mathrm{d}T = -\mathrm{d}t$,
$D(S, T)$ = amount of discrete dividend per share, assumed to be a deterministic function of S and T; to simplify the presentation and without loss of generality, we will assume a constant D paid at fixed intervals,
r = risk-free rate (assumed constant),
E = exercise price of the option,
α = instantaneous expected rate of return on the stock,
σ^2 = instantaneous variance of the rate of return on the stock.

Under mild assumptions Merton (1973a) shows that it is possible to write down the instantaneous rate of return on the stock (between dividend payment dates) as the stochastic differential equation

$$\mathrm{d}S/S = \alpha \mathrm{d}t + \sigma \mathrm{d}z, \tag{1}$$

where $\mathrm{d}z$ is a Gauss–Wiener process with zero mean and variance $\mathrm{d}t$.

Then, as shown by Black and Scholes (1973) and Merton (1973b), the partial

[2]This was also recognized by Merton (1973b).

[3]The option pricing model as developed by Black and Scholes (1973) and Merton (1973b) applies strictly only to call options where the net supply of securities is zero. This model, however, can be assumed to hold approximately for warrants in the case where the common shares corresponding to the warrants issued represent a small fraction of the total amount of common shares outstanding. To obtain an explicit pricing formula for warrants it is necessary to assume that the total value of the firm rather than the value of the common stock follows a Gauss–Wiener process. Work is currently being done in this area.

[4]The derivation for the case of continuous dividends payments can be found in Schwartz (1975).

differential equation governing the value of the option between dividend payment dates is

$$\tfrac{1}{2}\sigma^2 S^2 W_{SS} + rSW_S - W_T - Wr = 0. \tag{2}$$

The boundary conditions are given by:

(a) At maturity the value of the option will be equal to the exercise value, or zero if the former is negative,

$$W(S, 0) = \max\,[0, S-E]. \tag{3}$$

(b) At any time prior to maturity the option is worthless if the stock price is equal to zero,

$$W(0, T) = 0. \tag{4}$$

(c) At each date on which the stock goes ex-dividend a boundary condition is imposed to take into account the fact that the price of the stock will drop by the amount of the dividend and the fact that the option value for the ex-dividend stock price cannot be lower than the exercise value cum-dividend, because otherwise the option will be exercised.

Let T^+ and T^- denote the instants immediately before and after the stock goes ex-dividend respectively. (Recall that time to maturity, T, runs in the opposite direction to chronological time, t.) It is assumed that when the stock goes ex-dividend its price falls by the amount of the dividend, D. Then if the stock at time T^+ (cum dividend) is S, the option value at T^+, $W(S, T^+)$, is equal to the greater of the cum-dividend exercise value, $\max\,(0, S-E)$, and the ex-dividend option value $W(S-D, T^-)$. I.e.,

$$W(S, T^+) = \max\,[0, S-E, W(S-D, T^-)]. \tag{5}$$

Eq. (5) indicates that it may be profitable to exercise the option immediately before it goes ex-dividend if the value of the option, when the stock actually goes ex-dividend, is less than its exercise value prior to going ex-dividend. Fig. 1 shows the relationship between S and W at a dividend date.

The critical stock price, S_c, is defined here as the value of the stock cum-dividend for which the exercise value is equal to the option value when the stock goes ex-dividend. As can be seen in fig. 1, for values of the stock greater than S_c it will pay to exercise the option.

(d) When the stock price tends to infinity the partial derivative of the option value with respect to the stock price, W_S, tends to one,[5]

$$\lim_{S\to\infty} W_S(S, T) = 1. \tag{6}$$

[5]In between dividend dates $W > S-E$. Also $W < S$ and $W(S)$ is convex [see Merton (1973b)]. Therefore (6) obtains.

Partial differential equation (2) subject to boundary conditions (3), (4), (5), and (6) has no closed form solution. In the following section a general numerical solution is derived.

