THE EFFECT OF INSIDER BELIEFS ON INFORMED TRADE,
MARKET LIQUIDITY, AND PRICE EFFICIENCY

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Abstract

A one period model of a speculative market is analyzed in which a monopolistically
privately informed trader strategically exploits his information through trade with a
market maker. Both the informed trader and the market maker entertain doubts about
the uniqueness of the insider's information. I numerically solve for a linear approximate
equilibrium in which uncertainty about the number of informed traders in the market
plays a dual role. First, it acts as an additional source of noise in the market. Second, it
changes the implicit level of competition in the market. Depending on parameter values,
these two effects may impact equilibrium characteristics in like or opposite direction.
On balance I find: (i) the intensity of an insider's trading is monotonically decreasing in
the likelihood that his information is non-unique; (ii) market liquidity increases in the
insider's uncertainty and can decrease in the implied level of competition; (iii) expected
monopolist insider profits are higher on average than if the insider's information were
known to be unique; and (iv) prices tend to be more efficient than in the case of a
known monopolist insider.

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All errors are my own.

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1 Introduction

The manner in which private information is incorporated into security prices is of ongoing interest to researchers. Kyle (1985) provides an early and influential model in which an informed trader with unique and precise private information chooses to reveal his information slowly so as to optimize the tradeoff between profit accumulation and information revelation. Many extensions and refinements of Kyle's approach have deepened our understanding of informed trader behavior in increasingly rich settings.\(^1\) Still, some issues remain ambiguous. In particular, there is no conclusive answer to the fundamental question of how aggressively an informed trader trades on his information and hence how quickly and fully private information is impounded into prices. This paper provides further insight regarding the incorporation of private information into prices by relaxing the assumption that the number of informed traders is common knowledge.

Existing literature indicates many determinants of informed trader behavior. Among these are the number of trading rounds, the degree of agents' risk aversion, the number of informed traders, the homogeneity of informed traders' information, and the anonymity afforded by the trading mechanism to the informed traders. In Kyle (1985) a monopolistically informed trader maximizes expected profits by gradually revealing his information over all the trading rounds. Subrahmanyan (1991) shows that, relative to the risk neutral setting of Kyle, the intensity of informed trading is decreasing in the level of an informed trader's risk aversion. Subrahmanyan (1991) and Holden and Subrahmanyan (1992) show that in a single auction model with identical information among insiders the intensity of a single informed trader's order is decreasing in the number of informed traders, although the aggregate intensity of the informed order flow is increasing - a result analogous to a simple Cournot oligopoly model of product market competition.\(^2\) In a multiple auction setting, Holden and Subrahmanyan show that in early auctions insiders' aggregate trading intensity increases in the number of informed traders, while in later auctions it decreases. In the limit, traders are infinitely aggressive in the first auction and do not trade thereafter. This last result is overturned in Cao (1994) by allowing for differentially informed insiders. Lastly,

\(^1\)In addition to the papers cited in the text, extensions and applications of Kyle (1985) include Admati and Pfleiderer (1988), Kyle (1989), Chowdhry and Nanda (1991), Spiegel and Subrahmanyan (1992), and Foster and Viswanathan (1993), among others.

\(^2\)See Shubik (1980) for a discussion of oligopoly models based on linear demand schedules.
Corb (1993) examines a nonlinear equilibrium that results when the market maker sees the disaggregated order flow and the monopolist insider consequently tempers his orders when he has extreme information.

Clearly, the uniqueness of an informed trader's information is fundamental to how aggressively he trades. In the models discussed above, the information structure - including the number of informed traders and the correlations among their information - is common knowledge. However, not all the models specify how this knowledge is gained. Often the number of informed traders is exogenous (e.g., Kyle (1985), Cao (1994)). If the number of informed traders is not exogenous (e.g., in Subrahmanyam (1991) and Holden and Subrahmanyam (1992)), it is derived endogenously by assuming a fixed cost of information acquisition and equating the expected utility of becoming informed with the expected utility of remaining uninformed. Such a specification is deficient in several ways. First, there is no coordination mechanism. Which of the potentially informed are to become informed? Second, information must be costly. The models do not apply to situations in which agents may be costlessly endowed with information as might be the case, for example, with corporate insiders.

In this paper I allow uncertainty about the number of informed traders. I focus on the extreme case of a "doubtful" monopolist, a trader endowed with private information who is skeptical of his status as a monopolist. I adopt the simple single-period modeling framework of Kyle (1985), but my more general information structure precludes the analytic derivation of a simple linear equilibrium. Consequently I apply projection methods to numerically approximate the equilibrium. Surprisingly, over a broad range of parameter values, the approximated equilibrium is linear. By construction, this equilibrium satisfies necessary conditions that any true linear equilibrium would have to satisfy. I am therefore able to derive an analytic expression that is the sole candidate for a linear equilibrium of the model. I show that uncertainty about the number of informed traders in the market plays a dual role. First, it acts as an additional source of noise in the market. Second, it changes the implicit level of competition in the market. Depending on parameter values, these two effects may impact equilibrium characteristics in like or opposite direction. On balance I find: (i) the intensity of an insider's trading is monotonically decreasing in the likelihood that his information is non-unique; (ii) market liquidity increases in the insider's uncertainty and can decrease in the implied level of competition; (iii) expected monopolist insider profits are
higher on average than if the insider's information were known to be unique; and (iv) prices tend to be more efficient than in the case of a known monopolist insider. Surprisingly, on average a "doubtful" monopolist, relative to a "sure" monopolist, will earn higher profits even though he trades less aggressively yet reveals more of his information.

