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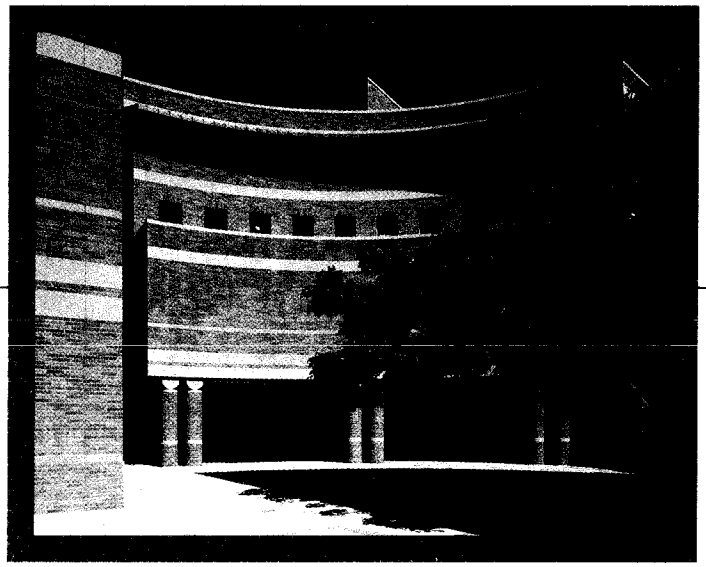

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Markovian Arbitrage-Free Models of the Term Structure of Interest Rates

First Draft: April 1994
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Abstract

This paper provides a general arbitrage-free model of interest rates in the spirit of Heath, Jarrow and Morton (1992). A characterization with an additional state variable is given, such that its joint process with the short rate is Markovian. This new state variable captures all the information in the history of interest rates that is relevant for pricing. For the models in this class, bond prices are obtained as a function of the two state variables. The Markovian character of this class of models greatly enhances their applicability for pricing of derivatives with numeric methods. Several parametric examples are given that fit stylized facts known about interest rate dynamics. Finally, a parametric version of the model is estimated in such a way that the state variables can be extracted.

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1 Introduction

Traditional term structure models take as given a short rate process and market prices of risk and then price interest rate sensitive securities, either in closed form or using numerical methods like binomial discretizations, simulations or numerical solutions of a PDE. These models can be obtained from general equilibrium, as in Cox, Ingersoll and Ross (CIR, 1985), or from exogenously given spot rate and market price of risk processes as in Vasicek (1977). The approach can be extended to include multiple sources of uncertainty by making the dynamics of the yield curve depend on a small number of state variables that summarize the information available in the economy. In any case, the basic building blocks of these models are state variables that are Markovian by construction. It is this characteristic that makes the models tractable for pricing. In particular, this is the reason why it is possible to formulate a PDE that must be satisfied by all interest rate derivatives.

There are, however, two important drawbacks to this approach. First, the models can only be made to fit the current observed term structure by making the spot rate dynamics depend on complex functions of time. And, second, the parameters of the model have to be estimated from a time series of bond prices using sophisticated econometric methods.

A new approach to interest rate modelling started with Ho and Lee (1986) and Heath, Jarrow and Morton (HJM, 1992). These models take as given some initial term structures of interest rates and forward rate volatilities - which are thus automatically fitted into the model - and, by imposing absence of arbitrage opportunities, work out their implications to yield curve dynamics. The relevant parameters of the model can in general be obtained by "inverting" bond option prices, much in the same way implied volatilities are extracted from stock option prices with the Black-Scholes model.

Unfortunately, this approach usually leads to a non-Markovian spot interest rate[1], which makes it difficult to obtain closed form pricing formulas in terms of a reduced set of state variables or even to use numerical pricing methods.

[1]This means that, in general, no finite dimensional set of state variables exists that captures the information necessary for pricing and that can be input into numerical pricing methods.

The only models in this class known until now to be Markovian have deterministic forward rate volatilities, thus leading to Gaussian interest rates - which is not very appealing.

The general non-Markovian nature of no-arbitrage models makes it impossible to obtain a PDE for pricing. The only method thus available for pricing derivatives is the construction of binomial trees. But, even with these, the possibilities are restricted. The binomial discretization method proposed by Nelson and Ramaswamy (1990) cannot be applied to non-Markovian processes, and the method of Amin and Bodurtha (1995) is computationally feasible only for a small number of time steps[2], thus making it little suited for the pricing of American claims or long maturity securities.

There is therefore a tradeoff in choosing between the two approaches to model interest rates. Our model resolves this tradeoff.

We start with two building blocks: no arbitrage - that is, we recognize that the currently observed term structure[3] has strong implications for the admissible dynamics of interest rates - and a set of state variables that synthesizes the information in the economy. We then obtain conditions on the shape of the term structure of volatilities that lead to Markovian interest rates. The interest rates are made Markovian by expanding the state space with a particular state variable that has the role of summarizing the information in the path of the term structure[4]. This variable and the spot interest rate are then sufficient to price bonds. For all the models in this class, bond prices are a function only of these two state variables, as opposed to depending on an infinite number of forward rates, which is the usual case in HJM. The Markovian nature of the state variables makes it possible to derive the PDE that all interest rate derivatives must satisfy and easily price derivatives by numerical methods.

It is an important characteristic of our model that the bond pricing equation

[2]The binomial tree that is constructed is non-recombinant and thus the number of nodes grows exponentially with the number of time intervals in the discretization. The tree approximates well the state space but not the time line.

[3]We retain the capability of fitting the initial term structure of previous no-arbitrage models.

[4]The condition on the volatilities of forward rates and the extension of the state space have been independently obtained by Ritchken and Sankarasubramanian (1995).

does not depend on the parameters of the state variable processes. Even if we make our two jointly Markovian state variables dependent on additional state variables, the latter will not show up in bond pricing formulas. Thus, whatever the complexity of their dynamics, the current values of the two state variables are sufficient statistics of the uncertainty relevant for bond pricing.

Although the restrictions that have to be imposed on the volatility of forward rates in order to obtain Markovian interest rates are quite strong, we still obtain a very large class of possible term structure dynamics. This means that we can have very rich dynamics for bond prices while still being able to price bonds with the same (simple) formula. This makes this class of models particularly interesting for empirical work.

We estimate a parsimonious version of our model from a panel data set of bond prices. We put the model in state space form and use the Kalman filter to simultaneously estimate the parameters of the model and filter the unobservable state variables. The estimation is done by pseudo maximum likelihood, using the true conditional moments of the state variables' processes and assuming that their transition density is Gaussian, when in fact it is unknown. Nevertheless, we retain consistency of the estimators and are able to obtain a consistent estimate of their covariance matrix.

The paper is organized as follows. In section 2, we rederive the results of HJM for completeness. Section 3 contains the condition for Markovian term structure dynamics and the pricing PDE. Section 4 gives several interesting parametric examples of models in our class. Section 5 presents the econometric method and the empirical results. Section 6 concludes the paper.

2 Arbitrage-Free Dynamics of Forward Rates

For completeness, this section reproduces results obtained in HJM. This is interesting on its own since we obtain the results in a different way. We model the pricing operator of the economy, thus showing that these models

are compatible with incomplete markets and also[5] how they can be obtained from general equilibrium.

Consider an economy which lives on a finite time interval $[0, T]$. Uncertainty is represented by an underlying probability space $(\Omega, \mathcal{F}, P)$, which is assumed to be complete. The probability measure P represents the common probability belief held by all agents. There is an N-dimensional standard Brownian motion $W(t)$ defined on the probability space $(\Omega, \mathcal{F}, P)$ and the time interval $[0, T]$, which is measurable with respect to the filtration that models information arrival to all agents of this economy.

We assume that no dynamic arbitrage trading strategy can be implemented by trading in the financial securities issued in the economy. This implies the existence of a strictly positive nominal state-price density, or pricing kernel, $M(t)$, that takes the form

$$\frac{dM(t)}{M(t)} = \mu(t)dt + \phi(t)'dW(t)$$

with μ and ϕ adapted processes in $\mathbb{R}$ and $\mathbb{R}^N$ such that the pricing kernel is well defined as an Ito process.

The pricing kernel is such that, under an adequate definition of the space of admissible trading strategies, the product $M(t)V(t)$, where $V(t)$ is the value process of any admissible self-financing trading strategy implemented by trading on financial securities, is a martingale. In particular, if buy-and-hold strategies are admissible, we know that, for any financial security that promises a nominal payoff $F(s)$ at some future date s, its nominal price, is

$$F(t) = \mathrm{E}_t \left[F(s) \frac{M(s)}{M(t)} \right] \tag{1}$$

We assume that among the securities issued in the economy there exists a nominal floating-rate bank account, or money market account. A security is referred to as a floating-rate bank account if it is "locally riskless". Thus,

[5] Since the connection between pricing operators and general equilibrium is well known in the literature.

the value at time t, of an initial investment of $B(0)$ units in the bank account that is continuously reinvested, is given by the following process

$$B(t) = B(0) \exp\left\{\int_0^t r(s)ds\right\}$$

where $r(t)$, the instantaneous nominal interest rate, is a measurable, adapted, strictly positive diffusion process.

