Christian Dippel  
Anderson School of Management, University of California, USA

Abstract  
The Groseclose and Snyder (1996) model is one of the best-known models of vote buying in legislatures. Although the logic of the model is compelling, it is not clear that its key propositions, derived in a continuous set-up, hold in finite legislatures. This is an important issue because many real-world legislatures are small and should be modeled as finite in order to make predictions on coalition formation in them. This paper makes two contributions. The main one is to show with full generality that the key propositions in the Groseclose and Snyder model do carry through into finite legislatures. Secondly, it clarifies the role that parameter restrictions played in previous work on this question by Banks (2000) which was not fully general.

Keywords  
committee voting; legislatures; vote buying

1. Introduction  
In the Groseclose and Snyder (1996) (GS96 hereafter) model of vote buying, two rival lobbies $X$ and $Y$ compete over the votes of a committee. Lobby $X$ moves first to build up a coalition, which lobby $Y$ can then ‘invade’ with counter-bribes. The two main propositions of GS96 can be summarized as follows. First, if lobby $X$ wins, they will in equilibrium make all bribed members of their coalition equally expensive for the rival lobby to invade with a counter-bribe (the ‘Leveling’ proposition). Secondly, in most environments the winning lobby optimally bribes a majority that is larger than minimal winning (the ‘Super-Majority’ proposition). The GS96 model of vote buying is one of the most well-known models in the literature on vote buying in legislatures. It is frequently cited as an explanation for the perceived prevalence of super-majority winning coalitions in situations where vote buying may be at play (Grossman and Helpman, 2002;
Persson, 1998; Wiseman, 2004). The GS96 model setup has also been the point of departure for several other important theoretical papers on vote buying in legislatures, such as Diermeier and Myerson (1999) and Dekel et al. (2009).

However, the GS96 model setup has a continuum of voters while many important real-world committees and legislatures are small.\(^1\) The UN Security Council is a pertinent example of a committee that is very important but small and for which there is compelling evidence of vote buying (Kuziemko and Werker, 2006). The large literature on lobbying votes in congressional committees is another case in point (Carsey and Rundquist, 1999; Ferejohn, 1974; Ray, 1981; Rundquist and Carsey, 2002). Two slightly frivolous but nonetheless important examples are the International Federation of Association Football (FIFA) and the International Olympic Committee (IOC)\(^2\) for both of which there is ample anecdotal evidence that vote buying plays a big role in choosing the location of, respectively, the Soccer World Cup and the Olympics.\(^3\)

When checking observed patterns of coalition formation for consistency with theories of vote buying, it is therefore important to know whether the insights of the GS96 model apply to small legislatures. This was recognized in Banks (2000) which, to the author’s knowledge, is to date the only paper that applied the GS96 model setup to finite legislatures. To characterize the equilibrium in his paper, Banks (2000) made restrictive assumptions on the preferences of legislators and the budgets of the vote-buying lobbies which ensured that the equilibrium bribed coalition would be flooded, which meant that every coalition member receives a bribe in equilibrium. With these auxiliary assumptions, Banks (2000) made two novel propositions that further characterized the optimal bribe schedule in this class of models. The first was that a preference shift towards the winning lobby by the marginal coalition member always weakly increases the winning lobby’s optimal coalition size. The second was that a preference shift away from the winning lobby by the marginal non-coalition member always weakly decreases the winning lobby’s optimal coalition size. These propositions are summarized as ‘the optimal coalition size is weakly increasing in the value voters place on the winning group’s preferred alternative’.

Notwithstanding the elegance of Banks (2000), its key assumption of floodedness of the winning coalition is clearly at odds with reality. For instance, while the United States may well bribe some members of the UN Security Council, it is unlikely that either France or Britain receive any bribes. Banks (2000) recognized this limitation, stating that ‘the results presented give a fairly complete picture of super-majority bribery under certain assumptions’, and that ‘an open question is the extent to which the results survive the weakening of these assumptions [for which] the corresponding analysis of the lobby’s optimal behavior will be considerably more intricate’.

