Possibility of dying as a unified explanation of
(i) why we discount the future,
(ii) get weaker with age, and
(iii) display risk-aversion

Bhagwan Chowdhry

January 26, 2007

1I thank Mark Garmaise and David Levine for reading the paper carefully and providing many useful comments. I am grateful to Eduardo Schwartz for help in formulating the problem elegantly in continuous time that allowed me to obtain the main results of the paper analytically. I also thank Michael Brennan, John Cochrane and Steve Ross for encouragement.
1 Introduction

A number of evolutionary theories have been proposed to explain the phenomenon of aging or senescence or why we get weak as we get older (see the review article by Gavrilov and Gavrilova, 2002). Economists have also begun to explore the biological basis of preferences, such as discounting of future consumption (Rogers, 1994), that are usually taken as primitive (see a comprehensive article by Robson, 2002). In this paper, I formulate a simple and parsimonious evolutionary model that shows that because most species face a possibility of dying because of external factors, called extrinsic mortality in the biology literature, it can simultaneously explain (i) why we discount the future, ii) get weaker with age,\(^1\) and (iii) display risk-aversion.

The central idea is as follows. I consider an organism that in principle can live and reproduce forever, \(i.e.,\) it faces no intrinsic mortality. The organism does, however, face external threats, such as predation, disease and cataclysmic events, that can potentially kill the organism because it does not possess infinite amount of resources to fight the external threats. Possibility of death directly implies that organism will discount the future. Moreover, if the organism has a flexibility in allocating some of its limited resources over time, then following an argument proposed in the Antagonistic Pleiotropy Theory of Williams (1957), the organism will disproportionately allocate resources to when it is young at the expense of having fewer resources when it is old. This is because it is more important to be alive when young, as being

\(^1\)Sozou and Symour (2003) analyzes interaction between aging and time preference.
alive when old is pre-conditioned on also being alive when young, whereas the converse is not true. Thus the implication suggesting why we age is nicely captured in Williams (1957):

...natural selection will frequently maximize vigor in youth at the expense of vigor later on and thereby produce a declining vigor (senescence) during adult life.

I show that the possibility of death also suggests that the organism will prefer a sure endowment of resources to an uncertain endowment of resources that has the same expected value. The intuition is that a decrease in resources has a disproportionately larger negative impact compared to an equal amount of increase in resources because the decrease is more likely to cause the organism to die and thereby permanently forgo the possibility of producing offspring in the future that will contribute to its genetic fitness. This intuition is similar to results in models of financial distress and bankruptcy in which firms with large amounts of intangible assets that might disappear in bankruptcy will tend to avoid financial distress by avoiding risky outcomes and by hedging.

2 The Model

Consider a living organism that maximizes its expected lifetime genetic contribution or genetic fitness:

\[ F_0 \equiv \max_{2^t} E \left[ \int_0^\infty f_i dt \right], \]
where \( x_t \) denotes the rate at which resources are expended by the organism to produce offspring and \( f_t \) represents the consequent rate of organism’s genetic contribution at any time \( t \geq 0 \). I want to make sure that risk-aversion is not induced by concavity in the production function and therefore I posit a simplest linear production function in which \( f_t \) depends only on the rate of expenditure \( x_t \), and is equal to the rate of expenditure, \( i.e., \)

\[
f_t = f(x_t) = x_t.
\]  

(1)

The mechanism that links expenditure of resources to production of offspring is through better health. In other words, when greater resources are spent by the organism, it leads to better overall health which in turn leads to a greater number of offspring that survive.

Suppose the organism is endowed with some initial stock of resources \( Z_0 \) to allocate over time. I do not, in this paper, analyze the mechanism by which the initial stock of resources is acquired. So, \( t = 0 \) in my formulation represents the time at which the organism can start reproducing. Essentially, what I am trying to formalize here is the notion that the organism has some flexibility, induced perhaps by genetic mutations over time, in allocating limited resources over time.