We further assume that riskless discount bonds of all maturities s in the interval $[0, T]$ trade in this economy. Let $P(t, s)$ denote the time t price of the s maturity bond, for all s in $[0, T]$ and t in $[0, s]$. We require that $P(s, s) = 1$, that $P(t, s) > 0$ and that $\partial P(t, s)/\partial s$ exists, for all s in $[0, T]$ and t in $[0, s]$.

The instantaneous forward rates at time t for all dates $s > t$, $f(t, s)$, are defined by

$$f(t, s) = -\frac{\partial \log P(t, s)}{\partial s}$$

which is the rate that can be contracted at time t fcr instantaneous riskless borrowing or lending at time s. We require that the initial term structure $f(0, s)$, for all s be differentiable. Note that, from the knowledge of the instantaneous forward rates for all maturities between time t and time s, the price at time t of a bond with maturity s can be obtained by

$$P(t, s) = \exp\left\{-\int_t^s f(t, y)dy\right\} \tag{2}$$

The spot interest rate at time t, $r(t)$, is the instantaneous forward rate at time t for date t,

$$r(t) = f(t, t)$$

We follow HJM in describing the dynamics of the term structure of interest rates by a family of stochastic processes representing forward rate movements. For all s in $[0, T]$, let

$$f(t, s) = f(0, s) + \int_0^t \alpha(v, s)dv + \int_0^t \sigma(v, s)'dW(v)$$

with α and σ are adapted processes in $\mathbb{R}$ and $\mathbb{R}^N$ such that the forward rate processes are well defined as Ito processes. The dynamics of the instantaneous interest rate are then given by

$$r(t) = f(t,t) = f(0,t) + \int_0^t \alpha(v,t)dv + \int_0^t \sigma(v,t)'dW(v)$$

We now find out what restrictions must the processes α and σ satisfy in order for no arbitrage opportunities to exist in bond trading. First, the product of the value of the bank account and the pricing kernel has to be a martingale. Therefore, the drift of the pricing kernel has to be equal to minus the instantaneous interest rate,

$$r(t) = -\mu(t)$$

Also, the product of the value of all discount bonds and the pricing kernel has to be a martingale. In order to obtain the constraints that this condition imposes on the coefficients of the forward rate processes, we start by writing the dynamics of the bond prices. We know that

$$\begin{aligned}
\ln P(t,s) &= -\int_t^s f(t,y)dy \\
&= -\int_t^s f(0,y)dy - \int_t^s \left(\int_0^t \alpha(v,y)dv\right) dy \\
&- \int_t^s \left(\int_0^t \sigma(v,y)'dW(v)\right) dy \\
&= \ln P(0,s) + \int_0^t \left(r(v) - \int_v^s \alpha(v,y)dy\right) dv \\
&- \int_0^t \left(\int_v^s \sigma(v,y)dy\right)' dW(v)
\end{aligned}$$

where the third equality obtains by applying a version of Fubini's theorem[6]. Now, from Ito's lemma

$$\begin{aligned}
\frac{dP(t,s)}{P(t,s)} &= \left[r(t) - \int_t^s \alpha(t,y)dy + \frac{1}{2}\left(\int_t^s \sigma(t,y)dy\right)'\left(\int_t^s \sigma(t,y)dy\right)\right] dt \\
&- \left(\int_t^s \sigma(t,y)dy\right)' dW(t) \qquad (3)
\end{aligned}$$

[6]See HJM, lemma 0.1.

Therefore, since for all s in $[0,T]$, the product $M(t)P(t,s)$ must be a martingale and hence have zero drift, we obtain the following no-arbitrage condition

$$\begin{aligned}\int_t^s \alpha(t,y)dy &= \frac{1}{2}\left(\int_t^s \sigma(t,y)dy\right)'\left(\int_t^s \sigma(t,y)dy\right) \\ &- \phi(t)'\left(\int_t^s \sigma(t,y)dy\right)\end{aligned}$$

Taking partial derivatives with respect to s

$$\alpha(t,s) = \sigma(t,s)'\left(\int_t^s \sigma(t,y)dy\right) - \phi(t)'\sigma(t,s) \tag{4}$$

which completely specifies the constraint that the coefficients of the forward rate processes must satisfy for there to be no arbitrage opportunities from trading in bonds. Arbitrage-free instantaneous forward rate processes must then take the form

$$\begin{aligned}f(t,s) &= f(0,s) + \int_0^t \sigma(v,s)'\left(\int_v^s \sigma(v,y)dy\right)dv \\ &- \int_0^t \phi(v)'\sigma(v,s)dv + \int_0^t \sigma(v,s)'dW(v)\end{aligned} \tag{5}$$

or, in differential form,

$$\begin{aligned}df(t,s) &= \left[\sigma(t,s)'\left(\int_t^s \sigma(t,y)dy\right) - \phi(t)'\sigma(t,s)\right]dt \\ &+ \sigma(t,s)'dW(t)\end{aligned} \tag{6}$$

The instantaneous interest rate process under the no-arbitrage condition is

$$\begin{aligned}r(t) &= f(0,t) + \int_0^t \sigma(v,t)'\left(\int_v^t \sigma(v,y)dy\right)dv \\ &- \int_0^t \phi(v)'\sigma(v,t)dv + \int_0^t \sigma(v,t)'dW(v)\end{aligned} \tag{7}$$

Finally, we can derive the dynamics of the instantaneous interest rate under the no-arbitrage condition,

$$dr(t) = df(t,s)|_{s=t} + (\partial f(t,s)/\partial s|_{s=t})\,dt$$

$$
\begin{aligned}
&= \left[f_2(0,t) + \int_0^t \sigma_2(v,t)' \left(\int_v^t \sigma(v,y)dy \right) dv \right. \\
&+ \int_0^t \sigma(v,t)'\sigma(v,t)dv - \int_0^t \phi(v)'\sigma_2(v,t)dv \\
&- \left. \phi(t)'\sigma(t,t) + \int_0^t \sigma_2(v,t)'dW(v) \right] dt \\
&+ \sigma(t,t)'dW(t) \qquad (8)
\end{aligned}
$$

where the subscript 2 denotes a partial derivative with respect to the second argument of the function. Note that the first term of the drift is the slope of the initial forward curve, the second and third terms depend on the history of the volatility process and the last term depends jointly on the history of the volatility process and the history of the Brownian motion. The two remaining terms depend on the market price of risk vector. It can easily be seen that, in general, the instantaneous interest rate process will not be Markovian.

We have thus been able to obtain the processes followed by the spot interest rate and forward rates of all maturities from the knowledge of the pricing kernel and the observation of the initial term structure[7].

3 Markovian Models

In this section, we investigate a Markovian characterization of the economy based on a reduced set of state variables that contain all the information relevant for the term structure of interest rates. In this way, we can obtain a variety of dynamics for interest rates that are consistent with the initial term structure and under which we are able to price bonds in closed form.

Let us assume that the volatility function of the forward rate processes is such that $\sigma_2(t,s) = k(s)\sigma(t,s)$, where k is an arbitrary deterministic scalar

[7]The opposite can be done to some extent. If we can estimate the drift and volatility functions of forward rates, we can use equation (4) and try to solve for $\phi(t)$. The problem is that this is a one-dimensional equation in N unknowns. We can thus only obtain a linear restriction that must be satisfied by the vector $\phi(t)$. The pricing kernel will only be completely identified if there is only one source of uncertainty in the economy.

function. Then, from (7) and (8), the dynamics of the instantaneous interest rate can be written as

$$dr(t) = [f_2(0,t) + k(t)r(t) - k(t)f(0,t) - \phi(t)'\sigma(t,t) + \varphi(t)]\,dt + \sigma(t,t)'dW(t)$$

where

$$\varphi(t) = \int_0^t \sigma(v,t)'\sigma(v,t)dv$$

with dynamics given by

$$d\varphi(t) = [\sigma(t,t)'\sigma(t,t) + 2k(t)\varphi(t)]\,dt$$

Therefore, $(r(t), \varphi(t))$ are jointly Markovian if $\sigma(t,t)$ depends only on $r(t)$. If $\sigma(t,t)$ depends also on additional state variables, then $(r(t), \varphi(t))$ will be jointly Markovian with them.

The condition for Markovian term structure dynamics, $\sigma_2(t,s) = k(s)\sigma(t,s)$, defines an ODE with fundamental solution $\sigma(t,s) = \sigma(t,t)\exp\{-\int_t^s k(u)du\}$, thus implying that $\sigma(t,s) = x(t)l(s)$, for any N-dimensional process $x(t)$ and any deterministic scalar function l. This is a special case of arbitrage-free dynamics of the spot interest rate such that the information in the path of the term structure that is relevant is contained in the state variable φ.

Note that the first term of the drift of the spot rate is the slope of the initial forward rate curve, the second and third terms reflect some kind of mean reversion to the initial forward rate, then there is a term reflecting the market price of risk, and finally, there is the variable that reflects the path information. It is interesting to consider the case of a flat initial yield curve. This implies that $f_2(0,t)$ is zero and $f(0,t)$ is constant. If, in addition, the forward volatility curve is exponentially dampened, $k(t)$ is constant, and the spot rate process displays the usual mean reversion. If the forward volatility curve is flat, $k(t)$ is zero, and the drift of the spot rate depends only on the market price of risk and the path variable.