The contribution of this paper is to undertake this ‘corresponding analysis’ and characterize the equilibrium in the GS96 setup with finite legislatures with full generality, using only basic game-theoretic and geometric arguments. By way of doing so, this paper also clarifies the role that the floodedness assumption plays in Banks (2000). In particular, I show that two novel propositions by Banks hold only under the floodedness assumption and not in general. Intuitively, the reason is that floodedness shuts off any effect of the changes in preferences of infra-marginal coalition members, leaving only comparative statics on the preferences at the margin, i.e. by the marginal coalition member and the marginal non-member.\(^4\)
In the following, Section 2 lays out the model and replicates the key proofs in finite legislatures with full generality. Section 3 shows that the two additional propositions in Banks (2000) hold under the ‘floodedness’ assumption. Section 4 concludes.

2. The GS96 model in finite legislatures

There are two rival lobbies $X$ and $Y$ and a committee of legislators (a legislature) with $M$ members. The two lobbies play a two-stage game in which lobby $X$ is the first mover and lobby $Y$ is the second mover. Only the two lobbies are players in this game. Committee members act non-strategically in that they simply choose the action that maximizes their one-shot payoff. A bill is up for a proposal and it can either be passed or not be passed. Label these two outcomes as Pro and Anti. The bill gets passed if a specific share of committee members votes in favor of it. Denote by $k$ the fraction of votes needed to pass a bill (e.g. one-half). Lobby $X$ prefers outcome Pro and $Y$ prefers Anti. Lobbies have valuations $W_X > 0$ and $W_Y > 0$ for seeing their preferred outcome occur. (Their valuations for the alternative are normalized to zero.) Committee members do not have preferences over outcomes but over how they vote. 5 Member $i$ receives utility $u_i^p$ from voting in favor of the bill and $u_i^d$ from voting against it. The key preference parameter for an agent is $v_i = u_i^p - u_i^d$. In the absence of bribes, member $i$ votes in favor of the bill if $v_i > 0$ and against it otherwise. (The indifferent member is assumed to vote against the bill.)

Lobby $X$ first offers a bribe schedule $x = (x_1, \ldots, x_M) \in \mathfrak{H}^M_+$. This vector specifies a bribe-offer to each coalition member if they vote in favor. Then lobby $Y$ offers a bribe schedule $y = (y_1, \ldots, y_M) \in \mathfrak{H}^M_+$ if they vote against. All bribe offers are contingent on actions but not on outcomes so that if legislator $i$ votes in favor, they receive $x_i$ regardless of whether the bill gets passed or not. After both bribe schedules are offered, the game ends and legislators vote. Members trade off bribe offers and preferences: if legislator $i$ receives bribe offers $x_i$ and $y_i$, they will vote in favor if $v_i + x_i > y_i$ and against otherwise. Without loss of generality, agents are ranked in decreasing order of $v_i$. Members can be divided into Pro-voters and Anti-voters such that

$$P(x, y) = \{i : v_i + x_i > y_i\} \quad \text{and} \quad A(x, y) = \{i : v_i + x_i \leq y_i\}.$$ 

This finite extensive game can be solved sequentially for its subgame perfect equilibrium (SPE) by first considering $Y$’s optimal response to a given $x$. To this end, define as $\lfloor kM \rfloor$ the largest integer smaller than $kM$ (i.e. one vote short of the number of votes required to get the proposal passed). Suppose Lobby $Y$ offered no bribes and $X$ offered some bribe schedule $x$. Then $X$’s coalition would be $P(x, 0)$. If $P(x, 0)$ was a winning coalition, then $|P(x, 0)| - \lfloor kM \rfloor$ is the number of legislators that $Y$ would have to bribe to invade $X$’s coalition. Call $v_i + x_i$ the defence of an agent in $P(x, 0)$ – it is the amount $Y$ needs to pay that particular legislator to invade them. In addition, let $\lambda(x)$ denote the set of agents that receive bribes from $X$ and $\mu(x)$ the set of $|P(x, 0)| - \lfloor kM \rfloor$ agents in $P(x, 0)$ with the lowest defence. This is the set of agents that $Y$ will optimally invade. Lobby $Y$’s optimal invasion strategy is then characterized as follows:

**Proposition 1.** For any given $x$, if $\sum_{i \in \mu(x)} (v_i + x_i) < W_Y$, then $Y$’s best response is to pay $y_i = v_i + x_i$ to $i \in \mu(x)$ and zero to all other members. If $\sum_{i \in \mu(x)} (v_i + x_i) \geq W_Y$, then $Y$’s optimal bribe offer is $y_i = 0$ to all agents.
Proposition 1 follows directly from the fact that $Y$ will never spend more money than she must. It uniquely identifies $Y$’s their equilibrium strategy in any SPE.6

Given $Y$’s optimal invasion strategy in the second stage, we can now characterize $X$’s equilibrium bribe schedule. The two lobbies’ relative strengths depend on their budgets and on the distribution of the preference parameter. Following GS96 and Banks (2000), I focus on the more interesting cases where $W_X$ is sufficiently large for $X$ to win. Define $\hat{i} = \max\{i : v_i > 0\}$, the highest indexed agent in $P(0, 0)$. Then there are four qualitatively different environments:

1. $\hat{i} \geq kM$ and lobby $Y$ cannot afford to invade even if $x = 0$, $\sum_{i \in \mu(0)} v_i > W_Y$.
2. $\hat{i} \geq kM$ and $X$ needs to pay positive bribes to prevent invasion. However, they can prevent invasion by building only the defence of legislators in $\mu(0)$, $\sum_{i \in \mu(0)} (v_i + x_i) > W_Y$.
3. $\hat{i} \geq kM$ and $X$ needs to pay positive bribes to prevent invasion. They need to pay bribes to members outside of $\mu(0)$ to prevent invasion.
4. $\hat{i} < kM$.

Cases 2 and 3 are distinct because in case 2 every bribe-dollar translates into an additional dollar spent leads to an additional dollar of invasion cost for $Y$ (as long as $x_i \leq v_{[kM]} - v_i$). By comparison, bribes to agents in $P(0, 0) \setminus \mu(0)$ do not raise $Y$’s invasion cost at all and are therefore wasted while bribes to agents in $A(0, 0)$ incur recruitment costs before making the agents vote in favor of the proposal so that not every bribe-dollar leads to an additional dollar of invasion cost. In case 2, we therefore have $\lambda(x) \leq \mu(x) = \mu(0)$ and any bribe schedule that satisfies $\sum_{i \in \mu(0)} (v_i + x_i) = W_Y$ is an equilibrium.

Cases 3 and 4 are more involved and take up the remainder of this section. In both cases $\mu(x) \leq \lambda(x)$. If $\mu(x) < \lambda(x)$ (i.e. $X$ bribes more people than $Y$ needs to invade), $X$’s optimal bribe schedule is characterized in Propositions 2 and 3 below. There are also corner solutions where $\mu(x) = \lambda(x)$ which are discussed at the end of the section.
Proposition 2 (Leveling). In any SPE in which \(\mu(x) \subset \lambda(x)\), \(X\) will pay a ‘leveling’ bribe schedule where the defence \((v_{i} + x_{i})\) of all agents in \(\lambda(x)\) is the same. In that case, \(X\)’s bribe payments are \(x_{i} = \left(\frac{W_{Y}}{\mu(x)} - v_{i}\right)\) for agents in \(\lambda(x)\) and \(x_{i} = 0\) for all others.