Notice that I have assumed no explicit discounting of future fitness in the organism’s objective function. However, I assume that the organism can die at any instant when it meets exogenous external threats (disease, predators, cataclysmic events etc.) that arrive randomly. Clearly, \( f_t = x_t = 0 \), if the organism is dead. The fact that the organism can die at any instant causes the organism to discount the future. With a linear production function
posed in (1), it will be optimal for the organism to expend all its initial resources $Z_0$ immediately to produce offspring unless being alive in the future conferred some additional advantage in terms of producing offspring in the future. I assume that in addition to the initial stock, resources are replenished continuously at a constant endowment rate of $\eta$ per unit of time. Thus, the level of resources at any given time, $Z_t$, follow the following equation:

$$dZ_t = (\eta - x_t)dt.$$ \hspace{1cm} (2)

The idea here is that as the organism consumes food and drink over time, it resuscitates and repairs the body, engenders cell growth, etc.

Let $\delta dt$ denote the probability of death over an infinitesimal period $dt$. Thus, $1-\delta dt$ represents the probability of surviving over the next infinitesimal period $dt$. I assume that $\delta$ depends on $x_t$. The organism is stronger if it expends resources at a greater rate and the stronger the organism is, the larger is the probability that it will survive the external threat and continue to live. Formally, I make the following assumption:

**Assumption 1**

$$\delta'(x_t) < 0$$ \hspace{1cm} (3)

which implies that the organism lowers its probability of death (increases its probability of survival) by expending resources at a greater rate $x_t$.

The organism’s maximization problem can thus be expressed as the following
dynamic optimization problem:  

$$\max_{x_t} \mathbb{E} \left[ \int_t^\infty x_s ds \right] \equiv F_t = \max_{x_t} x_t dt + [1 - \delta(x_t) dt](F_t + dF_t).$$  \hspace{1cm} (4)$$

It is easy to see that the fitness function $F$ will depend on the state variable $Z_t$ and thus $F_t = F(Z_t)$ and

$$dF_t = F'(Z_t) dZ_t = F'(Z_t)(\eta - x_t) dt.$$  

Substituting in the Bellman equation (4), we get

$$F_t = \max_{x_t} x_t dt + [1 - \delta(x_t) dt][F_t + F'(Z_t)(\eta - x_t) dt].$$

Rearranging terms, we get

$$F_t = \max_{x_t} x_t dt + [1 - \delta(x_t) dt][F_t + F'(Z_t)(\eta - x_t) dt] - \delta(x_t) F'(Z_t)(\eta - x_t) dt]$$

which, after cancelling $F_t$ from both sides and ignoring the last term

$$[\delta(x_t) F'(Z_t)(\eta - x_t) dt]$$

which is of second order, yields

$$0 = \max_{x_t} \left[ x_t + F'(Z_t)(\eta - x_t) - \delta(x_t) F(Z_t) \right] dt.$$

Though, a bang-bang solution for the above maximization problem, in which resources are expended instantly, is conceivable for some functions $\delta(x_t)$, I

\footnote{A formally equivalent formulation of the problem would be to posit the following stochastic process for $Z_t$:  
$$dZ_t = (\eta - x_t) dt - Z_t dq,$$

where $dq$ represents a jump process that takes on the value 1 with probability $\delta dt$ and value 0 with the probability $1 - \delta dt$. Applying Ito’s Lemma for jump processes and solving, is formally equivalent to the formulation presented here.}

\footnote{In general, $F$ may also depend on time $t$ but in our formulation the dependence on time enters only through the state variable $Z_t$.}

5
will restrict the analysis to those cases in which there is an interior solution to the maximization problem at least for some values of $Z_t$. A bang-bang solution for all values of $Z_t$ will imply that the organism expends all initial resources $Z_0$ immediately and thereafter expends only its endowment that is replenished at a continuous rate of $\eta$ per unit of time.

The first order condition for an interior maximum to the maximization problem is:

$$1 - F'(Z_t) - \delta'(x_t)F(Z_t) = 0.$$

The second order condition for an interior maximum is:

$$-\delta''(x_t)F(Z_t) < 0.$$

Thus I will assume that:

**Assumption 2**

$$\delta''(x_t) > 0$$

at least for some values of $x_t$.