My results are limited in that I do not explore how the skeptical insider behaves as auction frequency increases (in particular as auctions become continuous). Still the results suggest that markets may exhibit strong form efficiency in the limit. The significance of this conjecture is it applies even for a monopolistic holder of private information, provided the insider is not sure of his status as a monopolist.

The paper proceeds as follows. Section 2 describes the model. Section 3 describes the numerical methods employed to solve the model. Section 4 discusses the equilibrium, including a discussion of comparative statics. Section 5 concludes.

2 The Model

I choose the simplest modeling framework that is adequate for my purpose. The model is an adaptation of the single period model of Kyle (1985). A single risky asset is traded by three types of traders: one or two informed traders with perfect knowledge of the liquidation value of the risky asset; liquidity traders with exogenously determined needs for trade\(^3\); and a competitive market maker. All agents are risk neutral. Informed traders seek to maximize expected profit. The market maker sets price equal to her expectation of the liquidation value of the risky asset.

Adopting the notation of Kyle (1985), designate as \(\bar{v}\) the liquidation value of the risky asset. \(\bar{v}\) is supposed normally distributed with mean \(p_0\) and variance \(\Sigma_0\). \(\bar{u}\) represents the noise trader order flow. \(\bar{u}\) is distributed normal with zero mean and variance \(\sigma_u^2\).\(^4\) The number of informed traders is chosen by nature and distributed binomial. There are two

\(^3\)Spiegel and Subrahmanyam (1992) replace the liquidity traders with uninformed strategic maximizers who trade to hedge exogenous endowment shocks.

\(^4\)Assumptions regarding distributional characteristics and functional forms (e.g., of preferences) are adopted from the existing literature for purposes of comparing equilibrium characteristics. However, as will be discussed in Section 3, the numerical solution method employed is more broadly applicable an can accommodate more general specifications.
informed traders with \textit{ex ante} probability $q$ and one informed trader with \textit{ex ante} probability $1 - q$. $q$ is common knowledge. The quantity traded by informed trader $i = 1, 2$ is $\tilde{x}_i$. Each informed trader's strategy will be a function of his information and is written $\tilde{x}_i = X(\tilde{v})$, where symmetry allows excluding an $i$ subscript. The market maker sets price, $\tilde{p}$, based on the observed aggregate order flow $\tilde{\omega}$. Thus, we write $\tilde{p} = P(\tilde{\omega})$ where $\tilde{\omega} = \tilde{x}_1 + \tilde{I}_2 \tilde{x}_2 + \tilde{u}$ and $\tilde{I}_2$ is an indicator variable equal to 1 when there exist two informed traders and 0 when there exists a single informed trader.

Trading occurs in two stages. In stage one, the values of $\tilde{v}$ and $\tilde{u}$ are realized and the informed trader(s) choose(s) the quantity to trade, $\tilde{x}_i$, given $\tilde{v}$. In the second stage the market maker observes the aggregate order flow and sets the price at which she is willing to clear the market. Note that although informed traders receive perfect signals, observing $\tilde{v}$ without additional noise, the results would be qualitatively the same if the informed traders were to receive identical noisy signals (e.g., $\tilde{y} = \tilde{v} + \tilde{\epsilon}$). The important characteristic is that each informed trader believes there exists a (greater than zero probability) chance that another informed trader shares his signal. This, for example, would be the case if $\tilde{\epsilon}$ were discrete or if there existed only a finite number of potential signals.

In the one-period model an equilibrium is a trading strategy and price function pair $\{X(\tilde{v}), P(\tilde{\omega})\}$ such that:

1. The informed trader(s) maximize(s) profit:

   $$E[\tilde{\pi}(X^*(\tilde{v}), P(\tilde{\omega}))[\tilde{v}] \geq E[\tilde{\pi}(X(\tilde{v}), P(\tilde{\omega}))[\tilde{v}] \quad \forall \ X \neq X^*$$

   where $\tilde{\pi}(X(\tilde{v}), P(\tilde{\omega})) = (\tilde{v} - P(\tilde{\omega}))X(\tilde{v})$.

2. The market price is efficient:

   $$P(\tilde{\omega}) = E[\tilde{v}|\tilde{\omega}]$$

3 \ Numerical methods

While this model's conceptual departures from Kyle (1985) are small, there is a significant difference in the analytic tractability of the two models. Specifically, this model does not
support a fully linear equilibrium and analytic solutions for non-linear equilibria are allusive.\textsuperscript{5} I use the numerical approach of Judd (1992), Judd (1993) and Judd and Bernardo (1993) to approximate an equilibrium trading strategy and pricing policy in this setting.\textsuperscript{6}

The informed trader’s problem is to choose $\hat{\tilde{z}} = X(\tilde{v})$ to solve

$$\max_{\tilde{z}} = E_{u, I_2}(\tilde{v} - P(\tilde{\omega}))x_1[\tilde{v}].$$

The first order condition is

$$E_{u, I_2}(\tilde{v} - P(\tilde{\omega}) - x_1P_{x_1}(\tilde{\omega})|\tilde{v}] = 0. \tag{1}$$

The market maker sets price $\tilde{p} = P(\tilde{\omega})$ equal to the conditional expectation of the liquidation value of the risky asset

$$P(\tilde{\omega}) = E_{u, v, I_2}[\tilde{v} | \tilde{\omega}]$$

implying

$$E_{u, v, I_2}[\tilde{v} - P(\tilde{\omega}) | \tilde{\omega}] = 0. \tag{2}$$

The procedure for numerically solving (1) and (2) for approximations to the equilibrium $\{X(\tilde{v}), P(\tilde{\omega})\}$ is presented in the next section.

