PORTFOLIO OPTIMIZATION WITH MANY ASSETS:

THE IMPORTANCE OF SHORT-SELLING

Moshe Levy[*] and Yaacov Ritov[**]

May 2001

[*] School of Business Administration, The Hebrew University of Jerusalem, Jerusalem 91905, Israel. Tel: 972 2 5883219. Fax: 972 2 588 1341. mslm@mscc.huji.ac.il. On leave at UCLA Anderson School of Management, 110 Westwood Plaza, LA. CA. 90095-1481. lmoshe@anderson.ucla.edu.

[**] Department of Statistics, The Hebrew University of Jerusalem, Jerusalem 91905, Israel. Yaacov@mscc.huji.ac.il.

PORTFOLIO OPTIMIZATION WITH MANY ASSETS: THE IMPORTANCE OF SHORT-SELLING

ABSTRACT

We investigate the properties of mean-variance efficient portfolios when the number of assets is large. We show analytically and empirically that the proportion of assets held short converges to 50% as the number of assets grows, and the investment proportions are extreme, with several assets held in large positions. The cost of the no-shortselling constraint increases dramatically with the number of assets. For about 100 assets the Sharpe ratio can be more than doubled with the removal of this constraint. These results have profound implications for the theoretical validity of the CAPM, and for policy regarding short-selling limitations.

Keywords: *portfolio optimization, short-selling, CAPM.*

JEL Classification: *G11, G12, G18*.

I. Introduction

The Capital Asset Pricing Model (CAPM), which was developed by Sharpe [1964], Lintner [1965a], and Mossin [1966], is one of the cornerstones of modern finance. The CAPM is founded on Markowitz's mean-variance framework and on the assumptions of homogeneous expectations and no limitations on short-selling. Based on these (and other) assumptions, the CAPM derives a simple linear relation between risk and return, and predicts that the optimal mean-variance portfolio should coincide with the market portfolio.

As the CAPM allows for short-selling, it is possible, in principle, that the optimal mean-variance portfolio involves short positions in some stocks. However, this is in contradiction with the model's prediction that the optimal portfolio should coincide with the market portfolio. Thus, for the CAPM to be self-consistent, one must ensure that the optimal portfolio does not involve short positions. Indeed, several researchers attack this important problem by characterizing conditions that guarantee efficient portfolios with no short positions. Roll [1977] provides conditions which ensure that all positions in the global minimum-variance portfolio are positive (see also Rudd [1977]). Roll and Ross [1977] later show that while these conditions are sufficient, they are not necessary. Although Roll and Ross conclude that "…the prospect appears dim for general and useful qualitative results" (p. 265), by employing duality theory Green [1986] succeeds in finding general conditions ensuring the existence of a mean-variance efficient portfolio with no short positions.

While Green's conditions are intuitively appealing, empirical studies find that in typical mean-variance efficient portfolios many assets *are* held short. For example, Green and Hollifield [1992] compute the global minimum-variance portfolio for

different sets of 10 assets with empirically estimated parameters. They find that of the 90 sets of assets examined, 89 of the global minimum-variance portfolios involve short positions. Levy [1983] constructs the efficient frontier for a set of 15 stocks. He finds that throughout the efficient frontier 7-8 stocks, which constitute about 50% of the assets in the portfolio, are held short.[1] While several researchers suggest that these results may be due to measurement errors in the estimation of the assets' means and the covariance matrix (see, for example, Frost and Savarino [1986], [1988])[2], Levy shows that his results are robust even when he takes possible estimation errors into account, and when various estimation methods are employed. Are Levy's results coincidental, or are they general? If they are general, what is the reason for this result? Obviously, if it is a general result that a large proportion of the assets are held short in the mean-variance optimal portfolio, this constitutes a severe blow to the CAPM.

If the optimal mean-variance portfolio involves extensive short positions, this not only questions the self-consistency of the CAPM, but may also have very important practical policy implications. As many institutional investors are not allowed to hold short positions (either through explicit regulations or by the implicit threat of lawsuits), they are restricted to holding sub-optimal portfolios. How sub-optimal are these portfolios? In other words, what is the cost of the no-short restriction? (in terms of the Sharpe ratio, for example).

In this paper we empirically and theoretically investigate the properties of mean-variance efficient portfolios in markets with a large number of assets. Our main

[1] This is also consistent with the results of Pulley [1981], Kallberg and Ziemba [1983], and Kroll, Levy, and Markowitz [1984]. Jagannathan and Ma [2001] report similar results for the global minimum variance portfolio.

[2] See also Ledoit [1996], [1999] and Ledoit and Santa Clara [1998] for a discussion of covariance matrix estimation.

results are:

(1) When the number of assets is large, the proportion of assets held short in mean-variance efficient portfolios typically converges to 50%.

(2) The investment proportions are extreme: a small number of assets are held in large positions (long or short). (This is consistent with the findings of Green and Hollifield [1992]).

(3) The investment proportion in each stock is not directly related to any of the stock's intrinsic properties such as its mean, variance, or its average correlation (or covariance) with the other stocks. Rather, it depends on the exact composition of the market.

(4) The cost of the no-shortselling constraint is extremely high. For portfolios with many assets, the Sharpe ratio can be more than doubled by relaxing this constraint.

Result (1) implies that the optimal mean-variance portfolio can not coincide with the market portfolio, and therefore the CAPM can not be self-consistent. This problem is different in essence than previous criticisms of the CAPM, such as its testability (Roll [1977])[3], or the validity of its underlying assumptions (in particular the homogeneous expectation assumption, see Levy [1978], Merton [1987], and Markowitz [1991]). Result (1) implies that even if the model's assumptions *do* hold perfectly, in large markets the CAPM can not possibly hold because of this *theoretical*

[3] See also the discussion in Roll and Ross [1994] and Kandel and Stambaugh [1995].

internal inconsistency[4]. While there is an ongoing debate as to the empirical validity of the CAPM (see, for example, Lintner [1965c], Black, Jensen, and Scholes [1972], Miller and Scholes [1972], Levy [1978], Amihud, Christensen and Mendelson [1992], Fama and French [1992], and Jagannathan and Wang [1993]), according to result (1) the CAPM can not hold even in theory. Result (2) implies that the optimal mean-variance portfolio is very different than portfolios constructed by "naive diversification" strategies, such as the 1/n heuristic described by Benartzi and Thaler [2001]. Thus, naive diversification may be very sub-optimal; the theory of portfolio optimization is therefore of great practical importance. Result (3) implies that there is no straightforward way to characterize a stock as "good" or "bad" in the context of large portfolios. Even if a stock has a high expected return, a low variance, and a low average correlation with all other stocks, when the number of assets is large, there is no guarantee whatsoever that this stock will be held long in the optimal portfolio. Result (4) implies that the restriction on short-selling is *extremely* costly. This result has very important implications for policy-makers setting the regulations concerning short-selling.

The structure of the paper is as follows: The next section describes empirical results about mean-variance efficient portfolios. We find that approximately half of the assets are held short, and that investment proportions are extreme. Moreover, it is difficult to explain the proportion of a stock in the optimal portfolio in terms of any

[4] Of course, to avoid this problem one can find the efficient frontier and the optimal portfolio under the restriction of no shortselling. However, in this case the problem becomes analytically complicated and one needs to employ the critical-line algorithm developed by Markowitz [1956], [1987] (see also Elton, Gruber, and Padberg [1976], [1978], Alexander [1993], and Kwan [1997]). More importantly, when short-selling is not allowed the CAPM's linear risk-return relationship breaks down (see Markowitz [1990], Tobin [1990], and Sharpe [1991]). Our results are derived for general sets of parameters, thus, one could argue that the equilibrium parameters are endogenously determined by prices such that the optimal portfolio weights are all positive. In section V we show that this approach can not ensure positive portfolio weights, and therefore does not solve the CAPM's internal inconsistency problem.

simple characteristic of the stock. The cost of the no-short-sell constraint is analyzed in terms of the Sharpe ratio. Section III provides an intuition and a mathematical explanation for the results. Section IV discusses the robustness of the results to estimation errors. Section V concludes and discusses the implications of the results to the theoretical validity of the CAPM, and to policy-making regarding limitations on short-selling.