Proof: Label as \(\bar{d}, d\) the highest and lowest defence of any agents in \(\mu(x)\). Then \((v_{i} + x_{i}) \leq \bar{d}\) \(\forall i \in \lambda(x)\) because any \(i\) with \((v_{i} + x_{i}) > \bar{d}\) never gets invaded by \(Y\) and \(X\) would therefore be paying more than they have to if they built defences higher than \(\bar{d}\). Also, \((v_{i} + x_{i}) \geq d\) \(\forall i \in \lambda(x)\) by definition of \(d\) and the fact that \(\mu(x) \subseteq \lambda(x)\). It follows that \(d_{i} \in [d, \bar{d}]\) \(\forall i \in \lambda(x)\). Now suppose that \(\bar{d} \neq d\). Then \(d_{i} = \bar{d}\) \(\forall i \in \lambda(x) \setminus \mu(x)\). Then \(X\) could lift the defence of one agent with \(d_{i} = \bar{d}\) by \(\epsilon\) and lower the defence of the other \(|\mu(x)| - 1\) agents in \(\mu(x)\) by \(\epsilon/(|\mu(x)| - 1)\) which would keep both their defensive cost and \(Y\)’s invasion cost unchanged. However, \(X\) could then also reduce their bribes to the \(|\lambda(x)| - |\mu(x)|\) other agents in \(\lambda(x)\) by \(\epsilon/(|\mu(x)| - 1)\) so that they would reduce their total defensive cost. This implies that any bribe schedule with \(\bar{d} \neq d\) can not be in equilibrium. If, on the other hand, \(\bar{d} = d\), then there is no deviation that would reduce defensive costs. That the optimal bribe payment is \(x_{i} = \left(\frac{W_{Y}}{\mu(x)} - v_{i}\right)\) follows because this sets the cost of invasion to \(W_{Y}\).

Proposition 3 (Optimal coalition size). The size of \(X\)’s equilibrium coalition is uniquely determined in every SPE for cases 3 and 4. An optimal coalition size can be determined as follows. \(X\) first builds a minimal majority (bare minimal in case 4 or equal to \(|\mu(0)|\) in case 3) and then incrementally expands that majority until the marginal savings in defence cost are weakly smaller than the marginal recruitment cost \(−v_{i}\) for an additional coalition member. 

Proof: First, note that in case 3 \(\mu(x) \subset \lambda(x)\) or we would be in case 2. In case 4, \(\mu(x) \subset \lambda(x)\) is always true (because \(X\) starts with a minority). Proposition 2 then implies that the bribe schedule is initially leveling. Before proceeding with the proof, note Figures 1 and 2 which provide important graphical intuition for Proposition 3. In case 3, the cost of defending a majority can be separated into three distinct parts that correspond to three distinct areas in Figure 1. Area \(A\) consists of bribe payments to \(P(x, 0) \setminus \mu(x)\), area \(B\) is equal to the bribes that \(Y\) needs to pay to invade \(X\)’s majority and area \(C\) consists of the recruitment cost of bribing legislators to switch from \(A(0, 0)\) (whose ‘defence’ is part of area \(B\)). For case 4, Figure 2 shows a fourth area \(D\), which is the recruiting cost to reach a bare majority. Area \(B\) is fixed (at \((W_{Y} - \sum_{i \in \mu(0)} v_{i})\) in case 3 and \(W_{Y}\) in case 4) for any coalition size. Area \(D\) is also fixed. Contrariwise, area \(A\) is decreasing in \(|P(x, 0)|\) (as new coalition members reduce the required defences for existing ones) and area \(C\) – the recruiting cost of additional members – is increasing in \(|P(x, 0)|\). Given a leveling bribe schedule \(x\) and the resulting \(\mu(x)\), define \(x^{'}\) as the least-cost bribe schedule such that \(|\mu(x^{'})| = |\mu(x)| + 1\). Under the leveling strategy implied by Proposition 2, the marginal reduction in area \(A\) from moving from bribe schedule \(x\) to bribe schedule \(x^{'}\) has an upper and a lower bound:

\[
\left(\frac{W_{Y}}{|\mu(x)|} - \frac{W_{Y}}{|\mu(x)| + 1}\right)\left(|\lambda(x)| - |\mu(x)|\right) \geq A(x) - A(x^{'})
\]

\[
> \left(\frac{W_{Y}}{|\mu(x)|} - \frac{W_{Y}}{|\mu(x)| + 1}\right)\left(|\lambda(x^{'})| - |\mu(x^{'})|\right)
\]
The upper bound is attained when all coalition members in \( \lambda(x) \) are also in \( \lambda(x') \) in which case the upper and lower bounds are equal. On the other hand, if \( \lambda(x) \setminus \lambda(x') \neq \emptyset \), then some legislators no longer need to be bribed if \( X \) expands their coalition in which case there is a lower bound to the savings from this. This lower bound is never attained because at least some savings pertain from not having to bribe \( \lambda(x) \setminus \lambda(x') \) but the savings (reduction in \( A \)) from this can be arbitrarily close to the lower bound if \( v_i \) is arbitrarily close to \( W_Y |\mu(x)| \). Let \( x'' \) be the least-cost bribe schedule such that \( |\mu(x'')| = |\mu(x')| + 1 \). The lower bound of \( A(x) - A(x') \) (which is \( (W_Y / |\mu(x)|) - (W_Y / (|\mu(x)| + 1))(|\lambda(x')| - |\mu(x')|) \)) is larger than the upper bound of \( A(x') - A(x'') \) (which is \( (W_Y / (|\mu(x)| + 1)) - (W_Y / (|\mu(x)| + 2))(|\lambda(x')| - |\mu(x')|) \)). This implies that the savings from an additional recruit are decreasing in the coalition size. By comparison, the increase in area \( C \) from an additional recruit is obviously increasing in \( |P(x, 0)| \) because agents are ranked in decreasing order of \( v_i \). Decreasing savings and increasing costs of recruitment at the margin are then sufficient for a unique optimum coalition size.\(^7\)

Propositions 2 and 3 fully characterize \( X \)'s equilibrium bribe schedule in cases 3 and 4 when \( \mu(x) \subseteq \lambda(x) \). However, we can imagine that the ‘algorithm’ described in Proposition 3 leads to a corner solution where recruitment can be sufficiently cheap such that area \( A \) in Figures 1 and 2 disappears, in which case the optimal coalition size is determined at this step because further recruiting will not induce any savings but additional costs.\(^8\) In this case, the bribe schedule need not necessarily be leveling. Instead, the optimal bribe schedule is the same as that described for case 2: pay \( x_i \leq v_j \lfloor kM \rfloor - v_i \) to agents in \( \mu(x) \) and \( x_i = 0 \) to all other agents subject to the constraint that \( \sum_{\mu(x)}(v_i + x_i) = W_Y \).

From the algorithm described in Proposition 3, it is intuitive that super-majorities (majorities with \( |P(x, 0)| > \lfloor kM \rfloor + 1 \)) will be the rule rather than the exception in this model. Exceptions occur when \( W_Y \) is very small or recruitment costs are very high. In addition, it is easy to verify that the size of \( X \)'s coalition is weakly increasing in \( W_Y \). These two claims are not shown formally here.

![Figure 1. Lobby X's optimal bribe schedule in case 3.](attachment:image.png)
3. Additional propositions in Banks (2000)

While Banks (2000) also showed that the two main propositions in GS96 held in finite legislatures, he did so under the restrictive assumption that $v_i$ and $W_Y$ would be parameterized such that $X$’s optimal winning coalition is flooded in the sense that all of its members receive bribes, $P(x, 0) = \lambda(x)$. In the context of Proposition $3$, this means that the upper bound in (1) is always attained, effectively turning off one margin in $X$’s optimizing algorithm. Banks therefore characterized the vote-buying equilibrium only for a very specific subset of the solution space.

