R&D INVESTMENTS WITH COMPETITIVE INTERACTIONS

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Abstract. In this article we develop a model to analyze patent-protected R&D investment projects when there is (imperfect) competition in the development and marketing of the resulting product. The competitive interactions that occur substantially complicate the solution of the problem since the decision maker has to take into account not only the factors that affect her/his own decisions, but also the factors that affect the decisions of the other investors. The real options framework utilized to deal with investments under uncertainty is extended to incorporate the game theoretic concepts required to deal with these interactions. Implementation of the model shows that competition in R&D not only increases production and reduces prices, but also shortens the time of developing the product and increases the probability of a successful development. These benefits to society are countered by increased total investment costs in R&D and lower aggregate value of the R&D investment projects.

1. Introduction

Among all types of investment projects patent-protected R&D (research and development) investment projects pose one of the most difficult tasks for evaluators. The main reason for this is that there are multiple sources of uncertainty in R&D investment projects and that they interact in complicated ways. The problem is so complex that until recently it was not possible, even with numerical methods, to analyze them. The development of numerical simulation methods that deal with optimal stopping time problems (Longstaff and Schwartz 2001) has now made this possible.

R&D investment projects typically take a long time to complete and since there is a learning process about the R&D project as investments proceed, there is large uncertainty about the investment costs required for the R&D project. There is not only uncertainty about the total costs of the development, but also about the time it will take to complete the development. In essence, there is *learning while investing*. Moreover, during the *development phase* there exists a possibility that exogenous factors such as political or technical disasters can put an end to the R&D investment project. These type of *catastrophic events* are very common in R&D investment projects because of the long investment time horizon.

Once the development phase is completed, the resulting product is produced and marketed. During this *marketing phase* there is uncertainty about the demand for the product as well as the supply of competing products. Seen from the start of the development phase these uncertainties are magnified by the fact that it is not even clear what the exact product that comes out of the R&D investment project would be. In addition, if the resulting product is patent-protected and the patent is obtained during the development phase of the R&D investment project, there will be uncertainty not only about the level of the cash flows produced, but also about the duration of these cash flows since the starting date of the marketing phase is uncertain but the expiration date of the patent is fixed.

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The possibility of competing products during the marketing phase plays a crucial role for the R&D investment decisions during the development phase since also the competing products have to go through a similar development phase. Moreover, competition in the development phase feeds back into the marketing phase in the sense that the competitive interactions in the development phase may have the effect that some of the competitors terminate their R&D investment projects even before they complete their development.

In this article we develop a model to analyze patent-protected R&D investment projects that takes into account all the sources of uncertainty described above. In particular, we combine elements of real options theory with equilibrium concepts from game theory to study this problem where the R&D investment decisions of one player depend critically on the decisions of the other players. These competitive interactions affect the valuation problem both in the development phase and in the marketing phase. The possibility of an oligopolistic outcome in the marketing phase affects the decisions taken by the players in the development phase.

We have concretized our problem by taking as an example an R&D investment project from the pharmaceutical industry. This is a particularly interesting problem since the investments required to develop a new drug are in the magnitude of hundreds of millions of dollars and typically take more than ten years to complete. Moreover, these R&D investment projects are usually patent protected at a very early stage of the development phase. Without taking competitive interactions into account Schwartz and Moon (2000) and Schwartz (2001) have also studied R&D investment projects in the pharmaceutical industry using a real options framework. In this article we mainly focus on the competitive interactions between competing firms. In the monopoly situation the owner of the R&D investment project can assume that the probability distribution of the underlying is exogenously given, whereas in the oligopoly situation the decisions of all players affect this probability distribution. Hence, the probability distribution of the underlying becomes endogenous and it is therefore part of the equilibrium outcome.

Many of the aspects of our R&D investment problem have been analyzed separately in a number of articles in the literature. Grenadier and Weiss (1997) and Bernardo and Chowdhry (2002) concentrate on the experience obtained in the investment process, but do not consider competitive interactions. The idea is that the option to invest is also an option to get more experience with a certain technology, i.e. *learning by investing*, and that this should be taken into account when analyzing the optimal time to invest. The aspect of competition is considered by Williams (1993), who analyzes the competitive exercise of options to invest. The main point is that as more investors exercise their options, the less attractive it is for other investors to exercise their options because of a downward sloping demand curve. The aspect of competition and especially the problem of coordinating the investment behavior is further analyzed by Huisman and Kort (1999) and Huisman, Thijssen, and Kort (2001). Huisman and Kort (1999) argue that the perfect coordination between the competing investors assumed by Williams (1993) is not an equilibrium outcome without cooperation between investors; in a non-cooperative setting it can happen in equilibrium that more than one investors invest simultaneously. Huisman, Thijssen, and Kort (2001) generalize these results by allowing for mixed strategies by competing investors. The aspect of asymmetric competing firms is analyzed by Pawlina and Kort (2001). Smit and Ankum (1993) is the first article to combine sequential investment options with competitive issues. Their discrete two-period binomial model captures some of the same features as our model. A similar model but in continuous time is developed by Baldursson (1998), who shows that the problem can also be solved as a central-planner problem for a specifically engineered fictitious social

planner. Finally, Grenadier (2002) adds a *time-to-build* feature to this model. The models in the last three articles have in common that investors have a number of capacity options they can exercise. In deriving the optimal exercise strategy for these capacity options investors take into account both the impact that their own exercise strategy, as well as the exercise strategies of the other investors, have on the market. None of these models have the feature of a finite time horizon, which is essential to deal with patents with finite life. Since these models deal with capacity expansion, they do not distinguish between a development phase and a marketing phase, which is critical in R&D investment projects. Some of these models capture *learning by investing* in the sense that exercising an investment option reveals more information; Grenadier (2002) adds a *time-to-build* feature in the sense that it takes a certain amount of time from when the decision to exercise an option is taken and until the pay off is realized. But none of the models capture *learning while investing* in competitive markets in the sense that investments take time *and* information is revealed *while* investing, so that it can become optimal to abandon the investment project even before completion because of competitive interactions.

Grenadier (1999) and Lambrecht and Perraudin (1999) introduce asymmetric information issues in the competitive exercise of options to invest. In these one-investor-one-option models there are no compound option aspects. Grenadier (1999) shows that asymmetric information can lead to informational cascades. Lambrecht and Perraudin (1999) concentrate on preemption in winner-takes-it-all competitive investment games.

In our model we consider two firms which are investing in R&D for two different drugs targeted to cure the same disease, so that if both are successful they would have to share the same market. The fact that, if both are successful, they will obtain duopoly profits instead of monopoly profits in at least part of the marketing phase of the product, implies that during the development phase, each firm will take into account not only its own situation but also the situation of its competitor, to make its R&D investment decisions. The costs to completion of the R&D investment project for each firm are assumed to follow stochastic processes through time with two types of shocks, i.e. *technical shocks*, which are idiosyncratic to each firm, and *input cost shocks*, which are common to both firms. In addition, during the development phase there is a Poisson probability of *catastrophic events* for each R&D investment project in the sense that it may have to be terminated because of some terrible side effect in the clinical trials or other reasons. The *winning firm*, that is, the firm that first successfully completes the R&D investment project, starts receiving monopoly profits in the sale of the drug until the *losing firm* eventually completes the R&D investment project, at which point both firms share the duopoly profits from the sale of the drug. The demand for the drug is also stochastic and we assume that the *demand shocks* follow a geometric Brownian motion. We allow also for the input cost shocks, common to both R&D investment projects, to be correlated with the demand shocks since both can depend on general market conditions. The equilibrium investment and production strategies for both firms are derived in a Cournot-Nash framework. During the development phase we focus for each firm on the *optimal stopping time* to exit the R&D investment project which represents the optimal exercise of the option to abandon the R&D investment project. Note that the optimal exercise strategy for the abandonment option for one firm depends on the exercise strategy of the other firm and vice versa, so that the values of both R&D investment projects and the optimal exercise strategies have to be solved simultaneously.