We can now obtain bond prices in closed form. Replacing $\sigma(t,s) = x(t)l(s)$ in (5) and rearranging slightly,

$$\begin{aligned}(f(t,s) - f(0,s)) &= l(s)\int_0^t x(v)'x(v)\left(\int_v^s l(y)dy\right)dv - l(s)\int_0^t x(v)'\phi(v)dv \\ &+ l(s)\int_0^t x(v)'dW(v)\end{aligned}$$

We can thus write

$$\begin{aligned}\frac{1}{l(s)}\left(f(t,s)-f(0,s)\right) &- \int_0^t x(v)'x(v)\left(\int_v^s l(y)dy\right)dv = \frac{1}{l(t)}\left(f(t,t)-f(0,t)\right)\\ &- \int_0^t x(v)'x(v)\left(\int_v^t l(y)dy\right)dv\end{aligned}$$

and, making use of the fact that $\int_v^s l(y)dy - \int_v^t l(y)dy = \int_t^s l(y)dy$,

$$\frac{1}{l(s)}\left(f(t,s)-f(0,s)\right) = \frac{1}{l(t)}\left(r(t)-f(0,t)\right) + \left(\int_t^s l(y)dy\right)\left(\int_0^t x(v)'x(v)dv\right)$$

Replacing $\varphi(t) = l(t)^2 \int_0^t x(v)'x(v)dv$, we obtain

$$f(t,s) = f(0,s) + \frac{l(s)}{l(t)}\left(r(t)-f(0,t)\right) + \frac{l(s)}{l(t)^2}\left(\int_t^s l(y)dy\right)\varphi(t)$$

and thus, from (2),

$$P(t,s) = \frac{P(0,s)}{P(0,t)}\exp\left\{-\frac{\int_t^s l(y)dy}{l(t)}\left(r(t)-f(0,t)\right) - \frac{\left(\int_t^s l(y)dy\right)^2}{2l(t)^2}\varphi(t)\right\} \quad (9)$$

Although the previous equations give us closed form expressions for forward rates and bond prices, they involve relatively complex functionals of l which makes them difficult to interpret. These equations can be rewritten by noting that $l(s)/l(t) = \sigma_i(t,s)/\sigma_i(t,t)$, $i = 1, ..., n$, the ratio of any two corresponding elements in the volatility vectors of the forward rate and the short rate[8]. Then, our condition for a Markovian term structure implies that the ratio of any two forward rate volatilities has to be deterministic.

Now,

$$f(t,s) = f(0,s) + \frac{\sigma_i(t,s)}{\sigma_i(t,t)}\left(r(t)-f(0,t)\right) + \frac{\sigma_i(t,s)}{\sigma_i(t,t)^2}\left(\int_t^s \sigma_i(t,y)dy\right)\varphi(t)$$

Note that $\int_t^s \sigma_i(t,y)dy$ is the negative of the i-th component of the volatility vector of the s-maturity bond price process, as can be seen from (3). Denote

[8]Note that this ratio has to be equal for all $i = 1, ..., n$.

the volatility by $v_i(t,s)$, then, integrating by parts, $\int_t^s \sigma_i(t,y)v_i(t,y)dy = (1/2)v_i(t,s)^2$. Then

$$P(t,s) = \frac{P(0,s)}{P(0,t)} \exp\left\{ \frac{v_i(t,s)}{\sigma_i(t,t)} (r(t) - f(0,t)) + \frac{v_i(t,s)^2}{2\sigma_i(t,t)^2}\varphi(t) \right\}$$

Although we obtain a large class of models, there are important economic implications of the Markovian restriction. The term structure of volatilities is such that $l(s)/l(t) = \sigma_i(t,s)/\sigma_i(t,t)$ for any $i = 1,...,n$. In particular, for time homogeneity, there are only two possible cases. The first case has $l(s)$ constant, and thus all forward rates, and the spot rate, have the same volatility at a given time. In this case,

$$P(t,s) = \frac{P(0,s)}{P(0,t)} \exp\left\{ -(s-t)\,(r(t) - f(0,t)) - \frac{(s-t)^2}{2}\varphi(t) \right\} \tag{10}$$

The second case has $l(s) = e^{-as}$, where a is a constant, and we thus have exponential decay of volatilities along the maturity line. Here,

$$P(t,s) = \frac{P(0,s)}{P(0,t)} \exp\left\{ -\frac{1-e^{-a(s-t)}}{a} (r(t) - f(0,t)) - \frac{(1-e^{-a(s-t)})^2}{2a^2}\varphi(t) \right\} \tag{11}$$

There is one important case studied in the literature that is incompatible with the constraints we impose on the volatility curve. This is the case of forward rates defined by an SDE - where the volatility of a given forward rate depends on the level of the forward - described in HJM and Amin and Morton (1994). Note however that we can still make the volatility dependent on some number of forward rates appropriately defined[9].

In order to enrich the dynamics of the short rate, and thus of the bond prices, we can introduce additional state variables in our model. We can define a vector of state variables S to be a diffusion process satisfying a stochastic differential equation

$$dS(t) = \mu_S(t)dt + \sigma_S(t)'dW(t)$$

[9]See section 4.4. below.

In this setting, the market price of risk $\phi(t)$ and the volatility processes $\sigma(t,s)$ can be any functions of $S(t)$ and $r(t)$. Then, $(r(t), \varphi(t), S(t))$ will be jointly Markovian. Note that in the bond pricing formulas neither these additional state variables nor the parameters defining their processes show up directly. They only matter through $r(t)$ and $\varphi(t)$.

Finally, an interest rate derivative that pays a dividend stream at the rate $h(r(u), \varphi(u))$ and makes a terminal payment of $g(r(s), \varphi(s))$, for $t \leq u \leq s$ has value $F(r(t), \varphi(t), S(t), t)$ that must satisfy the following PDE[10]

$$\begin{aligned} F_t + F_r\left[f_2(0,u) + k(u)r(u) - k(u)f(0,u) + \varphi(u)\right] \\ +F_\varphi\left[\sigma(u,u)'\sigma(u,u) + 2k(u)\varphi(u)\right] + F_S\left[\mu_S(u) + \sigma_S(u)'\phi(u)\right] \\ +\frac{1}{2}F_{rr}\sigma(u,u)'\sigma(u,u) + \frac{1}{2}F_{SS}\sigma_S(u)'\sigma_S(u) \\ +F_{rS}\sigma(u,u)'\sigma_S(u,u) - Fr(u) + h = 0 \end{aligned}$$

with boundary condition $F(s) = g(r(s), \varphi(s))$. Derivatives can also be valued easily by simulation, using the risk neutral dynamics that underly the PDE above.

Note the importance of having a formula for bond prices when valuing bond derivatives by numerically solving the above PDE. If we wanted to value a derivative with a final payoff that depends on the price of a bond with a longer maturity - say a three month option on a 30 year bond - and did not have a formula for the bond price, we would have to carry the finite difference grid until the maturity of the bond. With a bond pricing formula, the grid only has to be taken until the maturity of the derivative - three months instead of 30 years.

4 Examples

The following are some examples of the class of models covered by our framework. Since it is what matters for pricing, we will only consider dynamics

[10] Where we drop the arguments of the functions h and F and its derivatives.

under the risk adjusted probability measure. We denote by W the Brownian motion under this equivalent probability measure. There should be no confusion with the Brownian motion under the true probability measure.

4.1 Deterministic Volatility Models

If $\sigma(t,s)$ is nonstochastic, φ collapses to a deterministic function of time and the spot rate is Markovian by itself. Two homogeneous cases, with one-dimensional uncertainty, are of interest: $\sigma(t,s) = \sigma_1$, a constant, and $\sigma(t,s) = \sigma_2 e^{-a(s-t)}$ with a and σ_2 constants. The first case is the well known continuous-time version of the Ho-Lee model, which corresponds to the so called extended Vasicek model and is studied in example 1 of HJM. In this case all forward rates have the same constant volatility, and a shock shifts all rates uniformly. The second example has the volatility decreasing exponentially with maturity, thus making short rates more volatile than long rates.

Example 2 of HJM studies a mixture of the two examples above. The Brownian motion driving the uncertainty of forward rates is two-dimensional and the two components of the volatility vector of forward rates are the ones described above. Shocks to the first Brownian motion affect the "level, while shocks to the second Brownian motion affect the "steepness of the forward curve.

Deterministic volatility models are the only Markovian arbitrage-free models known in the literature so far[11] and, as such, have been extensively studied both for closed-form pricing[12] and in empirical studies[13].

[11] In fact, Hull and White (1993) have shown that deterministic volatility is a necessary and sufficient condition for the spot interest rate to be Markovian (by itself).

[12] See, among many others, Jamshidian (1991), Flesaker (1993a) and Brace and Musiela (1994).

[13] See Flesaker (1993b).

4.2 Two-Factor Models

Apart from the deterministic volatility case, which is in a sense degenerate, the most parsimonious examples of our class of models are such that, with one dimensional Brownian motion, $\sigma(t, s; r(t)) = x(r(t), t)l(s)$.