II. Empirical Properties of Mean-Variance Efficient Portfolios

In this section we describe the main properties of empirical mean-variance efficient portfolios. We conduct this analysis by estimating the assets' parameters from the Center for Research in Security Prices (CRSP) monthly returns file, from the period January 1979 to December 1999. We randomly selected firms from the set of all CRSP firms, and then retained 200 of those firms selected with complete records over the entire period.[5] In our analysis we take the monthly risk-free rate as 0.32% (for an annual rate of 3.9%, see Ibbotson [2000]). However, the results reported below do not depend on the specific value of the risk-free rate. In this section we employ the sample estimates and do not consider sampling errors or shrinkage methods. These issues are discussed in section IV.

Result (1): Percentage of Assets Held Short

Figure 1 shows the percentage of stocks held short in the optimal portfolio as a function of the number of stocks in the portfolio, N. For each N we randomly draw a

[5] This is similar to the procedure employed by Green and Hollifield [1992]. However, as Green and Hollifield construct portfolios of up to 50 stocks, they require only a five year period of complete monthly records (in order for covariance matrix estimated from historical data to be non-singular, the number of time periods over which returns are observed must exceed the number of assets). We take a twenty-year period because we construct portfolios of up to 200 stocks. While this introduces a survivorship bias, we do not believe that this bias plays any significant role in our empirical analysis, and we obtain similar results in simulations where the expected returns and covariances are drawn randomly from some distributions, as described in the next section, rather than estimated empirically.

sub-set of N stocks out of our set of 200 stocks, we calculate the mean-variance optimal portfolio for this sub-set, and we record the number of assets held short in this portfolio[6]. We repeat this 10 times for each N (each time with a different sub-set of N stocks). Figure 1 reports the average proportion of stocks held short for each value of N. As the figure shows, the percentage of stocks held short in the optimal portfolios approaches 50% as the number of assets increases. This result is consistent with Levy [1983], who finds that in his sample of stocks only 1 stock on average is held short when the portfolio is constructed from 5 stocks, but when the portfolio is constructed from 15 stocks 7-8 stocks are held short.

(Insert Figure 1 About Here)

Result (2): Extreme Investment Proportions

While result (1) shows that about 50% of the stocks are optimally held short, we also find that some of these positions are rather extreme. We calculate the optimal mean-variance portfolio for the set of all of the 200 stocks and construct the distribution of portfolio weights. Figure 2 reports this distribution. The heavy line is the empirical distribution[7]. The light line is the best normal fit. As Figure 2 demonstrates, the normal distribution is a very good approximation for the

[6] Different assumptions regarding the terms of short selling are made by different models. The original CAPM, as well as Black [1972] and Merton [1972] assume that the short seller does not have to put up any initial margin and can use the short proceeds. Lintner [1965b] suggests that an amount of money equal to the short proceeds is put up as margin, with the short seller receiving the riskfree interest rate on both the short proceeds and the margin. He shows that when borrowing is allowed this framework is identical to having no margin requirements in terms of portfolio optimization. Dyl [1975] assumes that the short seller puts up initial margin that is less than the short proceeds, but receives no interest on the margin or on the short proceeds. Here we adopt the framework of the original CAPM, Lintner, Black, and Merton, for the sake of simplicity. See Markowitz [1987], Price [1989] and Weiss [1991] for a description of shortselling procedures in practice, and Alexander [1993] for an excellent review of alternative modeling assumptions and a unifying modeling framework for shortselling.

[7] The empirical distribution is obtained by employing a non-parametric density estimate with a Gaussian kernel and the "normal reference rule," (see Scott 1992, pg. 131).

distribution of optimal portfolio weights. This distribution implies that while most stocks are held in small proportions, a few stocks are held in very large positions (both long and short). This result is consistent with the findings of Green and Hollifield who analyze the portfolio weights in empirical minimum-variance portfolios. For portfolios of 50 assets they report an *average absolute* value of portfolio weights of up to 24%(!) (while "naive" diversification leads to weights of only 2%). The finding of extreme portfolio weights is in sharp contrast to notions of "naive" diversification.

(Insert Figure 2 About Here)

Result (3): Portfolio Weight and Stock Characteristics

What determines which stocks are held long and which are shorted in the optimal portfolio? In other words, what are the characteristics of a "good" stock, which one would like to hold long in the portfolio? While it is well-known that when stocks are correlated the optimal weight of each stock is affected by the parameters of all the other stocks (for example, see Merton [1972], Levy [1973], Roll [1977], and Stevens [1998]), it would seem intuitive that high mean return, low variance, and low correlations with the other stocks are desirable characteristics, and that stocks with these characteristics will *tend* to be held long. While this intuition may be helpful for small portfolios, when considering portfolios of many assets this intuition is misleading. When the number of assets in the portfolio is large, the very large number of cross-interactions are dominant, and there is no simple way to characterize "good" stocks, or to predict which stocks will have positive weights. Figure 3a shows the relationship between stocks' expected returns and their portfolio weights, in the mean-variance optimal portfolio of the 200 stocks. There is no clear relationship between

expected return and portfolio weight. Similarly, Figure 3b shows that low standard deviation does not imply positive portfolio weight. It is interesting to note, though, that small standard deviation generally implies more extreme positions in the stock (positive or negative). Finally, as Figure 3c shows, even average correlation with the other stocks is not related to the optimal portfolio weight (nor is the average covariance). While the figures show that the optimal investment proportion does not seem related to expected return, standard deviation, or average correlation when each of these factors is considered separately, one may suspect that a combination of these three factors may better explain the portfolio weight. However, multivariate regression of portfolio weight on these three variables reveals an R^2 of only 0.042 (adjusted R^2 of 0.027).

(Insert Figure 3 About Here)

Result (4): The Cost of the No-Shortselling Constraint

While the optimal mean-variance portfolio in a market with many assets involves short positions in about 50% of the assets, one may argue that this result is not necessarily significant in an economic sense, because it may be possible that there is a portfolio with no short positions which is only slightly sub-optimal. In order to address this issue of the economic cost of the short-selling restriction we calculate the Sharpe ratio of the mean-variance optimal portfolio (with shortselling), and compare it with the Sharpe ratio of the optimal portfolio constructed from the same assets, but under the restriction of no shortselling. We make the Sharpe ratio comparison for various portfolio sizes, N, where N is the number of assets in the portfolio. For each N we randomly select N stocks out of the 200 stocks, and we calculate the optimal

portfolios with and without shortselling, and their Sharpe ratios. Figure 4 reports the results.

(Insert Figure 4 About Here)

For portfolios with relatively few stocks, the difference in the Sharpe ratios is not very big, consistent with Sharpe's observation regarding the no-shortselling constraint that "… magnitudes of the departures from the implications of the original CAPM might be small" (see Sharpe [1991], p. 505). However, as the number of assets in the portfolio increases, the Sharpe ratio of the unrestricted portfolios grows at an almost steady rate, while the Sharpe ratio of the portfolios with the no-shortsell constraint almost levels off. For portfolios of a little over 100 stocks the Sharpe ratio can be more than doubled by removing the no-shortsell constraint. This is clearly a tremendous economic difference.