3.1 Projection methods

We will solve for an approximate equilibrium in which the true trading strategy and pricing function are approximated by finite-degree polynomials which converge to the true functions as the degree of the polynomial approximations approaches infinity. Specifically, take

$$\hat{x}(v) = \sum_{i=0}^{N_x} a_i H_i(v), \quad \text{and}$$

$$\hat{p}(\omega) = \sum_{i=0}^{N_p} b_i H_i(\omega) \tag{4}$$

\textsuperscript{5}In this model the market maker’s conditioning information is distributed mixed-normal, with a known mean and unknown variance. The mixing distribution of the variance is binomial. It is the form of this mixing distribution that precludes a linear equilibrium. Choosing the appropriate natural conjugate distribution for the mixing distribution (i.e., a gamma distribution) would enhance the tractibility but does not conform to the economic motivation for the model.

\textsuperscript{6}See Appendix 1 for a brief overview of the general approach.
where $H_i(\cdot)$ is the degree $i$ Hermite polynomial and $N(\cdot)$ is the degree of the polynomial approximation. Hermite polynomials are chosen because we have normally distributed random variables, but other choices exist and the general approach is well-defined for arbitrary distributional assumptions.\(^7\)

Our goal is to determine the $N_x + N_p$ unknown coefficients $(a_i, b_i)$ to satisfy (1) and (2) where (3) and (4) have been substituted in for $X(\tilde{v})$ and $P(\tilde{w})$, respectively. This is accomplished by imposing an equal number of projection conditions derived from the equilibrium relations (1) and (2). The resulting system of $N_x + N_p$ equations in $N_x + N_p$ unknowns can be solved for the coefficients.\(^8\)

An arbitrary number of projection conditions can be derived from (1) and (2) by exploiting the following fact: $Z = \mathbb{E}[Y|X]$ if and only if $Z$ is a function of $X$ and $\mathbb{E}[(Y-Z)f(X)] = 0$ for all continuous bounded functions $f$. That is, $Z$ is the conditional expectation of $Y$ given $X$ if and only if the prediction error, $Y - Z$, is uncorrelated with any function of the conditioning variable $X$. The conditional expectation is therefore equivalent to an infinite number of unconditional expectations. Approximate the conditional expectation of the informed trader’s first-order condition by $N_x$ unconditional expectations

$$E_{u,v,\tilde{v}}[(\tilde{v} - \tilde{p}(\tilde{w}) - \tilde{x}_1(\tilde{v})\tilde{p}_{x1}(\tilde{w}))H_i(\tilde{v})] = 0, \quad i = 0, \ldots, N_x \tag{5}$$

where $H_i(\tilde{v})$ is the $i$th order Hermite polynomial.\(^9\) Likewise, approximate the conditional expectation of the market efficiency condition by $N_p$ unconditional expectations

$$E_{u,v,\tilde{v}}[(\tilde{u} - \tilde{p}(\tilde{w}))H_i(\tilde{w})] = 0, \quad i = 0, \ldots, N_p \tag{6}$$

The system of integral equations represented by the above projection conditions is solved numerically using Gauss-Hermite quadrature formulae to represent the integrals and using

\(^7\)See Appendix 1 for elaboration on this point.

\(^8\)Alternatively, I could over-identify the system by imposing a greater number of projection conditions and determine the unknown coefficients by minimizing a loss function defined with respect to the projection conditions. This is analogous to GMM estimation and could potentially exhibit better convergence properties than the exact-identification method described. I am not aware of a comparison of the relative merits of the two approaches and leave it for further research.

\(^9\)Using the first $n$ elements of the basis as the projection conditions is known as the Galerkin method. Other projection conditions could be chosen. See Judd (1998) for a discussion.
a combination of finite-difference and secant methods to solve the system of nonlinear equations.\textsuperscript{10}

3.2 Computing expected profits and price efficiency

Given the estimates \( \hat{\pi}(\hat{\varpi}) \) and \( \hat{\varpi}(\hat{\varpi}) \) I calculate expected profits as

\[
E[\pi(\hat{\varpi}, \hat{\varpi})] = E[(\hat{\varpi} - \hat{\varpi}(\hat{\varpi}))\hat{\pi}(\hat{\varpi})]
\]

using Gauss-Hermite quadrature.

