Numbers 101: Growth Rates and Interest Rates

A growth rate is a numerical measure of the rate of expansion or contraction of a quantity over time. Suppose Swift Sales Co. reported revenue of $100 million last year. This year, revenue grew by 12%. More precisely, revenue this year was $100+100\times0.12=112$ million. Normally we simplify the calculation slightly: if there is 12% growth, we know that the *growth ratio* is 1.12 and so we calculate this year's revenue as $100\times1.12=112$ million.

Consider a different problem. Suppose this year revenue is $252 million and last year is was $194 million. What was the growth rate? We obtain the *growth ratio* by dividing the ending quantity by the beginning quantity:

$$\text{growth ratio} = \frac{\text{ending quantity}}{\text{beginning quantity}} = \frac{252}{194} = 1.299\,.$$

The *growth rate* is the *growth ratio* less one. We often express rates as percentages—multiplying by 100 to move the decimal point to a more convenient place. Thus, the growth rate is $1.299-1=.299=29.2\%$. You may have learned that growth rates are gotten by looking at the ratio of the "growth" (the difference between the ending and beginning values) to the beginning value. This way of doing the calculation is not wrong, but requires an extra calculation. In the case above, the amount of growth is 252-194 = 58, so the growth rate is 58/194 = 0.299 = 29.9%.

Combining and Compounding

Suppose that we are given five years of revenue data (Table 1) for Swift Sales Co. and asked for the average annual rate of revenue growth. For each year we can calculate an annual growth rate just as we did in the section above. Note that the growth rate for 1996 cannot be calculated because we do not have 1995 data. The five years of data give us four growth rates.

One might think that the average annual growth rate is just the average of these four numbers. However, such a calculation is normally considered to be an error! The proper (or expected) procedure is to take the *geometric average* of the *growth ratios*. That is because growth compounds, it doesn't add. Grow at 20% for two years and the total growth ratio is **not** 1.4. Instead it is $1.2\times1.2=1.44$.

TABLE 1
Revenue and Growth Rate

Year	Revenues $ Millions	Annual Growth Rate
1996	176	
1997	185	5.11%
1998	180	-2.70%
1999	194	7.78%
2000	252	29.90%

A regular average of n numbers is obtained by summing them and dividing by n. By contrast, a geometric average of n numbers is obtained by multiplying them all together and taking the n^{th} root. For example, the geometric average of the three numbers 2, 4, and 6 is $\left[2\times4\times6\right]^{1/3}=48^{1/3}=3.63$.

The four multipliers in Table 1 are 1.0511, 0.973, 1.0778, and 1.299. To obtain the geometric average, multiply them all together and take the fourth root. You get $1.4318^{1/4}=1.0939$. That is the average *growth ratio*. So the average annual *growth rate* is 9.39%.

Actually, things are not really this complicated. The total growth ratio is simply the ratio of the last value of sales to the first value of sales—the intermediate values don't matter. To get the total growth ratio you only have to divide the ending revenue by the starting revenue. That is, 252/176 = 1.4318. So, we really don't have to multiply all the multipliers together—we only have to get the total multiplier by looking at the starting and ending values, then take its fourth root (four years of growth), and finally subtract 1.

Exercise: In 1990, Zapata's revenues were \$30 million. In 2000, revenues were \$300 million. What was the average annual growth rate in revenues?

Answer: Between 1990 and 2000 there are 10 years of growth. Take the 10th root of the overall multiplier: $\left[300/30\right]^{0.1}=10.^{0.1}=1.2589$. The average annual growth rate was 1.2589-1 = .2589 = 25.89%.

Monthly Growth Rates

In the above problems we have taken several years of data and asked for the average annual growth rate. A parallel problem is to work between monthly (or quarterly, or daily) growth rates and annual growth rates. For example, suppose we take Swift Sales' projected growth of 29.9% for the coming year. What is the implied monthly (compound) growth rate? The multiplier is 1.299 and we are asking for the number which when multiplied by itself 12 times gives 1.299: the twelfth root. Thus, $1.299^{1/12}=1.022\Rightarrow 2.2\%$. So, monthly growth of 2.2% compounds to 29.9% growth in a year.

From the above, it should become clear that working with growth rates involves learning to work with the growth multipliers (1+growth) and taking roots. The principle we just used to move between annual and monthly growth rates applies equally well to weekly or daily growth rates.[1]

Interest Rates

Interest rates, by the way, are growth rates. Everything just said so far about growth rates applies to interest rates. And what we are about to say in this section applies to growth rates as well.