III. Explanation of the Results

This section provides a mathematical and intuitive explanation for the empirical results reported in previous studies and in the preceding section. These results are shown to be a general property of mean-variance optimal portfolios when the number of assets is large. Graphical analysis may provide some insight and intuition. Figure 5 shows Markowitz's mean-variance plane, and the efficient frontier derived for a set of N stocks. Now, suppose that we add a new stock, stock N+1. One can think of the possible portfolios of the N+1 stocks as combinations of portfolios of the N "old" stocks and the new stock. In order for the new stock to have a non-zero portfolio weight in any efficient portfolio, one of these combinations must improve the "old" efficient frontier. However, notice that if no shortselling is allowed and most covariances are positive, combinations of the new stock with existing portfolios are most likely to be interior to the frontier (see solid lines in Figure 5). Thus, if no

shortselling is allowed and there are many assets in the market, adding one more asset does not typically extend the efficient frontier, and the weight of this newly added stock in efficient portfolios will typically be 0. This is consistent with the results of Figure 4, showing that when shortselling is not allowed, at some stage the Sharpe ratio almost does not increase as more stocks are being added.

(Insert Figure 5 About Here)

However, when shortselling is allowed, the situation is very different. In this case, one can extend the efficient frontier by considering combinations of the new stock with other portfolios, in which either the new stock, or the portfolio of "old" stocks are held short. As Figure 5 shows, for a "typical" stock both cases are similarly likely (see dotted lines). Thus, for a given riskfree rate, we would expect that every newly added stock has roughly the same probability of being positively or negatively weighted. Hence, when the number of stocks is large, we can expect efficient portfolios to have about half of the stocks held short, as indeed observed in Figure 1.

For a mathematical explanation of the empirical findings described in the previous section, one has either to assume a specific covariance matrix, or alternatively to derive statistical results for certain covariance matrix classes. In what follows we take both approaches. First, we analyze the equal pairwise correlation case advocated by Elton and Gruber [1973] and Schwert and Seguin [1990], and we prove that this covariance structure generally leads to about half of the assets being held short and to extreme portfolio weights. In the second approach we consider covariance matrices drawn at random from some ensemble, and prove that the Sharpe ratio increases indefinitely with the number of assets when shortselling is allowed, but levels-off when shortselling is not allowed.

Equal Correlations

Consider the Elton-Gruber [1973] case in which all pairwise correlations are identical. For simplicity, assume first that all variances are identical. For N stocks, the covariance matrix in this case is an NxN matrix with σ^2 on the diagonal and $\rho\sigma^2$ elsewhere. The inverse of this matrix is $a = \frac{1}{\sigma^2}\frac{1+(N-2)\rho}{1+(N-2)\rho-(N-1)\rho^2}$ on the diagonal, and $b = \frac{1}{\sigma^2}\frac{-\rho}{1+(N-2)\rho-(N-1)\rho^2}$ elsewhere. Denoting excess returns by μ, the (unscaled) investment proportion of stock i in the optimal portfolio is: $\omega_i = a\mu_i + \sum_{\substack{j=1 \\ j\neq i}}^{n} b\mu_j = (a-b)\mu_i + Nb\bar{\mu}$, where $\bar{\mu} = \frac{1}{N}\sum_{j=1}^{N}\mu_j$ is the mean value of the assets' expected returns (for the scaled proportions one has to divide ω_i by $\sum_{j=1}^{n}\omega_j$).

Writing a and b explicitly, we find that an asset is held short if $\mu_i < \frac{\bar{\mu}}{1+((1-\rho)/N\rho)}$.

For small values of N this condition may not hold for any of the assets. However, when the number of assets is large, each stock is held short if its expected return is slightly smaller than the average expected return $\bar{\mu}$ (and held long if it is larger than this average). If the distribution of excess returns is not very skewed, this implies that about half of the stocks are held short in the optimal portfolio. Notice that this result holds for any value of the correlation ρ, as long as it is not 0 or 1.

In addition, the portfolio weights are extreme in the sense that they do not become smaller as the number of assets increases. To see this, recall that the scaled portfolio

weights are given by $\frac{\omega_i}{\sum_{j=1}^{n}\omega_j}$. As $N \to \infty$ ω_i converges to $\frac{\mu_i - \bar{\mu}}{\sigma^2(1-\rho)}$. The denominator, $\sum_{j=1}^{n}\omega_j$, converges to $\frac{\bar{\mu}}{\sigma^2\rho}$. Thus, the scaled portfolio weight of asset i converges to $\frac{\mu_i - \bar{\mu}}{\bar{\mu}}\frac{\rho}{1-\rho}$, and does not get small even when the number of assets becomes very large. For example, if $\rho = 0.5$ and a stock has an expected return 20% higher than the average expected return, this stock will have a weight of 20% in the optimal portfolio, even if there are thousands of other stocks in the portfolio.

In the case where stocks have different variances, the arguments are very similar. In this case the covariance matrix can be written as:

$$C = \begin{bmatrix} \sigma & 0 & \cdots & & 0 \\ \vdots & \sigma_2 & & & \\ & & \ddots & & \vdots \\ & & & & 0 \\ 0 & & \cdots & 0 & \sigma_n \end{bmatrix} \begin{bmatrix} 1 & \rho & \cdots & & \rho \\ \rho & 1 & & & \\ \vdots & & \ddots & & \vdots \\ & & & & \rho \\ \rho & & \cdots & \rho & 1 \end{bmatrix} \begin{bmatrix} \sigma & 0 & \cdots & & 0 \\ \vdots & \sigma_2 & & & \\ & & \ddots & & \vdots \\ & & & & 0 \\ 0 & & \cdots & 0 & \sigma_n \end{bmatrix}$$

and when N is large stock i will be held short if $\frac{\mu_i}{\sigma_i} < \frac{1}{n-1}\sum_{\substack{j=1 \\ j\neq i}}^{n}\frac{\mu_j}{\sigma_j}$. Again, if the distribution of $\frac{\mu}{\sigma}$ is not very skewed, we would expect about half of the assets to be held short[8].

Kandel [1984] shows that for any set of N-1 assets one can mathematically construct an Nth asset such that the mean-variance optimal portfolio is positively

[8] As σ is bounded by 0, if the distribution of $\frac{\mu}{\sigma}$ is skewed, it is probably positively skewed, implying that even *more* than 50% of the assets will be held short.

weighted (see Theorem 1, p. 67 in Kandel [1984], and Green and Hollifield [1992] p. 1066). While such an Nth asset always exists mathematically, in large markets this asset may be very a-typical and unrealistic. To see this, consider the 200 assets randomly selected from the CRSP file as described in section II. What are the characteristics of the 201st asset which makes the optimal portfolio positively weighted (say, with an investment proportion of 1/201 in each asset)? Following the procedure in Kandel for characterizing this asset (p. 67), we find that the added asset should have a monthly standard deviation of at least 642%.[9] The expected monthly return of this asset is 64,020% (!). Thus, while it is always possible to *mathematically construct* an Nth asset which makes the optimal portfolio positively weighted, this does not imply that it is reasonable to expect the existence of a positively weighted optimal portfolio for a *general* (or *empirical*) set of parameters.

Random Covariance Matrix Analysis

Obviously, any specific set of covariances and expected returns determines a specific set of optimal portfolio weights. However, one can derive general properties of optimal portfolios by making some assumptions on the space of covariance matrices and taking a statistical approach[10]. Namely, one can assume that covariance matrices are drawn randomly from some ensemble, and derive statistical results regarding optimal portfolio weights. This approach is commonly employed in physics when one is not interested (or does not know) the parameters of a specific system, but rather one tries to make a statement about the properties of "typical"

[9] As Kandel shows, the added N^{th} asset is not unique. We report here the added 201st asset with the minimal possible standard deviation. The other possibilities involve even more extreme parameters.

[10] Longstaff, Santa-Clara, and Schwartz [2001a, 2001b] recently employ a similar approach to investigate the covariance structure among forwards.

systems of a certain type (see, for example, Wigner [1951], Carmeli [1983], and Mehta [1991]). Below we conduct such an analysis to show that the Sharpe ratio increases indefinitely with the number of assets when shortselling is allowed, but levels-off when shortselling is not allowed (as found empirically and reported in Figure 4). We would like to stress that while Theorem 1 below assumes a certain covariance structure, extensive numerical analysis indicates that the results are quite general.

Theorem 1:

Consider an $N \times N$ *covariance matrix given by:* $C = M + \alpha \underline{1}\underline{1}'$*, where* M *is a standard Wishart matrix with* d<N *degrees of freedom,* $\underline{1}$ *is a vector of 1's, and* α *is a positive constant. Let the vector of excess returns be non-degenerate. As the number of assets grows to infinity (*$N \to \infty$*) the Sharpe ratio grows indefinitely when shortselling is allowed, but levels-off when shortselling is not allowed.*

Comment 1: The Wishart matrix M can be written as $X'X$, where X is a $d \times N$ matrix with random N(0,1) elements. This construction implies a linear factor structure (see, for example, Carmeli [1983]).

Comment 2: α is the average covariance.

Comment 3: M is symmetric positive definite, and therefore so is C.

Proof:

Denote the vector of excess returns by μ. If shortselling is allowed, the unscaled optimal investment proportions vector, ω, is given by $C^{-1}\mu$ (the scaled proportions are given by $\omega/(\underline{1}'\omega)$, see Merton [1972]; since we are deriving the Sharpe ratio, it does not make a difference if one is working with the scaled or unscaled proportions).

The expected return is given by $\mu'\omega$ or $\mu'\mathrm{C}^{-1}\mu$. Notice, however, that μ can be written as:

$$\mu = \mathrm{C}\omega = \mathrm{M}\omega + \alpha(\underline{1}'\omega)\underline{1}. \tag{1}$$

Rearranging we have:

$$\mathrm{M}\omega = \mu - \alpha(\underline{1}'\omega)\underline{1}. \tag{2}$$

Multiplying by M^{-1} we obtain:

$$\omega = \mathrm{M}^{-1}\mu - \alpha(\underline{1}'\omega)\mathrm{M}^{-1}\underline{1}, \tag{3}$$

or:

$$\underline{1}'\omega = \underline{1}'\mathrm{M}^{-1}\mu - \alpha(\underline{1}'\omega)\underline{1}'\mathrm{M}^{-1}\underline{1}. \tag{4}$$

Rearranging eq.(4) yields:

$$\underline{1}'\omega = \frac{\underline{1}'\mathrm{M}^{-1}\mu}{\alpha\underline{1}'\mathrm{M}^{-1}\underline{1}+1} \tag{5}$$

Substituting this expression for $\underline{1}'\omega$ in eq.(3), we have:

$$\omega = \mathrm{M}^{-1}\mu - \alpha\frac{\underline{1}'\mathrm{M}^{-1}\mu}{\alpha\underline{1}'\mathrm{M}^{-1}\underline{1}+1}\mathrm{M}^{-1}\underline{1}. \tag{6}$$

Thus, the expected return, $\mu'\mathrm{C}^{-1}\mu$ or $\mu'\omega$ is given by: (7)

$$\mu'\omega = \mu'\mathrm{M}^{-1}\mu - \alpha\frac{(\underline{1}'\mathrm{M}^{-1}\mu)^2}{\alpha\underline{1}'\mathrm{M}^{-1}\underline{1}+1} = \frac{\alpha\left(\mu'\mathrm{M}^{-1}\mu\underline{1}'\mathrm{M}^{-1}\underline{1} - (\underline{1}'\mathrm{M}^{-1}\mu)^2\right) + \mu'\mathrm{M}^{-1}\mu}{\alpha\underline{1}'\mathrm{M}^{-1}\underline{1}+1}.$$

Since the eigenvalues of the Wishart matrix M are bounded away from 0 and are bounded from above (Silverstein [1986], Mehta [1991] p. 75), both expressions $\mu'\mathrm{M}^{-1}\mu$ and $\underline{1}'\mathrm{M}^{-1}\underline{1}$ are of order N, where N is the number of assets (and dimension of the vectors μ and $\underline{1}$). The difference $\mu'\mathrm{M}^{-1}\mu\underline{1}'\mathrm{M}^{-1}\underline{1} - (\underline{1}'\mathrm{M}^{-1}\mu)^2$, which is positive by the Schwartz inequality, is therefore of order N^2. Hence, the expected return $\mu'\omega$ is of order N.

The standard deviation of the optimal portfolio is $\sqrt{\omega' C \omega}$. As $\omega = C^{-1}\mu$ we have:

$$\sigma = \sqrt{\omega' C \omega} = \sqrt{\mu' C^{-1} C C^{-1} \mu} = \sqrt{\mu' C^{-1} \mu} = \sqrt{\mu' \omega} , \qquad (8)$$

which is the square root of the expression in eq.(7), and is therefore of the order of $\sqrt{N}$. Thus, when the number of assets, N, is large, and shortselling is allowed, the Sharpe ratio is of order $\frac{N}{\sqrt{N}} = \sqrt{N}$, and it grows indefinitely with the number of assets.

In contrast, if shortselling is not allowed, the Sharpe ratio levels off:

$$\max_{\omega > 0} \frac{\omega' \mu}{\sqrt{\omega' C \omega}} \le \frac{\omega' \mu}{\sqrt{\omega' M \omega + \alpha (\underline{1}' \omega)^2}} \le \frac{\omega' \mu}{\sqrt{\alpha} \underline{1}' \omega} \le \frac{1}{\sqrt{\alpha}} \max \mu \ . \qquad (9)$$

Q.E.D.

While the above analysis makes an assumption regarding the structure of the covariance matrix, numerical simulations show that the results are very general, and hold under a variety of other covariance structures. For example, Figure 6 displays the results of numerical simulations in which the correlation between any two stocks is drawn randomly from a uniform distribution. After the correlation matrix is drawn this way, we check whether it is positive definite- if it is not we reject it and draw another matrix. The results in Figure 6 are obtained with the following parameters: ρ_{ij} is drawn from a uniform distribution on the segment [0.4, 0.5][11], the excess return μ_i is drawn from a uniform distribution on the segment [0, 0.2], and the standard deviation σ_i is drawn from a uniform distribution on the segment [0.2, 0.5]. As

[11] We choose a relatively narrow range for the correlations, because otherwise, as the number of assets becomes large, it becomes increasingly difficult to generate positive definite matrices by the procedure described above.

Figure 6 shows, the results we find empirically are also obtained in the simulation analysis, and they are therefore not likely to be due to empirical estimation errors or to a survivorship bias. Similar numerical results are obtained with various parameter values and distributional assumptions.

(Insert Figure 6 About Here)

IV. Robustness to Estimation Error

The analysis up to this point, as well as the CAPM framework, assume that the expected returns and covariances are known. In practice, however, these parameters are not given, and have to be estimated, which typically involves some estimation error. To what extent do the above results carry through when estimation error is involved?

In order to deal with estimation errors it is common to employ shrinkage estimators, or to impose portfolio weight constraints to avoid extreme positions. In a recent innovative paper Jagannathan and Ma (2001) that these two approaches are in fact very closely linked. The effect of employing portfolio weight constraints depends on two main factors: the magnitude of the estimation error, and the difference in performance between the optimal and the constrained portfolios with the "true" parameters. If this performance difference is large, and the estimation errors are small, imposing constraints is likely to hurt performance, and vice versa. Jagannathan and Ma (2001) employ simulations to investigate the benefits of the constraints as a function of the estimation error. Here we take a similar approach.