While the problem is initially formulated in continuous time, it is solved using a discrete time approximation. Since there is no closed form solution to the complex problem we formulate, we solve the problem using numerical simulation methods. We apply an extended version of the least-squares approach proposed by Longstaff and Schwartz (2001) for valuing American options, to determine the optimal stopping time for both firms, taking into account the competitive interactions.

For comparative purposes, when we report the results of the analysis for the duopoly situation, we also report the corresponding results for the monopoly situation. The monopoly situation corresponds closely to the real option problem solved by Schwartz (2001).

In reporting the results we mainly concentrate on the *symmetric case*, that is, when both R&D investment projects are identical in the duopoly situation. Though the computer program we have developed to solve the problem numerically is able to handle a great deal of generality, most of the interesting insights of the model can be better observed in the symmetric case. Also, comparisons with the monopoly situation are more meaningful in this case. Without loss of generality, we concentrate on the case where the patents for both competitive drugs expire at the same date. If, on the other hand, the patents have different expiration dates, there is no value in the second patent protection when the first patent expires since generic drugs related to the first drug will be introduced and be able to compete with the second drug.

The model provides some interesting results with potentially important policy implications. As expected, the value of the R&D investment project to the monopolist is higher than the aggregate value of the R&D investment projects for both duopolists since both have to share the same demand. The amount produced, however, is on average higher for the duopolists, not only because when both are producing simultaneously they produce a larger amount (at a lower price), but also because the probability that at least one of the duopolists eventually produces is higher than the probability that the monopolist produces, and on average the time until the first project is completed is shorter. Thus, even though the total costs to R&D are higher in the duopoly situation and the value of the R&D investment projects is lower, the amount produced and the probability of actually producing is higher and the average time to develop the project is shorter. Hence, if the objective of the policy maker or regulator is to promote the production of the largest possible amount of drugs at the lowest possible price in the shortest period of time, competition in R&D accomplishes this objective. The model presented can also be used to derive other policy implications such as the effect of subsidies or drug price and/or quantity commitments on the amount of R&D investments.

The article is organized as follows. Section 2 presents the model and derives the Cournot-Nash type equilibrium. Section 3 explains the numerical solution procedure used in the implementation of the model. Section 4 describes the numerical results and performs sensitivity analysis of these results with respect to key parameters of the model. Finally, Section 5 summarizes the article and provides some concluding remarks.

2. The Model

We assume that two firms are each investing in R&D for a drug that is targeted to cure the same disease. Both firms take out a patent on their specific drug at date zero based on their earlier (pre patents) R&D.[1] The two patents are based on different molecules and will lead to different drugs, but both drugs are targeted to cure the same disease. Before the drugs can be marketed, some post patents R&D must be conducted (further research, development, testing, clinical trials, etc.). When the first drug is marketed, the drug will

[1]The assumption that both firms take out their patents at the same date is not important. The game could also start at the date when the second firm takes out its patent.

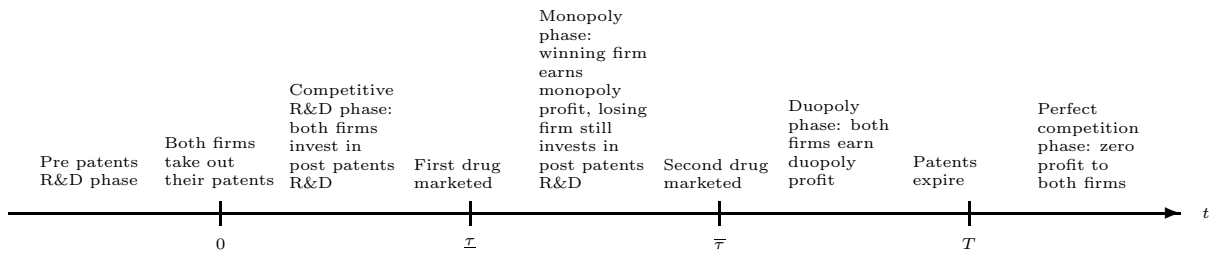


FIGURE 1. Time line of our model.

be protected from competition by the patent so that the owner would be able to earn a monopoly profit until, eventually, the second firm markets its competing drug. When that second drug is marketed, the two firms will still be protected from further competition by their two patents. Hence, the two firms have the only two drugs for this disease and will therefore be able to earn a duopoly profit.[2] This situation continues until the two patents expire. When this happens, we assume that generics will flood the market and drive all profits to zero in a perfect competitive market setting.[3] The whole time line of our model is summarized in Figure 1. The important decision variables for our two firms are the post patents R&D investment/abandonment decisions. That is, based on the information of both the firm's own and its competitor's estimated remaining R&D investments and forecasts of the demand for the drug, each firm must consider whether it is worthwhile for it to continue investing in R&D or whether it should abandon its R&D investment project. In order to solve that problem we first have to develop a model for the consumption market where the drug is eventually going to be sold.

We start by modeling the market for drugs for a given disease. We assume that the price of the drug, denoted P_t, at any given date, t, is given by

$$P_t = Y_t Q(q_t),$$

when the date t instantaneous production rate is q_t. Here Y is an exogenously given stochastic process that models *demand shocks* to the model. That is, Y captures stochastic shocks that change the demand of the drug, e.g., epidemics, acts of terror, development of vaccines, non-anticipated alternative drugs, etc. We assume Y follows a geometric Brownian motion under an equivalent martingale measure, $\mathbb{Q}$, i.e.[4]

$$dY_t = \mu_y Y_t dt + \sigma_y Y_t dW_t^y, \qquad Y_0 = 1,$$

where μ_y and σ_y are given constants parameterizing the drift and volatility of the demand shocks and W^y is a standard Brownian motion under an equivalent martingale measure, $\mathbb{Q}$.[5] $Q(\cdot)$ is the inverse demand function for the drug (except for Y_t) and we assume it has the following form

$$Q(q) \equiv a e^{-bq^2}, \qquad q \geq 0, \tag{1}$$

[2]Note that here we have abstracted from the fact that one of the drugs may be more efficient than the other and, thus, may capture a larger share of the market.

[3]It would be easy to introduce some terminal value to the R&D investment projects at the expiration of the patent period.

[4]Since in this article we pursue the valuation and the optimal R&D investment/abandonment strategies, we only need to model our stochastic processes under an equivalent martingale measure, $\mathbb{Q}$.

[5]Formally, define the probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ and a filtration, $\mathbf{F} \equiv \{\mathcal{F}_t\}_{t\in[0,T]}$, which we will concretize later, that fulfills the usual conditions. All stochastic processes we define in this article, including Y and W^y, are implicitly assumed to be adapted to $\mathbf{F}$.

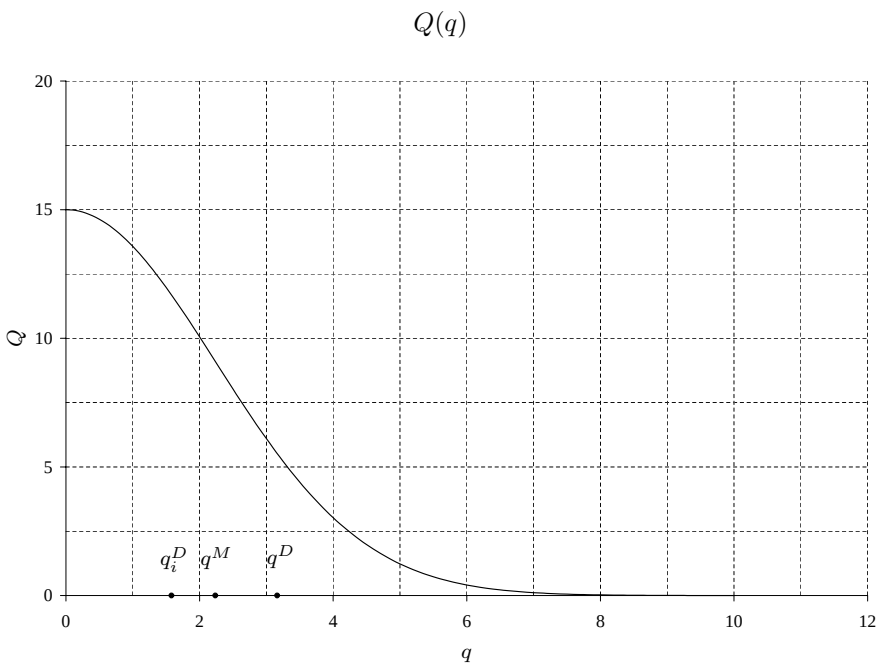


FIGURE 2. The inverse demand function, $Q(\cdot)$, from equation (1) for $a = 15$ and $b = 0.1$. The optimal production rates, q^M and q_i^D, for the monopoly and the duopoly situations are marked together with the total production rate in the duopoly case, q^D.

where a and b are positive constants. We have chosen this specific form of the inverse demand function since it gives internal optimal solutions even without variable production cost rates both for the monopoly and duopoly supply situations.[6] For $a = 15$ and $b = 0.1$ we have depicted the inverse demand function, $Q(\cdot)$, in Figure 2.