3. Solution algorithm

To solve partial differential equation (2) subject to its boundary conditions by numerical methods, partial derivatives are approximated by finite differences.[6]

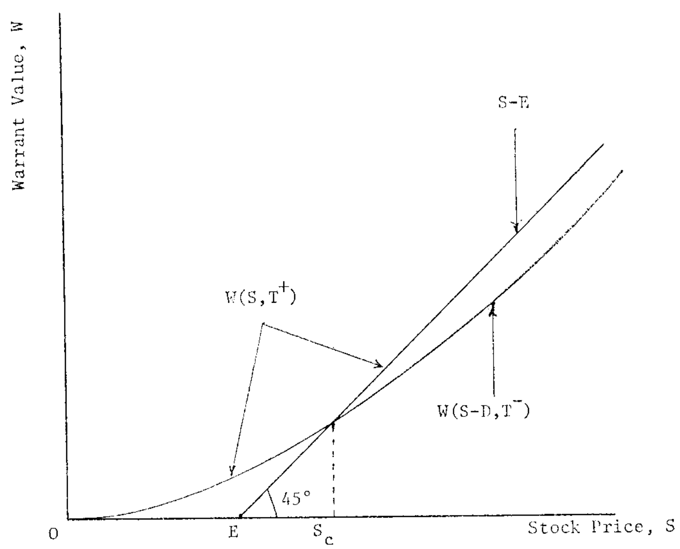

Fig. 1. Boundary condition at a dividend date.

The partial derivative of W with respect to S at the point (S, T), $W_S(S, T)$, can be approximated by

$$W_S(S, T) = [W(S+h, T) - W(S, T)]/h, \tag{7}$$

where h is the size of the discrete step in the value of S.

Eq. (7) is called the 'forward difference' approximation. $W_S(S, T)$ can also be approximated by

$$W_S(S, T) = [W(S, T) - W(S-h, T)]/h. \tag{8}$$

Eq. (8) is called the 'backward difference' approximation. In fig. 2, the slopes of the indicated chords represent the forward and backward difference approximations for $W_S(S, T)$.

[6]For a detailed explanation of the method, see McCracken and Dorn (1964) and Forsythe and Wasow (1960).

The numerical procedure requires that we consider a finite number of discrete values of the variables involved. Thus, for S we consider $n+1$ discrete values ($i = 0, \ldots, n$) such that the difference between two consecutive values is the step size h. Then, if the lowest value of S is zero we can use the following notation to represent the discrete values that S can take:

$$S_i = ih, \qquad i = 0, \ldots, n. \tag{9}$$

Likewise, we consider $m+1$ discrete values for T ($j = 0, \ldots, m$) such that the difference between two consecutive values is the step size in time to maturity of

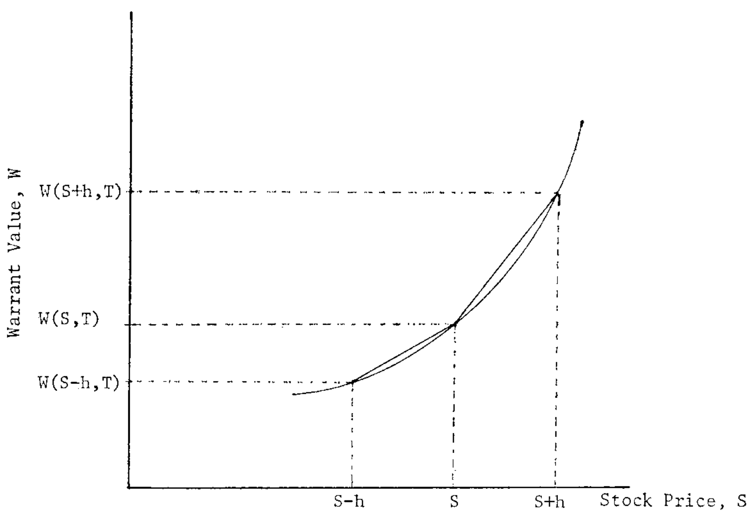

Fig. 2. Forward and backward differences. The slope of the chord between $[S, W(S, T)]$ and $[S+h, W(S+h, T)]$ is the forward difference approximation, and the slope of the chord between $[S-h, W(S-h, T)]$ and $(S, W(S, T)]$ is the backward difference approximation of $W_S(S, T)$.