As shown in the preceding sections, when the number of assets is large, the difference in performance between the optimal and the constrained portfolios with the "true" parameters is very large. Hence for large portfolios we expect the unconstrained portfolio to outperform the constrained portfolio even when substantial

estimation error is involved. We investigate this issue numerically by randomly drawing a "true" covariance matrix and "true" excess returns as described in section III. Then we create the "estimated" or observed parameters by adding estimation error, or "noise", to the true parameters. Specifically, we take

$$\mu_i^0 = \mu_i(1+\tilde{\varepsilon}_\mu) \tag{10}$$

$$\sigma_i^0 = \sigma_i(1+\tilde{\varepsilon}_\sigma) \tag{11}$$

where μ and σ are the "true" parameters, the superscript 0 denotes the observed parameters, and $\tilde{\varepsilon}_\mu$ and $\tilde{\varepsilon}_\sigma$ are error terms which are normally distributed N(0, σ_ε^2), and are independent of each other and across assets. Given the observed parameters, the optimal portfolio weights, ω^0, (with and without shortselling) are derived. We calculate the actual performance of the constructed portfolios by employing the "true" parameters, in terms of the Sharpe ratio:

$$\frac{\omega^{0\prime}\mu}{\sqrt{\omega^{0\prime}C\omega^0}} .$$

The performance of the portfolios with and without shortselling as a function of the estimation error σ_ε is described in Figure 7. The number of assets is 100, and the Sharpe ratios reported are averaged over 10 independent simulations at each error level. For comparison, the figure also describes the performance of a "naive" diversification portfolio with a proportion $\frac{1}{100}$ in each asset. For low error levels ($\sigma_\varepsilon \approx 0$) the observed parameters are close to the "true" parameters, and shortselling dramatically increases the portfolio performance, as reported previously in Figures 4 and 6. As the error increases, the Sharpe ratio of both portfolios (with and without shortselling) declines, and the advantage of shortselling also decreases, as reported in

Jagannathan and Ma. Notice, however, that even with a significant level of error (e.g. $\sigma_\varepsilon = 30\%$) employing shortselling results in doubling the Sharpe ratio.

(Insert Figure 7 About Here)

V. Summary and Discussion

In this paper we investigate the properties of mean-variance efficient portfolios in markets with a large number of assets and a general return and covariance structure. Our main results are:

(1) The proportion of assets held short in mean-variance efficient portfolios converges to 50% as the number of assets increases.

(2) The investment proportions are extreme: several assets are held in very large positions (long or short).

(3) The investment proportion in each stock is not directly related to any simple intrinsic characteristic of the stock such as its mean, variance, or its average correlation (or covariance) with the other stocks. Thus, in the context of large portfolios it is not straightforward to characterize a "good" stock.

(4) The cost of the no-shortselling constraint is extremely high. For large portfolios the Sharpe ratio can be more than doubled by relaxing this constraint.

These results are obtained empirically under various estimation methods, analytically, and in numerical simulations. Thus, the results seem to be fundamental properties of mean-variance efficient portfolios in large markets.

The first result reveals a severe theoretical inconsistency of the CAPM. On the one hand, the model predicts that the optimal mean-variance portfolio coincides with the market portfolio; On the other hand, result (1) states that for large markets about half of the assets are held short in the optimal portfolio, which means that the market

portfolio can not possibly coincide with the optimal portfolio. As Green and Hollifield [1992] and Kandel [1984] show, there *are* conditions which ensure a mean-variance efficient portfolio with no short positions. However, result (1) indicates that for large markets these conditions may be very unlikely to hold.

One line of defense for the CAPM could be based on the notion that in equilibrium prices are determined such that the optimal portfolio has all weights positive. Specifically, according to Lintner's approach to the CAPM companies' end-of-period value distributions are given, and market prices "adjust and readjust" until in equilibrium the vector of asset prices yields a vector of expected returns and a covariance matrix which lead to the linear SML relation between beta and expected return (see Lintner [1965a] p. 598). Taking this approach, one may hope that the price vector can be determined such that not only the SML holds, but in addition the optimal portfolio is positively weighted. However, this line of defense is problematic for at least two reasons. First, as Nielsen [1988] elegantly shows, Lintner's approach leads to the problem of multiple CAPM equilibria. In order to reduce the infinite set of possible equilibria, specific preferences and initial endowments have to be considered. However, this does not automatically ensure a unique equilibrium. Even with specific assumptions regarding preferences and endowments one can have multiple equilibria, a unique equilibrium, or no equilibrium at all. Second, even if the preferences and endowments are such that there is a unique equilibrium, this does not solve the more severe problem of short positions in the optimal mean-variance

portfolio.[12] Thus, it seems that this line of defense does not save the model from its internal inconsistencies. One promising avenue which may offer a solution to this problem is the segmented market approach of Levy [1978], Merton [1987], Markowitz [1990] and Sharpe [1991]. According to the segmented market approach investors may hold only a limited number of assets due to transaction costs, asymmetric information, or various biases. In this case each investor may hold several assets short in his portfolio, while the market portfolio can have all positive weights.

The second and third results of this paper state that the optimal mean-variance portfolio is very different from "naively diversified" portfolios. As the forth result shows, this difference is economically very significance. This implies that naively diversifying between many stocks or between several mutual funds (as many investors do) is very sub-optimal. Thus, there is great practical and economic value to the portfolio optimization taught to us by Markowitz.

The cost of the no-shortselling constraint depends on the magnitude of the estimation error (Jagannathan and Ma [2001]), and on the number of assets. When the number of assets is large the cost of this constraint may be tremendous. Although

[12] For example, consider an economy with a single constant-absolute-risk-aversion investor with risk aversion parameter a. Given a riskfree rate r and a market portfolio with expected return $\overline{R}_m$ and variance σ_m^2, such an investor optimally invests $T_0 = \frac{\overline{R}_m - r}{\sigma_m^2 a}$ dollars in the market portfolio. Lintner [1965a] defines the market price of risk, γ, as $\gamma = \frac{\overline{R}_m - r}{\sigma_m^2 T_0}$. Thus, $\gamma = a$, and in this case the market price of risk is just the investor's risk aversion parameter. To see that negative prices are possible in this framework, assume, for example, a=2, and a market with a riskfree rate of 10% and two risky assets with $\overline{P}_1 = 1, \overline{P}_2 = 10, \overline{\sigma}_1 = 0.4, \overline{\sigma}_2 = 2, \rho_{12} = 0.5$, where all of these relate to end-of-period values (recall that in this framework the firms' end-of-period value distributions are given, and today's prices simultaneously determine the expected returns and covariances). Let there be one share of each risky asset. Employing Lintner's formulas for equilibrium prices (Lintner, 1965a, eq.17 on pg. 600) with a market price of risk of 2, we obtain a price of -$0.109 for asset 1, and $1.091 for asset 2. While it is counter-intuitive to have an asset with a negative price, this results from the CAPM framework with normal distributions, which implies that negative terminal values are possible.

short-selling may have a speculative and therefore risky connotation, used responsibly in a large portfolio context it implies the exact opposite. Funds that are not allowed to sell short because of “safety considerations” could reduce their risk by more than half while maintaining the same expected return if they were allowed to sell short.

REFERENCES

Alexander, G. J., 1993, Short selling and efficient sets, *Journal of Finance* 48, 1497-1506.

Amihud, Y., B. J. Christensen and H. Mendelson, 1992, Further evidence on the risk-return relationship, working paper, Stanford University.

Benartzi, S. and R. Thaler, 2000, Naive diversification strategies in defined contribution saving plans, *American Economic Review*, forthcoming.

Black, F., 1972, Capital market equilibrium with restricted borrowing, *Journal of Business* 45, 444-454.