A simple example that covers a great many models studied in the literature[14] is $x(r(t)) = \sigma r(t)^\gamma$, for constant σ and γ. When $\gamma = 0$ we have the Vasicek volatility, when $\gamma = 1$ we have versions of the Brennan and Schwartz (1977) volatility, when $\gamma = 1/2$ we have the CIR volatility and when $\gamma = 3/2$ we have Richard's (1994) volatility. For the case of exponentially declining forward rate volatilities,

$$dr(t) = [f_2(0,t) + a(f(0,t) - r(t)) + \varphi(t)]\,dt + \sigma r(t)^\gamma dW(t)$$

where

$$\varphi(t) = \int_0^t \sigma^2 e^{-2a(t-v)} r(v)^{2\gamma} dv$$

with dynamics given by

$$d\varphi(t) = [\sigma^2 r(t)^{2\gamma} - 2a\varphi(t)]dt$$

Bond prices are then given by (11).

In the particular case of square root volatilities, $\gamma = 1/2$, pricing of some derivatives is possible in closed form since the joint distribution of $e^{r(s)}$ and $e^{\varphi(s)} = e^{\sigma^2 \int_t^s r(v)dv}$, for some future s, is computable[15]. Note also that, under the true probability measure, it is convenient to make the price of risk proportional to the square root of the spot rate, in order to get make the drift linear in the state variables. This is indeed the choice of CIR.

4.3 Stochastic Volatility Models

This example of stochastic volatility[16] of forward rates envolves the introduction of an additional state variable. We consider the case of equal forward

[14] This example covers all cases studied by Chan, Karolyi, Longstaff and Sanders (1992)

[15] See Beaglehole and Tenney (1991).

[16] Inspired by the models in Heston (1993) and Longstaff and Schwartz (1992).

rate volatilities for simplicity. Let

$$dr(t) = [f_2(0,t) + \varphi(t)]\,dt + \sqrt{v(t)}dZ_1(t)$$

where the instantaneous variance follows a square root process

$$dv(t) = (\delta^2 - 2\beta v(t))dt + 2\delta\sqrt{v(t)}dZ_2(t)$$

and Z_1 is instantaneously correlated with Z_2, with constant correlation ρ. This can be obtained by making $(Z_1(t), Z_2(t)) = A(W_1(t), W_2(t))$, with

$$A = \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{pmatrix}$$

4.4 Forward-Factor Models

In this example the state variables are themselves forward rates[17]. Note that we use forward rates with fixed time to maturity, instead of the fixed maturity case usually considered, in order to be able to separate the volatility function in both time arguments.

Consider now a vector of additional state variables which are forward rates with fixed time to maturity $F(t) = (f(t, t+\theta_1), f(t, t+\theta_2), ..., f(t, t+\theta_n))$ for constant θ_1,θ_2,...,θ_n. We can now consider forward rate volatilities of the form $\sigma(t, s; F(t)) = x(F(t), t)l(s)$. Then, the dynamics of the state variables are given, for $i = 1, 2, ..., n$, by

$$\begin{aligned} dr(t) &= df(t,s)|_{s=t} + (\partial f(t,s)/\partial s|_{s=t})\,dt \\ &= \left[f_2(0,t) + \frac{l'(t)}{l(t)}(r(t) - f(0,t)) + \varphi(t)\right] dt \\ &+ l(t)x(F(t),t)'dW(t) \end{aligned}$$

and

$$df(t, t+\theta_i) = df(t,s)|_{s=t+\theta_i} + \left(\partial f(t,s)/\partial s|_{s=t+\theta_i}\right) dt$$

[17] In the spirit of El Karoui and Lacoste (1992) and Duffie and Kan (1993)

$$
\begin{aligned}
&= \left[f_2(0, t+\theta_i) + \frac{l'(t+\theta_i)}{l(t+\theta_i)} (f(t, t+\theta_i) - f(0, t+\theta_i)) \right. \\
&+ \left. l(t+\theta_i) \left(\int_t^{t+\theta_i} l(y) dy \right) x(F(t), t)' x(F(t), t) + \frac{l(t+\theta_i)^2}{l(t)^2} \varphi(t) \right] dt \\
&+ l(t+\theta_i) x(F(t), t)' dW(t)
\end{aligned}
$$

where

$$
\varphi(t) = l(t)^2 \int_0^t x(F(v), v)' x(F(v), v) dv
$$

with dynamics

$$
d\varphi(t) = \left[l(t)^2 x(F(t), t)' x(F(t), t) + 2 \frac{l'(t)}{l(t)} \varphi(t) \right] dt
$$

Bond prices are still given by (9).

5 Econometric Analysis

We see, from (10), (11) and the examples given in the previous section, that our bond pricing formulas are compatible with a variety of bond price dynamics. If we think of our class of models in state-space form, we see that we have the same measurement equations for bond prices for a variety of possible transition equations for the state variables. This is then an obviously very rich class of models to do empirical work with.

In this section we estimate by pseudo maximum likelihood a very parsimonious model in our class, the square root process shown in section 4.2., and extract the underlying state variables with the Kalman filter.

5.1 Econometric Method

We follow Chen and Scott (1993), Zheng (1994) and Geyer and Pichler (1995) in using the Kalman filter[18] to study term structure models. The dynamics

[18] A useful reference is Harvey (1989). See also Watson and Engle (1983).

of the state variables for the model we want to estimate are given by[19]

$$dr(t) = \left[f_2(0,t) + a(f(0,t) - r(t)) - \phi(t)\sigma\sqrt{r(t)} + \varphi(t)\right] dt + \sigma\sqrt{r(t)}dW(t)$$

and

$$d\varphi(t) = [\sigma^2 r(t) - 2a\varphi(t)]dt$$

For convenience, we make $\phi(t) = \lambda\sqrt{r(t)}$, thus making the drift of the spot rate linear in itself.

A particularly convenient way to specify the functional form to fit the initial yield curve is the parametrization given by Nelson and Siegel (1987)[20]. The function they propose is, using our notation,

$$f(0,t) = \beta_0 + \beta_1 e^{-t/\gamma} + \frac{\beta_2}{\gamma} t e^{-t/\gamma}$$

Then,

$$\int_t^s f(0,u)du = \beta_0(s-t) - [(\beta_1 + \beta_2)\gamma + \beta_2 s]e^{-s/\gamma} + [(\beta_1 + \beta_2)\gamma + \beta_2 t]e^{-t/\gamma}$$

These two formulas can be easily replaced in the bond pricing equation (11), more conveniently stated in terms of continuously compounded yields

$$\begin{aligned} R(t,s) &= \frac{1}{s-t}\int_t^s f(0,u)du - \frac{1-e^{-a(s-t)}}{a(s-t)} f(0,t) \\ &+ \frac{1-e^{-a(s-t)}}{a(s-t)} r(t) + \frac{(1-e^{-a(s-t)})^2}{2a^2(s-t)}\varphi(t) \end{aligned} \tag{12}$$

An important restriction imposed by modelling the term structure of volatilities as exponentially decreasing is that the two factor loadings depend on

[19] Here we have to consider the dynamics under the true probability measure since that is the measure of the uncertainty in the bond prices observed.

[20] Note that this parametric form is particularly suited for our purposes. First, it can produce the variety of shapes usually observed for the yield curve: Nelson and Siegel found it to explain more that 95% of the variation of yields across maturities. Second, the function for the instantaneous forward rate is the solution to a second order ODE, which is consistent with forward rates being forecasts of spot rates generated by a differential equation. And, third, there are only four parameters to be estimated.

a single parameter a. A less parsimonious version of our model would relax this assumption, undoubtedly obtaining a better fit.

We start by writing our model in state space form

$$R_t = M_R(y_t) + \epsilon_t$$

$$y_t = M_y(y_{t-1}) + \nu_t$$

where R_t is the vector of n bond yields[21] observed at time t and y_t is the vector of two state variables $r(t)$ and $\varphi(t)$. $M_R(y_t)$ is a vector of formulas like (12), which are linear in the state variables. The error term ϵ_t is serially and cross-sectionally uncorrelated and uncorrelated with ν_t, which is consistent with its interpretation as a measurement error. Its covariance matrix, U, is a constant, h^2, times the Identity matrix. $M_y(y_{t-1})$ is a two-dimensional vector of conditional means of the state variables. Finally, ν_t has conditional mean zero and conditional covariance matrix $V_y(y_{t-1})$, the conditional variance of the state variables' process. $M_y(y_{t-1})$ and $V_y(y_{t-1})$ are derived in the appendix.

Write $M_R(y_t) = A_t + B_t y_t$, $M_y(y_{t-1}) = a_t + b_t y_{t-1}$ and $V_y(y_{t-1}) = Q_t$ and define the forecasting equation for the state variables

$$\hat{y}_{t|t-1} = a_t + b_t \hat{y}_{t-1|t-1} \tag{13}$$

where $\hat{y}_{0|0}$ is the unconditional mean of the state variables[22] and the updating equation is

$$\hat{y}_{t|t} = \hat{y}_{t|t-1} + \hat{\Sigma}_{t|t-1} B_t' H_t^{-1} u_t \tag{14}$$

The yield forecast error, u_t, is defined as

$$u_t = R_t - M_R(\hat{y}_t)$$

and the matrix H_t is obtained as

$$H_t = B_t \hat{\Sigma}_{t|t-1} B_t' + U$$

[21] It is not necessary to have the same number of yields at all points in time. We procede as if this number was constant to keep the notation simple.