Under this assumption, Banks (2000) proved two additional propositions: first, that a preference shift of the marginal bribed legislator toward $X$, that is an increase in $v_{|P(x,0)|}$, always weakly increases $X$’s optimal coalition size; and second, that a preference shift away from $X$ by the marginal non-bribed legislator, that is a decrease in $v_{|P(x,0)|+1}$, weakly decreases $X$’s optimal coalition size. Banks (2000) showed this to be true regardless of what happens to the preferences $v_i$ of any other legislators, a very surprising result.

Intuitively, Banks obtained these results because in his set-up the floodedness assumption not only keeps the second margin (savings in per-head defensive costs from additional recruits) constant for any additional recruit into $X$’s coalition but also makes this margin invariant to any changes $v_i$ of infra-marginal coalition members. Relaxing Banks’ assumptions allows both margins to vary for changes in the preferences $v_i$ of any legislator. This makes Banks’ propositions no longer hold. To see this with a counter-example, suppose we are in case $3$, where $v_{|P(x,0)|} < 0$ so that the legislator indexed by $|P(x, 0)|$ would not vote for $X$ without bribes. Now suppose that there is a preference shift such that $v_{|P(x,0)|}$ increases by a small amount but that this is accompanied by a very large increase in $v_i$ for all $i < \lfloor kM \rfloor$. Then we would switch into an environment of case $2$ where there are no savings in per-capita bribes to be had at all from additional recruits so that $X$ optimally does not waste bribes to convince legislators to change their voting behavior and instead only pays bribes to legislators in $P(0, 0)$ that would vote for them without bribes. On the other margin, we can imagine that a downward shift in $v_{|P(x,0)|+1}$
is accompanied by a much larger downward shift in $v_i$ for all $i < \lfloor kM \rfloor$ which would actually make the optimal coalition size larger than before. It is therefore not possible to generate comparative statics from changes in $v_{i(x,0)}$ and $v_{i(x,0)+1}$ alone that would hold regardless of any changes in $v_i$ of other legislators.

4. Conclusion

This paper’s main contribution is to characterize the optimal bribe schedule in the Groseclose and Snyder (1996) model of vote buying in committees in a finite legislature set-up with full generality, confirming the two main results of the model that there is a unique solution that will typically be characterized by a ‘leveling strategy’ and a super-majority winning coalition. A second contribution is to revisit some additional propositions made in work by Banks (2000) and to show how those propositions hinge on assumptions made on the environment which limit the analysis to the subset of the solution space where all members of a coalition in a legislature receive bribes at all times.

Acknowledgements

I am grateful to Martin Osborne and Benjamin Nyblade for insightful comments.

Notes

1. Only their last proposition, which shows that super-majorities should be a common outcome, pertains to a finite legislature but assumes a very specific form of the legislature’s preference distribution.
2. The FIFA committee has 23 members, the IOC committee 15.
4. The importance of the floodedness assumption is further illustrated by the fact that, with this assumption, Banks also provided a counterexample to one of the propositions of GS96 while, in response, Groseclose and Snyder (2000) demonstrated the validity of their original proposition under the assumption of ‘non-floodedness’.
5. For a discussion of this assumption, see Dekel et al. (2008, 2009) who argue that voters in committees and legislatures are best modeled as caring about how they vote while voters in elections are best modeled as caring about the outcome of the vote.
6. Note that there are ‘trivial’ positive equilibrium bribe offers in the SPE where $Y$ cannot win in which $Y$ makes positive offers to agents in $P(x,0)$ that are smaller than $x_i + v_i$, so that agents do not take them up and $Y$ still ends up making zero payments.
7. There could be a ‘trivial’ special case with two adjacent optimal coalition sizes because savings and costs from an additional member are exactly equal.
8. In case 4, area $A$ can only disappear when $\hat{i} = \lfloor kM \rfloor$ and $v_{\lfloor kM \rfloor}$ is large.

References