the option, k. Then, if the lowest value of T is zero we can use the following notation to represent the discrete values that T can take:

$$T_j = jk, \qquad j = 0, \ldots, m. \tag{10}$$

Using this notation, the values of W for the discrete values of S and T can be written as

$$W(S, T) = W(S_i, T_j) = W(ih, jk) = W_{i,j}. \tag{11}$$

To obtain a better approximation of W_S we use the average between the forward difference (7) and the backward difference (8). Using the notation defined in (9), (10) and (11), W_S can then be written as

$$W_S = [W_{i+1,j} - W_{i-1,j}]/2h. \tag{12}$$

The approximation for W_{SS}, using a forward difference, is

$$W_{SS}(S, T) = [W_S(S+h, T) - W_S(S, T)]/h. \tag{13}$$

If forward differences are now substituted for W_S, the result would be biased in the forward direction. In order to avoid this effect, backward differences are used. Substituting backward differences for $W_S(S+h, T)$ and $W_S(S, T)$ in (13), simplifying, and using the notation introduced in (9), (10) and (11), W_{SS} can be written as

$$W_{SS} = [W_{i+1,\,j} - 2W_{i,\,j} + W_{i-1,\,j}]/h^2. \tag{14}$$

The partial derivative of W with respect to T is approximated by

$$W_T = [W_{i,\,j} - W_{i,\,j-1}]/k. \tag{15}$$

Notice that in (15) we use the backward difference for W_T because we want to relate the value of the option at time j to its value at time $j-1$.[7]

Substituting (9), (13), (14) and (15) into (2) we obtain

$$\tfrac{1}{2}\sigma^2(ih)^2 \frac{W_{i+1,\,j} - 2W_{i,\,j} + W_{i-1,\,j}}{h^2} + r(ih)\frac{W_{i+1,\,j} - W_{i-1,\,j}}{2h}$$

$$-\frac{W_{i,\,j} - W_{i,\,j-1}}{k} - rW_{i,\,j} = 0. \tag{16}$$

Rearranging terms we obtain

$$a_i W_{i-1,\,j} + b_i W_{i,\,j} + c_i W_{i+1,\,j} = W_{i,\,j-1}, \tag{17}$$

$$i = 1, \ldots, (n-1), \quad j = 1, \ldots, m,$$

where

$$a_i = \tfrac{1}{2}rki - \tfrac{1}{2}\sigma^2 ki^2,$$

$$b_i = (1+rk) + \sigma^2 ki^2,$$

$$c_i = -\tfrac{1}{2}rki - \tfrac{1}{2}\sigma^2 ki^2.$$

[7]As will be seen shortly, the numerical procedure works by solving a system of linear equations which give the values of the option at time j as a function of the values of the option at time $j-1$. Knowing the value of the option at maturity ($j = 0$) for different stock prices, the method proceeds backward in chronological time (increasing time to expiration) by a step wise process. Eq. (15), when introduced into (2), is the only expression that relates option values at different times to expiration.

By reducing the step sizes h and k any desired degree of accuracy in the solution can be achieved,[8] but at the expense of increased computational cost. m and n represent the number of steps in the time dimension and stock value dimension respectively; the former is chosen to correspond to the maturity of the option under consideration, while the latter must be sufficiently large for the boundary condition (6) to be well approximated at the maximum stock value considered.