If there is only one firm which has monopolistic supply of the drug at date t, this firm would simply control the production rate, q_t, to maximize the instantaneous profit rate, Π_t, given by

$$\Pi_t \equiv P_t q_t = Y_t a q_t e^{-bq_t^2}.$$

Note again that for simplicity we have assumed that the variable direct production cost rate is zero. The optimal monopoly production rate at date t, q^M, is easily derived as

$$q^M = \frac{1}{\sqrt{2b}},$$

[6]For simplicity we assume that variable production costs are zero, because this significantly simplifies our analysis. Basically, the only role for the production function, Q, is to provide two different production levels, one for the situation where there is only one producer, the monopoly situation, and one for the situation where there are two producers, the duopoly situation. Production costs would only matter for the decision of how much to produce when the drug is marketed. If there are positive production costs, the production function, Q, should just be altered so that it gives the two optimal production rates as solutions when the production costs are included in the optimization and so that the corresponding function values are the profit rates. The whole analysis can then be carried out the same way as it is in the article. In the pharmaceutical industry variable production cost rates have little importance relative to R&D investment costs. That is, variable production cost rates can be neglected from the problem without any significant alterations of the qualitative conclusions from our analysis.

which is independent of t. Since in the monopoly case there is only one firm producing, this production rate will also be the total production rate at any date. It implies a monopoly price at date t of

$$P_t^M = \frac{a}{\sqrt{e}} Y_t$$

and a monopoly profit rate at date t of

$$\Pi_t^M = \frac{a}{\sqrt{2be}} Y_t.$$

The superscript M indicates monopoly.

If there are two firms, indexed one and two, competing for selling drugs to cure the same disease at date t, we assume that these two firms compete in a Cournot competitive fashion. In order to calculate the corresponding market equilibrium we first would have to calculate the two firms' response functions. Given firm $j \in \{1, 2\}$ has set its production rate at date t to q_{jt}, consider the problem of finding the optimal production rate for the other firm, which is indexed $i = 3 - j$,[7] at date t. Given firm $j \in \{1, 2\}$ has set its production rate at date t to q_{jt}, firm $i = 3 - j$ should maximize its instantaneous profit rate at date t as a function of its own production rate, q_{it},

$$\Pi_{it} \equiv P_t q_{it} = Y_t a q_{it} e^{-b(q_{it}+q_{jt})^2}.$$

The response function for firm i's production rate at date t is easily derived as

$$q_{it}^*(q_{jt}) = \sqrt{\frac{bq_{jt}^2 + 2}{4b}} - \frac{q_{jt}}{2}.$$

By symmetry we know that the response functions for both firms are identical. The unique Nash equilibrium production rate at date t in a Cournot duopoly setting is then the (unique) fix point of the function

$$q(q) \equiv \sqrt{\frac{bq^2 + 2}{4b}} - \frac{q}{2},$$

which is again independent of t. Hence, the equilibrium production rate at date t for each of the two firms can easily be derived as

$$q_i^D = \frac{1}{2\sqrt{b}}, \qquad i \in \{1, 2\}.$$

Hence, the total duopoly production rate will be

$$q^D = \sum_{i=1}^{2} q_i^D = \frac{1}{\sqrt{b}},$$

the duopoly price at date t will be

$$P_t^D = \frac{a}{e} Y_t,$$

and the duopoly profit rate at date t to each of the two firms will be

$$\Pi_{it}^D = \frac{a}{2e\sqrt{b}} Y_t, \qquad i \in \{1, 2\}.$$

The superscript D indicates duopoly. Note that the total production rate at date t has increased by a factor $\sqrt{2} \approx 1.41$ from $\frac{1}{\sqrt{2b}}$ in the monopoly case to $\frac{1}{\sqrt{b}}$ in the duopoly case and at the same time the price has dropped by a factor $\sqrt{e} \approx 1.65$ and total profit rates have dropped by a factor $\sqrt{\frac{e}{2}} \approx 1.17$, see Figure 2.

[7]There are exactly two firms in our model, indexed one and two. Hence, if one firm has index $j \in \{1, 2\}$, the other firm must be indexed $3 - j$.

If there is perfect competition, standard microeconomic arguments give that the profit of each (identical) firm is driven to zero. In our model we have assumed that variable production cost rates are zero so this means that the sum of the production rates for all the firms would converge to infinity and the corresponding equilibrium price for the drug would be zero. That is,

$$q^{PC} \equiv \infty,$$
$$P_t^{PC} = 0,$$

and

$$\Pi_{it}^{PC} \equiv 0, \qquad i \in \mathbb{N}.$$

The superscript PC indicates perfect competition.

This characterizes the situation our two firms will face when their respective R&D investment projects eventually develop into a drug that can be marketed. That is, in real options terms we have characterized the underlying security. However, in order to develop the drug the firms have to go through an uncertain phase of R&D. At date zero when the firms take out their patents, each of the two firms has an estimate of the costs of the remaining R&D investments, K_{10} and K_{20}, that they each still have to conduct. These estimates of remaining R&D investment costs are assumed to be public information.[8] At any given date t the estimated remaining R&D investment costs for firm $i \in \{1, 2\}$ is given by the stochastic variable, K_{it}. For tractability we assume that the whole process of past and present estimated remaining R&D investment costs, $\{(K_{1s}, K_{2s})\}_{s\in[0,t]}$ as well as the past and present values of the demand shock process, $\{Y_s\}_{s\in[0,t]}$, are public information. As long as firm i has not yet abandoned its R&D investment project, the stochastic process, K_i, for $i \in \{1, 2\}$, develops over time under an equivalent martingale measure, $\mathbb{Q}$, according to the stochastic differential equations

$$dK_{it} = -I_i dt + \gamma_i \sqrt{I_i K_{it}} dz_t^i + \mu_{ik} K_{it} dt + \sigma_{ik} K_{it} dW_t^k. \tag{2}$$

Here z^1, z^2, and W^k are standard Brownian motions under $\mathbb{Q}$. The first term in equation (2) reflects the rate at which the firm invests in R&D for the drug at date t.[9] Since the decision to continue investing in R&D is an irreversible decision, the current investment rate, I_i, must at any date t be non-negative. Furthermore, since it takes time to conduct R&D, the current investment rate, I_i, must at any date t be finite. The second term in equation (2) reflects the uncertain nature of the R&D process itself over time due to technical uncertainty. The more R&D investments the firm estimates it still has to conduct and the higher the current R&D investment rate is, the more uncertainty will be revealed per time unit. Moreover, we assume that these type of *technical shocks* are independent between the two firms and also independent of the demand shocks and the R&D input cost shocks. That is, z^1 and z^2 are independent of each other and also independent of W^y and W^k.[10] γ_i is a firm specific volatility parameter measuring the size of technical shocks. The two last terms in equation (2) reflect that the estimated remaining R&D investment costs vary

[8]In this article we have abstracted from the interesting issues arising from asymmetric information, and concentrated our attention on capturing the competitive interactions.

[9]Purely for expositional simplicity we have assumed that the investment rate of firm i is a constant, I_i. In our numerical implementation of our model, cf. Section 3, it could as well have been a deterministic function of time or even a deterministic function of the current values of the governing state variables.