As is standard in the literature, price efficiency is measured by the variance of \( \hat{\varpi} \) conditional on the price. I use the following definition to calculate price efficiency:

\[
\Sigma_1(p) = \text{var}(\hat{\varpi}|\hat{p}) = E[\hat{\varpi}^2|\hat{p}] - (E[\hat{\varpi}|\hat{p}])^2
\]

We know by market efficiency that \( E[\hat{\varpi}|\hat{p}] = \hat{p} \). To calculate \( E[\hat{\varpi}^2|\hat{p}] \) I use projection methods as before. Specifically, define \( s_2(p) = E[\hat{\varpi}^2|\hat{p}] \). Then

\[
E[\hat{\varpi}^2 - s_2(\hat{p})|\hat{p}] = 0
\]

and

\[
E[(\hat{\varpi}^2 - s_2(\hat{p}))f(\hat{p})] = 0
\]

for all bounded \( f \) by definition of the conditional expectation. Approximate \( s_2(p) \) with the polynomial

\[
\hat{s}_2(p) = \sum_{i=0}^{N_s} c_i H_i(p).
\]

The unknown coefficients \( c_i \) are determined by choosing \( N_s \) projection conditions

\[
E[(\hat{\varpi}^2 - s_2(\hat{p}))H_i(\hat{p})] = 0, \quad i = 0, \ldots, N_s
\]

and solving the resulting system of \( N_s \) nonlinear equations in \( N_s \) unknowns as before. My estimate of price efficiency is

\[
\hat{\Sigma}_1(\hat{p}) = \hat{s}_2(\hat{p}) - \hat{p}^2.
\]

\textsuperscript{10}These are standard numerical methods covered in most introductory texts. See, for example, Atkinson (1989) or Hildebrand (1974).
I am also interested in the *ex ante* expected price efficiency $E[\text{var}(\tilde{v} | \tilde{p})]$. It is calculated using

\[
E[\text{var}(\tilde{v} | \tilde{p})] = \text{var}(\tilde{v}) - \text{var}(E[\tilde{v} | \tilde{p}])
\]

\[
= \Sigma_0 - \text{var}(\tilde{p})
\]

\[
= \Sigma_0 - (E[\tilde{p}^2] - (E[\tilde{p}])^2)
\]

\[
\approx \Sigma_0 - E[\tilde{p}^2] + (E[\tilde{p}])^2
\]

where the expectations are approximated using Gauss-Hermite quadrature.

4 Results

4.1 A check on the method

There are two values of $q$ for which I can solve the model analytically, namely $q = 0$ and $q = 1$. These correspond to the results of Kyle (1985) and Holden and Subrahmanyan (1992) (with their parameter $M = 2$), respectively. I check my methodology and programming by solving these cases numerically and comparing my approximate solutions with the known analytic solutions. Table 1 gives results for several sets of parameter values. Panel A presents results for the equilibrium trading strategy and expected insider profits and panel B presents results for the equilibrium pricing function and expected price efficiency.

In all cases the numeric and analytic equilibria are identical up to six decimal places. The numerical procedure was performed using third order polynomial approximations and correctly set all quadratic and cubic coefficients to zero. Estimates of expected profit are slightly less precise, matching to six decimal places in most cases and to five decimal places otherwise. Estimates of expected price efficiency are the least precise, in a few cases matching only to three decimal places. Overall, the results in Table 1 indicate that the procedure yields accurate approximations to true equilibria. In appendix 2 I comment briefly on what can be said about the accuracy of numerical equilibrium approximations in cases where analytic solutions are not available for comparison (i.e., the usual circumstance).
4.2 A check on the second order condition

Because the informed trader is risk-neutral, satisfying his first-order condition does not guarantee that he has satisfied his second-order condition. Furthermore, we haven’t made use of the insider’s second-order condition anywhere in the problem formulation or solution. Therefore, we verify ex post that the insider is indeed maximizing expected profits and not minimizing them. It will be shown in the next section that the model produces a linear approximate-equilibrium in which the market maker sets price as a linear function of the observed order flow. This makes verifying the second-order condition straightforward: The insider’s expected profit is

\[ E[(\hat{\tilde{u}} - P(\hat{\omega}))x|\hat{\tilde{v}}] \]

and we wish to show that in equilibrium this is concave in \( x \). In equilibrium we have

\[ P(\omega) = p_0 + \lambda \omega \]

for some \( \lambda \). Therefore

\[
E[(\hat{\tilde{u}} - P(\hat{\omega}))x|\hat{\tilde{v}}] = E[(\hat{\tilde{u}} - p_0 - \lambda(\omega + \hat{I}x' + \hat{\omega}))x|\hat{\tilde{v}}]
\]

\[ = q(v - p_0 - \lambda(x + x'))x + (1 - q)(v - p_0 - \lambda x)x \]

\[ = (v - p_0 - \lambda q x')x - \lambda x^2 \]

The second derivative with respect to \( x \) is \( -\lambda \) which is negative for \( \lambda > 0 \).