Black, F., M.C. Jensen, and M. Scholes, 1972, The Capital Asset Model: Some Empirical Tests, in Michael C. Jensen, ed., *Studies in the Theory of Capital Markets*, New York.

Carmeli, M., 1983, *Statistical Theory of Random Matrices*, Marcel Dekker, New York.

Dyl, E. A., 1975, Negative betas: The attractions of selling short, *Journal of Portfolio Management* 3, 74-76.

Elton, E. J., and M. J. Gruber, 1973, Estimating the dependence structure of share prices – implications for portfolio selection, *Journal of Finance* 28, 1203-1232.

Elton, E. J., M. J. Gruber, and M. W. Padberg, 1976, Simple criteria for optimal portfolio selection, *Journal of Finance* 31, 1341-1357.

Fama, E. and K. French, 1992, The cross-section of expected stock returns, *Journal of Finance* 47, 427-465.

Frost, P. A. and J. E. Savarino, 1986, An empirical Bayes approach to efficient portfolio selection, *Journal of Financial and Quantitative Analysis* 21, 293-306.

Frost, P. A. and J. E. Savarino, 1988, For better performance: Constrain portfolio weights, *Journal of Portfolio Management* 14, 29-34.

Green, R. C., 1986, Positively weighted portfolios on the minimum-variance frontier, *Journal of Finance* 41, 5, 1051-1068.

Green, R. C. and B. Hollifield, 1992, When will mean-variance efficient portfolios be well diversified? *Journal of Finance* 47, 1785-1809.

Ibbotson Associates, 2000, *Stocks, Bonds, Bills and Inflation,* (Chicago, Illinois).

Jagannathan, R., and Z. Wang, 1993, The CAPM is alive and well, Federal Reserve Bank of Minneapolis Staff Report 165.

Jagannathan, R., and T. Ma, 2001, Risk reduction in large portfolios: a role for portfolio weight constraints, Northwestern University working paper.

Kallberg, J. G. and W. T. Ziemba, 1983, Comparison of alternative utility functions in portfolio selection problems, *Management Science* 29, 1257-1276.

Kandel, S., 1984, On the exclusion of assets from tests of the mean variance efficiency of the market portfolio, *Journal of Finance* 39, 63-75.

Kandel, S., and R. F. Stambaugh, 1995, Portfolio inefficiency and the cross-section of expected returns, *Journal of Finance* 50, 157-184.

Kroll, Y., H. Levy and H. Markowitz, 1984, Mean-variance versus direct utility maximization, *Journal of Finance* 39, 47-61.

Kwan, C. Y., 1997, Portfolio selection under institutional procedures for short selling: Normative and market-equilibrium considerations, *Journal of Banking and Finance* 21, 369-391.

Ledoit, O., 1996, A well-conditioned estimator for large dimensional covariance matrices, UCLA working paper.

Ledoit, O., 1999, Improved estimation of the covariance matrix of stock returns with an application to portfolio selection, UCLA working paper.

Ledoit, O., and P. Santa-Clara, 1999, Estimating large conditional covariance matrices with an application to international stock markets, UCLA working paper.

Levy, H., 1973, The demand for assets under conditions of risk, *Journal of Finance*, 28, 79-96.

Levy, H., 1978, Equilibrium in an imperfect market: a constraint on the number of securities in the portfolio, *American Economic Review* 68, 643-658.

Levy, H., 1983, The capital asset pricing model: theory and empiricism, *Economic Journal* 93, 145- 165.

Lintner, J., 1965a, Security prices, risk, and the maximal gains from diversification, *Journal of Finance*, 20(4), 587-615.

Lintner, J., 1965b, The valuation of risk assets and the selection of risky investments in stock portfolios and capital budgets, *Review of Economics and Statistics* 47, 13-37.

Lintner, J., 1965c, Security prices and risk: the theory of comparative analysis of AT&T and leading industrials, paper presented at the conference on the economics of regulated public utilities, Chicago.

Longstaff, F., Santa-Clara, P., and E. Schwartz, 2001a, Throwing away a billion dollars: the cost of suboptimal exercise in the swaptions market, *Journal of Financial Economics*, forthcoming.

Longstaff, F., Santa-Clara, P., and E. Schwartz, 2001b, The relative valuation of caps and swaptions: theory and empirical evidence, *Journal of Finance*, forthcoming.

Markowitz, H., 1952, Portfolio selection, *Journal of Finance* 7, 77-91.

Markowitz, H., 1956, The optimization of a quadratic function subject to linear constraints, *Naval Research Logistics Quarterly* 3, 111-133.

Markowitz, H., 1959, *Portfolio Selection: Efficient Diversification of Investments*, (John Wiley and Sons, Somerset, NJ).

Markowitz, H., 1987, *Mean-Variance Analysis in Portfolio Choice and Capital Markets*, (Basil Blackwell Inc., New York).

Markowitz, H., 1990, Risk adjustment, *Journal of Accounting, Auditing and Finance*, 213-225.

Mehta, M.L., 1991, *Random Matrices*, Academic Press, Boston.

Merton, R. C., 1972, An analytic derivation of the efficient portfolio frontier, *Journal of Financial and Quantitative Analysis* 7, 1851-1872.

Merton, R. C., 1987, A simple model of capital market equilibrium with incomplete information, *Journal of Finance* 42, 483-510.

Mossin, J., 1966, Equilibrium in a capital asset market, *Econometrica* 35, 768-783.

Nielsen, L. T., 1988, Uniqueness of equilibrium in the classical CAPM, *Journal of Financial and Quantitative Analysis* 23, 329-336.

Price, K, 1989, Long on safety: don't sell short selling short, *Barron's*, November 6, 16.

Pulley, L. B., 1981, General mean-variance approximation to expected utility for short holding periods, *Journal of Financial and Quantitative Analysis* 16, 361-373.

Roll, R., 1977, A critique of the asset pricing theory's tests; Part I: On past and potential testability of the theory, *Journal of Financial Economics* 4, 129-176.

Roll, R. and S. A. Ross, 1977, Comments on qualitative results for investment proportions, *Journal of Financial Economics* 5, 265-268.

Roll, R. and S. A. Ross, 1994, On the cross-sectional relation between expected returns and betas, *Journal of Finance* 49, 101-121.

Rudd, A., 1977, A note on qualitative results for investment proportions, *Journal of Financial Economics* 5, 259-263.

Schwert, G.W., and P. J. Seguin, 1990, Heteroskedasticity in stock returns, *Journal of Finance* 45, 1129-55.

Scott, D. W., 1992, *Multivariate Density Estimation*, New York: Wiley.

Sharpe, W., 1964, Capital asset prices: a theory of market equilibrium under conditions of risk, *Journal of Finance* 19, 425-442.

Sharpe, W., 1966, Mutual fund performance, *Journal of Business*, 119-138.

Sharpe, W., 1991, Capital asset prices with and without negative holdings, *Journal of Finance* 46, 489-509.

Silverstein, J.W., 1986, Eigenvalues and eigenvectors of large dimensional sample covariance matrices*, Contemporary Mathematics* 59, 153—159.

Stevens, G., 1998, On the inverse of the covariance matrix in portfolio analysis, *Journal of Finance* 53, 1821-27.

Tobin, J., 1990, Discussion of risk adjustment, *Journal of Accounting, Auditing and Finance*, 229-234.

Weiss, G., 1991, The long and short of short-selling, *Business Week*, June 10, 106-108.

Wigner, E. P., 1951, On the statistical distribution of the widths and spacings of nuclear resonance levels, *Proceedings of the Cambridge Philosophical Society* 47, 790.