A real interest rate is the growth rate of money invested or loaned at that interest rate. If a bank paid a real 6% interest rate, your deposits would grow at 6% per annum (ignoring taxes). If a bank gave you a loan at a 10% interest rate, the balance outstanding would grow at 10% per annum if you didn't make any payments to reduce the balance.

There is an extra complication that arises in the arithmetic of interest rates. The custom in banking and finance is to state "nominal" rather than real interest rates. An example should

[1] What happens when we divide time into very tiny parts? That is called continuous growth. It is that problem which produces the definition of *e*, the base of natural logarithms. If the annual growth rate is *g*, then the continuous rate of growth is $g_c=\log_e(1+g)$.

make this clear. A bank quotes a mortgage loan at a nominal 8% rate. Mortgage interest is computed monthly. What the bank really means is that they will charge interest of (8/12)% each month—a monthly interest rate of 0.667%. See the trick? They don't take the twelfth root; instead they divide by 12. In this case, the real annual rate of interest is

$$(1+0.00667)^{12}-1=0.08304\,.$$

So you are really paying 8.3% interest, not 8%. This is what credit card companies have to reveal when the fine print shows the APR interest (annual percentage rate).

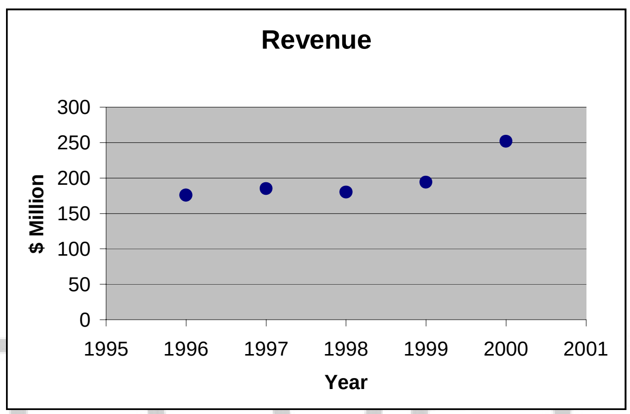
To be fair, the same thing happens on the deposit side. If the bank pays 5% interest, you have to look at the compounding. If it is compounded daily, even over weekends, then the real interest you are getting is

$$(1+0.05/365)^{365}-1=0.0513=5.13\%\,.$$

Estimating Multi-Period Growth Rates (advanced)

In the section on compounding (page 2) we calculated the average 4-year growth rate of the data in Table 1 by using the endpoints. That is the average growth rate was obtained by dividing the ending value by the starting value, taking the fourth root, and subtracting one. Now look at Exhibit 1. It shows a graph of the revenue data from Table 1.

EXHIBIT 1
Graph of Data From Table 1



The average growth gotten by using the end points is one way of working with this data. Another way is to fit a growth trend-line to the data. The difference is subtle but very real. The average growth ratio approach takes, works just with the endpoints. We have shown that this is equivalent to taking the geometric average of each year-to-year growth ratio. The growth trend approach, by contrast, assumes a *constant* underlying real growth rate. Each year's observation is seen as being that growing trend-line plus random error. In the first case, the growth rate are assumed to vary randomly from year to year. In the second case, the growth rate is assumed to be constant and the errors in observation are random.

Technically, to fit a growth trend line we need to take logarithms of the data and regress the logs against the years. The fitted relationship will have the functional form

$$R=AG^{t-1996}$$

Luckily, we can get all this done with one simple Excel worksheet functions. The worksheet function **LOGEST** does the job. It fits the above formula to the data using least squares, returning the estimated value of G. Applying this to the data in Table 1 we could write =LOGEST({176,185,180,194,252}) in the Excel spread sheet, or more simply

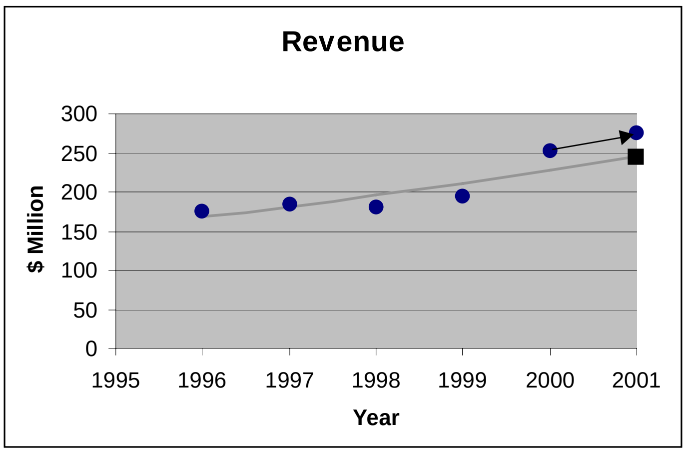
=LOGEST(C7:C11), where the range holds the data (See Exhibit 3). This function returns the value 1.0795, so the estimated constant growth rate underlying this data is 7.95%.[2]

Recall that on page 1 we noted averaging year-to-year growth rates was normally considered an error and that you get average growth rates by working with just the end-points. Well, LOGEST is *the right* way to approach averaging year-to-year growth rates. Instead of averaging the year-to-year growth rates, it averages the logarithms of the year-to-year growth ratios.

Note that this result is much smaller than the 9.39% gotten by working just with the end-points. Which is right? It depends upon which model you think best describes the real world process that generated the data in the first place. Is there a constant underlying growth trend and the data appear as errors off this trend line, or are the growth rates themselves real but random?

The difference between these two models shows up sharply when you are making a forecast. What will next year's sales be? Look at Exhibit 2. The upper forecast (dot) is gotten by taking the last data point ($252 million) and extrapolating from it at the average annual growth rate of 9.39%. The bottom forecast (square) is gotten by extrapolating the constant growth best-fit trend-line. The dot marks the value $276 million, the square marks the value $246 million.

EXHIBIT 2
TWO FORECASTS



<u>The lower forecast is actually below the previous year's data</u>! Even though we are looking at a growth process, we expect next year to be lower than this year. Not because growth has gone down, but because we think last year's observation was abnormally high. It is not that we think that last year's *growth* was abnormally high—we think the growth is always the same. We think that last year's error above the trend-line was abnormally high. So which model do you believe? Take it from me, if you use give the CEO the box forecast she won't like it! On the other hand, if it's your business, it's more important to be right than popular.

If you are working with stock-price data, there is a lot of research to indicate that you want to use the random growth model, NOT the constant growth model. That's what the term "random walk" refers to. The idea is that the stock price sums up all that is known—there is no "underlying" trend-line or other hidden state that helps us make better forecasts. However, if you are working with revenue data, it may well be that the constant growth model is better. Revenues are influenced each year by accidents, special orders, plant closings, strikes, etc. These can be thought of as random shocks that take revenue above or below its "natural" level. If that is the case, then the trend-line model could be better.[3]

[2] The function has other capabilities as well. It can do regressions with several multiplicative growth ratios and can return the constants and statistics as well. See Excel Help for details.

[3] You could compare the mean-square errors of these two forecasting methods. In Statistics you will learn how to set-up and work this kind of problem.

To get the extrapolated trend-line forecast we also use a simple Excel worksheet function. It is possible to work with the outputs of LOGEST and build the forecast, but it is easier to use the Excel function GROWTH. This function takes three array inputs: (1) the past known “y” variables (the things we think are growing, (2) the past known “x” variables (the year numbers or any series that increases by 1 each year (e.g., 1,2,3,4,…), and (3) the new “x” variables for which we want a forecast. Suppose the year data are in cells B7:B11 and the revenue data are in cells C7:C11 (See Exhibit 3). The function {=GROWTH($C$7:$C$11,$B$7:$B7$11,C7:C11)} will generate the best fit to the known data. In this case the “forecast” is for the past data. That is, the forecast “x” variables ($C$7:$C$11) are the same as the variables used in the fit. In other words, we are asking for the best-fit trend-line. Note that this is an *array function* and is entered by typing control-shift-enter to establish it as a function returning an array of results.

If you want the forecast, you have to feed the function new “x” variables outside the range of the old known “x” variables. The graph in Exhibit 2 was made with the function {=GROWTH($C$7:$C$11,$B$7:$B$11,C7:C12)}. Note that the last range specified now covers cell C12, which holds the number 2001. The array result extends to generate the forecast. If you only wanted the forecast and not the fitted data, you would use the specification =GROWTH(C7:C11,B7:B11,C12).

EXHIBIT 3

	B	C	D	E	F	G
3						
4				Constant	Constant	
5		Revenues	Annual	Growth	Growth	
6	Year	$ Millions	Growth Rate	Model	Trendline	
7	1996	176			168	
8	1997	185	5.11%		181	
9	1998	180	-2.70%		196	
10	1999	194	7.78%		211	
11	2000	252	29.90%	7.95%	228	
12	2001				246	
13						
14						
15						
16						
17						

=LOGEST(C7:C11)-1

=GROWTH(C7:C11,B7:B11,B12) [forecast]