[22] Derived in the appendix.

We let $\hat{\Sigma}_{0|0}$ be the unconditional covariance matrix of the state variables[23] and define the mean squared error associated with the forecasts

$$\hat{\Sigma}_{t|t-1} = b_t \hat{\Sigma}_{t-1|t-1} b_t' + Q_t \tag{15}$$

and the mean squared error associated with the updates

$$\hat{\Sigma}_{t|t} = \hat{\Sigma}_{t|t-1} - \hat{\Sigma}_{t|t-1} B_t' H_t^{-1} B_t \hat{\Sigma}_{t|t-1} \tag{16}$$

The Kalman filter iterates equations (13) to (16).

If we assume u_t to be normally distributed, we can write the log-likelihood function[24] of our model as

$$L = -\frac{nT}{2} \ln 2\pi - \sum_{t=1}^{T} \frac{1}{2} (\ln |H_t| + u_t' H_t^{-1} u_t)$$

Although we know that u_t is not normal, we can still use the estimator that results from the maximization of the function above, calling it a pseudo maximum likelihood estimator and relying on the asymptotic properties shown in Gourieroux, Monfort and Trognon (1984)[25]. Finally, a consistent estimate of the covariance matrix of the parameters can be obtained as in White (1982).

Given some consistent parameter estimates, we can filter the series $\hat{y}_{t|t}$ as a consistent estimate of the path of the factors of the model.

5.2 Data

The data set used to estimate the model consists of ten daily continuously compounded zero coupon bond yields for US Treasuries. From the beginning of 1990 to the end of 1994, our data set includes 1252 observations of yields with maturities of one, three and six months, and one, one and a half, two, four, six, ten and fifteen years.

[23] Also derived in the appendix.

[24] The gradient and expected Hessian are given in Watson and Engle (1983).

[25] Alternatively, it is easy to see that the pseudo maximum likelihood method can be interpreted as an implementation of GMM with a particular metric, as is done by Chen and Scott (1993). The standard results of Hansen (1992) for GMM can then be used.

We need to use zero coupon yields in order for the measurement equation in our state space model to be linear in the state variables, which would not be the case for coupon bond yields. The problem is that we cannot observe zero coupon bond prices for the relevant maturities and that even most of the discount bonds that do trade do not have sufficient liquidity.

We overcome this problem by constructing discount bond prices from observed coupon bond prices. For each day in the sample, we collect from Datastream averages of bid and ask prices for the most recently issued three, six and twelve month bills, two, three, four, five, seven and ten year notes and the current bond with no option features. We also include data on the bill with approximately one month to maturity. We obtain the discount bond prices from the coupon bond data using the procedure in Coleman, Fisher and Ibbotson (1992): assuming that the forward rate curve is piecewise linear and minimizing the sum of squared pricing errors. We use eight different linear splines for the estimation of the forward curve at each date. Computing the zero coupon yield curve in this way introduces an error in the discount yields that further justifies the error in the measurement equation.

To give a feel for the data, Figure 1 provides the zero coupon yield surface for the entire sample period and Table 1 gives summary statistics. The yield curve starts out very flat at the level of 8%. The level then decreases steadily until it reaches 4% in the second half of 1993, increasing again to 7% at the end of the sample. The slope remaines flat until mid 1991, when it increases to a difference between short and long rates of around 4% in the begining of 1992 and stays at these levels until the end of the sample. There is thus a variety of yield curve shapes in the sample. Another fact worth mentioning is that yield volatilities measured by standard deviations decrease with the maturity.

5.3 Parameter Estimation and Filtration of State Variables

The results of the maximum likelihood estimation are presented in Table 2. Time is measured in years so that all the parameter values are expressed on an annual basis. All the estimates are statistically significant.

The estimate for the parameter a implies a ratio between the volatilities of the ten year forward and the spot rate of around 30% which is in accordance with casual observation. The parameter estimates that refer to the initial forward curve give an initial forward curve that is indeed similar to the curves observed for the begining of the sample. In particular, the implied initial spot rate, $\beta_0 + \beta_1$, is 7.36%, which is very close to the short term rates in the begining of the sample. The volatility parameter, σ, has a value that is consistent with the estimates of Pearson and Sun (1994)[26]. The risk premium parameter λ is estimated to be larger than in Pearson and Sun, but the comparison is more difficult here since our spot rate has a very different drift from theirs. Finally, the condition derived in the appendix for stationarity of the first two moments is satisfied by our parameter estimates.

Given the parameter estimates, we can extract the series of the two state variables. Figure 2 plots the first factor, the spot rate, r, against the six month rate. The closeness of the two series is striking. Figure 3 plots the second factor, φ, against the integral it is defined to be

$$\varphi(t) = \int_0^t \sigma^2 e^{-2a(t-v)} r(v) dv$$

The graph shows the computed series of the discretized version of this integral. Again, the two series are very close.

Table 3 gives summary statistics for the extracted factors. Three comments are worth making. First, both factors are very persistent, showing autocorrelations for one month intervals that are higher than 97%. Second, the distribution of both factors has third and fourth moments that imply non-normal shapes. Third, the two factors are highly negatively correlated.

A different way to identify the factors is to look at their estimated loadings in the yield equation. This is done in Figure 4, which can be interpreted in the spirit of Litterman and Scheinkman (1991). The first factor is mostly responsible for the level of the yield curve, while the second explains the steepness. The curvature is not very well explained by either factor. It

[26] Their model has two factors and, thus, two volatilities. If we compute the square root of the sum of the two squared volatility estimates, we obtain a number of 0.21, which is quite close to our volatility estimate.

thus seems that the curvature of the yield curve is mostly explained by the deterministic part of the yield equation.

Finally, we look at the contribution of each factor to explaining the variability of the yields within sample. Table 4 gives the results of regressions of yields on a constant and the two factors. The R^2 of all these regressions is between 95% and 100%, thus showing that the model captures well the evolution through time of the term structure.

5.4 Yield Errors

We can assess the quality of the fit of the model by looking at the difference between the actual yields and the yields implied by the model

$$\tilde{R}_t = A_t + B_t \hat{y}_{t|t}$$

Figure 5 plots, for selected dates, the estimated yield curve versus the observed yields and Table 5 provides information about the statistical characteristics of the errors. The overall mean yield error is of minus three basis points, which is not statistically significant, showing that the filtered state variables are not biased. The mean squared yield error is 21 basis points, which is small but, from a trading perspective, economically significant. The magnitude of the yield errors seems to vary with maturity, with both the short and long end of the curve being comparatively worse explained.

Two facts are important. The first is that the errors are highly autocorrelated, which they should not be if the factors fully explained the dynamics of yields through time. The second fact is that the errors are non-normal, showing that the normality assumption underlying the estimation method is not valid[27].

[27] The normality test shown in Table 5 is from Bowman and Shenton (1975) and is distributed as a Chi-square with two degrees of freedom. See Harvey (1990).

6 Conclusion

We develop a class of term structure models under no arbitrage such that bonds and interest rate derivatives depend on two state variables that are jointly Markovian. One of these state variables is the instantaneous interest rate, while the other captures the information in the history of interest rates that is relevant for pricing. The condition for the Markovian representation of the term structure is that the diffusion function of forward rates be separable in its two time arguments, with only the current time part dependent on stochastic factors. Although restrictive, this class of models is shown to include several interesting parametric examples.

For all the models in this class, we obtain bond pricing formulas that depend only on the two state variables. Furthermore, the pricing formulas depend only on the parameters that fit the current term structures of interest rates and volatilities and not on the parameters of the dynamics of the state variables. The pricing formulas do not even depend on any additional state variables on which the original two factors may depend.

The Markovian nature of the term structure in our class of models is particularly important for the pricing of derivatives. In contrast to the traditional arbitrage-free models, we are able to derive the PDE that must be satisfied by all interest rate dependent contingent claims. Pricing derivatives by simulation or binomial approximation is also greatly enhanced by the formulation with state variables.

Finally, we estimate a parametric version of our model by pseudo maximum likelihood and extract the implied factors using a Kalman filter. The model seems to explain very well the dynamics and the cross section of the term structure, with reasonably small yield errors. There is however considerable persistence remaining in the yield errors, showing that not all the dynamic characteristics of the term structure are captured by the model.

Appendix

In this appendix we derive the first two conditional and unconditional moments of the two state variables.