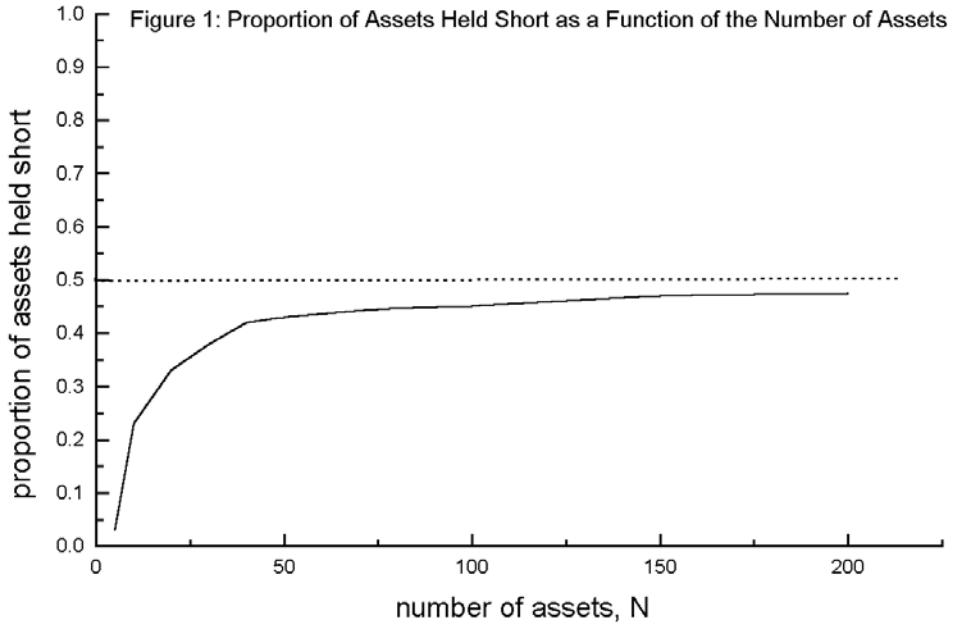


Figure 1: Proportion of Assets Held Short as a Function of the Number of Assets
proportion of assets held short
number of assets, N
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0
50
100
150
200

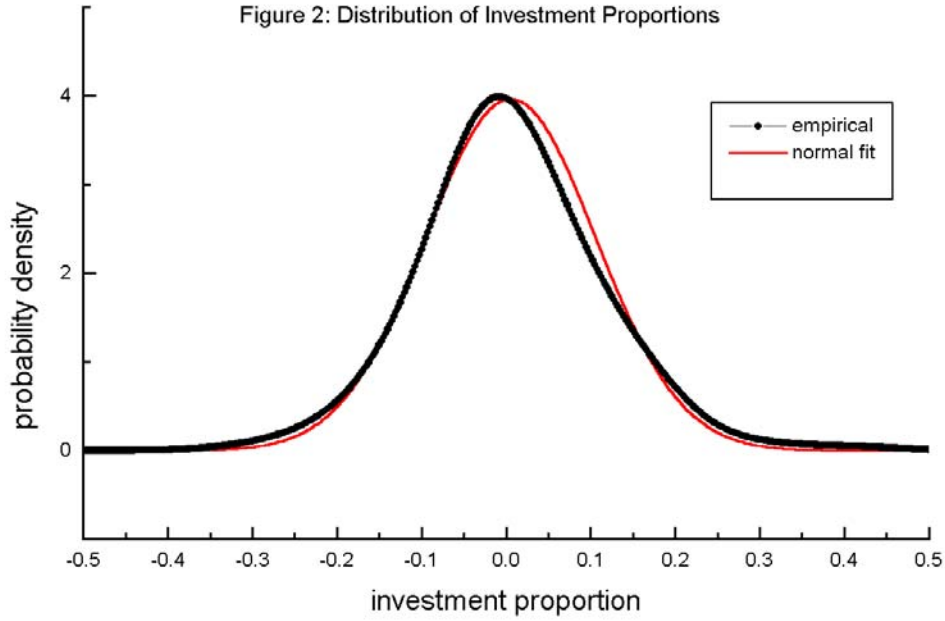


Figure 2: Distribution of Investment Proportions
empirical
normal fit
probability density
investment proportion
4
2
0
-0.5
-0.4
-0.3
-0.2
-0.1
0.0
0.1
0.2
0.3
0.4
0.5

Figure 3A: Investment Proportion as a Function of the expected Return

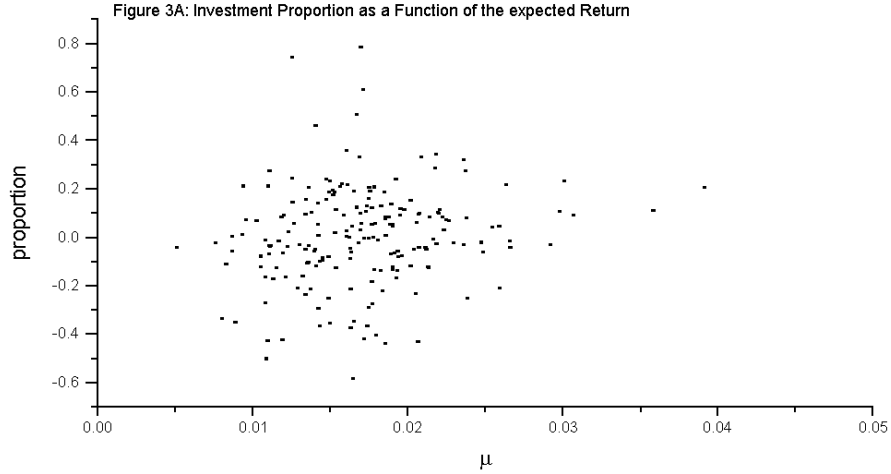


Figure 3B: Investment proportion as a Function of the Standard Deviation

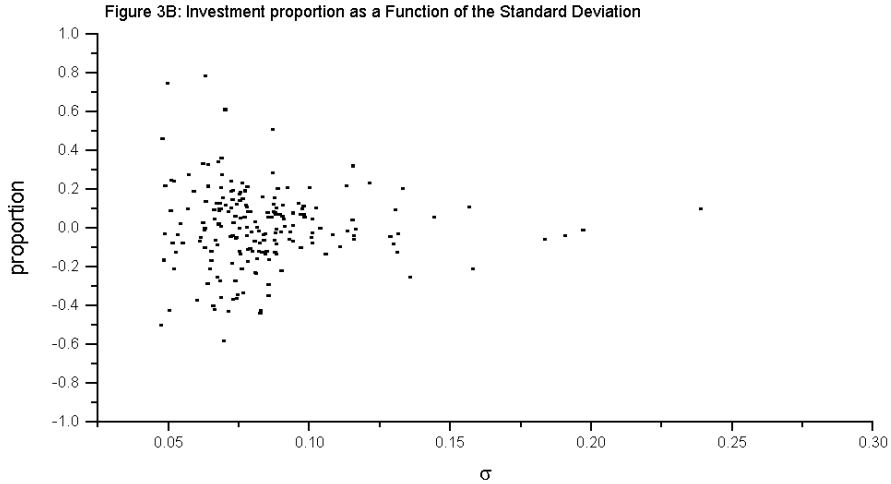
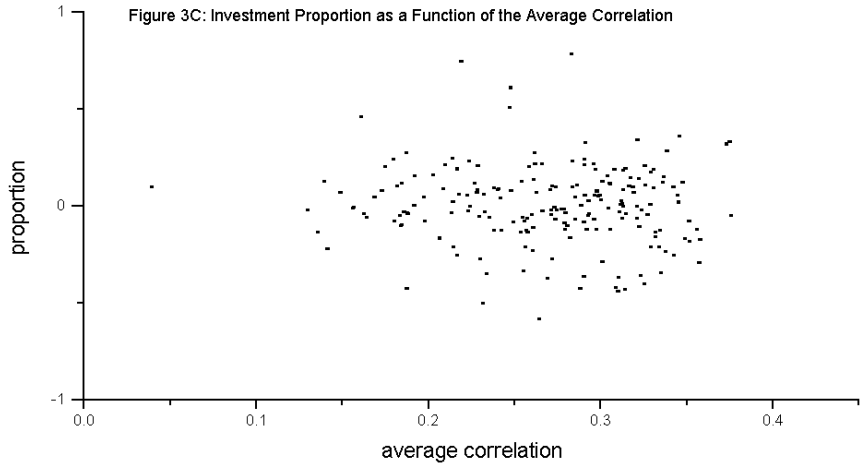