The boundary conditions can be written in finite difference form:

(a) At expiration (3) can be written as

$$W_{i,0} = ih - E, \qquad \text{for} \quad i \geqq E/h, \tag{18}$$
$$W_{i,0} = 0, \qquad \text{for} \quad i < E/h.$$

(b) At any time prior to maturity (4) can be written as

$$W_{0,j} = 0, \qquad j = 0, 1, \ldots, m. \tag{19}$$

(c) On a dividend date the exercise option gives rise to (5) which can be written as

$$W_{i,j^+} = W_{i-D/h,\, j^-}, \qquad \text{for} \quad W_{i-D/h,\, j^-} \geqq ih - E, \tag{20}$$
$$W_{i,j^+} = ih - E, \qquad \text{for} \quad W_{i-D/h,\, j^-} \leqq ih - E.$$

The values of W_{i,j^+}, for $i = 0, \ldots, D/h$, are equal to zero because the stock (and the option) will be worthless after the payment of the dividend.

The critical stock price (if it exists), $S_c = zh$, is defined for the value of $i = z$, for which

$$W_{z-D/h,\, j^-} = zh - E, \tag{21}$$

or

$$z = (W_{z-D/h,\, j^-} + E)/h.$$

(d) For 'high' stock values (6) can be approximated by

$$W_{n,j} - W_{n-1,j} = h, \qquad \text{for} \quad j = 0, \ldots, m. \tag{22}$$

For any given value of j (17) constitutes a set of $(n-1)$ linear equations in $(n+1)$ unknowns, $W_{i,j}(i = 0, \ldots, n)$. The remaining two equations come from

[8]By setting $D = 0$, the numerical solution can be compared with the analytical solution obtained using the Black–Scholes formula. When this was done [see Schwartz (1975)], the solutions differed always by less than 0.3 percent for a stock price step of \$0.25 and time step of 0.25 months.

boundary conditions (19) and (22). The resulting set of $(n+1)$ linear equations enable us to solve for $W_{i,j}$ in terms of $W_{i,j-1}$. Since $W_{i,0}$ is given by (18) the whole set of $W_{i,j}$ may be generated by repeated solution of this set of equations, taking into account the boundary condition (20) imposed by the exercise option at each dividend date.

In this section we have shown how the numerical solution to the partial differential equation starts from the expiration date of the option where the boundary conditions are known and, in the spirit of dynamic programming, by a step wise process proceeds to compute the value of the option for different stock prices at increasing times to expiration.

4. Some numerical results

The American Telephone and Telegraph (ATT) warrant maturing on May 15, 1975 was selected as an example to illustrate the methods described in the preceding sections. This section reports the effects of variation in selected parameters on warrant valuation and on the relationship between critical stock prices and time to maturity.

The ATT warrant was issued in April 1970, entitling the holders to subscribe to 31,375,540 shares of common stock (which represented 5.71 percent of the outstanding number of shares) at a price of $52.00 a share beginning November 15, 1970 and up to May 15, 1975. The parameters for the ATT warrant, as of November 1970, are:

Variance rate	0.0017 per month,[9]
Risk-free rate	0.0637 per annum,
Quarterly indicated dividend	$0.65,
Exercise price	$52.00.

4.1. The early exercise issue

When the ATT warrants expired on May 15, 1975 only 3.1 million warrants (about 10 percent of the outstanding issue) had been exercised.[10] At the time of expiration the stock price was fractionally below the exercise price. The first question we address is whether or not the majority of investors behaved rationally in not exercising the warrants.

It will be recalled that an investor should exercise his warrant on an ex-dividend date only if the stock price exceeds some critical value. Using the above mentioned data this critical stock price for the ATT warrant is infinite for all dividend dates but the final dividend before expiration. This result should not be

[9]The variance of the monthly rate of return was computed using 60 monthly observations from July 1965 to June 1970.

[10]For details see the *Wall Street Journal* of May 19, 1975.

surprising. Merton (1973b) has shown that if a stock pays a constant continuous dividend rate, d, the sufficient condition for no premature exercising of the warrant is given by

$$d < Er. \tag{23}$$

With a dividend of \$2.60 per year (5 percent of the exercise price) condition (23) is satisfied for the continuous equivalent of our problem. Only on the last dividend date, one and a half months before expiration, does the discreteness of the problem become significant because the effective dividend rate increases substantially.[11,12]

The critical stock price for the final dividend date was \$54.75. At that time the stock was actually selling for \$49 so that the optimal strategy was clearly not to exercise since not only was the stock price below the critical stock price, it was also below the exercise price (\$52.00). It is to be presumed that the ten percent of the warrants actually exercised were exercised at an earlier period when the stock price exceeded the exercise price.

4.2. Comparative static analysis

To illustrate the comparative statics of critical stock prices, selected parameters of the ATT example were varied. Fig. 3 shows how the critical stock

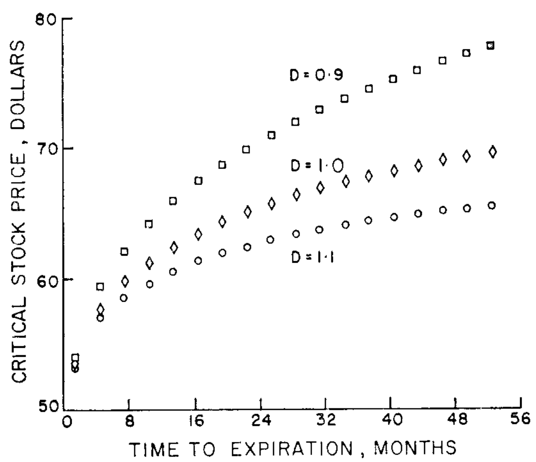

Fig. 3. Critical stock prices as a function of dividends and time to expiration. Variance rate = 0.0017 per month; risk free-rate = 6.37 percent per annum.

[11]This is because \$0.65 of dividend for one and a half months represents a dividend rate of \$5.20 per year or 10 percent of the exercise price.

[12]The increments used in the computation were \$0.65 for the stock price step and 0.25 months for the time step. Note that for accuracy the stock price step should be a factor of the dividend.

price varies as a function of time to maturity for three different assumed dividends. To obtain interesting results the value of the dividends selected are such that condition (23) is not satisfied. Were condition (23) satisfied the critical stock price would again be infinite except for the final dividend. Fig. 4 and fig. 5 show the same relationship between critical stock price and time to maturity for different assumed variance rates and interest rates, respectively.

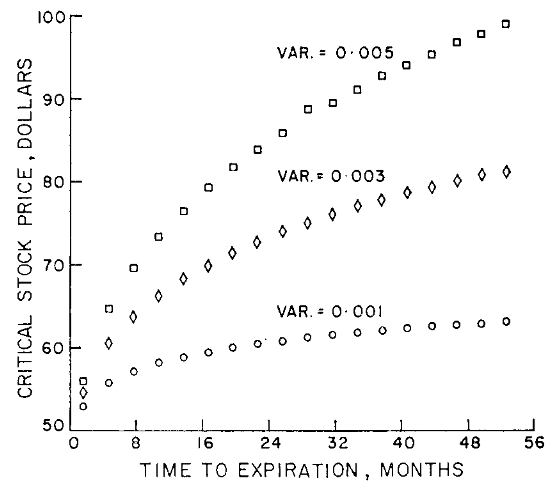

Fig. 4. Critical stock prices as a function of the variance rate and time to expiration. Risk-free rate = 6.37 percent per annum, dividends = $1.00 per quarter.

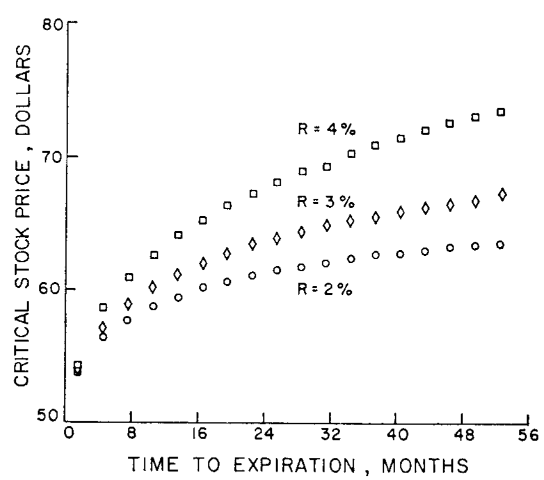

Fig. 5. Critical stock prices as a function of the riskless rate and time to expiration. Variance rate = 0.0017 per month, dividends = $0.65 per quarter.