This assumption ensures that an interior maximum exists.

**Proposition 1** The optimal rate of expenditure $x(Z_t)$ decreases over time and the probability of death at any instant $\delta(x(Z_t))$ increases over time.

**Proof**: Let $x(Z_t)$ denote the optimal policy function that satisfies the first order condition (6). Substituting the optimal policy function $x(Z_t)$ in (5), we get:

$$0 = x(Z_t) + [\eta - x(Z_t)]F'(Z_t) - \delta(x(Z_t))F(Z_t).$$  

6
Differentiating (8) with respect to $Z_t$ and using the Envelope Theorem:

$$0 = [\eta - x(Z_t)]F''(Z_t) - \delta(x(Z_t))F'(Z_t). \quad (9)$$

Differentiating the first order condition (6) with respect to $Z_t$ gives the following comparative statics result (suppressing the dependence of $\delta$ on $x(Z_t)$ and those of functions $F$ on $Z_t$ for notational clarity):

$$x'(Z_t) = \frac{-F'' - \delta'F'}{\delta''F}. \quad (10)$$

Now,

$$\frac{dx}{dt} = x'(Z_t)\frac{dZ_t}{dt}.$$  

Substituting from (10) and (2), we get:

$$\frac{dx}{dt} = \frac{-F'' - \delta'F'}{\delta''F}(\eta - x).$$

Substituting for $F''$ from (9), multiplying and dividing the above by $F$, we get

$$\frac{dx}{dt} = \frac{-F'}{\delta''F^2}[\delta F + (\eta - x)\delta'F].$$

Since (8) can be rewritten as $\delta F = \eta - [\eta - x](1 - F')$, substituting it above and simplifying, we get

$$\frac{dx}{dt} = \frac{-F'}{\delta''F^2}[(\eta - (\eta - x)(1 - F' - \delta'F)].$$

Substituting $1 - F' - \delta'F = 0$ from the first order condition (6), we get

$$\frac{dx}{dt} = \frac{-F'}{\delta''F^2} \eta < 0$$

since $F'$ is positive from (6) and the fact that $\delta'$ is negative from Assumption 1. As $x$ decreases over time, it follows that $\delta$ increases over time because it is decreasing in $x$. \[ \square \]
The result in Proposition 1 formally shows that it is optimal for the organism to expend more resources when young and less resources as it gets older. Since the level at which resources are spent also determines the probability of death and survival, it follows that a bias towards greater expenditure of resources when young also implies that the organism is healthier when it is young and gets weaker, in the sense that it is more likely to succumb to external threats and die at any instant, as it gets older.

**Proposition 2** The optimal rate of expenditure \(x(Z_t)\) exceeds the rate of replenishment \(\eta\):

\[
\eta < x(Z_t)
\]

and consequently the level of resources \(Z_t\) decreases over time.

**Proof:** If the rate of expenditure \(x(Z_t) < \eta\), and (from Proposition 1) it continues to fall over time, it cannot possibly be a maximizing strategy as it can be shown to be inferior to a feasible strategy of choosing \(x(Z) = \eta\). With \(x(Z_t) < \eta\), \(\delta(x(Z_t))\) will be strictly greater than \(\delta(\eta)\) and thus the lifetime genetic contribution will be smaller because the rate of expenditure \(x(Z_t)\) is strictly smaller than \(\eta\) at all times and the probability of death at any point is strictly larger.\(^4\) Clearly, \(x(Z) = \eta\) could not be an interior optimum either for any \(Z_t > 0\) because by expending a little more resources in any period, the organism can increase its lifetime production while at the same time decrease its probability of death.

\(^4\)The solution in which \(x(Z_t) < \eta\) will violate a technical transversality condition because \(Z_t\) will grow unboundedly large over time.
Since $\eta < x(Z_t)$ it follows from (2) that $Z_t$ depletes over time.\[\]

Since the expected life-span, and fitness at any instant $F$ is increasing in $Z_t$ which depletes over time from Proposition 2, it follows that as the organism gets older, its remaining life-span decreases monotonically.