We start by the conditional expectation. Denote the vector that piles $E_t r(s)$ and $E_t \varphi(s)$, for $t < s$, by Y_1. Then, this vector must satisfy the ODE

$$\frac{dY_1(s)}{ds} = A_1 Y_1(s) + Q_1(s)$$

with initial condition $Y_1(t) = (r(t), \varphi(t))'$, where

$$A_1 = \begin{pmatrix} -a - \lambda\sigma & 1 \\ \sigma^2 & -2a \end{pmatrix}$$

and $Q_1(s) = (g(s), 0)'$ where

$$g(s) = k_1 + k_2 e^{-k_4 s} + k_3 s e^{-k_4 s}$$

The solution of the ODE that is to be replaced in $M_y(y_{t-1})$, substituting s by t and t by $t-1$, is given by[28]

$$Y_1(s) = S_1 X_1(s)$$

where

$$X_1(s) = e^{\Lambda_1(s-t)} X(t) + e^{\Lambda_1(s-t)} \int_t^s e^{-\Lambda_1(u-t)} P_1(u) du$$

with

$$X_1(t) = S_1^{-1} Y_1(t)$$
$$P_1(t) = S_1^{-1} Q_1(t)$$

Thus,

$$X_1(s) = e^{\Lambda_1(s-t)} X_1(t) + I_1(s)$$

[28] In order to make system computationally easier to solve, we first orthogonalized the differential equations and then solved the resulting system of decoupled ODE's.

where

$$
\begin{aligned}
I_1(1)(s) &= \frac{\sigma^2}{\sqrt{\Delta}}\Bigg[\frac{k_1}{\Lambda_1(1,1)}(e^{\Lambda_1(1,1)(s-t)}-1) \\
&+ \frac{k_2}{k_4+\Lambda_1(1,1)}e^{-k_4 t}(e^{\Lambda_1(1,1)(s-t)}-e^{-k_4(s-t)}) \\
&+ \frac{k_3}{(k_4+\Lambda_1(1,1))^2}e^{-k_4 t}\Big((1+(k_4+\Lambda_1(1,1))t)e^{\Lambda_1(1,1)(s-t)} \\
&\quad -(1+(k_4+\Lambda_1(1,1))s)e^{-k_4(s-t)}\Big)\Bigg]
\end{aligned}
$$

and

$$
\begin{aligned}
I_1(2)(s) &= -\frac{\sigma^2}{\sqrt{\Delta}}\Bigg[\frac{k_1}{\Lambda_1(2,2)}(e^{\Lambda_1(2,2)(s-t)}-1) \\
&+ \frac{k_2}{k_4+\Lambda_1(2,2)}e^{-k_4 t}(e^{\Lambda_1(2,2)(s-t)}-e^{-k_4(s-t)}) \\
&+ \frac{k_3}{(k_4+\Lambda_1(2,2))^2}e^{-k_4 t}\Big((1+(k_4+\Lambda_1(2,2))t)e^{\Lambda_1(2,2)(s-t)} \\
&\quad -(1+(k_4+\Lambda_1(2,2))s)e^{-k_4(s-t)}\Big)\Bigg]
\end{aligned}
$$

The unconditional first moments can be obtained by making $s \to \infty$, under the condition that $\sqrt{\Delta} - \mu < 0$, for stationarity. Then

$$
Y(\infty) = \begin{pmatrix} -\frac{k_1}{\Lambda_1(1,1)} \\ 0 \end{pmatrix}
$$

Note that the unconditional mean for the spot rate is positive.

In all of the above, S_1 is a matrix with columns equal to the eigenvectors of A_1

$$
S_1 = \begin{pmatrix} \frac{a-\lambda\sigma+\sqrt{\Delta}}{2\sigma^2} & \frac{a-\lambda\sigma-\sqrt{\Delta}}{2\sigma^2} \\ 1 & 1 \end{pmatrix}
$$

and Λ_1 is the diagonal matrix of the eigenvectors of A_1

$$
\Lambda_1 = \begin{pmatrix} \frac{\sqrt{\Delta}-\mu}{2} & 0 \\ 0 & -\frac{\sqrt{\Delta}+\mu}{2} \end{pmatrix}
$$

Finally, the following simplifications where made

$$k_1 = a\beta_1$$

$$k_2 = \frac{(a\gamma - 1)\beta_1 + \beta_2}{\gamma}$$

$$k_3 = \frac{(a\gamma - 1)\beta_2}{\gamma^2}$$

$$k_4 = \frac{1}{\gamma}$$

$$\mu = 3a + \lambda$$

$$\Delta = (a - \lambda\sigma)^2 + 4\sigma^2$$

The second conditional moments involve the solution of a system of three ODE's. Denote the vector that piles $\mathrm{E}_t r^2(s)$, $\mathrm{E}_t r(s)\varphi(s)$ and $\mathrm{E}_t \varphi^2(s)$, for $t < s$, by Y_2. Then, this vector must satisfy the ODE

$$\frac{dY_2(s)}{ds} = A_2 Y_2(s) + Q_2(s)$$

with initial condition $Y_2(t) = (r^2(t), r(t)\varphi(t), \varphi^2(t))'$, where

$$A_2 = \begin{pmatrix} -2a - 2\lambda\sigma & 2 & 0 \\ \sigma^2 & -\mu & 1 \\ 0 & 2\sigma^2 & -4a \end{pmatrix}$$

and

$$Q_2(s) = \begin{pmatrix} (2f(s) + \sigma^2)\mathrm{E}_t r(s) \\ f(s)\mathrm{E}_t \varphi(s) \\ 0 \end{pmatrix}$$

The solution of the ODE is given by

$$Y_2(s) = S_2 X_2(s)$$

where

$$X_2(s) = e^{\Lambda_2(s-t)} X_2(t) + e^{\Lambda_2(s-t)} \int_t^s e^{-\Lambda_2(u-t)} P_2(u) du$$

with

$$X_2(t) = S_2^{-1} Y_2(t)$$
$$P_2(t) = S_2^{-1} Q_2(t)$$

Thus,

$$X_2(s) = e^{\Lambda_2(s-t)} X_2(t) + e^{\Lambda_2(s-t)} \begin{pmatrix} I_2(1)(s) \\ I_2(2)(s) \\ I_2(3)(s) \end{pmatrix}$$

where, for $i = 1, 2, 3$

$$I_2(i)(s) = \int_t^s e^{-\Lambda_2(i,i)(u-t)} P_2(i)(u) du$$

The second conditional moments can be obtained from the above, after some simple but tedious algebra, making

$$\begin{pmatrix} \mathrm{Var}_t[r(s)] \\ \mathrm{Cov}_t[r(s), \varphi(s)] \\ \mathrm{Var}_t[\varphi(s)] \end{pmatrix} = Y(s) - \begin{pmatrix} (\mathrm{E}_t r(s))^2 \\ \mathrm{E}_t r(s) \mathrm{E}_t \varphi(s) \\ (\mathrm{E}_t \varphi(s))^2 \end{pmatrix}$$

These are the elements to be replaced in the matrix $V_y(y_{t-1})$. The unconditional moments can be obtained by again making $s \to \infty$. The condition for stationarity is the same as for the first moments.

In the above,

$$\Lambda_2(s) = \begin{pmatrix} -\mu & 0 & 0 \\ 0 & (-\mu + \sqrt{\Delta}) & 0 \\ 0 & 0 & (-\mu - \sqrt{\Delta}) \end{pmatrix}$$

is the diagonal matrix of the eigenvalues of A_2, and

$$S_2(s) = \begin{pmatrix} 1 & \frac{(a-\lambda\sigma)^2 + (a-\lambda\sigma)\sqrt{\Delta} + 2\sigma^2}{2\sigma^4} & \frac{(a-\lambda\sigma)^2 + (a-\lambda\sigma)\sqrt{\Delta} + 2\sigma^2}{2\sigma^4} \\ -\frac{a-\lambda\sigma}{2} & \frac{a-\lambda\sigma+\sqrt{\Delta}}{2\sigma^2} & \frac{a-\lambda\sigma-\sqrt{\Delta}}{2\sigma^2} \\ -\sigma^2 & 1 & 1 \end{pmatrix}$$

is the matrix of the eigenvectors of A_2.

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Table 1
Summary Statistics for Treasury Yields

	Yield to Maturity									
	1m	3m	6m	1y	1.5y	2y	4y	6y	10y	15y
Mean	0.0453	0.0483	0.0504	0.0526	0.0556	0.0573	0.0631	0.0674	0.0713	0.0736
Std. Dev.	0.0170	0.0175	0.0174	0.0170	0.0164	0.0156	0.0137	0.0119	0.0099	0.0095
$\rho(1)$	0.9975	0.9996	0.9996	0.9995	0.9979	0.9981	0.9972	0.9966	0.9953	0.9949
$\rho(30)$	0.9743	0.9836	0.9799	0.9737	0.9653	0.9645	0.9396	0.9506	0.9415	0.9427
Skew.	0.7188	0.6296	0.5340	0.4203	0.3333	0.3021	0.0392	-0.0087	-0.1806	-0.2379
Kurt.	-0.8257	-0.9785	-1.0715	-1.1730	-1.1968	-1.1967	-1.1752	-1.2092	-1.0226	-0.8820

Table 2
ML Parameter Estimates

	ln L = -44,168	
	Estimate	Std.Error
a	0.1154	0.0288
β_0	0.0252	0.0037
β_1	0.0484	0.0047
β_2	-0.0001	0.0000
γ	39.4220	4.1236
λ	0.3010	0.0455
σ	0.2011	0.0181
h	0.0026	0.0004