The effect of dividends on warrant values is illustrated in fig. 6 for three different dividends 54 months prior to maturity (on November 15, 1970). For a dividend of \$1.1 per quarter the warrant curve approaches the exercise line because one and a half months later the warrant is exercised if the stock price surpasses \$65.38 (see fig. 3). Naturally, the warrant curve for a zero-dividend corresponds to the original solution obtained by Black and Scholes.

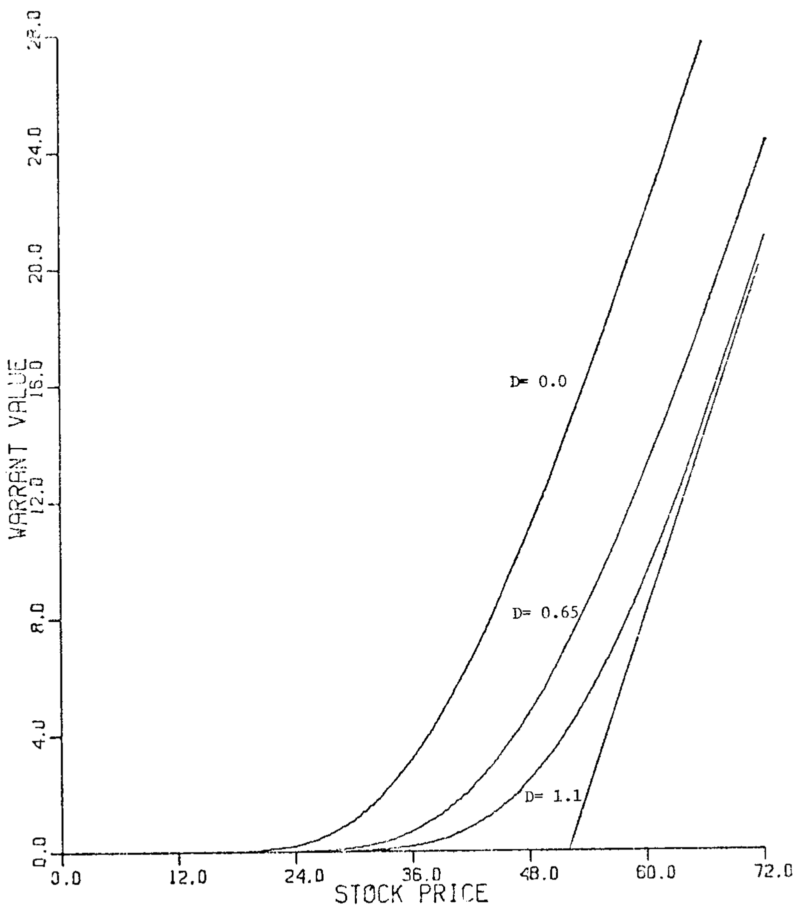

Fig. 6. Warrant values as a function of stock prices for different dividends. Variance rate = 0.0017 per month, risk-free rate = 6.37 percent per annum.

4.3. Comparison of market and model values

The final analysis on the ATT warrant consists of a comparison between the actual warrant price in the market and the theoretical value obtained by three different procedures:

(i) the numerical solution with discrete dividends presented in this study,[13]
(ii) the Black and Scholes formula, disregarding any dividend payment,[14]
(iii) the Black and Scholes formula, assuming a constant dividend yield (a dividend proportional to the value of the stock).[15]

To avoid the problem of establishing exactly when the stock goes ex-dividend, the warrant value is estimated using the three above mentioned methods and compared with the market price just midway between two dividend payments, during the period when exercising was permitted, November 15, 1970 to November 15, 1974.

For the warrant value estimates the actual riskless interest rate, the indicated quarterly dividend, and the closing stock price for the particular date were used; but the same variance rate estimated from the period 1965 to 1970 was used. These data are tabulated in table 1. In table 2 the warrant market price is compared with the three theoretical prices at the selected dates.

According to these results the two formulations that take into account dividend payments give consistently lower values than those in the market.

Table 1
Data used in the estimation of theoretical ATT warrant prices.

Date	Time to expiration (months)	Riskless rate (% per year)	Quarterly dividend ($)	Stock price ($)
Nov. 16, 1970	54	6.37	0.65	45.0
Feb. 16, 1971	51	5.31	0.65	52.375
May 14, 1971	48	6.02	0.65	47.125
Aug. 16, 1971	45	6.39	0.65	44.5
Nov. 15, 1971	42	5.50	0.65	42.25
Feb. 15, 1972	39	5.51	0.65	44.0
May 15, 1972	36	5.69	0.65	42.5
Aug. 15, 1972	33	5.92	0.65	41.875
Nov. 15, 1972	30	6.03	0.70	50.25
Feb. 15. 1973	27	6.61	0.70	51.125
May 15, 1973	24	6.78	0.70	53.125
Aug. 15, 1973	21	7.75	0.70	47.5
Nov. 15, 1973	18	6.96	0.70	47.5
Feb. 15, 1974	15	6.77	0.77	51.75
May 15, 1974	12	8.24	0.77	47.0
Aug. 15, 1974	9	8.64	0.77	42.35
Nov. 15, 1974	6	7.65	0.85	47.5

[13]In the solution procedure it was assumed that the actual indicated quarterly dividend would remain constant up to the expiration of the warrant.

[14]See Black and Scholes (1973).

[15]See Merton (1973b). The formula actually applies only to an European warrant because for high stock prices there is always a positive probability of premature exercising. The dividend yield used was computed by dividing the actual indicated dividend per year by the closing stock price for the particular date.

Table 2
Comparison between ATT warrant market price and three estimated values for selected dates.

Time to expiration (months)	Warrant prices ($)			
	Market price	Numerical method	B–S no dividends	B–S with dividends
54.000	8.000	3.270	8.580	2.600
51.000	11.500	6.090	12.550	5.380
48.000	9.675	3.720	8.730	3.090
45.000	9.000	2.630	6.710	2.120
42.000	7.125	1.430	4.210	1.090
39.000	7.675	1.810	4.750	1.460
36.000	6.875	1.310	3.610	1.030
33.000	5.675	1.090	3.000	0.850
30.000	7.500	3.890	7.590	3.500
27.000	7.250	4.400	8.000	4.040
24.000	6.675	5.420	8.960	5.070
21.000	4.875	2.440	4.640	2.200
18.000	4.750	1.930	3.630	1.750
15.000	4.375	3.410	5.540	3.180
12.000	2.750	1.240	2.280	1.120
9.000	1.500	0.160	0.340	0.130
6.000	1.250	0.570	0.950	0.510

It is interesting to note the similarity between the numerical solution and the Black and Scholes solution for constant dividend yield, although in all cases the former gives a value closer to the market price. The Black and Scholes solution without considering dividends is much closer to the market price.

There are three possible reasons which may account for the fact that model prices differ from market prices:

(i) The model is not an accurate description of the market. The assumptions made are too restrictive.

(ii) The model gives the 'right' values, but the market either 'under' or 'over-values' the option. If this is the case, there should be some profit opportunities.

(iii) The model is an accurate description of the market, but the historical variance used in the model is not the same variance used by the market to price the option.

This paper has offered a procedure for valuing call options on dividend paying stocks. Further empirical research is required to determine whether possible discrepancies are accounted for by model limitations or by market inefficiencies. The example presented in this section is intended only to be illustrative.

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