The derivation of the results in Propositions 1 and 2 allows me to interpret the observed phenomenon of aging and senescence as an interior solution to a formal dynamic maximization problem. But importantly, I now show that the interior solution also implies an additional restriction on preferences that can be interpreted as risk-aversion.

**Lemma 1** At the interior maximum, the fitness function $F(Z_t)$ is concave in $Z_t$ and $-\frac{F''(Z_t)}{F'(Z_t)}$ increases over time.

**Proof**: From the first order condition (6), we know that $F'(Z_t) > 0$ which from (9) implies that

$$[\eta - x(Z_t)]F''(Z) > 0.$$ 

From Proposition 2, since $\eta - x(Z_t) < 0$, it follows that

$$F''(Z_t) < 0.$$ 

Rearranging (9), we get

$$\frac{F''(Z_t)}{F'(Z_t)} = \frac{\delta(x(Z_t))}{x(Z_t) - \eta}.$$ 

Since $\delta$ is increasing over time and $x$ is decreasing over time from Proposition 1, it follows that $-\frac{F''(Z_t)}{F'(Z_t)}$ increases over time.\[\]
The concavity of the fitness function $F(Z_t)$ allows me to interpret that the organism will be risk-averse. The first interpretation follows directly in a case in which the organism faced the choice of initial level of resources $Z_0$. The organism, if offered with two uncertain bundles of $Z_0$ with same expected values, will prefer the one with smaller variance. However, in the present formulation, once $Z_0$ is determined, the level of resources $Z_t$ evolve deterministically as long as the organism is alive because the endowment is replenished at a constant rate. I now show that allowing the replenishing endowment process to evolve stochastically with a process

$$\eta dt + \sigma dw$$

where $dw$ represents the standard diffusion process, and assuming a specific death probability function

**Assumption 3**

$$\delta(x_t) = \exp(-x_t)$$

implies risk-aversion in the following formal sense:

**Proposition 3** The concavity of the fitness function $F(Z_t, \sigma^2)$ in $Z_t$ implies that it is decreasing in $\sigma^2$, i.e.,

$$\frac{\partial}{\partial \sigma^2} F(Z_t, \sigma^2) < 0$$

as $\sigma^2 \to 0$.

**Proof**: For the stochastic process described in (12), equation corresponding to (8) is (suppressing the dependence of the optimal expenditure function $x$
and the fitness function $F$ on $Z_t$ and $\sigma^2$):

$$\delta F = \eta - [\eta - x](1 - F') + \frac{1}{2} \sigma^2 F''.$$  \hfill (14)

The first order condition (6), using (13), can be rewritten as:

$$1 - F' + \delta F = 0.$$  \hfill (15)

Substituting for $(1 - F')$ from (15) into (14) and rearranging we get:

$$\delta F = \frac{\eta + \frac{1}{2} \sigma^2 F''}{1 + x - \eta}.$$  \hfill (16)

Taking natural logs of (16) and differentiating with respect to $\sigma^2$, we get

$$-\frac{\partial x}{\partial \sigma^2} + \frac{1}{F} \frac{\partial F}{\partial \sigma^2} = \frac{\frac{1}{2} F'' + \frac{1}{2} \sigma^2 \frac{\partial}{\partial \sigma^2} F''}{\eta + \frac{1}{2} \sigma^2 F''} - \frac{1}{(1 + x - \eta)} \frac{\partial x}{\partial \sigma^2}.$$  

Letting $\sigma^2 \to 0$, and rearranging terms, we get

$$\frac{1}{F} \frac{\partial F}{\partial \sigma^2} = \frac{\eta}{1 + x - \eta} \frac{\partial x}{\partial \sigma^2}.$$  \hfill (17)

Notice that the concavity of the function $F$ implies that the first term on the RHS of the above equation is negative. We do not know the sign of $\frac{\partial x}{\partial \sigma^2}$ but we will see that if it were positive, it will not be big enough to make $\frac{1}{F} \frac{\partial F}{\partial \sigma^2}$ positive.