[10]This assumption is not essential, but it simplifies the development of the model.

not only because of technical shocks but also because of general uncertainty in the surrounding market, e.g., labor costs, input costs to the R&D process, etc. We assume that these *input cost shocks* are the same for both firms; thus it is the same Brownian motion, W^k, that enters into both firms' estimated remaining R&D investment cost processes. Moreover, W^k may be correlated with W^y to reflect that the general market conditions are also related to the demand of the drug.[11] That is, we assume

$$d\langle W^y, W^k\rangle_t = \rho_{yk} dt.$$

The drift terms μ_{ik} and volatility terms σ_{ik} parameterize the uncertainty in the surrounding market, which may be different for the two firms. For example, firm specific expected increases in labor costs and input costs over time is parameterized via μ_{ik}. At date zero when the firms take out their patents, their estimated remaining R&D investment costs are of course positive, so $K_{i0} > 0$, $i \in \{1, 2\}$. The specification of the development of the estimated remaining R&D investment costs from equation (2) is very similar to the specifications used in Pindyck (1993), Schwartz and Moon (2000), and Schwartz (2001).[12]

Schwartz and Moon (2000) and Schwartz (2001) consider also the possibility of *catastrophic events*. This reflects the fact that besides costs uncertainty and demand uncertainty there is also a risk that the R&D investment project can simply fail for other reasons independent of how much the firm invests in it and independent of how high the demand for the drug will be. It may be that the clinical trials reveal that the drug has some terrible side effects, it may turn out that it simply is not technically feasible to develop the drug, it may be that the government prohibits certain classes of drugs, etc. We model this type of catastrophic events as two Poisson processes, denoted Q_1 and Q_2, one for each firm, with intensities λ_1 and λ_2. These two Poisson processes are independent of each other and also independent of the other three governing state variables, K_1, K_2, and Y. For tractability we also assume that past and present values of the Poisson processes, $\{(Q_{1s}, Q_{2s})\}_{s\in[0,t]}$, are public information.

We have depicted illustrative sample paths of the two estimated remaining R&D investment cost processes, K_1 and K_2, in Figure 3 in an example where both firms are exactly equal (the *symmetric case*): both firms have at date zero estimated remaining R&D investment costs of 100 ($K_{10} = K_{20} = 100$) and both invest 10 per year in R&D ($I_1 = I_2 = 10$). Both firms face technical shock volatility of 20% ($\gamma_1 = \gamma_2 = 0.2$) and equal drift and volatility parameters of the input cost shocks to the R&D investment project of zero and 10% ($\mu_{1k} = \mu_{2k} = 0$ and $\sigma_{1k} = \sigma_{2k} = 0.1$). In these sample paths we have assumed that the R&D investment projects continue until the corresponding estimated remaining R&D investment cost processes, K_1 and K_2, hit zero. The competitive R&D phase (marked **R&D** in Figure 3) takes place in the time period from date zero and until the first process hits zero around date 10.6. The monopoly phase (marked **M** in Figure 3)

[11]Both positive and negative correlations as well as no correlation are economically plausible. A positive correlation could be explained by a higher than expected demand for the drug if the general economy booms, which would then also lead to higher than expected input costs to the R&D investment project. This would, e.g., be the case for a drug like Insulin. A negative correlation could be explained by a higher than expected demand for the drug if the general economy ends up in a recession. This would be the case for a drug like Prozac. Naturally, there are also cases where there is no connection between the demand for the drug and the general state of the economy. In our main numerical examples in Section 4 we use a small negative correlation, but we also perform sensitivity analysis with respect to this correlation parameter.

[12]It should be pointed out that the models in these articles are formulated as stochastic optimal control problems, whereas our problem is formulated as an optimal stopping time problem. The optimal solutions to these stochastic optimal control problems are typically bang-bang solutions and therefore they are very similar to the solution obtained by solving an optimal stopping time problem. However, the optimal stopping time solution does not allow for costless temporary shut-down of the R&D investment project. Since we are dealing here with a finite time horizon, the option to temporary shut down is not important and, in addition, probably unrealistic for a drug development project.

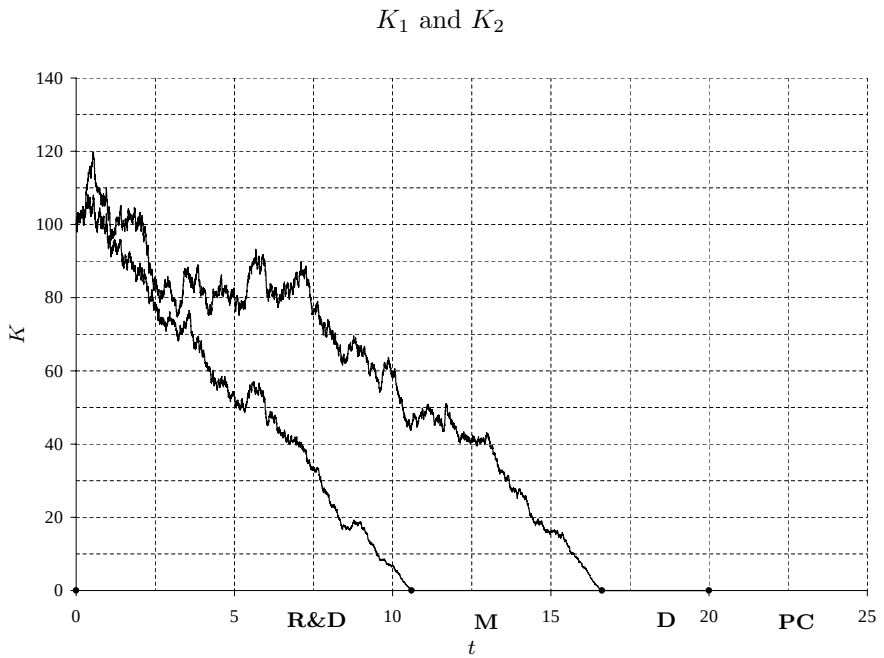


FIGURE 3. Illustrative sample paths of the two estimated remaining R&D investment cost processes, K_1 and K_2, from equation (2) for $K_{10} = K_{20} = 100$, $I_1 = I_2 = 10$, $\gamma_1 = \gamma_2 = 0.2$, $\mu_{1k} = \mu_{2k} = 0$, and $\sigma_{1k} = \sigma_{2k} = 0.1$. In these sample paths we have assumed that both R&D investment projects continue until their corresponding estimated remaining R&D investment cost processes, K_1 and K_2, hit zero. The competitive R&D phase (marked **R&D**) takes place in the time period from date zero and until the first process hits zero around date 10.6. The monopoly phase (marked **M**) takes place in the time period from when the first process hits zero around date 10.6 and until the second process hits zero around date 16.6. Finally, the duopoly phase (marked **D**) takes place in the time period from when the second process hits zero around date 16.6 and until the patents expire at date T, which is 20 years in this example. After date T (20 years) the perfect competition phase (marked **PC**) takes over.

takes place in the time period from when the first process hits zero around date 10.6 and until the second process hits zero around date 16.6. Finally, the duopoly phase (marked **D** in Figure 3) takes place in the time period from when the second process hits zero around date 16.6 and until the patents expire at date T, which is 20 years in this example. After date T (20 years) the perfect competition phase (marked **PC** in Figure 3) takes over. Cf. Figure 1 for the complete time line of our model. The parameter values used to create Figure 3 are identical to the ones that we will use in our numerical examples in Section 4. Note that the four phases of our model, the competitive R&D phase, the monopoly phase, the duopoly phase, and the perfect competition phase, are defined based solely on the development of the two estimated remaining cost processes, K_1 and K_2. Because of optimal abandonment of the R&D investment project and/or the occurrence of catastrophic events, it may very well be the case that there is only one firm (or even no firms) producing drugs in the duopoly phase. Similar things can happen in the other phases. The names of the different phases are based on what would have happened if there were no abandonment and the catastrophic

events never occurred. The reader should only use the names of the different phases to be able to distinguish the four phases of the model and not necessarily as a statement of what type of economic activity that will occur in these phases.