4.3 Discussion of the equilibrium

I compute an approximate equilibrium for the model using the methods of section 3. My baseline parameter values are \( p_0 = 10, \sigma_0 = 1, \) and \( \sigma_\alpha^2 = 1 \). Unless noted otherwise, all approximations are made with third-degree polynomials (\( N_x = N_p = N_\alpha = 3 \)) and integrals are numerically evaluated using seven quadrature nodes. For a broad range of parameter values I find a linear approximate-equilibrium in which the insider’s trading strategy and the market maker’s pricing function are linear in their information.\(^{11}\) I am unable to verify

\(^{11}\)In rare cases my algorithms do not converge to an approximate equilibrium or converge to (mildly) non-linear equilibria. Further research is required to determine if these results derive from a lack of robustness of the methodology or if they are indicative of the existence of multiple equilibria or other economic phenomena. In this paper I focus on a discussion of the linear approximate equilibrium.
analytically that a true linear equilibrium exists.\textsuperscript{12} Nevertheless, we can gain some analytical insight to the numerical results by re-examining the projection conditions we have imposed.

Represent the linear approximate equilibrium by

\[
x(v) = \beta_0 + \beta_1 v \\
p(\omega) = \lambda_0 + \lambda_1 \omega
\]

Then from the informed trader's first-order condition we determine

\[
\beta_0 = -\frac{\lambda_0}{\lambda_1(2 + q)} \\
\beta_1 = \frac{1}{\lambda_1(2 + q)}
\]

We solve for \(\lambda_0\) and \(\lambda_1\) by using the market efficiency projection conditions of (6) with \(H_i(\hat{\omega})\) equal to 1 and \(\omega\). For \(H_i(\hat{\omega}) = 1\) we have:

\[
0 = E \left[ v - \lambda_0 - \lambda_1 \left( (1 + I_2) \frac{v - \lambda_0}{\lambda_1(2 + q)} + u \right) \right] \\
= \frac{p_0 - \lambda_0}{(2 + q)}
\]

implying \(\lambda_0 = p_0\). For \(H_i(\hat{\omega}) = \omega\) we have:

\[
0 = E \left\{ \left[ v - \lambda_0 - \lambda_1 \left( (1 + I_2) \frac{v - \lambda_0}{\lambda_1(2 + q)} + u \right) \right] \left( (1 + I_2) \frac{v - \lambda_0}{\lambda_1(2 + q)} + u \right) \right\} \\
= E \left\{ \frac{(v - p_0)^2(1 + I_2)}{\lambda_1(2 + q)} - \frac{(v - p_0)^2(1 + I_2)^2}{\lambda_1(2 + q)^2} - \lambda_1 u^2 \right\} \\
= \frac{\Sigma_0(1 + q)}{\lambda_1(2 + q)} - \frac{(v - p_0)^2(1 + I_2)^2}{\lambda_1(2 + q)^2} - \lambda_1 \Sigma_0
\]

implying

\[
\lambda_1 = \frac{\Sigma_0^{1/2} \sqrt{1 + q^2}}{\sigma_u(2 + q)}. \tag{7}
\]

The approximate trading strategy and pricing function are therefore:

\[
x(v) = \frac{\sigma_u(v - p_0)}{\Sigma_0^{1/2} \sqrt{1 + q^2}} \tag{8} \\
p(\omega) = p_0 + \frac{\Sigma_0^{1/2} \sqrt{1 + q^2}}{\sigma_u(2 + q)} \omega \tag{9}
\]

\textsuperscript{12}See Appendix 2 for a brief discussion of the analytic intractability of the model.
For $q = 0, 1$ these equations correspond with the Kyle (1985) and Holden and Subrahmanyam (1992) results, respectively. For $0 < q < 1$ our numerical results agree with these equations to six or more significant digits. Of course, the above findings seem to be compelling evidence that a true linear equilibrium exists, but as noted above I have been unable to construct a proof.

4.4 Comparative statics

In this section I focus on the comparative statics with respect to $q$. One sees from equations (8) and (9) that comparative statics with respect to $\Sigma_0$ and $\sigma_u$ are identical to those in Kyle (1985).

Figure 1 graphs the market liquidity parameter with respect to $q$. Market liquidity is represented by the inverse of the market liquidity parameter $\lambda_1$. Market liquidity is lowest when there is a single informed trader in the market ($q = 0$) and highest when $q = 0.5$, not $q = 1$ as may be expected. That is, the market is most liquid for intermediate values of $q$. This occurs because uncertainty with respect to the number of informed traders in the market affects market liquidity in two ways. First, as $q$ increases from 0 to 1, it becomes more likely that there are two informed traders competing with one another in the market. The market maker recognizes that an increased likelihood of competition among insiders causes insiders to trade less aggressively (individually). The market maker responds by making the price less sensitive to trading volume, providing increased liquidity. Second, uncertainty about the number of informed traders acts as an additional source of "noise" in the market. For $q = 0$ or 1 we know that market liquidity is proportional to $\sigma_u$, the "noise" in the market due to liquidity traders. From the market maker's perspective $I_2$ adds to the noise in the market. The variance of $I_2$ is $q(1 - q)$, which attains a maximum value at $q = 0.5$. This second effect accounts for the non-monotonicity of market liquidity in $q$, with the maximum at 0.5. As a result as $q$ increases from 0 to 0.5 the market is becoming both more competitive and noisier; but as $q$ increases from 0.5 to 1 market competitiveness continues to increase while market noise decreases.

