A Proof of Theorem 3

The $k \times 1$ random vector $\tilde{f} = (\tilde{f}_j)_{j=1,\ldots,k}$ contains factors. The $n \times k$ matrix $B = (\beta_{ij})_{j=1,\ldots,k}^{i=1,\ldots,n}$ contains factor loadings. $\Psi = \text{Var}[\tilde{f}]$ is the $k \times k$ variance-covariance matrix of factors. The $k \times 1$ vector $\tau = (\tau_j)_{j=1,\ldots,k}$ contains factor risk premia. The $n \times n$ matrix $\Omega = (\omega_{ij})_{i,j=1,\ldots,n} = (\text{Cov}[\tilde{e}_i, \tilde{e}_j])_{i,j=1,\ldots,n}$ contains variances and covariances of residuals. Consider the regression of factors on stocks:

$$\tilde{f}_j = m_{j1}\tilde{x}_1 + \ldots + m_{jn}\tilde{x}_n + \tilde{\eta}_j,$$

where the projection residual $\tilde{\eta}_j$ is uncorrelated with asset returns. The coefficients $m_{ki}$ are weights of factor-mimicking portfolios. $M = (m_{ji})_{j=1,\ldots,k}^{i=1,\ldots,n}$ is the $k \times n$ matrix of weights of factor-mimicking portfolios. The $k \times 1$ random vector $\tilde{\eta} = (\tilde{\eta}_j)_{j=1,\ldots,k}$ contains the residuals of the projection of factors onto returns. $\Theta = \text{Var}[\tilde{\eta}]$ is its $k \times k$ variance-covariance matrix. We have: $\Psi = M\Sigma M' + \Theta$.

$\tilde{f}_* = M\tilde{x}$ is the $k \times 1$ random vector of returns on factor-mimicking portfolios. $\Psi_* = \text{Var}[\tilde{f}_*]$ is the variance-covariance matrix of $\tilde{f}_*$. We have: $\Psi_* = M\Sigma M'$ and $\Psi = \Psi_* + \Theta$, hence $\Psi \succ \Psi_*$, where the symbol $\succ$ represents the ordering between symmetric matrices. A useful implication is that $\Psi \Psi_*^{-1} \Psi - \Psi \succ 0_k$, where $0_k$ is the $k \times k$ null matrix.

The coefficients of the regression of stock returns on $\tilde{f}_*$ are:

$$B_* = \text{Cov}[\tilde{x}, \tilde{f}_*']\text{Var}[\tilde{f}_*]^{-1}$$

$$= \text{Cov}[\tilde{x}, \tilde{f}_*']\text{Var}[\tilde{f}_*]^{-1}$$

$$= \text{Cov}[\tilde{x}, \tilde{f}_*']\Psi_*^{-1}$$


\[ = \text{Cov}[\tilde{x}, \tilde{f}'] \Psi^{-1} \Psi_s^{-1} \]
\[ = B \Psi \Psi_s^{-1} \]  \hspace{1cm} (18)

Let \( \Omega_* \) denote the covariance matrix of the residuals of the projection of stock returns on \( \tilde{f}_* \). We have: \( \Sigma = B \Psi B' + \Omega = B \Psi, B'_* + \Omega_* \). Therefore:

\[ \Omega - \Omega_* = B \Psi, B'_* - B \Psi B' \]  \hspace{1cm} (19)

\[ = (B \Psi \Psi_s^{-1}) \Psi_s (B \Psi \Psi_s^{-1})' - B \Psi B' \]  \hspace{1cm} (20)

\[ = B \Psi \Psi_s^{-1} \Psi B' - B \Psi B' \]  \hspace{1cm} (21)

\[ = B (\Psi \Psi_s^{-1} \Psi - \Psi) B' \]  \hspace{1cm} (22)

\[ = 0_n. \]  \hspace{1cm} (23)

Therefore the largest eigenvalue of \( \Omega \) exceeds the largest eigenvalue of \( \Omega_* \). This completes the proof of Theorem 3.

\section*{B Proof of Theorem 5}

Decompose the covariance matrix of stock returns \( \Sigma \) into eigenvalues and eigenvectors: \( \Sigma = U \Lambda U' \). The diagonal elements of the \( n \times n \) matrix \( \Lambda \) are the eigenvalues \( \lambda_1, \ldots, \lambda_n \) of \( \Sigma \), and the off-diagonal elements are equal to zero. The eigenvalues are sorted in descending order. The column vectors of the \( n \times n \) orthogonal matrix \( U \) are the eigenvectors \( u_1, \ldots, u_n \) of \( \Sigma \).

Let \( U_j \) denote the \( n \times k \) matrix containing the first \( k \) columns of \( U \). The \( k \) factors are the returns on the portfolios whose weights are the column vectors of \( U_j \): \( \tilde{f} = U_j' \tilde{x} \). The matrix of
betas is: \( \mathbf{B} = \Sigma \mathbf{U}_j (\mathbf{U}_j' \Sigma \mathbf{U}_j)^{-1} = \mathbf{U}_j. \)

The \( k \times 1 \) random vector \( \vec{\tau} = (\vec{\tau}_j)_{j=1,\ldots,k} \) contains estimates of risk premia based on \( t \) iid observations. The variance-covariance matrix of estimated risk premia is: \( \text{Var}[\vec{\tau}] = \text{Var}[\vec{\epsilon}] / t = \mathbf{U}_j' \Sigma \mathbf{U}_j / t. \) It is the diagonal matrix containing the top \( k \) eigenvalues of \( \Sigma \) divided by \( t \).

The beta pricing equation with estimated risk premia is:

\[
E[\vec{x}_i] \approx \sum_{j=1}^{k} \beta_{ij} \vec{\tau}_j
\]  

(24)

The expected sum of squared deviations from Equation (24) is:

\[
E \left[ \sum_{i=1}^{n} \left( E[\vec{x}_i] - \sum_{j=1}^{k} \beta_{ij} \vec{\tau}_j \right)^2 \right] = \sum_{i=1}^{n} \left( E[\vec{x}_i] - \sum_{j=1}^{k} \beta_{ij} \vec{\tau}_j \right)^2 + \sum_{i=1}^{n} \text{Var} \left[ \sum_{j=1}^{k} \beta_{ij} \vec{\tau}_j \right] \quad (25)
\]

\[
= \sum_{i=1}^{n} \left( E[\vec{x}_i] - \sum_{j=1}^{k} \beta_{ij} \vec{\tau}_j \right)^2 + \sum_{i=1}^{n} \sum_{j=1}^{k} \beta_{ij}^2 \text{Var} \left[ \vec{\tau}_j \right] \quad (26)
\]

\[
= \sum_{i=1}^{n} \left( E[\vec{x}_i] - \sum_{j=1}^{k} \beta_{ij} \vec{\tau}_j \right)^2 + \sum_{j=1}^{k} \text{Var} \left[ \vec{\tau}_j \right] \quad (27)
\]

\[
= \sum_{i=1}^{n} \left( E[\vec{x}_i] - \sum_{j=1}^{k} \beta_{ij} \vec{\tau}_j \right)^2 + \sum_{j=1}^{k} \frac{\lambda_j}{t}. \quad (28)
\]

This completes the proof of Theorem 5.

C Proof of Theorem 6

The \( n \times 1 \) vector \( \mathbf{e} \) contains the endowment of traded assets. Let \( \widehat{\mathbf{h}} \) denote the future value of all non-traded assets. The representative agent has risk aversion \( A \). Her utility maximization problem is equivalent to: \( \max_{\mathbf{w}} \mathbf{w}' \mathbf{\mu} - \frac{1}{2} A (\mathbf{w}' \Sigma \mathbf{w} + 2 \mathbf{w}' \text{Cov}[\mathbf{x}, \widehat{\mathbf{h}}]) \), where the \( n \times 1 \) vector \( \mathbf{w} = (w_i)_{i=1,\ldots,n} \)
contains portfolio weights. Let \( h = \Sigma^{-1} \text{Cov}[x, \tilde{h}] \) denote the portfolio whose future value most closely mimics the future value of non-traded assets. The solution to the representative agent’s utility maximization problem is: \( w = -h + \Sigma^{-1} \mu/A \). The agent first hedges her exposure to non-traded risk by shorting portfolio \( h \), and then adds a mean-variance efficient position. The market clearing condition is \( w = e \), therefore in equilibrium: \( \mu = A \Sigma (e + h) \).

The maximum squared Sharpe measure in the market is: \( \bar{S}^2 = \mu' \Sigma^{-1} \mu = A^2 (e' \Sigma e + 2e' \Sigma h + h' \Sigma h) \). The squared Sharpe measure of the market portfolio is: \( S^2_M = (\mu' e)^2 / (e' \Sigma e) = A^2 (e' \Sigma e + e' \Sigma h)^2 / (e' \Sigma e) \). Therefore we have:

\[
\frac{\bar{S}^2}{S^2_M} - 1 = \frac{(e' \Sigma e) (h' \Sigma h) - (e' \Sigma h)^2}{(e' \Sigma e + e' \Sigma h)^2}.
\]  
(29)

The numerator on the right hand side of Equation (29) is no greater than \((e' \Sigma e)(h' \Sigma h)\). In Theorem 6, it is assumed that the covariance between traded and non-traded assets is non-negative, which ensures that \( e' \Sigma h \geq 0 \). Therefore the denominator on the right hand side of Equation (29) is at least as great as \((e' \Sigma e)^2\). It implies that:

\[
\frac{\bar{S}^2}{S^2_M} - 1 \leq \frac{h' \Sigma h}{e' \Sigma e}.
\]

(30)

The variance of the value of non-traded assets \( v_{NT} \) is even higher than the variance of its projection onto the market \( h' \Sigma h \). Noting that \( v_T = e' \Sigma e \) completes the proof of Theorem 6.
References