The drug developed by firm $i \in \{1,2\}$ is marketed as soon as the corresponding (estimated) remaining R&D investment cost process, K_i, hits zero unless either an optimal abandonment decision has been taken earlier on or catastrophic events have occurred to the R&D investment project earlier on.[13] In order to keep track of when this happens we introduce some stopping times.[14] Define τ_i to reflect when firm i's product will be marketed, $i \in \{1,2\}$, if its project is still alive, i.e., if neither an optimal abandonment decision has been taken earlier on nor catastrophic events have occurred to the R&D investment project earlier on. As a first attempt we can specify this as

$$\inf\{t \geq 0 | K_{it} = 0\}.$$

However, the patents the firms take out at date zero have a certain life span, normally twenty years, which we here denote T. If none of the two firms have been able to market a drug within that life span, they will not be able to derive any profits from their R&D effort. There are many possible scenarios leading to that conclusion. One of them is that if they continue their R&D effort even after date T and eventually market their drug, an instant later the generics are ready with competing drugs because the patents have already expired. So they will not be able to derive any profits from their R&D effort. A more likely scenario is the following: since it would never be optimal to continue the R&D investment projects after date T, the R&D investment project will be abandoned at date T at the latest, and very likely much earlier. Hence, there will be no drugs marketed and therefore no generics either. Again, they will not be able to derive any profits from their R&D effort. Thus, our stopping times are only interesting when they are strictly smaller than T since there can be derived no profits after date T. That is, we would like to refine our definition of τ_i, $i \in \{1,2\}$ to[15]

$$\tau_i \equiv \min\big\{\inf\{t \geq 0 | K_{it} = 0\}, T\big\}.$$

Hence, the monopoly phase starts at date $\underline{\tau}$ defined as

$$\underline{\tau} \equiv \min\{\tau_1, \tau_2\},$$

and the duopoly phase starts at date $\overline{\tau}$ defined as

$$\overline{\tau} \equiv \max\{\tau_1, \tau_2\}.$$

As long as the firms are still investing in R&D, they can decide to abandon their R&D investment project if they find that it is not profitable to continue. We will denote the stopping time when the firms stop investing in their R&D investment projects for economic reasons by ν_i, $i \in \{1,2\}$.[16] Surely, they will stop investing no later than when their R&D investment project is completed, hence $\nu_i \leq \tau_i$. The event $\{\nu_i = \tau_i\}$ now means

[13] We place estimated in parentheses because when the (estimated) remaining R&D investments are exactly zero, they are not just estimates any more, they are truly zero: the drug *is* ready.

[14] Formally, a stochastic variable, τ, is a stopping time related to the filtration $\mathbf{F}$ if the event $\{\tau \leq t\} \in \mathcal{F}_t$, for all $t \in [0,T]$. Moreover let $\mathcal{S}(\mathbf{F})$ denote the set of all stopping times related to the filtration $\mathbf{F}$. For the rest of the article the filtration $\mathbf{F}$ will be the filtration generated by the governing state variables, i.e. $\mathcal{F}_t = \sigma\{(Y_s, K_{1s}, K_{2s}, Q_{1s}, Q_{2s}) | s \in [0,t]\}$.

[15] We do not include the catastrophic events into our stopping times since these are much more efficiently dealt with explicitly by multiplying the relevant expressions with the probability that the catastrophic events will occur under an equivalent martingale measure, $\mathbb{Q}$.

[16] We still do not include catastrophic events into our stopping times, cf. footnote no. 15.

that the firm did not find it optimal to abandon its R&D investment project before completion, whereas the event $\{\nu_i < \tau_i\}$ means that the firm did find it optimal to abandon its R&D investment project before completion.

We are now ready to more formally set up the objectives of the two firms. Define the winning firm as the firm which, if its project is alive, markets its drug at the entrance date into the monopoly phase and let w denote the index of the winning firm. That is,

$$w \equiv \begin{cases} 1, & \tau_1 = \underline{\tau}, \\ 2, & \tau_1 \neq \underline{\tau}. \end{cases}$$

Moreover, let l denote the index of the losing firm, i.e., the firm which, if its project is alive, markets its drug at the entrance date into the duopoly phase.[17] That is,

$$l \equiv 3 - w.$$

In order to find the values of the two firms' R&D investment projects as well as their optimal R&D investment/abandonment strategies we have to value their projects in all three phases of our model starting from the last phase, i.e. the duopoly phase. At the entrance date into the duopoly phase there are three possible situations: there can be either two, one, or no projects alive to be marketed. If there are still two projects alive to be marketed at the entrance date into the duopoly phase, the firms will compete in the usual Cournot fashion. At any given date t in the duopoly phase, i.e. $t \in [\overline{\tau}, T)$, the total value to each of the two firms of all cash flows after that date can be derived as[18]

$$\begin{aligned} V^{D2}(Y_t, t) &\equiv E^{\mathbb{Q}}\left[\int_t^T e^{-r(s-t)}\Pi_{is}^D ds \;\middle|\; \mathcal{F}_t\right] \\ &= \frac{a}{2e\sqrt{b}} E^{\mathbb{Q}}\left[\int_t^T e^{-r(s-t)} Y_s ds \;\middle|\; \mathcal{F}_t\right] \\ &= \frac{a}{2e\sqrt{b}} \int_t^T e^{-r(s-t)} E^{\mathbb{Q}}[Y_s|\mathcal{F}_t] ds \\ &= \frac{a}{2e\sqrt{b}} \int_t^T e^{-r(s-t)} Y_t e^{\mu_y(s-t)} ds \\ &= \frac{a}{2e\sqrt{b}} Y_t e^{(r-\mu_y)t} \int_t^T e^{-(r-\mu_y)s} ds \\ &= \frac{a}{2(r-\mu_y)e\sqrt{b}} Y_t e^{(r-\mu_y)t}\left(e^{-(r-\mu_y)t} - e^{-(r-\mu_y)T}\right) \\ &= \frac{a}{2(r-\mu_y)e\sqrt{b}}\left(1 - e^{-(r-\mu_y)(T-t)}\right) Y_t, \qquad t \in [\overline{\tau}, T), \end{aligned} \tag{3}$$

where r is the riskless interest rate, which we for simplicity assume is constant. Note that the value will only depend on the value of the state variable Y and the date t. The two state variables measuring the

[17]Note that in the event that the two firms' estimated remaining R&D investment cost processes hit zero exactly at the same instant in time or none of them hit zero before the patent expires at date T, firm one would be called the winning firm and firm two would be called the losing firm. But as we will see in the derivation of the objective functions below in these two special cases, there will be no difference between the winning firm's and the losing firm's objective functions. Hence, it does not really matter which of the two we assign as the winning firm and which we assign as the losing firm in these two special cases.

[18]Since we have already developed all our stochastic processes under an equivalent martingale measure, $\mathbb{Q}$, the value of all future profits and costs can be calculated by just summing (integrating) all the expected cash flows (cash flow rates) discounted using the riskless interest rate.

estimated remaining R&D investment costs, K_1 and K_2, are already zero so they are not relevant any more. The superscript $D2$ indicates that this is the project value in the duopoly phase if there are still two projects alive, i.e. if both projects have survived the catastrophic events and none of them have been abandoned for economic reasons.

If one of the firms is hit by catastrophic events or if one of the firms abandons its R&D investment project prior to the duopoly phase, the other firm would be able to earn a monopoly profit even in the duopoly phase. At any given date t in the duopoly phase, i.e. $t \in [\overline{\tau}, T]$, the total value to the surviving firm of all cash flows after that date can similarly be derived as

$$\begin{aligned} V^{D1}(Y_t, t) &\equiv E^{\mathbb{Q}}\left[\int_t^T e^{-r(s-t)}\Pi_s^M ds \;\middle|\; \mathcal{F}_t\right] \\ &= \frac{a}{(r-\mu_y)\sqrt{2be}}\big(1 - e^{-(r-\mu_y)(T-t)}\big)Y_t, \qquad t \in [\overline{\tau}, T). \end{aligned} \tag{4}$$

The superscript $D1$ indicates that this is the surviving project value in the duopoly phase if only one of the projects is alive.