Figure 2 graphs the trading intensity of each informed trader and the expected total trading intensity with respect to $q$. The expected total insider order is $(1 + q)$ times the insider order. An informed trader's order is monotonically decreasing in $q$ and the total
expected order is monotonically increasing in $q$. These first order effects are in line with intuition based on drawing parallels with Cournot duopoly results. There also exist second order effects. Informed traders, like the market maker, respond differently to the two roles played by uncertainty about their number: they trade less aggressively on their information as the likelihood of competition increases (i.e., as $q$ rises) while at the same time using the uncertainty to their best advantage in "hiding" their order flow. These effects oppose one another as $q$ rises from 0 to 0.5 and reinforce one another as $q$ rises from 0.5 to 1. This accounts for the concavity in $q$ of insider behavior. Insiders recognize that the market maker, upon seeing a large order, will compare the likelihood that $v$ is intermediate and there are two insiders with the likelihood that $v$ is large and there is a single insider. The higher the likelihood ratio of the former to the latter, the less the market maker will adjust the price. In such cases, when an insider trades heavily (contingent on being the only insider) the market maker will use the information in the heavy order flow disproportionately more to update her beliefs about $q$ than to update her beliefs about $v$. In other words, a high realization of $v$ coupled with a low $q$ creates the most favorable environment for insiders to "hide" behind the additional market noise created by $J_2$. As a result, insiders scale back their trading relatively less as $q$ rises from 0 to 0.5 than they do as $q$ approaches 1.

Total expected insider trading therefore increases most rapidly for $q$ near 0. It is my conjecture that increasing the number of trading rounds will increase the concavity of insiders' trading behavior with respect to $q$ and steepen the expected total insider order flow curve at $q = 0$. It remains for future research to determine whether or not in the limit, as auctions become continuous, a monopolistic insider might be induced to behave in a strong-form efficient manner (i.e., immediately revealing his information rather than trading strategically on it) for any $q > 0$.

Figure 3 summarizes the affect varying $q$ has on expected insider profits. 3A is the expected profit for individual informed traders and 3B is the expected total profit for all informed traders. Each is nonlinear in $q$ and conforms well with intuition. Expected insider profits decrease monotonically in $q$. Expected profits are always less conditional on there being two informed traders than conditional on there being a single informed trader. From an informed trader's perspective there is an ex post bad state of the world and an ex post good state of the world. Given the bad state of the world, expected insider profits are monotonically increasing in $q$. This makes perfect sense given that $q$ is the ex ante
probability of the bad state occurring. Therefore as \( q \) increases, insider behavior is better 
optimized relative to the bad state and expected profits increase when the bad state occurs.
Interestingly, this same logic does not apply when the good state occurs. Given the good 
state of the world, expected insider profits are maximized for \( q \approx 0.32 \). As discussed earlier,
for lower values of \( q \) insiders essentially gamble on being the only insider. That is, they 
trade more aggressively than the probability of competition dictates in the hopes that they 
will be the only insider but that the market maker will mitigate her price adjustment in 
the (mistaken) belief that there are two insiders. Here we see the upside of this gamble.
The true monopolist actually benefits from his (and more importantly, the market maker’s) 
uncertainty about the uniqueness of his information.

Figures 4A and 4B depict \( E[\text{var}(\bar{v}|\bar{p})] \) and \( \text{var}(\bar{v}|\bar{p}) \), respectively. Changes in expected 
price efficiency are larger for values of \( q \) for which insiders trade relatively more aggressively 
as discussed above. Here we also see clear evidence of the dual role of \( q \). With respect to 
market liquidity we interpreted \( q \) – because it determines the standard deviation of \( I_2 \) – as 
parameterizing how much additional noise there is in the market due to uncertainty about 
the number of informed traders. If \( q \) only influenced market noise (as does \( \sigma_n^2 \)) we would 
expect price efficiency to be unaffected by changes in \( q \). Informed traders would simply scale 
up or down their orders proportional to the change in the market noise and price efficiency 
would remain as before. We see from figure 4A that this is not the case. It is clear that 
changes in \( q \) impact the equilibrium by changing both the amount of noise in the market 
and the implicit level of competition in the market. More competition (imagined or real) 
on average creates more efficient prices.

Figure 4B shows that although prices are more efficient on average for increasing \( q \), 
there are significant regions (in \( p-q \) space) where prices are less efficient than those when 
\( q = 0 \). Specifically, prices are relatively inefficient for low values of \( q \) and moderate values 
of \( p \) (e.g. less than one standard deviation away from \( p_0 \)). As discussed above, this is the 
region where insiders are using the additional noise in the market to their best advantage 
in gaming the market maker. Prices are less efficient in this region because it is the region 
of highest ambiguity with respect to whether the informed order flow is informationally or 
competitively driven.
5 Conclusion

We are aware (Holden and Subrahmanyam (1992), Cao (1994)) that competition causes informed traders with similar or identical information to mitigate their strategic behavior. In previous models the characteristics and distribution of information among agents was known. I have studied the impact of uncertainty in the distribution of private information. I focused on the case of a monopolistically informed trader who is skeptical of the uniqueness of his information. I find that such skepticism (when shared by the market maker) plays a dual role in the market by both increasing the level of implicit competition and acting as an additional source of noise in the market. I find that uncertainty with respect to the number of informed traders in the market:

1. Increases market liquidity by introducing an additional source of noise in the market.

2. Decreases the intensity of each individual’s informed trading, but increases the total expected level of informed trade by increasing the implicit level of competition in the market.