We first establish that for $Z = 0$, $x = \eta$ and thus

$$\frac{1}{F} \frac{\partial F}{\partial \sigma^2} = \frac{\eta}{\eta} < 0.$$  

Now consider the rate at which the above changes as $Z$ increases by considering the sign of

$$\frac{\partial}{\partial Z} \left[ \frac{1}{F} \frac{\partial F}{\partial \sigma^2} \right].$$
If it is negative, it implies that $F$ is decreasing in $\sigma^2$ for all values of $Z > 0$. However, if it is positive, it implies that

$$0 < \frac{\partial}{\partial Z} \left( \frac{1}{F} \frac{\partial F}{\partial \sigma^2} \right) = \frac{\partial}{\partial \sigma^2} \left[ 1 + x + \frac{1}{2} \sigma^2 F'' \right] = \frac{\partial}{\partial \sigma^2} \left[ \frac{F'}{F} \right]. \tag{18}$$

From (15) and (16), it follows that

$$F' = 1 + \frac{\eta + \frac{1}{2} \sigma^2 F''}{1 + x - \eta} = 1 + x + \frac{1}{2} \sigma^2 F'' \tag{19}$$

Dividing (19) by (16), we get

$$\frac{F'}{F} = \frac{1 + x + \frac{1}{2} \sigma^2 F''}{\eta + \frac{1}{2} \sigma^2 F''} \delta.$$

Taking the natural log of the above, we get

$$\ln \frac{F'}{F} = \ln(1 + x + \frac{1}{2} \sigma^2 F'') + \ln \delta - \ln(\eta + \frac{1}{2} \sigma^2 F'').$$

Differentiating with respect to $\sigma^2$ and letting $\sigma^2 \to 0$, we get

$$\frac{\partial}{\partial \sigma^2} \left[ \ln \frac{F'}{F} \right] = \frac{1}{1 + x} \left( \frac{\partial x}{\partial \sigma^2} + \frac{1}{2} F'' \right) - \frac{\partial x}{\partial \sigma^2} - \frac{1}{\eta} F''.$$

From (18), $\frac{\partial}{\partial \sigma^2} \left[ \frac{F'}{F} \right] > 0$, which implies that the LHS of the above equation must also be positive. Thus, rearranging the terms on the RHS of the above equation, it follows that

$$\frac{1}{2} F'' \left( \frac{1}{1 + x} - \frac{1}{\eta} \right) - \frac{\partial x}{\partial \sigma^2} \left( 1 - \frac{1}{1 + x} \right) > 0.$$

Rearranging terms, the above condition can be rewritten as

$$\frac{\partial x}{\partial \sigma^2} < -\frac{1}{2} F'' \frac{(1 + x - \eta)}{x}.$$
Multiplying both sides by \( \frac{x-\eta}{1+x-\eta} \), we get
\[
\left[ \frac{x-\eta}{1+x-\eta} \right] \frac{\partial x}{\partial \sigma^2} < -\frac{1}{2} \frac{F''(x-\eta)}{x}.
\]

Rearranging terms, we get
\[
\frac{\frac{1}{2} F''}{\eta} + \left[ \frac{x-\eta}{1+x-\eta} \right] \frac{\partial x}{\partial \sigma^2} < \frac{1}{2} \frac{F''}{x} < 0.
\]

Since from (17), the LHS of the above condition is \( \frac{1}{F} \frac{\partial F}{\partial \sigma^2} \), it follows that \( F \) is decreasing in \( \sigma^2 \) for all values of \( Z > 0 \).

3 Concluding Remarks

I proposed a simple evolutionary model in which the possibility of death precipitated by external causes such as predation, disease or cataclysmic events can simultaneously generate time preference, senescence and risk-aversion and provide testable restrictions among these. The model can be exploited, using numerical simulations perhaps, to understand variations in time-discounting, aging and risk-aversion, across species at a given point in time and variations across time for a given species, by varying exogenous parameters of the death probability function and of the stochastic process for endowment replenishment. Furthermore differences between genders could also be examined by positing different production functions. For instance, one might conjecture that the production function for females might be concave rather than linear resulting in greater risk-aversion and perhaps slower aging. I leave it to future research to explore these issues in greater detail.
References:


