#13-95

Hedging Long Maturity Commodity Commitments With Short-Dated Futures Contracts

July 1995

Michael J. Brennan

and

Nicholas I. Crew
The Anderson School of Management
University of California, Los Angeles, CA 90095

Finance Working Paper
sponsored by

CMS CAPITAL MANAGEMENT SCIENCES

Hedging Long Maturity Commodity Commitments with Short-dated Futures Contracts

M.J. Brennan[1]
N.I. Crew[2]

July, 1995

[1] Irwin and Goldyne Hearsh Professor of Banking and Finance, University of California, Los Angeles, and Professor of Finance, London Business School.
[2] Doctoral student, University of California, Los Angeles.

Abstract

The problem of hedging a long term commitment to deliver a fixed amount of a commodity by trading in short term futures contracts is analyzed. Conditions under which a simple or tailed stack and roll stategy will yield a perfect hedge are developed, and the relation between the 'rollover gains' from a stack and roll policy and the economic returns from the policy is clarified. The stack and roll policy presumes very restricted variation in the behavior of futures prices. Models of Brennan (1987, 1991) and Gibson and Schwartz (1990) allow for a richer range of behavior in futures prices by introducing a stochastic model of the dynamics of the commodity convenience yield. The futures hedging strategies implied by these models are developed and compared with the stack and roll hedge and a minimum variance hedge, using monthly price data on NYMEX light oil futures contracts.

Hedging Long Maturity Commodity Commitments with Short-dated Futures Contracts

The Metallgesellschaft incident of December 1993 has led to interest in the related issues of the extent to which long dated commitments to deliver (or receive) a fixed amount of a commodity can be hedged by rolling over a series of short term futures contracts, and how this can best be accomplished[1]. In this paper we analyze the hedging problem in terms that are familiar to economists, and consider the ability of different models of futures prices to yield trading strategies in short dated futures contracts that provide effective hedges for long-term commitments.

In a series of papers, Culp and Miller (1994, 1995) analyze this problem with special reference to Metallgesellschaft, and present numerical examples that suggest that a perfect hedge is possible. As we shall show, their analysis rests on the implicit assumption that the convenience yield on the underlying commodity is deterministic, an assumption that is in conflict with the data. They argue correctly that the funding required to meet the variation margin on the hedge is an inappropriate measure of the interim success of the strategy, but their analysis, and that of Edwards and Canter(1995), is presented in terms of 'rollover gains', a concept which is foreign to economists and which, we shall argue, does not correspond to the economic concept of gain or profit. In Section I we show how the final profit on a commitment matched by a 'stack-and roll' hedge is related to the sum of the 'rollover gains' over the life of the commitment, but show that the rollover gain in any single period bears no relation to the economic profit earned in that period.

---

[1] See Culp and Miller (1994, 1995),Edwards and Canter (1995), Mello and Parsons (1995), Neuberger (1995), and Ross (1995).

The difficulty in measuring the interim success of a hedging program is that in general there exists no market price for the underlying commitment that is being hedged. It is therefore necessary to have a *model* of how the price is determined, both in order to determine the appropriate hedge strategy, and to measure the current position at a point in time prior to the maturity of the commitment[2]. Such models of futures or forward prices[3] have been developed by Brennan (1987)[4] and Gibson and Schwartz (1990).[5] These models and the hedging strategies implied by them are presented in Section II.

In Section III we describe the data that we use in our empirical evaluations of hedging effectiveness. In Section IV we evaluate the hedge strategies derived from the models of futures prices, by examining their ability to provide trading strategies in short dated futures contracts to hedge forward commitments in oil, with maturities corresponding to those of extant long-dated traded futures contracts. We choose these particular commitments to hedge because the availability of the long dated futures price allows us to calculate the present value of the commitment each period, which permits us to measure the periodic hedging errors under the strategies. Of course, we would prefer to be able to assess directly the ability of the strategies to hedge much longer-term commitments[6]; while we are not able to do this,

---

[2] Strictly speaking, prior to the period in which the commitment corresponds to the longest available futures contract.

[3] In this paper we assume that interest rates are non-stochastic so that futures and forward prices are the same. See Cox, Ingersoll and Ross (1981).

[4] Later published as Brennan (1991).

[5] Garbade (1993) presents a model which is a special case of Brennan (1991).

[6] The commitments undertaken by Metallgesellschaft had maturities of 5-10 years.

our results for commitments of maturities up to 24 months should provide useful indications of the efficacy of these strategies, since the evidence is that the futures price curve is essentially flat for longer maturities. For comparison, we also analyze the stack and roll hedge proposed by Culp and Miller(1994, 1995), and a minimum variance hedge suggested by Edwards and Canter (1995). We find that hedge strategies that take account of stochastic variation in the convenience yield lead to monthly hedging errors with a standard deviation which is only about 25% of that obtained from the simpler stack and roll strategy.

I

Rollover Gains and the Profits on a Hedged Commitment

Consider a firm that makes a forward commitment to deliver one unit of a commodity, say a barrel of oil, at time T, in return for a payment of \$K at that time. The present value of the profit of entering into this commitment is $Ke^{-rT} - PV_0(P_T)$, where r is the interest rate, P_T is the spot price at time T and $PV_t(P_T)$ denotes the present value at time t of the (uncertain) future amount P_T to be received at time T. If this commitment is held to maturity, uncertainty about P_T means that the total profit realized from entering into and meeting the commitment will be uncertain. This uncertainty could in principle be eliminated by entering into a futures contract of maturity T. However, in practice, there is no market for long term futures contracts. One suggested solution is to roll over a series of long positions in a short maturity futures contract which, without loss of generality, we shall take as the nearby futures contract: such a hedging strategy is often referred to as a 'stack-and roll' strategy.

To analyze the profit under a stack and roll strategy it will be convenient initially to assume that the interest rate is zero. Let $F_{t,\tau}$ denote the futures price at time t for delivery at time $t+\tau$. Then π_t, the profit that is realized at time t from holding the commitment from time t-1, matched by an offsetting long position of n 1-period futures contracts, is equal to the change in the value of the futures position less the change in the present value of the delivery commitment:

$$\pi_t = -[PV_t(P_T) - PV_{t-1}(P_T)] + n[F_{t,0} - F_{t-1,1}] \tag{1}$$

The cumulative profit from entering into the commitment and hedging it in this fashion till maturity is obtained by summing equation (1) over $t=1,..,T$, and adding the expression for the profit realized when the contract is entered into, $Ke^{-rT} - PV_0(P_T)$. Recognizing that $F_{T,0} = PV_T(P_T) = P_T$, the cumulative profit, Π, when the hedge ratio, n, is set equal to unity, may be written as:

$$\Pi = K - F_{0,1} + \sum_{t=1}^{T-1} (F_{t,0} - F_{t,1}) \tag{2}$$

The quantity $(F_{t,0} - F_{t,1})$ is referred to by Culp and Miller (1994, 1995) and Edwards and Canter (1995) as the 'rollover gain'; thus the cumulative profit is the difference between the contract price K and the one-period forward price, $F_{0,1}$, plus the cumulative rollover gains realized over the life of the strategy. The intuition behind the expression is that if it were possible to enter a futures contract at time 0 at a price $F_{0,1}$ and hold the contract until maturity at time T when its price would be equal to the spot price, the realized profit would simply be $K - F_{0,1}$, the gains on the future contract exactly offsetting changes in the price of the underlying commodity. The stack and roll strategy does this, except for the price 'gaps' that arise when one contract is closed out and another is entered into; it is these price gaps that give rise to the rollover gains or losses.

Equation (2) is no more than an accounting identity, and although it is tempting to identify the amount of the rollover gain realized in a period as the part of the final aggregate profit that is realized in that period, such an identification is wrong. Note first that the rollover gain does not appear in expression (1) for the economic profit realized in the period

ending at time t; hence the rollover gain is not the same as the economic profit. In fact, economic gains and losses in futures markets can arise only from *changes* in the prices of the *same* futures contract, and not from *differences* between the prices of *different* futures contracts at a point in time which is what the 'rollover gain' is.

Thus far we have assumed that the interest rate is zero. To allow for a non-zero interest rate, r, it is necessary to 'tail' the hedge by adjusting the number of futures contracts, so that n, the number of one-period futures contracts entered into at time t is equal to $e^{-r(T-t-1)}$.[7] Thus with a tailed stack and roll hedge, the second term in equation (1) is multiplied by $e^{-r(T-t)}$, and the aggregate realized profit will still be given by equation (2).

It is apparent that the profit from a tailed stack and roll hedge will be riskless only if the *rollover gains* are certain when the contract is entered into. This requires that the basis between the price of the one period futures contract and the maturing contract be predictable. Figure 1 shows the end of month rollover gains for the 2 month NYMEX light oil futures contract[8] between 1983 and 1994. It is apparent that the rollover gains are highly volatile and, while the mean is positive, it is only \$0.14 per barrel per month, and the standard deviation is \$0.34. Moreover, since the serial correlation of the rollover gains is 0.76, it would be rash to suppose that the rollover risk could be diversified away over time

---

[7] More precisely, to account for daily settlement, the number of contracts should be adjusted daily to e^{-ry}, where y is the exact number of years to maturity of the commitment from the following day. The formula in the text implicitly assumes that settling up takes place only at the maturity of each futures contract; that is, it treats the futures as short-dated forwards.

[8] The rollover gain is defined as the difference between the price of the nearby futures contract and the next shortest contract, and corresponds to the gains from a strategy of going long in the second nearby contract and rolling over at the end of every month.

for a long-term commitment. Figure 2 shows the corresponding rollover gains for a 3 month contract.[9] The mean of the monthly gains is now \$0.27, and the standard deviation is \$0.65, while the serial correlation is 0.79. As these figures show, there is considerable uncertainty about future rollover gains, and therefore about the final realized profit from a stack and roll hedging strategy.

Equation (1), adjusted for the tailing of the hedge by setting n equal to $e^{-r(T-t-1)}$, implies that the periodic *economic profit* will be riskless only if the change in the future value of the amount to be delivered under the contract (i.e. the change in the implicit futures or forward price for delivery at time T) is equal to the change in the price of the short-dated futures contract. This requires that the implied futures price curve shift up and down in a parallel fashion[10]. Figure 3 plots the futures price curves for the ends of alternate years from 1983 to 1994. It is apparent that the assumption of parallel shifts is not a good one, even within the limited maturity range of traded futures contracts: in some periods the term structure of futures prices slopes up, while in others it slopes down. In the following section we will consider two models of the behavior of futures prices that allow for non-parallel shifts in the term structure of futures prices, and derive the hedging strategies that correspond to them.

---

[9] Defined as the one month futures price less the three month futures price.

[10] The reader will note the analogy with the assumption of parallel shifts in the yield curve that underlies the derivation of the duration model of bond price hedging. Indeed, the stack and roll policy has much in common with a duration matched bond price hedge in Treasury Bill futures.

II

Futures Prices and Hedging with Stochastic Convenience Yields

The convenience yield of a commodity is defined as the flow of services that accrues from possession a physical inventory but not to the owner of a contract for future delivery. The marginal convenience yield includes both the reduction in costs of acquiring inventory, and the value of being able to profit from temporary local (or grade specific) shortages of the commodity through ownership of an additional unit of inventory. The profit may arise either from local price variations, or from the ability to maintain a production process despite local shortages of a raw material. The convenience services yielded by an inventory depend upon the identity of the individual storing it; however, competition between potential storers will ensure that in equilibrium the marginal convenience yield net of storage costs will be equalized across all storers. Then, assuming that there exists a positive inventory of the commodity, the relation between spot and futures prices will reflect this marginal net convenience yield in a manner which we will now develop[11].

It will be helpful to make explicit the dependence of the futures price on the current spot price. Therefore, let $F(P,t,\tau)$ denote the futures price at time t for delivery at $t+\tau$, when the current spot price is P, and let $PVC(t,\tau)$ denote the present value of the marginal net convenience yield over the interval $t,..,t+\tau$. Then the relation between the futures price, the current spot price and the convenience yield may be written:

[11] Previous authors who have discussed the convenience yield include Kaldor(1939), Working (1948, 1949), Brennan (1958), Telser (1958), and Fama and French (1987).

$$F(P,t,\tau)e^{-r\tau} = P - PVC(t,\tau) \tag{3}$$

The left hand side of (3) is the present value of a forward purchase commitment: this is equal to the current spot price less the value of the convenience services that would be available to a storer having physical possession of a marginal unit of the commodity up to the maturity of the forward commitment.

It follows from (3) that the rollover gain from a tailed stack-and-roll strategy is given by:

$$e^{-r(T-t)}[F(P,t,0) - F(P,t,1)] = e^{-r(T-t)}PVC(t,1) \tag{4}$$

Therefore, unless the convenience yield is deterministic so that PVC(t, 1) is known, the rollover gain will be uncertain. Thus, except in this special and unlikely case in which the futures price curve shifts only in a parallel fashion, a stack and roll strategy does not provide a perfect hedge.

In order to model the behavior of the structure of futures prices implied by (3) it is necessary to model the behavior of the marginal convenience yield. The (marginal net) convenience yield depends on the level of inventories; the higher the current level of inventories, the less will merchants and manufacturers be willing to pay to have an additional unit on hand. Since spot prices are also likely to be associated with the level of inventories, a natural simplifying assumption is that the instantaneous rate of convenience yield is a function of the spot price, C(P), where P is the spot price. The simplest assumption is that the convenience yield is proportional to the spot price:

The Constant (proportional) Convenience Yield Model

If the convenience yield is *proportional* to the current spot price: C(P) = cP, then it is not difficult to show[12] that the term structure of futures prices is given by:

$$F(P,t,\tau) = Pe^{(r-c)\tau} \tag{5}$$

and the present value at time t of the uncertain future amount $P_{t+\tau}$ to be received at $t+\tau$ is

$$PV_t(P_{t+\tau}) = e^{-r\tau}F(P,t,\tau) = Pe^{-c\tau} \tag{6}$$

The only source of uncertainty in this setting is the spot price P. The derivative of the present value of a commitment to deliver one barrel of oil in T periods with respect to the spot price is e^{-cT}, and the derivative of a τ period futures price with respect to P is $e^{(r-c)\tau}$. Therefore to hedge the T period delivery commitment it is necessary to take a long position in $n = e^{-[r\tau + c(T-\tau)]}$ futures contracts.

However, as the results of Brennan (1991) show, the constant proportional convenience yield model is too simple to be descriptive of the behavior of futures prices for most commodities. Therefore, we turn to a more realistic model:

The Brennan Autonomous Convenience Yield Model

Brennan (1987, 1991) assumes that C, the instantaneous net marginal rate of convenience yield measured in dollars per unit of inventory per period, follows the simple mean-reverting process:

---

[12] See Brennan and Schwartz (1985).

$$dC = \alpha(m - C)dt + \eta dz_c \tag{7}$$

where $\alpha > 0$, is the speed of adjustment, m is the long run mean rate of convenience yield, and dz_C is the increment to a standard Gauss-Wiener process. This assumption about the behavior of the convenience yield is motivated by the consideration that if the convenience yield is high because inventories are low, storage firms will tend[13] to have an incentive to increase their investment in inventories which, in turn, will tend to reduce the convenience yield. The commodity spot price is assumed to follow the exogenously given stochastic process:

$$\frac{dP}{P} = \mu dt + \sigma(P,C)dz_P \tag{8}$$

where dz_P is the increment to a Gauss-Wiener process and $dz_P dz_C = \rho dt$, and μ, the expected rate of price change, may be stochastic.

Under these assumptions, the futures price will depend upon the current instantaneous rate of convenience yield, as well as the current spot price, so we write it as $F(P,C,\tau)$. It is shown in the Appendix that the futures price under the Brennan model is given by:

$$F(P,C,\tau) = (P - PVC(C,\tau))e^{r\tau} \tag{9}$$

where

and $m^* \equiv m - \lambda^*/\alpha$, where λ^* is a risk adjustment parameter, so that m^* is the risk-adjusted

[13] Only tend to because the investment decisions of storage firms will depend upon the expected rate of change in the commodity price as well as upon the convenience yield.

$$PVC(C,\tau) = \frac{m^*}{r}(1 - e^{-r\tau}) - \frac{m^* - C}{\alpha + r}(1 - e^{-(\alpha+r)\tau}) \tag{10}$$

mean rate of convenience yield.

Consider now the problem of hedging a commitment to deliver one barrel of oil at a date T periods in the future. The present value of the commitment may be written as:

$$PV_0(P_T) = F(P,C,T)e^{-rT} = P - PVC(C,T) \tag{11}$$

Using equations (10) and (11), the derivatives of the present value of the commitment with respect to P and C are:

$$\begin{aligned} \frac{dPV}{dP} &= 1 \\ \frac{dPV}{dC} &= \frac{-1}{\alpha + r}\left(1 - e^{-(\alpha + r)T}\right) \end{aligned} \tag{12}$$

while, using equations (9) and (10), the derivatives of the price of a futures contract with maturity, τ, are:

$$\begin{aligned} \frac{dF}{dP} &= e^{r\tau} \\ \frac{dF}{dC} &= \frac{-1}{\alpha + r}\left(e^{r\tau} - e^{-\alpha\tau}\right) \end{aligned} \tag{13}$$

In order to hedge a commitment of maturity T it is necessary to hold a portfolio of futures contracts with the same sensitivities to P and C. Using equations (12) and (13), the hedge portfolio consists of n_1 and n_2 contracts of maturities τ_1 and τ_2 respectively where n_1 and n_2 are the solutions to:

$$n_1 e^{-\alpha\tau_1} + n_2 e^{-\alpha\tau_2} = 1$$

$$n_1 e^{-\alpha\tau_1} + n_2 e^{-\alpha\tau_2} = e^{-(\alpha+r)T} \tag{14}$$

The Gibson-Schwartz Model

Gibson and Schwartz (1990) follow Brennan in assuming a mean-reverting process for the instantaneous convenience yield; however, they define the instantaneous convenience yield in terms of dollars per *dollar* of inventory per period. This implies that C(P) is written as δP, where δ follows the mean-reverting process:

$$d\delta = k(m_\delta - \delta)dt + \xi dz_\delta \tag{15}$$

with $dz_\delta dz_P = \rho dt$. Arbitrage arguments similar to those developed above imply that the forward or futures price, $F(P, t, \tau)$ can be written as[14]

$$F(P,\delta,\tau) = P\exp\{-\delta(1 - e^{-k\tau})/k + A\tau + \frac{1}{2}v^2\tau\} \tag{16}$$

where:

[14] Jamshidian and Fein (1990) appear to have been the first to develop a closed form expression for the futures price in this model.

$$v^2(\tau) = (\sigma_P^2 + \xi^2/k^2 + 2\rho\sigma_P\xi/k)\tau + \xi^2(1 - e^{-2k\tau})/2k^3 + 2\xi(\sigma_P\rho - \xi/k)(1 - e^{-k\tau})/k^2 \tag{17}$$

and

$$A(\tau) = (r - \frac{1}{2}\sigma_P^2 - \tilde{m})\tau + \tilde{m}(1 - e^{-k\tau})/k \tag{18}$$

$$\tilde{m} = m_\delta - \lambda_\delta\xi/k \tag{19}$$

The hedge portfolio for a commitment of maturity T is found by equating the derivatives of the present value of the commitment with respect to the spot price P and the convenience yield rate, δ, to the corresponding derivatives of the value of the hedge portfolio. Using equations (11) and (16), this yields the following equations for the numbers of futures contracts of maturities τ_1 and τ_2:

$$\begin{aligned} n_1F(P,\delta,\tau_1) + n_2F(P,\delta,\tau_2) &= e^{-rT}F(P,\delta,T) \\ n_1(1 - e^{-k\tau_1})F(P,\delta,\tau_1) + n_2(1 - e^{-k\tau_2})F(P,\delta,\tau_2) &= (1 - e^{-kT})F(P,\delta,T)e^{-rT} \end{aligned} \tag{20}$$

The solutions to equations (14) and (20) determine the composition of the portfolio of futures contracts of maturities τ_1 and τ_2 which will hedge a fixed commitment of maturity T, when futures prices are described by the Brennan, and the Gibson and Schwartz, models of the convenience yield, respectively. Note that to determine the hedge portfolio weights in the Brennan model it is necessary to estimate only the single, speed of adjustment parameter, α. For the Gibson-Schwartz model it is necessary to estimate the speed of adjustment parameter,

δ, and forward price of the commitment being hedged, $F(P, \delta, T)$.[15] This is estimated from equation (16) using the current estimated value of δ and an estimate of the risk-adjusted mean of the convenience yield process, $\mathfrak{m}$. In Section IV below we shall describe the estimation of the speed of adjustment parameters for the two models and the mean parameter, $\mathfrak{m}$, for the Gibson-Schwartz model, and investigate the effectiveness of their implied hedges empirically.

[15] Recall that we wish to evaluate a technique for hedging that does not rely on observation of the commitment value (though our evaluation of the technique does).

III

Data

The data that were used to evaluate the hedging strategies were end of month settlement prices on futures contracts for light crude oil quoted on NYMEX for the period March 1983 to December 1994[16]. We label the contracts as 'nearby', '2 month' , '3 month', etc., these designations corresponding roughly to the remiaing time to maturity of the contracts. Table 1 reports the average of the number of of days to maturity of each of the contracts as of the end of each month of the sample period. Note that the longer maturity contracts only became available towards the end of our sample period, and even then were not available each month; as a result, the number of observations for the 21 and 24 month contracts is only 17 and 15. Continuously compounded interest rates for monthly maturities of 1 to 12 months were taken from the Fama T-Bill files for the end of each month of the sample period; the rates for intermediate maturities were linearly interpolated according to the number of days. Interest rates for maturities beyond 12 months were estimated by assuming that the yield curve was flat beyond 12 months.

For each month from April 1985 to December 1994, α, the speed of adjustment parameter of the Brennan convenience yield model was estimated as follows. The value of the instantaneous convenience yield, c, for each of the prior months was approximated from the prices of the two nearby futures contracts by assuming that the rate of convenience yield

[16] On November 30, 1990 there was a limit move in futures prices which affected all but the nearby contract. Therefore we substituted the prices for November 29th.

was constant over this interval[17]. From this time series of instantaneous convenience yields, α, the speed of adjustment parameter of the stochastic process (7), was estimated using the exact discrete model corresponding to (7). Note that the number of time series observations used to estimate α increses each month as one more historical observation becomes available; the smallest number of observationsis 25 in April 1985.

Similarly, for each month from March 1985 to December 1994, k, the speed of adjustment parameter of the Gibson-Schwartz model, was estimated over the previous months, using estimates of the instantaneous convenience yield constructed under the assumption that δ was constant till the maturity of the second nearest futures contract.[18] Again, the number of time series observations used to estimate δ increases from a minimum of 24 in March 1985. The time series of estimates of α and δ are plotted in Figure 4. The estimates show considerable volatility in the early part of the sample period, reflecting in part the small number of observations used in estimation. By 1989 the estimates of the parameters for both models settledown in the neighbourhood of 0.3 which correponds to a half-life of 2.3 months for deviations of the convenience yield from its long run mean.

It is also necessary to estimate m to implement the hedge for the Gibson-Schwartz model. This was done for each month by finding the value that minimized the sum of the

---

[17] This is similar to the procedure followed by Gibson and Schwartz (1990). Note that with a constant convenience yield, C, the futures price is given by $F(P, C, \tau) = Pe^{r\tau} + (C/r)(1 - e^{r\tau})$. Subtracting two adjacent futures prices yields an estimate of C.

[18] Under the G&S model the expression for the futures price when the convenience yield is constant is $F(P, t, \tau) = Pe^{(r-\delta)\tau}$. The ratio of the two nearby futures prices yields a simple estimate of δ.

squared price prediction errors for the 3-6 month contracts using equations (16)-(18).

IV

Empirical Results

The hedge properties of four basic strategies were evaluated using the oil futures price data. These strategies are the Brennan, the Gibson-Schwartz, the stack and roll, and the Edwards-Canter minimum variance strategy. Two variants of the Brennan and Gibson-Schwartz strategies were implemented: the first using 2 and 3 month futures to construct the hedge portfolio, and the second 2 and 6 month futures.[19] Two variants of the stack and roll and Edwards-Canter hedges were implemented: the simple version and the tailed version. These latter hedges were all implemented using the 2 month futures contract. The hedges were designed to hedge fixed commitments with maturities ranging from 6 to 24 months, and the present values of these commitments were computed each month using the observed futures prices and interest rates for these maturities. All the hedges were revised monthly and the monthly hedge errors were calculated as described below.

Specifically, for each month t, for each liability maturity τ, the hedging error under each of the policies was computed as follows. First, the present value of the liability under the commitment at the beginning of the month was computed by discounting the futures price for the corresponding maturity:

[19] By a τ month contract we mean the contract which is τ^{th} closest to maturity: this will have a maturity which does not exceed τ months. See Table 1.

$$L_{t-1,T} = e^{-r_{t-1,T}T} F_{t-1,T} \tag{21}$$

where $r_{t-1,T}$ is the continuously compounded T period interest rate at the beginning of month t, $F_{t-1,T}$ is the T period futures price at the beginning of the month, and $L_{t-1,T}$ is the value of the liability. The change in the present value of the liability over the month is defined by $\Delta L_{t,T} \equiv L_{t,T-1} - L_{t-1,T}$. Suppose that the hedge consists of n_1 and n_2 futures contracts of maturities τ_1 and τ_2. Then the change in the hedger's wealth over the month, from hedging a T period liability, $\Delta W_{t,T}$ is given by

$$\Delta W_{t,T} = -\Delta L_{t,T} + n_1 \Delta F_1 + n_2 \Delta F_2 \tag{22}$$

where ΔF_s is the change in the τ_s period futures price during month t. Finally, the hedging error, $E_{t,T}$, is defined as the difference between the change in wealth and the riskless return on the present value of the liability:

$$E_{t,T} = \Delta W_{t,T} - (e^{r_{t-1,1}} - 1) L_{t-1,T} \tag{23}$$

For the Brennan and Gibson-Schwartz models, n_1 and n_2 were calculatd from equations (14) and (20) respectively. For the simple stack and roll strategy, $n_1 = 1$, $n_2 = 0$, while for the tailed stack and roll strategy $n_1 = e^{-rT}$. For the Edwards and Canter strategy, $n_1 = \beta$, where β is the coefficient from the regression of the change in the six month futures price on the change in the nearby futures price estimated over all prior months; for the tailed version the coefficient is multiplied by e^{-rT}.[20]

The means and standard deviations of the monthly hedging errors, measured in dollars

[20] The reason that we use the six month contract to calculate β is that we wish to assess the effectiveness of the hedge for a commitment whose market value cannot be observed.

per barrel of oil committed, for the various strategies are reported in Table 2 for commitment maturities of 6-24 months at 3 month intervals. In reviewing this table it is important to remember that these are the errors from hedging the same *fixed maturity* commitment each month.

First we observe that out to 18 months the standard deviation of the hedge errors increases monotonically with the maturity of the commitment; there is a discrete drop in the standard deviations beyond 18 months but this reflects the fact that the 21 and 24 month contracts have been available (and therefore the hedge errors could be calculated) only since December 1990, and even since then these contracts are not traded every month, so that the numbers of observations for these maturities are only 17 and 15, as compared wth 54 for the 18 month maturity. Overall, for all maturities, the worst performing strategy is the simple stack and roll strategy, followed by the tailed version of the strategy. The next worst strategies are generally the Edwards and Canter 'minimum variance' strategies. The best performing strategy throughout is the Gibson-Schwartz strategy executed in the 2 and 6 month futures contracts, and the second best is the Brennan strategy in the same contracts. The Gibson-Schwartz strategy has a standard deviation of $0.40 per month for the 18-month maturity, compared with $1.66 for the simple stack and roll. Overall, it appears that substantial gains in hedging efficiency can be achieved by taking account of the variability in the convenience yield.

Ironically, the ranking in terms of the mean errors for the 18 month maturity is almost exactly reverse the ranking in terms of standard deviations! However, we are not inclined to accept these results at face value, but rather attribute to chance the fact that, for

example, the mean error of the Stack and Roll hedge during this period was $0.01. Figures 5a-f show the performance of the different hedge strategies by aggregating over time the monthly hedging errors for selected strategies. The first three figures relate to the 12 month commitment hedge for which there are 85 monthly observations. The flat parts of the figures correspond to periods when there was no 12 month future outstanding to allow calculation of a hedge error. Figure 5a shows that the Stack and Roll and Edwards and Canter hedges performed similarly whether tailed or not, but that the hedge errors quickly cumulated to $5-10 per barrel. Figure 5b shows the Brennan and Gibson and Schwartz hedges. When implemented in the 2 and 3 month futures they perform badly. The reason for this is that the sensitivity of these futures prices to changes in the convenience yield is not very different; as a result it is necessary to take large offsetting positions in the two maturities to hedge out the convenience yield sensitivity, and this increases the importance of model specification and estimation errors.[21] By comparison, the 2 and 6 month hedge performs very well except in the second half of 1990 when the cumulative error of G&S(2, 6 months) increases by about $2.50. However, this was the time of the Iraqi invasion of Kuwait, an extremely turbulent period in oil markets as shown in Figure 7. The spot price more than doubled from $17.05 in June to $39.50 in September. Figure 5c compares the tailed Stack and Roll hedge with the Gibson and Schwartz (2, 6 months) on the same scale, and shows the substantial gains in effectiveness made possible by the G&S strategy.

---

[21] For example, to hedge a 12 month liability the average position for the G&S 2 and 3 month strategy would be a short position of 2.3 contracts in the 2 month maturity and long 3.2 contracts in the 3 month maturity. The corresponding hedge using 2 and 6 month contracts would be only short 0.3 2 month contracts and long 1.2 6 month contracts.

Figures 5d-f repeat a similar analysis for hedging an 18 month commitment. Again, the Stack and Roll and Edwards and Canter hedges perform similarly, and the Brennan and Gibson and Schwartz (2,6 month) hedges are also similar, and perform well except during the period of extreme price turbulence in 1990. Figures 5g-h relate to a hedge for a 24 month commitment. Here there are only 15 monthly price changes to be hedged, so that the results cannot be regarded as in any way definitive. However, we note that the cumulative error of the G&S (2, 6 month) hedge is less than $1.00, which is about one quarter that of the tailed Stack and Roll hedge.

Figures 6a and 6b show the hedge portfolios for the Edwards and Canter, Brennan and Gibson and Schwartz (2, 6 month) strategies for a 12 month commitment. The Edwards and Canter strategy takes a long position of about 0.7 barrels in the 2 month maturity; the volatility of the hedge in the first few months reflects the small number of observations used to estimate the hedge ratio. Note that the stack and roll hedge (before tailing) takes a long position in one barrel per barrel of commitment. The Brennan, and Gibson and Schwartz, strategies take very similar positions - roughly 1.3 barrels long in the 6 month maturity and 0.2 barrels short in the 2 month maturity. There is relatively little variation in the hedge position over time (once the initial volatility due to estimation error is passed), despite the wide variation we have noted in the level and slope of the futures pricing curves

V

Conclusion

In this paper we have developed two new models for constructing hedges for long term commodity commitments using short term futures contracts, and have compared their performance with that of the simpler Stack and Roll and minimum variance hedges, using monthly data on the prices of light crude oil. We find that the models of Brennan and of Gibson and Schwartz perform significantly better than the two simpler models, the Gibson and Schwartz model reducing the standard deviation of the hedge error by about 75% relative to the Stack and Roll strategy.

Our analysis suffers from two limitations, one empirical and one theoretical. On the empirical side, we have had to limit the maturity of the commitments whose hedges were analyzed to the maturity of the longest available futures contract. Clearly it would be desirable to test the effectiveness of the strategies in hedging longer term commitments if data on the prices of the these could be obtained on a periodic basis. A significant limitation of our theoretical models is the assumption that interest rates are deterministic. For longer time horizons, uncertainty about future interest rates becomes important for the hedge strategies, and therefore it would be desirable to extend the Brennan and Gibson-Schwartz models to allow for stochastic interest rates. We leave this for subsequent work.

Appendix

Applying Ito's Lemma to the futures price, $F(P,C,\tau)$,

$$dF = \left[-F_\tau + \frac{1}{2}F_{PP}\sigma^2P^2 + F_{PC}\rho\sigma\eta + \frac{1}{2}F_{CC}\eta^2\right]dt + F_P dP + F_C dC \qquad \text{(A-1)}$$

Consider a storage firm that invests one dollar in an inventory of the commodity, hedging its investment by shorting $(PF_P)^{-1}$ futures contracts. The return on this hedged investment, including the convenience yield is:

$$P^{-1}\left[C - F_P^{-1}\left[-F_\tau + \frac{1}{2}F_{PP}\sigma^2P^2 + F_{PC}\rho\sigma\eta + \frac{1}{2}F_{CC}\eta^2 + F_C\alpha(m - C)\right]\right]dt \qquad \text{(A-2)}$$
$$- (PF_P)^{-1}F_C\eta\, dz_C$$

The investment is not riskless because of the influence of the stochastic convenience yield on the futures price. We assume that the risk premium associated with any asset which is perfectly (positively) correlated with the stochastic change in the convenience yield is proportional to the standard deviation of the return on the asset. Then the equilibrium expected return on the above portfolio may be written as $r - (PF_P)^{-1}F_C\lambda\eta$, where λ is a constant of proportionality. Equating this to the drift term in (A-2) and rearranging, we obtain the following partial differential equation for the futures price:

$$\frac{1}{2}F_{PP}\sigma^2P^2 + F_{PC}\rho\sigma\eta + \frac{1}{2}F_{CC}\eta^2 + F_P(rP - C) \qquad \text{(A-3)}$$
$$F_C(\alpha(m - C) - \lambda\eta) - F_\tau = 0$$

The solution to this equation may be written as:

$$F(P,C,\tau) = (P - PVC(C,\tau))e^{r\tau} \tag{A-4}$$

where

$$PVC(C,\tau) = \frac{m^*}{r}(1 - e^{-r\tau}) - \frac{m^* - C}{\alpha + r}(1 - e^{-(\alpha+r)\tau} \tag{A-5}$$

and $m^* \equiv \lambda^*/\alpha$.

References

Brennan, M.J., 1987, The cost of convenience and the pricing of commodity contingent claims, Working Paper, Columbia Futures Center.

Brennan, M.J., , 1991, The price of convenience and the valuation of commodity contingent claims, in D. Lund and B. Oksendal (Editors), *Stochastic Models and Option Values*, New York, North Holland.

Brennan, M.J., and E.S. Schwartz, 1985, Evalauting natural resource investments, *Journal of Business*, 58, 133-155.

Cox, J.C., J.E. Ingersoll, and S.A. Ross, 1981, The relation between forward prices and futures prices, *Journal of Financial Economics*, 9,321-346.

Culp, C.L., and M.H. Miller, 1994, Hedging a flow of commodity deliveries with futures: lessons from Metallgesellschaft, *Derivatives Quarterly*, 1, 7-15.

Culp, C.L., and M.H. Miller, 1995, Metallgesellschaft and the economics of synthetic storage, *Journal of Applied Corporate Finance*, 7, 62-76.

Edwards, F., and M. Canter, 1995, The collapse of Metallgesellschaft: unhedgeable risks, poor hedging strategy, or just bad luck?, *Journal of Futures Markets*, forthcoming.

Fama E.F., and K.R. French, 1987, Commodity futures prices: some evidence on forecast power, premiums and the theory of storage, *Journal of Business*, 60, 55-74.

Garbade, K.D., 1993, A two-factor, arbitrage-free, model of fluctuations in crude oil futures prices, *The Journal of Derivatives*, 1, 86-97.

Gibson, R., and E.S. Schwartz, 1990, Stochastic convenience yield and the pricing of oil contingent claims, *Journal of Finance*, 45, 959-976.

Jamshidian, F., and M. Fein, 1990, Closed-form solutions for oil futures and European options in the Gibson-Schwartz model: a note, Working Paper, Merrill Lynch Capital Markets.

Kaldor, N., 1939, Speculation and economic stability, *Review of Economic Studies*, 7, 1-27.

Mello, A.A., and J.E. Parsons, Maturity structure of a hedge matters: lessons from the Metallgesellschaft debacle, *Journal of Applied Corporate Finance*, 8, 106-120.

Neuberger, A., 1995, How well can you hedge long term exposures with multiple short-term futures contracts?, unpublished manuscript, London Business School.

Ross, S.A., 1995, Hedging long run commitments: exercises in incomplete market pricing, preliminary draft.

Telser, L.G., 1958, Futures trading and the storage of cottonand wheat, *Journal of Political Economy*, 66, 233-244.

Working, H., 1948, Theory of the inverse carrying charge in futures markets, *Journal of Farm Economics*, 30, 1-28.

Working, H., 1949, The theory of the price of storage, *American Economic Review*, 39, 1254-1262.