3. Decreases the expected profit of an informed trader, but increases the expected profit of an (ex post) unwarrantedly skeptical monopolistically informed trader.

4. Increases the average price efficiency in the market, while significantly decreasing the price efficiency in certain well-specified situations.

These results are consistent with the conjecture that a monopolist insider’s uncertainty about the uniqueness of his information may, in the limit as the frequency of auctions increases, induce the monopolist to fully reveal his information almost immediately. Further research is required to confirm if this is so.
A Methodology overview

This appendix provides a brief introduction to the general projection method approach used in this paper. The approach is well known and well documented. (See, e.g., Judd (1992), Judd (1993) and Judd and Bernardo (1993).) This appendix is freely adapted from the referenced sources and no original insights are provided.

Suppose the solution to an economic problem requires determining the zero of an operator equation

$$\mathbb{K}(f) = 0$$  \hspace{1cm} (A1)

where $f : D \subset \mathbb{R}^N \rightarrow \mathbb{R}^M$, $\mathbb{K} : B_1 \rightarrow B_2$, and $B_1$ and $B_2$ are normed vector spaces of functions. In this paper the function domain $D$ contains the state variables $(\bar{u}, \bar{I}, \bar{u})$, the unknown function $f$ is a vector containing the trading strategy and price function, and the operator equation in (A1) represents the first order and market efficiency conditions.

The first step is to choose a class of functions with which to approximate solutions. A common choice, and the one made here, is to assume that the approximate solution, $\hat{f}$, is a linear combination of simple (e.g., basis) functions. Therefore, we choose a basis set of functions $\Phi = \{\phi_i\}_{i=1}^\infty$ that spans $B_1$ and adopt an appropriate measure of distance between functions (i.e., an inner product $\langle \cdot, \cdot \rangle$ over $B_2$) so that we can measure the accuracy of our approximations. In addition, if $\mathbb{K}$ is not directly computable, we choose a computable approximation $\hat{\mathbb{K}}$. $\mathbb{K}$ may not be directly computable, for example, if it involves analytically non-tractable integrals requiring numerical approximations. This is the case in my paper.

Since $f$ satisfies (A1), we seek an $\hat{f}$ which makes $\hat{\mathbb{K}}(\hat{f})$ “close” to zero. “Closeness” is with respect to the inner product previously chosen. For a given degree of approximation $n$, the approximate solution is of the form $\hat{f} = \sum_{i=1}^n a_i \phi_i$. Therefore, determining the optimal $\hat{f}$ becomes a matter of finding the $n$-dimensional vector $a$ that makes $\hat{\mathbb{K}}(\hat{f})$ closest to zero. In general, increasing $n$ yields better approximations and we choose the smallest $n$ which yields an acceptable approximation. Several aspects of the above procedure warrant further discussion.
A.1 The choice of a basis and inner product

Several considerations guide the choice of a basis. To economize on computational overhead basis elements should be easy to compute and “efficient” for the problem at hand. “Efficiency” implies that useful approximations are derived using few elements, i.e., with small $n$. Two characteristics contributing to efficiency are: (i) basis elements should have a form similar to the solution, and (ii) basis elements should be orthogonal with respect to the inner product chosen. Similarity of basis elements to the solution are typically of a general nature such as ensuring that the basis elements have the same range, smoothness, and periodicity (or lack of same) as the solution. The motivation for choosing an orthogonal basis is similar to the motivation for choosing uncorrelated explanatory variables in a regression - nonorthogonal bases will reduce numerical accuracy just as multicollinear regressors will enlarge confidence intervals. The inner product chosen should be sensitive to errors we care about (e.g., in the tails versus near the mean). We generally use inner products of the form

$$\langle f(x), g(x) \rangle \equiv \int_D f(x)g(x)w(x)dx$$

for some weighting function $w(x) \geq 0$.\textsuperscript{13}

In this paper I have chosen $w(x) = e^{-x^2}$ as the inner product weighting function and Hermite polynomials as the basis. $w(x) = e^{-x^2}$ is a natural choice for the weighting function in settings with normally distributed random variables. Hermite polynomials are orthogonal with respect to this weighting function so they too are a natural choice in settings with normally distributed random variables. The first and second elements of the Hermite polynomials are $H_0(x) = 1$ and $H_1(x) = 2x$, respectively, and the remaining members are given by the recursive relation

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x).$$

A.2 Multidimensional bases

Many problems involve more than one state variable and require the use of multidimensional basis functions. If $\{\phi_i(x)\}_{i=1}^\infty$ is a basis for functions of one variable, then the set of all pairwise products $\{\phi_i(x)\phi_j(y)\}_{i,j=1}^\infty$ is the tensor product basis for functions of two

\textsuperscript{13}Other inner products are feasible. See Judd (1998) for a discussion of this issue.
variables. The definition is similarly extended to $n$-dimensional bases. A disadvantage of tensor product bases is the number of elements increases exponentially with the dimensionality. We can avoid this by restricting ourselves to the smaller set of complete polynomials, which grows only polynomially with degree. This restriction yields approximations that are nearly as good as those formed from the full tensor product basis. The motivation for the use of the complete set of polynomials is the multidimensional version of Taylor’s Theorem:

Suppose $f : \mathbb{R}^{n+1} \to \mathbb{R}$ and is $C^{k+1}$. Then for $x^0 \in \mathbb{R}^{n+1}$

$$f(x) = f(x_0) + \sum_{i=0}^{n} \frac{\partial f}{\partial x_i}(x^0)(x_i - x^0_i) + \frac{1}{2} \sum_{i=0}^{n} \sum_{j=0}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x^0)(x_i - x^0_i)(x_j - x^0_j) + \cdots$$

$$+ \frac{1}{k!} \sum_{i_1=0}^{n} \cdots \sum_{i_k=0}^{n} \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}(x^0)(x_{i_1} - x^0_{i_1})(x_{i_k} - x^0_{i_k}) + O\left(\|x - x^0\|^{k+1}\right)$$

Taylor’s Theorem provides an approximation to $f$ near the point $x^0$. The terms used in the $k$th degree Taylor series expansion are analogous to those used in a $k$th degree complete polynomial. Namely, the $k$th degree Taylor series expansion uses the functions in

$$P_k = \left\{ x_0^{i_0} x_1^{i_1} \cdots x_n^{i_n} \mid \sum_{l=0}^{n} i_l \leq k, \ 0 \leq i_0, i_1, \ldots, i_n \right\}$$

That is, $P_k$ is the complete set of polynomials of total degree $k$ or less over $\mathbb{R}^{n+1}$. It contains far fewer elements than the set containing all the elements of the tensor product of $n$ polynomials of degree $k$: $T_k = \left\{ x_0^{i_0} x_1^{i_1} \cdots x_n^{i_n} \mid 0 \leq i_0 \leq k, \ 0 \leq i \leq n \right\}$. Taylor’s Theorem tells us that, asymptotically, many of the additional elements in the tensor product basis are adding little in terms of the quality of our approximation.
B Analytic intractability

This appendix briefly illustrates how remarkable it is that the approximate equilibrium is linear and indicates the difficulty in solving the model analytically.

It is easy to show that given a linear pricing function an insider’s optimal trading strategy is linear. Furthermore, based on well-known properties of conditionally normal random variables, it can be shown that in a linear equilibrium $\lambda$ must satisfy

$$
\lambda = \frac{\Sigma_0^{1/2}}{\sigma_u} \sqrt{(q + q\hat{q} + 2\hat{q} - 3) + \sqrt{(q + q\hat{q} + 2\hat{q} - 3)^2 + 8(2q - q\hat{q} - 2\hat{q} + 2)}}
$$

(B1)

where $\hat{q} = E[I_2|\omega]$ and is calculated as the likelihood ratio:

$$
\hat{q} = \frac{q(2\pi \sigma_1^2)^{-1/2} \exp\left[-\frac{1}{2} \left( \frac{\omega}{\sigma_1} \right)^2 \right]}{q(2\pi \sigma_1^2)^{-1/2} \exp\left[-\frac{1}{2} \left( \frac{\omega}{\sigma_1} \right)^2 \right] + (1 - q)(2\pi \sigma_0^2)^{-1/2} \exp\left[-\frac{1}{2} \left( \frac{\omega}{\sigma_0} \right)^2 \right]}
$$

where

$$
\sigma_1^2 = \frac{4\Sigma_0}{\lambda^2(2 + q)^2} + \sigma_u^2
$$

$$
\sigma_0^2 = \frac{\Sigma_0}{\lambda^2(2 + q)^2} + \sigma_u^2
$$

Equation (B1) presents a very non-linear relationship between $\lambda$ and $\omega$ and substituting in equation (7) for $\lambda$ does not trivially reduce (B1) to an identity in $\omega$. I therefore classify my numerically approximated equilibrium as a linear approximate equilibrium and consider equations (8) and (9) to be an acceptably accurate representation of the equilibrium (e.g., for purposes of deriving comparative statics, etc.). In this context I view an approximate equilibrium as an acceptable description of the true equilibrium if agents can not materially (in an economic sense) improve their welfare by adopting more sophisticated strategies (i.e., strategies closer to the “true” equilibrium strategies). In this model, more sophisticated strategies involve using higher order polynomials to construct equilibrium strategies. The results show that, even when given the opportunity to adopt quadratic or cubic strategies, informed traders and the market maker “choose” to restrict themselves to (almost exactly) linear functions. In this sense, one can view the analytic representation of the linear approximate-equilibrium as a sufficient description of the “true” equilibrium.
REFERENCES


Table 1
This table compares analytical with numerical solutions for several sets of parameter values for which analytical solutions are known. All approximations are made using third-order Hermite polynomials and five quadrature nodes. Panel A gives comparisons for trading strategy coefficients and expected profit. Trading strategy coefficients apply to simple polynomials of the insiders information $v$. Panel B gives comparisons for pricing function coefficients and expected price efficiency. Pricing function coefficients apply to simple polynomials of the market makers information $o$. For each set of parameter values the numerical results are listed above the analytical results.

### Panel A

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### Panel B

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Figure 1

Liquidity Parameter w.r.t. $q$

Figure 2

Informed Trading Intensity w.r.t. $q$

- Individual insider trade
- $E[\text{total informed trade}]$