Table 3
Summary Statistics for Factors

	Factors	
	r	ϕ
Mean	0.0469	0.0107
Std. Dev.	0.0180	0.0015
$\rho(1)$	0.9995	0.9999
$\rho(30)$	0.9765	0.9955
Skew.	0.4709	-1.7303
Kurt.	-1.0408	2.3106
Corr.	-0.83652	

Table 4
Within-Sample Regressions of Yields on Factors

	Yield to Maturity									
	1m	3m	6m	1y	1.5y	2y	4y	6y	10y	15y
α	0.0312	0.0230	0.0118	0.0009	-0.0031	-0.0036	-0.0097	0.0007	0.0080	0.0155
Std.Dev.	0.0014	0.0010	0.0007	0.0009	0.0009	0.0008	0.0013	0.0009	0.0012	0.0015
β_1	0.7868	0.8690	0.9323	0.9779	0.9859	0.9602	0.9249	0.8234	0.7124	0.6652
Std.Dev.	0.0084	0.0056	0.0043	0.0052	0.0050	0.0047	0.0076	0.0054	0.0068	0.0084
β_2	-2.0878	-1.4294	-0.4743	0.5534	1.1652	1.4862	2.7452	2.6142	2.7870	2.5099
Std.Dev.	0.1023	0.0677	0.0520	0.0631	0.0601	0.0575	0.0918	0.0654	0.0832	0.1026
R^2	97.01%	99.39%	99.27%	98.87%	98.90%	98.88%	96.30%	97.52%	96.22%	94.51%

Table 5
Summary Statistics for Yield Errors

	Yield to Maturity									
	1m	3m	6m	1y	1.5y	2y	4y	6y	10y	15y
Mean	-0.0020	0.0000	0.0007	0.0004	0.0009	0.0006	-0.0004	-0.0004	-0.0007	0.0009
Std. Dev.	0.0023	0.0022	0.0016	0.0019	0.0018	0.0018	0.0022	0.0020	0.0024	0.0032
$\rho(1)$	0.9455	0.9719	0.9517	0.9669	0.9016	0.9308	0.9649	0.9307	0.9490	0.9693
$\rho(30)$	0.6246	0.7001	0.6476	0.8208	0.7049	0.7927	0.7184	0.6685	0.6720	0.7972
Skew.	-0.9877	-0.3260	-0.3789	0.3737	0.4123	0.9094	0.4016	0.1051	-0.2507	-0.8119
Kurt.	2.0911	-0.9111	-0.1294	1.2388	2.2463	0.9833	-0.6225	-0.7345	-0.0752	-0.0853
Normality	431.69	65.49	30.84	109.20	298.68	222.99	53.87	30.45	13.41	137.93