Figure 3C: Investment Proportion as a Function of the Average Correlation



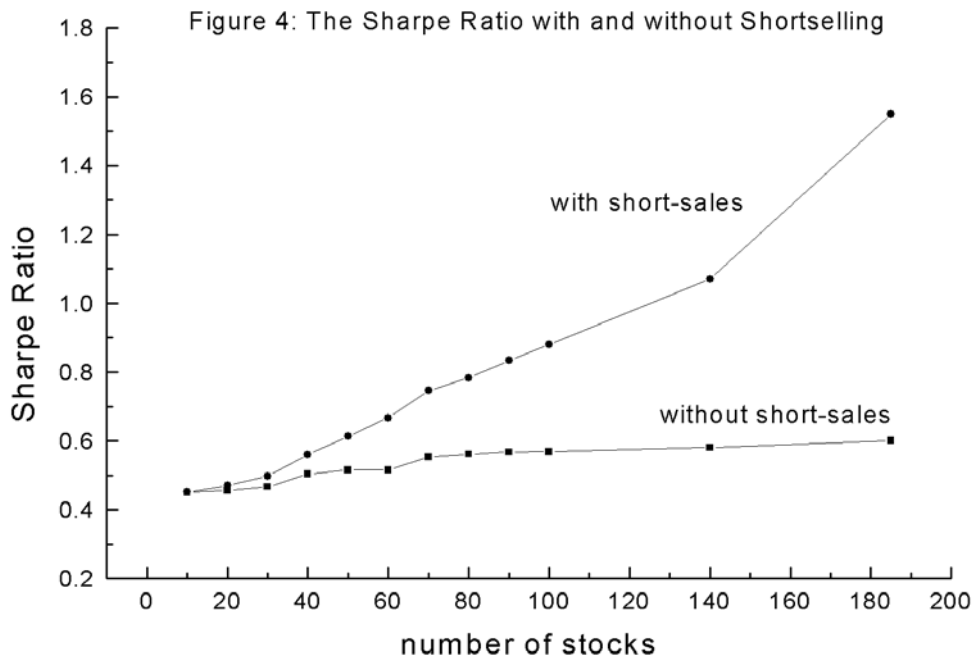


Figure 4: The Sharpe Ratio with and without Shortselling
Sharpe Ratio
1.8
1.6
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0
20
40
60
80
100
120
140
160
180
200
number of stocks
with short-sales
without short-sales

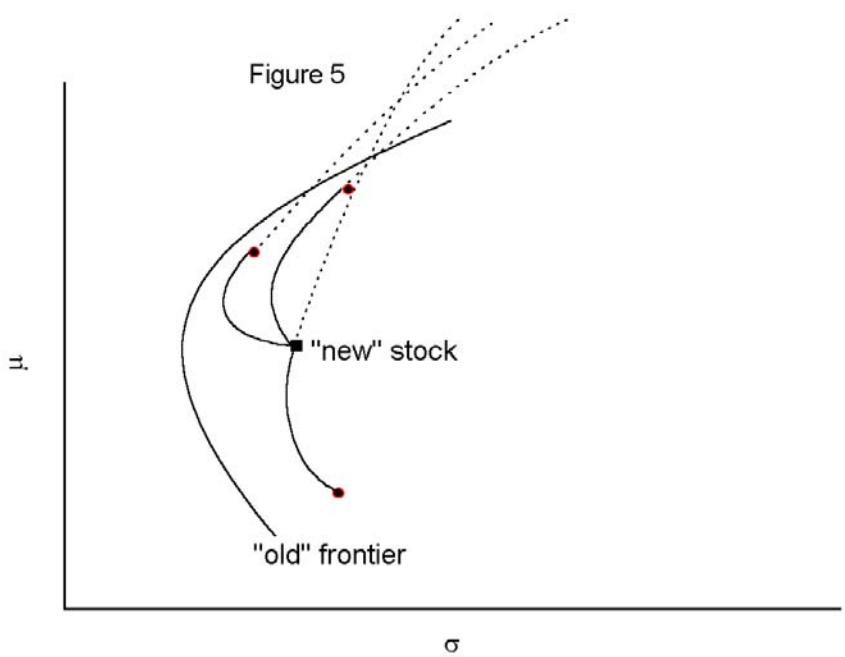
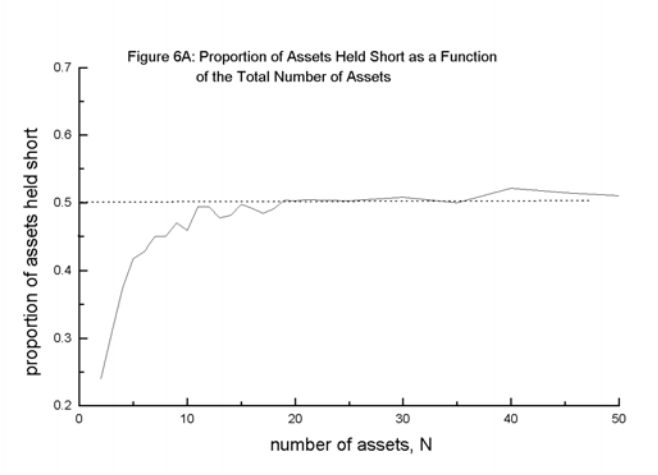
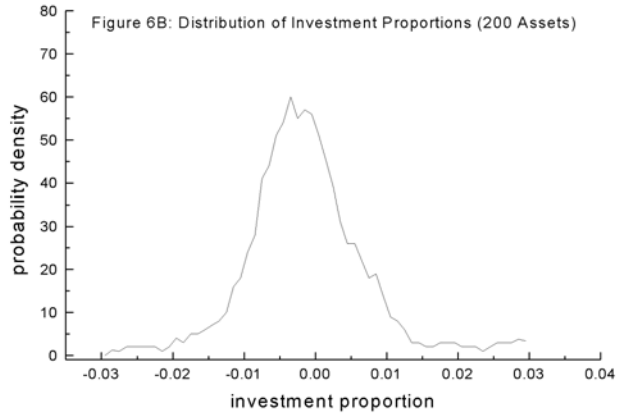


Figure 5
μ
σ
"new" stock
"old" frontier

Figure 6: Simulation Results





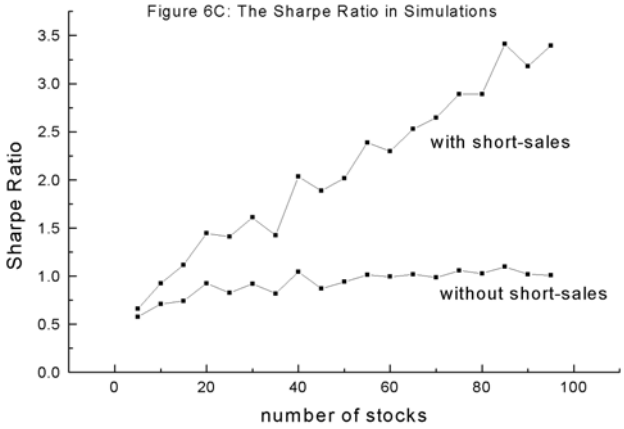


Figure 7: Results with Estimation Error