If none of the two projects are alive in the duopoly phase, obviously no profits will be made and the value is therefore zero.

In the monopoly phase, i.e. from date $\underline{\tau}$ to date $\overline{\tau}$, the winning firm makes a monopoly profit (if its project is still alive) while the losing firm is (perhaps) still investing in R&D. For this period we will have to separate the calculations of the values of the two firms. If the losing firm's project is still alive, it is still investing in R&D and, therefore, it is still exposed to catastrophic events. The conditional probability (under an equivalent martingale measure, $\mathbb{Q}$) that the losing firm is hit by catastrophic events during a period from date t to date s in the monopoly phase, given that its project was alive at date t is $1 - e^{-\lambda_l(s-t)}$. Similarly, the conditional probability (under an equivalent martingale measure, $\mathbb{Q}$) that it is not hit by catastrophic events throughout the period from date t to date s in the monopoly phase, given that its project was alive at date t is $e^{-\lambda_l(s-t)}$. The winning firm, on the other hand, is no longer exposed to catastrophic events since it has already completed its R&D investment project at the entrance date into the monopoly phase. However, the objective function of the losing firm depends on whether or not the winning firm's project is alive at the entrance date into the monopoly phase, since this determines whether the losing firm will be earning a monopoly or a duopoly profit when its R&D investment project is eventually completed. If the winning firm's project is still alive at the entrance date into the monopoly phase, then, at any given date t in the monopoly phase, i.e. $t \in [\underline{\tau}, \overline{\tau})$, the total value to the losing firm (if its project is alive) of all cash flows after

that date can be derived as

$$
\begin{aligned}
V_l^{M2l}(Y_t, K_{lt}, t) &\equiv \max_{\nu_l \in \mathcal{S}(\mathbf{F})} E^{\mathbb{Q}}\Bigg[-\int_t^{\nu_l} e^{-\lambda_l(s-t)} e^{-r(s-t)} I_l ds \\
&\qquad + 1_{\{\nu_l = \overline{\tau}\}} e^{-\lambda_l(\overline{\tau}-t)} e^{-r(\overline{\tau}-t)} V^{D2}(Y_{\overline{\tau}}, \overline{\tau}) \,\Bigg|\, \mathcal{F}_t \Bigg] \\
&= \max_{\nu_l \in \mathcal{S}(\mathbf{F})} E^{\mathbb{Q}}\Bigg[-\int_t^{\nu_l} e^{-(r+\lambda_l)(s-t)} I_l ds \\
&\qquad + \frac{a}{2(r-\mu_y)e\sqrt{b}} e^{-(r+\lambda_l)(\overline{\tau}-t)} \times \\
&\qquad \left(1 - e^{-(r-\mu_y)(T-\overline{\tau})}\right) 1_{\{\nu_l = \overline{\tau}\}} Y_{\overline{\tau}} \,\Bigg|\, \mathcal{F}_t \Bigg], \qquad t \in [\underline{\tau}, \overline{\tau}).
\end{aligned} \tag{5}
$$

Note that the value will only depend on the value of the state variable Y, the estimated remaining R&D investment costs for the losing firm, K_l, and the date t. The state variable measuring the estimated remaining R&D investment costs for the winning firm, K_w, is already zero and therefore not relevant any more. The superscript $M2l$ indicates that this is the losing firm's value in the monopoly phase if the winning firm's project is still alive. Note the two terms in equation (5): the first term is the losing firm's R&D investment costs in the monopoly phase after date t and until it is either hit by catastrophic events, it decides to abandon its R&D investment project, or until its R&D investment project is completed; the second term is the losing firm's share of the duopoly profit in the duopoly phase in the event that the losing firm is not hit by catastrophic events in the period from date t and until the entrance date into the duopoly phase and it does not decide to abandon its R&D investment project before completion. In equation (5) we use a so-called indicator function of the form 1_A, where A is some event. This function takes the value one if the event, A, is true and zero otherwise. The value in equation (5) is the result of a maximization problem, since the losing firm should decide at each instant in time whether to continue investing in R&D or to abandon the R&D investment project. This decision must at each date be taken based on the available information, i.e. the past and current values of the governing state variables. That is, the R&D investment/abandonment strategy must be a stopping time related to the filtration $\mathbf{F}$. We have indicated this restriction in equation (5) by requiring ν_l to be a member of the set $\mathcal{S}(\mathbf{F})$. Denote the optimal R&D investment/abandonment strategy for the problem in equation (5) as ν_{lt}^{2*}. Note that the optimal R&D investment/abandonment strategy will depend on the valuation date t in the problem in equation (5), i.e., it is the (date t) optimal R&D investment/abandonment strategy for the rest of the monopoly phase, *given* that the losing firm has not yet abandoned its R&D investment project at date t. The superscript $2*$ indicates that this is the (date t) optimal R&D investment/abandonment strategy, given that the winning firm's project is still alive at that date.

Intuitively the optimal stopping time problem in equation (5) can be solved by dynamic programming. The boundary condition is given by the value at the entrance date into the duopoly phase. That is,

$$
V_l^{M2l}(Y_{\overline{\tau}}, 0, \overline{\tau}) = V^{D2}(Y_{\overline{\tau}}, \overline{\tau}). \tag{6}
$$

The optimal stopping time problem is solved by starting with the boundary condition and then going backward in time in the usual dynamic programming fashion. That is, we solve for the value of the R&D investment project at date t (in the monopoly phase) conditional on that we have already solved for the value at any later date s. Let $V_l^{M2l}(Y_s, K_{ls}, s)$ denote the total value at date s in the monopoly phase to

the losing firm (if its project is still alive) of all cash flows after date s when it follows the optimal stopping time rule. The value at date t to the losing firm, if it continues investing in its R&D investment project at date t, can then (intuitively) be written as

$$\hat{V}_l^{M2l}(Y_t, K_{lt}, t) = E^{\mathbb{Q}}\big[-e^{-(r+\lambda_l)dt} I_l dt + e^{-(r+\lambda_l)dt} V_l^{M2l}(Y_t + dY_t, K_{lt} + dK_{lt}, t + dt) \;\big|\; \mathcal{F}_t\big]. \tag{7}$$

If $\hat{V}_l^{M2l}(Y_t, K_{lt}, t)$ is positive, the losing firm should continue investing at date t, otherwise it should abandon its R&D investment project. That is,

$$V_l^{M2l}(Y_t, K_{lt}, t) = \max\big\{\hat{V}_l^{M2l}(Y_t, K_{lt}, t), 0\big\}. \tag{8}$$

To make this method rigorous in continuous time, we must derive a partial differential equation to solve for $\hat{V}_l^{M2l}$, assuming that the firm continues investing, and at each instant in time check whether its value is non-negative. As soon as it becomes negative, it is time to abandon the R&D investment project. This is the same as the standard solution method normally applied to value an American option in a Black-Scholes setting. Details can be found in Appendix A.