Figure 1
Sample Yield Surface

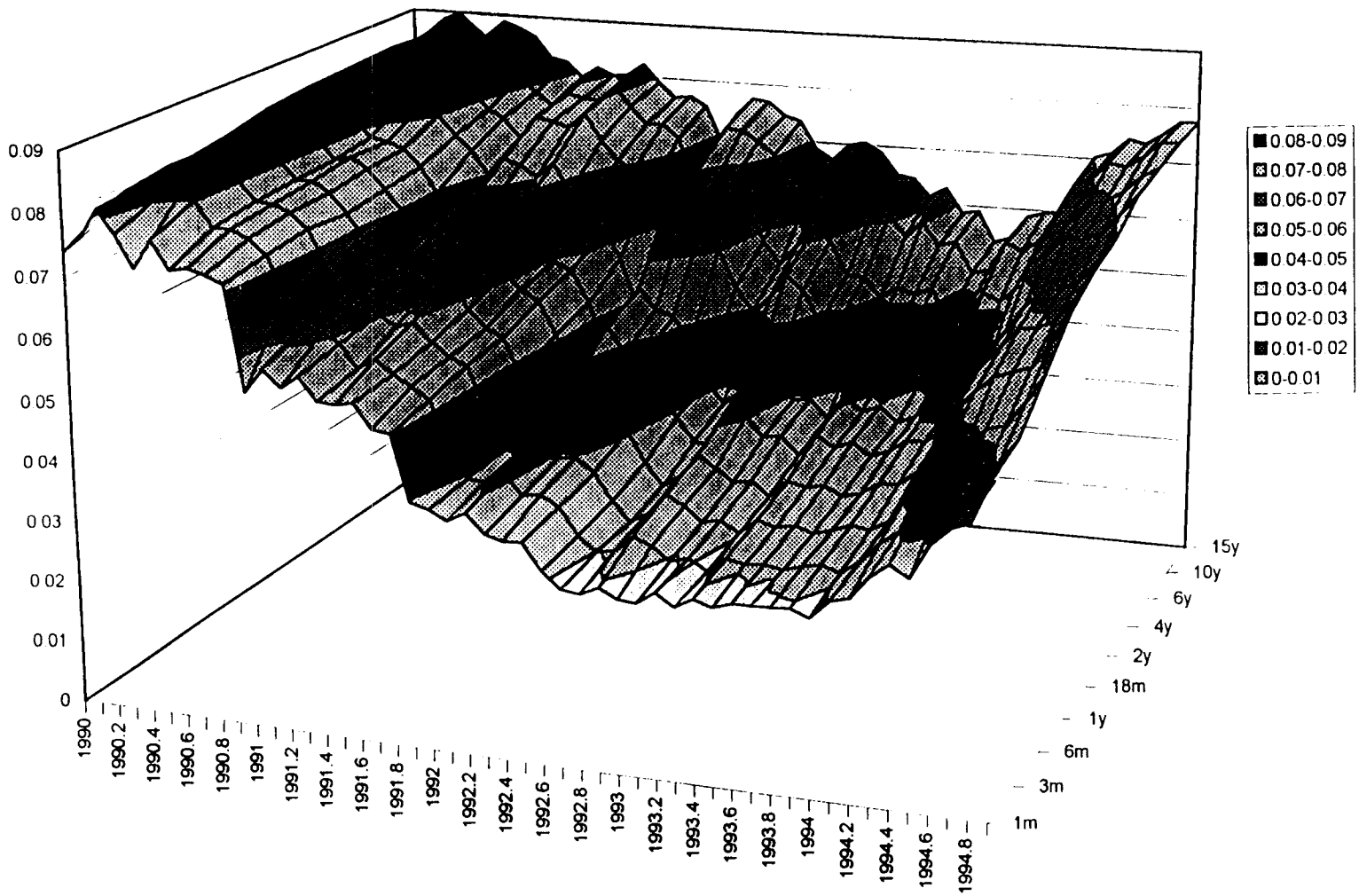

Figure 2
Instantaneous Interest Rate vs 6 Month Interest Rate

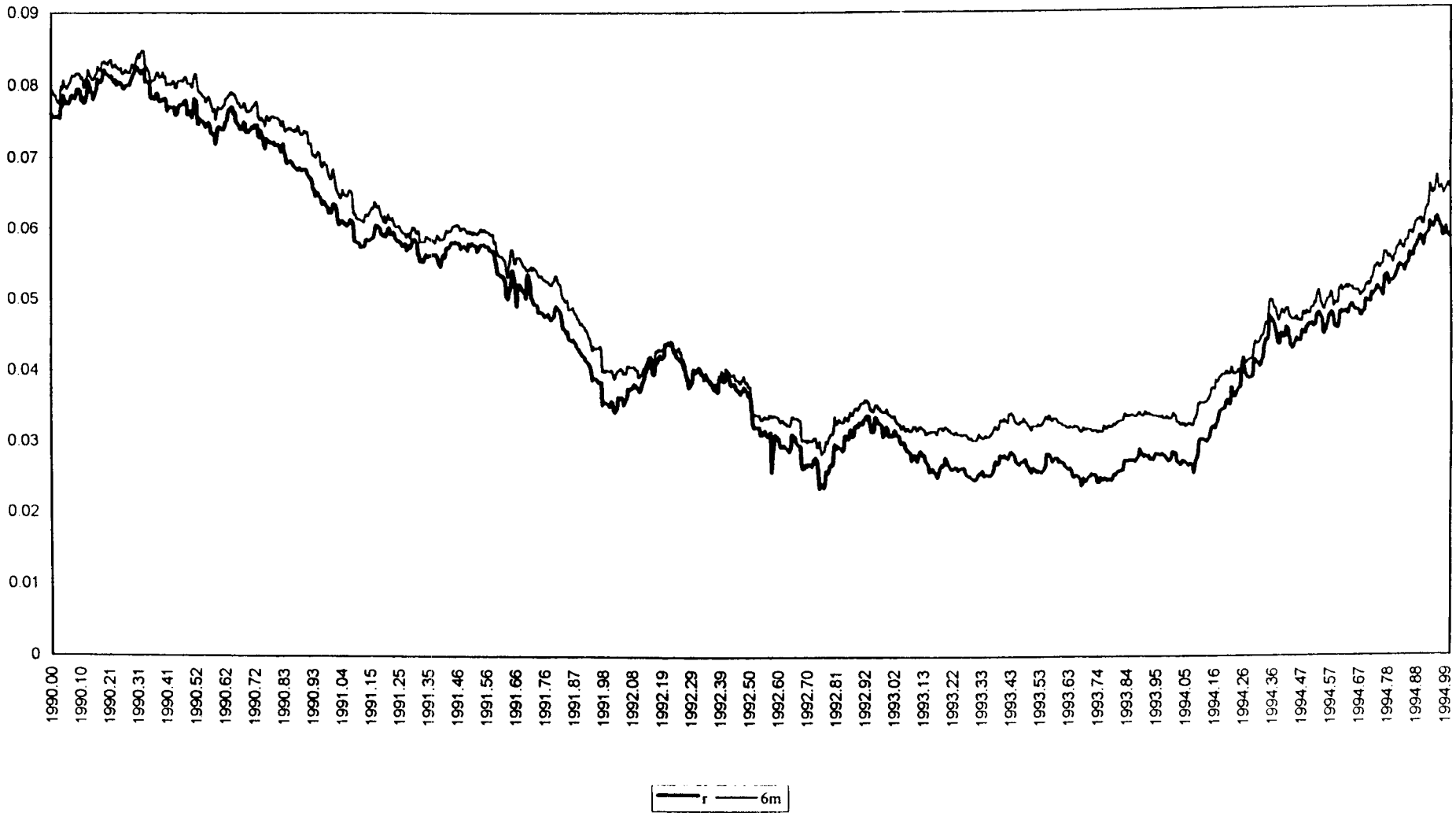

Figure 3
Path Variable vs Integral of Past Variance

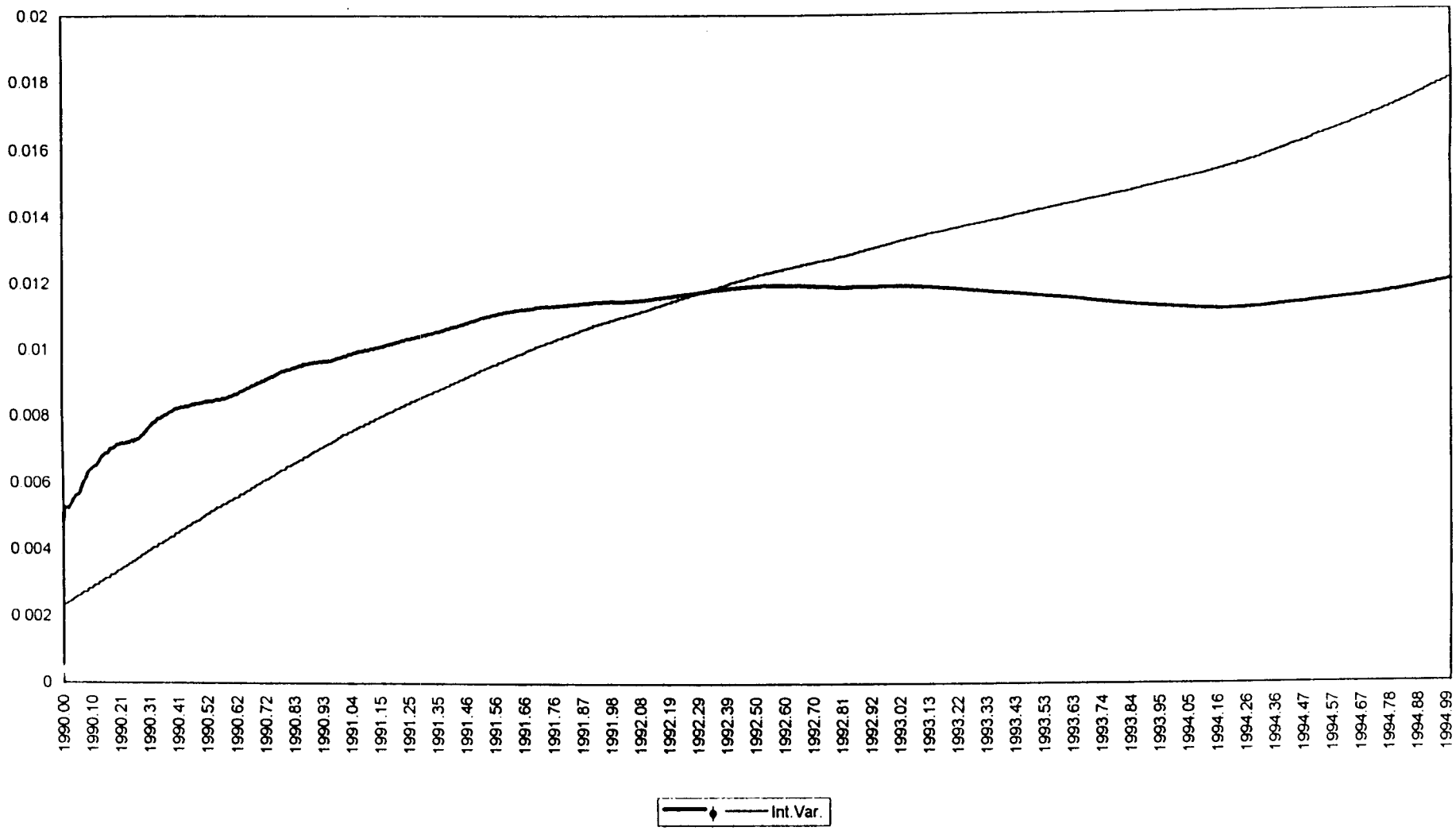

Figure 4: Factor Loadings

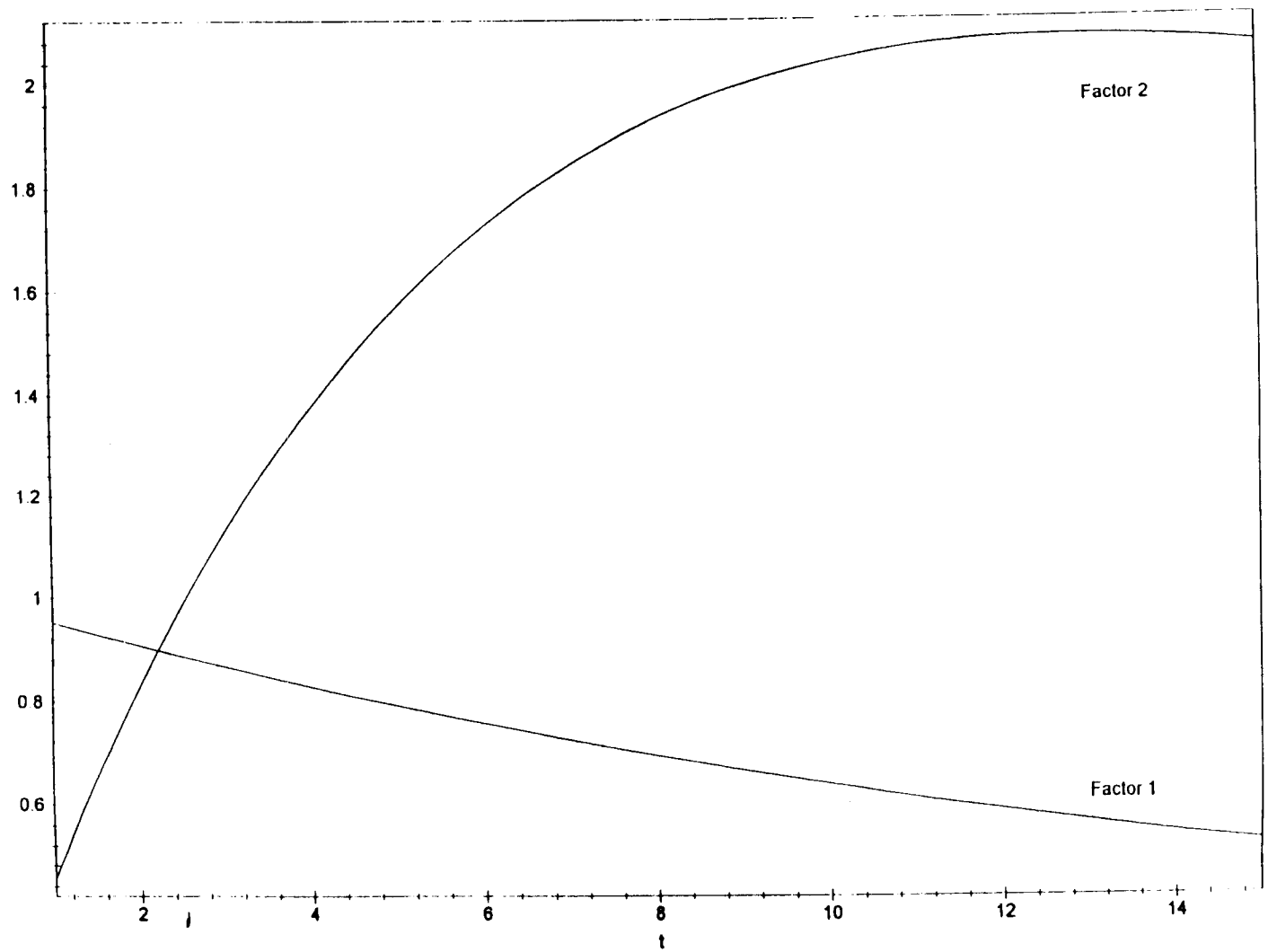

Figure 5

Estimated Yield Curves vs Actual Yields for Selected Dates