A Proof of Theorem 3

The $k \times 1$ random vector $\widetilde{\mathbf{f}} = (\widetilde{f}_j)_{j=1,\ldots,k}$ contains factors. The $n \times k$ matrix $\mathbf{B} = (\beta_{ij})_{\substack{i=1,\ldots,n \\ j=1,\ldots,k}}$ contains factor loadings. $\mathbf{\Psi} = \mathrm{Var}[\widetilde{\mathbf{f}}]$ is the $k \times k$ variance-covariance matrix of factors. The $k \times 1$ vector $\boldsymbol{\tau} = (\tau_j)_{j=1,\ldots,k}$ contains factor risk premia. The $n \times n$ matrix $\mathbf{\Omega} = (\omega_{ij})_{i,j=1,\ldots,n} = (\mathrm{Cov}[\widetilde{e}_i, \widetilde{e}_j])_{i,j=1,\ldots,n}$ contains variances and covariances of residuals. Consider the regression of factors on stocks:

$$\widetilde{f}_j = m_{j1}\widetilde{x}_1 + \ldots + m_{jn}\widetilde{x}_n + \widetilde{\eta}_j, \tag{13}$$

where the projection residual $\widetilde{\eta}_j$ is uncorrelated with asset returns. The coefficients m_{ki} are weights of factor-mimicking portfolios. $\mathbf{M} = (m_{ji})_{\substack{j=1,\ldots,k \\ i=1,\ldots,n}}$ is the $k \times n$ matrix of weights of factor-mimicking portfolios. The $k \times 1$ random vector $\widetilde{\boldsymbol{\eta}} = (\widetilde{\eta}_j)_{j=1,\ldots,k}$ contains the residuals of the projection of factors onto returns. $\mathbf{\Theta} = \mathrm{Var}[\widetilde{\boldsymbol{\eta}}]$ is its $k \times k$ variance-covariance matrix. We have: $\mathbf{\Psi} = \mathbf{M}\mathbf{\Sigma}\mathbf{M}' + \mathbf{\Theta}$.

$\widetilde{\mathbf{f}}_* = \mathbf{M}\widetilde{\mathbf{x}}$ is the $k \times 1$ random vector of returns on factor-mimicking portfolios. $\mathbf{\Psi}_* = \mathrm{Var}[\widetilde{\mathbf{f}}_*]$ is the variance-covariance matrix of $\widetilde{\mathbf{f}}_*$. We have: $\mathbf{\Psi}_* = \mathbf{M}\mathbf{\Sigma}\mathbf{M}'$ and $\mathbf{\Psi} = \mathbf{\Psi}_* + \mathbf{\Theta}$, hence $\mathbf{\Psi} \succ \mathbf{\Psi}_*$, where the symbol $\succ$ represents the ordering between symmetric matrices. A useful implication is that $\mathbf{\Psi}\mathbf{\Psi}_*^{-1}\mathbf{\Psi} - \mathbf{\Psi} \succ \mathbf{0}_k$, where $\mathbf{0}_k$ is the $k \times k$ null matrix.

The coefficients of the regression of stock returns on $\widetilde{\mathbf{f}}_*$ are:

$$\begin{aligned}
\mathbf{B}_* &= \mathrm{Cov}[\widetilde{\mathbf{x}}, \widetilde{\mathbf{f}}_*']\mathrm{Var}[\widetilde{\mathbf{f}}_*]^{-1} && (14)\\
&= \mathrm{Cov}[\widetilde{\mathbf{x}}, \widetilde{\mathbf{f}}']\mathrm{Var}[\widetilde{\mathbf{f}}_*]^{-1} && (15)\\
&= \mathrm{Cov}[\widetilde{\mathbf{x}}, \widetilde{\mathbf{f}}']\mathbf{\Psi}_*^{-1} && (16)
\end{aligned}$$

$$= \text{Cov}[\tilde{\mathbf{x}}, \tilde{\mathbf{f}}']\boldsymbol{\Psi}^{-1}\boldsymbol{\Psi}\boldsymbol{\Psi}_*^{-1} \quad (17)$$

$$= \mathbf{B}\boldsymbol{\Psi}\boldsymbol{\Psi}_*^{-1} \quad (18)$$

Let $\boldsymbol{\Omega}_*$ denote the covariance matrix of the residuals of the projection of stock returns on $\tilde{\mathbf{f}}_*$. We have: $\boldsymbol{\Sigma} = \mathbf{B}\boldsymbol{\Psi}\mathbf{B}' + \boldsymbol{\Omega} = \mathbf{B}_*\boldsymbol{\Psi}_*\mathbf{B}_*' + \boldsymbol{\Omega}_*$. Therefore:

$$\boldsymbol{\Omega} - \boldsymbol{\Omega}_* = \mathbf{B}_*\boldsymbol{\Psi}_*\mathbf{B}_*' - \mathbf{B}\boldsymbol{\Psi}\mathbf{B}' \quad (19)$$

$$= (\mathbf{B}\boldsymbol{\Psi}\boldsymbol{\Psi}_*^{-1})\boldsymbol{\Psi}_*(\mathbf{B}\boldsymbol{\Psi}\boldsymbol{\Psi}_*^{-1})' - \mathbf{B}\boldsymbol{\Psi}\mathbf{B}' \quad (20)$$

$$= \mathbf{B}\boldsymbol{\Psi}\boldsymbol{\Psi}_*^{-1}\boldsymbol{\Psi}\mathbf{B}' - \mathbf{B}\boldsymbol{\Psi}\mathbf{B}' \quad (21)$$

$$= \mathbf{B}(\boldsymbol{\Psi}\boldsymbol{\Psi}_*^{-1}\boldsymbol{\Psi} - \boldsymbol{\Psi})\mathbf{B}' \quad (22)$$

$$\succ \mathbf{0}_n. \quad (23)$$

Therefore the largest eigenvalue of $\boldsymbol{\Omega}$ exceeds the largest eigenvalue of $\boldsymbol{\Omega}_*$. This completes the proof of Theorem 3.

B Proof of Theorem 5

Decompose the covariance matrix of stock returns $\boldsymbol{\Sigma}$ into eigenvalues and eigenvectors: $\boldsymbol{\Sigma} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}'$. The diagonal elements of the $n \times n$ matrix $\boldsymbol{\Lambda}$ are the eigenvalues $\lambda_1, \ldots, \lambda_n$ of $\boldsymbol{\Sigma}$, and the off-diagonal elements are equal to zero. The eigenvalues are sorted in descending order. The column vectors of the $n \times n$ orthogonal matrix $\mathbf{U}$ are the eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_n$ of $\boldsymbol{\Sigma}$.

Let $\mathbf{U}_j$ denote the $n \times k$ matrix containing the first k columns of $\mathbf{U}$. The k factors are the returns on the portfolios whose weights are the column vectors of $\mathbf{U}_j$: $\tilde{\mathbf{f}} = \mathbf{U}_j'\tilde{\mathbf{x}}$. The matrix of

betas is: $\mathbf{B} = \Sigma \mathbf{U}_j (\mathbf{U}_j' \Sigma \mathbf{U}_j)^{-1} = \mathbf{U}_j$.

The $k \times 1$ random vector $\widehat{\boldsymbol{\tau}} = (\widehat{\tau}_j)_{j=1,\ldots,k}$ contains estimates of risk premia based on t iid observations. The variance-covariance matrix of estimated risk premia is: $\mathrm{Var}[\widehat{\boldsymbol{\tau}}] = \mathrm{Var}[\widetilde{\mathbf{f}}]/t = \mathbf{U}_j' \Sigma \mathbf{U}_j / t$. It is the diagonal matrix containing the top k eigenvalues of Σ divided by t.

The beta pricing equation with estimated risk premia is:

$$\mathrm{E}[\widetilde{x}_i] \approx \sum_{j=1}^{k} \beta_{ij} \widehat{\tau}_j \tag{24}$$

The expected sum of squared deviations from Equation (24) is:

$$\begin{aligned}
\mathrm{E}\left[\sum_{i=1}^{n}\left(\mathrm{E}\left[\widetilde{x}_i\right] - \sum_{j=1}^{k}\beta_{ij}\widehat{\tau}_j\right)^2\right] &= \sum_{i=1}^{n}\left(\mathrm{E}\left[\widetilde{x}_i\right] - \sum_{j=1}^{k}\beta_{ij}\tau_j\right)^2 + \sum_{i=1}^{n}\mathrm{Var}\left[\sum_{j=1}^{k}\beta_{ij}\widehat{\tau}_j\right] && (25)\\
&= \sum_{i=1}^{n}\left(\mathrm{E}\left[\widetilde{x}_i\right] - \sum_{j=1}^{k}\beta_{ij}\tau_j\right)^2 + \sum_{i=1}^{n}\sum_{j=1}^{k}\beta_{ij}^2\mathrm{Var}\left[\widehat{\tau}_j\right] && (26)\\
&= \sum_{i=1}^{n}\left(\mathrm{E}\left[\widetilde{x}_i\right] - \sum_{j=1}^{k}\beta_{ij}\tau_j\right)^2 + \sum_{j=1}^{k}\mathrm{Var}\left[\widehat{\tau}_j\right] && (27)\\
&= \sum_{i=1}^{n}\left(\mathrm{E}\left[\widetilde{x}_i\right] - \sum_{j=1}^{k}\beta_{ij}\tau_j\right)^2 + \sum_{j=1}^{k}\frac{\lambda_j}{t}. && (28)
\end{aligned}$$

This completes the proof of Theorem 5.

C Proof of Theorem 6

The $n \times 1$ vector $\mathbf{e}$ contains the endowment of traded assets. Let $\widetilde{h}$ denote the future value of all non-traded assets. The representative agent has risk aversion A. Her utility maximization problem is equivalent to: $\max_{\mathbf{w}} \mathbf{w}'\boldsymbol{\mu} - \frac{1}{2}A(\mathbf{w}'\Sigma\mathbf{w} + 2\mathbf{w}'\mathrm{Cov}[\widetilde{\mathbf{x}}, \widetilde{h}])$, where the $n \times 1$ vector $\mathbf{w} = (w_i)_{i=1,\ldots,n}$

contains portfolio weights. Let $\mathbf{h} = \Sigma^{-1}\text{Cov}[\tilde{\mathbf{x}}, \tilde{h}]$ denote the portfolio whose future value most closely mimics the future value of non-traded assets. The solution to the representative agent's utility maximization problem is: $\mathbf{w} = -\mathbf{h} + \Sigma^{-1}\boldsymbol{\mu}/A$. The agent first hedges her exposure to non-traded risk by shorting portfolio $\mathbf{h}$, and then adds a mean-variance efficient position. The market clearing condition is $\mathbf{w} = \mathbf{e}$, therefore in equilibrium: $\boldsymbol{\mu} = A\Sigma(\mathbf{e} + \mathbf{h})$.

The maximum squared Sharpe measure in the market is: $\overline{S}^2 = \boldsymbol{\mu}'\Sigma^{-1}\boldsymbol{\mu} = A^2(\mathbf{e}'\Sigma\mathbf{e} + 2\mathbf{e}'\Sigma\mathbf{h} + \mathbf{h}'\Sigma\mathbf{h})$. The squared Sharpe measure of the market portfolio is: $S_M^2 = (\boldsymbol{\mu}'\mathbf{e})^2/(\mathbf{e}'\Sigma\mathbf{e}) = A^2(\mathbf{e}'\Sigma\mathbf{e} + \mathbf{e}'\Sigma\mathbf{h})^2/(\mathbf{e}'\Sigma\mathbf{e})$. Therefore we have:

$$\frac{\overline{S}^2}{S_M^2} - 1 = \frac{(\mathbf{e}'\Sigma\mathbf{e})\,(\mathbf{h}'\Sigma\mathbf{h}) - (\mathbf{e}'\Sigma\mathbf{h})^2}{(\mathbf{e}'\Sigma\mathbf{e} + \mathbf{e}'\Sigma\mathbf{h})^2}. \tag{29}$$

The numerator on the right hand side of Equation (29) is no greater than $(\mathbf{e}'\Sigma\mathbf{e})(\mathbf{h}'\Sigma\mathbf{h})$. In Theorem 6, it is assumed that the covariance between traded and non-traded assets is non-negative, which ensures that $\mathbf{e}'\Sigma\mathbf{h} \geq 0$. Therefore the denominator on the right hand side of Equation (29) is at least as great as $(\mathbf{e}'\Sigma\mathbf{e})^2$. It implies that:

$$\frac{\overline{S}^2}{S_M^2} - 1 \leq \frac{\mathbf{h}'\Sigma\mathbf{h}}{\mathbf{e}'\Sigma\mathbf{e}}. \tag{30}$$

The variance of the value of non-traded assets v_{NT} is even higher than the variance of its projection onto the market $\mathbf{h}'\Sigma\mathbf{h}$. Noting that $v_T = \mathbf{e}'\Sigma\mathbf{e}$ completes the proof of Theorem 6.

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