If the winning firm's project is no longer alive at the entrance date into the monopoly phase,[19] then at any given date t in the monopoly phase, i.e. $t \in [\underline{\tau}, \overline{\tau})$, the total value to the losing firm (if its project is alive) of all cash flows after that date can similarly be derived as

$$\begin{aligned} V_l^{M1l}(Y_t, K_{lt}, t) &\equiv \max_{\nu_l \in \mathcal{S}(\mathbf{F})} E^{\mathbb{Q}}\Bigg[-\int_t^{\nu_l} e^{-(\lambda_l + r)(s-t)} I_l ds \\ &\qquad + 1_{\{\nu_l = \overline{\tau}\}} e^{-(\lambda_l + r)(\overline{\tau} - t)} V^{D1}(Y_{\overline{\tau}}, \overline{\tau}) \;\Bigg|\; \mathcal{F}_t\Bigg] \\ &= \max_{\nu_l \in \mathcal{S}(\mathbf{F})} E^{\mathbb{Q}}\Bigg[-\int_t^{\nu_l} e^{-(r+\lambda_l)(s-t)} I_l ds \\ &\qquad + \frac{a}{(r-\mu_y)\sqrt{2be}} e^{-(r+\lambda_l)(\overline{\tau}-t)} \times \\ &\qquad \big(1 - e^{-(r-\mu_y)(T-\overline{\tau})}\big) 1_{\{\nu_l = \overline{\tau}\}} Y_{\overline{\tau}} \;\Bigg|\; \mathcal{F}_t\Bigg], \qquad t \in [\underline{\tau}, \overline{\tau}). \end{aligned} \tag{9}$$

Here the superscript $M1l$ indicates that this is the losing firm's project value in the monopoly phase if the winning firm's project is no longer alive. In this case we denote the date t optimal R&D investment/abandonment strategy for the problem in equation (9) as ν_{lt}^{1*}. Here the superscript $1*$ indicates that this is the (date t) optimal R&D investment/abandonment strategy, given that the winning firm's project is no longer alive at that date. This optimal stopping time problem can be solved in the same way as sketched in equation (7). Details can be found in Appendix A.

In the valuation of the winning firm's future cash flows we must take into account both the fact that the losing firm is (perhaps) still investing in R&D and is therefore still exposed to catastrophic events, and the fact that it will follow the just derived optimal R&D investment/abandonment strategy. At any given date t in the monopoly phase the winning firm (if its project is alive) can observe whether or not the losing firm's project is still alive. This observation is important for the valuation of the winning firm's future cash flows since this indicates whether there is still uncertainty about whether the losing firm will eventually complete its R&D investment project or not. At any given date t in the monopoly phase, i.e. $t \in [\underline{\tau}, \overline{\tau})$, if the losing

[19]Note again that the winning (losing) firm's project refers to the project that would have been completed first (last) if it is neither abandoned for economic reasons nor hit by catastrophic events.

firm's project is still alive at that date, the total value to the winning firm of all cash flows after that date can be derived as,

$$
\begin{aligned}
V_w^{M2w}(Y_t, K_{lt}, t) \equiv E^{\mathbb{Q}}\Bigg[& \int_t^{\overline{\tau}} e^{-r(s-t)}\Pi_s^M ds + 1_{\{\nu_{lt}^{2*}=\overline{\tau}\}} e^{-\lambda_l(\overline{\tau}-t)} e^{-r(\overline{\tau}-t)} V^{D2}(Y_{\overline{\tau}}, \overline{\tau}) \\
& + \Big(\big(1 - e^{-\lambda_l(\overline{\tau}-t)}\big) 1_{\{\nu_{lt}^{2*}=\overline{\tau}\}} + 1_{\{\nu_{lt}^{2*}<\overline{\tau}\}} \Big) e^{-r(\overline{\tau}-t)} V^{D1}(Y_{\overline{\tau}}, \overline{\tau}) \;\Big|\; \mathcal{F}_t \Bigg] \\
= E^{\mathbb{Q}}\Bigg[& \frac{a}{\sqrt{2be}} \int_t^{\overline{\tau}} e^{-r(s-t)} Y_s ds \\
& + e^{-(r+\lambda_l)(\overline{\tau}-t)} \frac{a}{2(r-\mu_y)e\sqrt{b}} \big(1 - e^{-(r-\mu_y)(T-\overline{\tau})}\big) 1_{\{\nu_{lt}^{2*}=\overline{\tau}\}} Y_{\overline{\tau}} \\
& + \Big(\big(1 - e^{-\lambda_l(\overline{\tau}-t)}\big) 1_{\{\nu_{lt}^{2*}=\overline{\tau}\}} + 1_{\{\nu_{lt}^{2*}<\overline{\tau}\}} \Big) e^{-r(\overline{\tau}-t)} \times \\
& \frac{a}{(r-\mu_y)\sqrt{2be}} \big(1 - e^{-(r-\mu_y)(T-\overline{\tau})}\big) Y_{\overline{\tau}} \;\Big|\; \mathcal{F}_t \Bigg] \\
= \frac{a}{\sqrt{2be}} E^{\mathbb{Q}}\Bigg[& \int_t^{\overline{\tau}} e^{-r(s-t)} Y_s ds \;\Big|\; \mathcal{F}_t \Bigg] \\
+ \frac{a}{2(r-\mu_y)e\sqrt{b}} E^{\mathbb{Q}}\Bigg[& e^{-(r+\lambda_l)(\overline{\tau}-t)} \big(1 - e^{-(r-\mu_y)(T-\overline{\tau})}\big) 1_{\{\nu_{lt}^{2*}=\overline{\tau}\}} Y_{\overline{\tau}} \;\Big|\; \mathcal{F}_t \Bigg] \\
+ \frac{a}{(r-\mu_y)\sqrt{2be}} E^{\mathbb{Q}}\Bigg[& e^{-r(\overline{\tau}-t)} \big(1 - e^{-(r-\mu_y)(T-\overline{\tau})}\big) \times \\
& \Big(\big(1 - e^{-\lambda_l(\overline{\tau}-t)}\big) 1_{\{\nu_{lt}^{2*}=\overline{\tau}\}} + 1_{\{\nu_{lt}^{2*}<\overline{\tau}\}} \Big) Y_{\overline{\tau}} \;\Big|\; \mathcal{F}_t \Bigg], \qquad t \in [\underline{\tau}, \overline{\tau}).
\end{aligned}
\tag{10}
$$

The superscript $M2w$ indicates that this is the winning firm's project value in the monopoly phase if the losing firm's project is still alive. Note the three terms in equation (10): the first term is the winning firm's monopoly profit from date t and until the end of the monopoly phase; the second term is the winning firm's share of the duopoly profit in the duopoly phase in the event that the losing firm is not hit by catastrophic events in the period from date t and until the entrance date into the duopoly phase and the losing firm does not abandon its R&D investment project before completion; finally the third term is the winning firm's monopoly profit in the duopoly phase in the event that the losing firm is either hit by catastrophic events before the entrance date into the duopoly phase or the losing firm finds it optimal to abandon its R&D investment project before completion.

Finally, if the losing firm's project is no longer alive, the winning firm will be able to make a monopoly profit until its patent expires. Hence, we can value its profit in the same way as in the duopoly phase. That is, at any given date t in the monopoly phase, i.e. $t \in [\underline{\tau}, \overline{\tau})$, if the losing firm's project is no longer alive at that date, the total value to the winning firm of all cash flows after that date can be derived as,

$$
V^{M1w}(Y_t, t) \equiv V^{D1}(Y_t, t) = \frac{a}{(r-\mu_y)\sqrt{2be}} \big(1 - e^{-(r-\mu_y)(T-t)}\big) Y_t, \qquad t \in [\underline{\tau}, \overline{\tau}),
\tag{11}
$$

even though, strictly speaking, V^{D1} is not defined to be used in the monopoly phase. Note that the value will only depend on the value of the state variable Y, and the date t. The state variable measuring the estimated remaining R&D investment costs for the losing firm, K_l, is no longer relevant since that firm has either been hit by catastrophic events or has abandoned its R&D investment project before completion. The

superscript $M1w$ indicates that this is the winning firm's project value in the monopoly phase if the losing firm's project is no longer alive.

In the competitive R&D phase before any of the two drugs are marketed, i.e. from date zero to date $\underline{\tau}$, the two firms are competing to market their drug before the competitor markets its drug. Therefore, the actual R&D investment/abandonment decision itself is somewhat trickier in this phase. In the monopoly phase there was no element of competitive interaction between the two firms in the R&D investment decision itself: the losing firm simply considers, given that the winning firm is already making a monopoly (production) profit in the monopoly phase, whether it is more profitable to continue investing in R&D or whether it is more profitable to abandon its R&D investment project. In the competitive R&D phase, when both firms are still investing in R&D, there is a competitive interaction element to the R&D investment/abandonment strategy. There is a clear advantage to the firm which markets its drug before its competitor. Moreover, it may very well be the case that the value to one of the firms of all future cash flows is negative, if the other firm continues investing in R&D, but positive if the other firm abandons its R&D investment project. Hence, the R&D investment/abandonment strategy may not only depend on the valuation of the firm's own R&D investment project. That is, we will have to use the same kind of reasoning in order to find the optimal R&D investment/abandonment strategies for both of the firms as we did in the derivation of the duopoly production rates, i.e., a Cournot-Nash type equilibrium. Therefore, we must investigate at any given date t the two firms' response functions in terms of their date t optimal R&D investment/abandonment decision, as a reaction to their competitor's given date t R&D investment/abandonment decision. To solve that problem we first need to know the date t value of the project as well as the optimal R&D investment strategy if the other firm is hit by catastrophic events or if it abandons its R&D investment project for economic reasons. This is a standard optimal stopping time problem that can be solved by the same methods as applied for the losing firm in the monopoly phase. That is, at any given date t in the competitive R&D phase, i.e. $t \in [0, \underline{\tau})$, if the other firm's project is no longer alive, the total value to firm i of all cash flows after that date can be derived as,

$$
\begin{aligned}
V_i^{R\&D1}(Y_t, K_{it}, t) \equiv \max_{\nu_i \in \mathcal{S}(\mathbf{F})} E^{\mathbb{Q}}\Bigg[& -\int_t^{\nu_i} e^{-(r+\lambda_i)(s-t)} I_i ds \\
& + 1_{\{\nu_i = \tau_i\}} e^{-(r+\lambda_i)(\tau_i - t)} V^{M1w}(Y_{\tau_i}, \tau_i) \,\Bigg|\, \mathcal{F}_t \Bigg], \qquad t \in [0, \underline{\tau}).
\end{aligned} \tag{12}
$$

In equation (12) we have exploited the fact that V^{D1} and V^{M1w} are identical, even though, strictly speaking, they are not defined on the same time intervals, cf. equation (11). Note that the value will only depend on the value of the state variable Y, the estimated remaining R&D investment costs for the firm itself, K_i, and the date t. The state variables measuring the estimated remaining R&D investment costs for the competing firm, K_j, for $j = 3 - i$, is no longer relevant since that firm has either been hit by catastrophic events or has abandoned its R&D investment project before completion. The superscript $R\&D1$ indicates that this is the project value in the competitive R&D phase if the other firm's project is no longer alive. Note the two terms in equation (12): the first term is firm i's R&D investment costs from date t and until it is either hit by catastrophic events, it decides to abandon its R&D investment project, or until its R&D investment project is completed; the second term is firm i's monopoly profit in the event that its project is completed. In this case we denote the date t optimal R&D investment/abandonment strategy for the problem in equation (12) as ν_{it}^{1*}. Again the superscript $1*$ indicates that this is the (date t) optimal R&D investment/abandonment

strategy, given that the other firm's project is no longer alive at that date. This optimal stopping time problem can be solved in the same way as sketched in equation (7). Details can be found in Appendix A.

Now that we know the date t value of firm i's R&D investment project if the other firm decides to abandon its R&D investment project, we need to derive the date t value of firm i's R&D investment project if the other firm continues investing in its R&D investment project in order to find the best response function. In order to derive the date t value of firm i's R&D investment project if the other firm continues investing in its R&D investment project, we must take into account both the fact that both firms are exposed to catastrophic events and the fact that both firms follow R&D investment strategies that are Cournot-Nash equilibria at any later date $s \geq t$ in the competitive R&D phase. Therefore, in the case where both firms are still investing in their R&D investment projects, we cannot just write the objective function of firm i at date t as a simple optimal stopping time problem as we have done in the monopoly phase for the losing firm and in the competitive R&D phase when there is only one project left. Because of the competitive interactions we are only able to derive the objective function as a solution to a dynamic programming problem. If both projects are alive in the competitive R&D phase, the boundary conditions for the project values are given by the value at the entrance date into the monopoly phase. That is,[20]

$$V_w^{R\&D2}(Y_{\underline{\tau}}, K_{l\underline{\tau}}, 0, \underline{\tau}) = V_w^{M2w}(Y_{\underline{\tau}}, K_{l\underline{\tau}}, \underline{\tau}) \tag{13}$$

and

$$V_l^{R\&D2}(Y_{\underline{\tau}}, K_{l\underline{\tau}}, 0, \underline{\tau}) = V_l^{M2l}(Y_{\underline{\tau}}, K_{l\underline{\tau}}, \underline{\tau}). \tag{14}$$

The valuation problem in the competitive R&D phase is then solved by going backwards in time in the usual dynamic programming fashion. That is, we solve for the value of the R&D investment project at date t (in the competitive R&D phase) conditional on that we have already solved for the value at any later date s. Let $V_i^{R\&D2}(Y_s, K_{1s}, K_{2s}, s)$ denote the total value to firm i at date s of all cash flows after date s in the competitive R&D phase if both projects are still alive and both firms follow R&D investment strategies that are Cournot-Nash equilibria at any later date $u \geq s$ in the competitive R&D phase. From equation (12) we have, in addition, derived the total value at date s in the competitive R&D phase to firm i of all cash flows after date s when it follows the optimal stopping time rule in the case where the other firm abandons its R&D investment project. This value is denoted $V_i^{R\&D1}(Y_s, K_{is}, s)$. The total value at date t in the competitive R&D phase to firm i (if its project is still alive) of all cash flows after date t, if both firms continue investing

[20] In this derivation of the boundary conditions we have implicitly assumed that firm two is the winning firm and that firm one is the losing firm. If the opposite situation is the case, then $K_{l\underline{\tau}}$ and zero should be interchanged in the arguments of both $V^{R\&D2}$s on the left hand sides of the equations.

		Firm two	
		Continues investing	Abandons
Firm one	Continues investing	$\hat{V}_1^{R\&D2}(Y_t, K_{1t}, K_{2t}, t), \hat{V}_2^{R\&D2}(Y_t, K_{1t}, K_{2t}, t)$	$V_1^{R\&D1}(Y_t, K_{1t}, t), 0$
	Abandons	$0, V_2^{R\&D1}(Y_t, K_{2t}, t)$	$0, 0$

TABLE 1. Normal form representation of the R&D investment game with competitive interactions between the two firms at date t in the competitive R&D phase. The table contains the date t value to each of the two duopolists of all future cash flows after that date (firm one's value before the comma and firm two's value after the comma) given that both firms follow the decisions indicated in the margin of the box at date t and that both firms follow R&D investment strategies that are Cournot-Nash equilibria at any later date $s \geq t$ in the competitive R&D phase.

in their R&D investment projects at date t, can then (intuitively) be written as

$$\begin{aligned}\hat{V}_i^{R\&D2}(Y_t, K_{1t}, K_{2t}, t) = E^{\mathbb{Q}}\Big[& -e^{-(r+\lambda_i)dt} I_i dt \\ & + e^{-(r+\lambda_i+\lambda_j)dt} V_i^{R\&D2}(Y_t + dY_t, K_{1t} + dK_{1t}, K_{2t} + dK_{2t}, t + dt) \\ & + e^{-(r+\lambda_i)dt}\big(1 - e^{-\lambda_j dt}\big) \times \\ & V_i^{R\&D1}(Y_t + dY_t, K_{it} + dK_{it}, t + dt) \;\Big|\; \mathcal{F}_t\Big].\end{aligned} \tag{15}$$

The superscript $R\&D2$ indicates that this is the firm's value in the competitive R&D phase if the other firm's project is still alive.

To find the Cournot-Nash type equilibrium R&D investment/abandonment decisions at date t for the two firms in the competitive R&D phase, we will have to consider the game in Table 1. Table 1 contains the date t value to each of the two duopolists of all future cash flows after that date (firm one's value before the comma and firm two's value after the comma) given that both firms follow the decisions indicated in the margin of the box at date t and that both firms follow R&D investment strategies that are Cournot-Nash equilibria at any later date $s \geq t$ in the competitive R&D phase. Take, e.g., the upper right corner of the box in Table 1. The two numbers, $V_1^{R\&D1}(Y_t, K_{1t}, t)$, 0, indicate that if firm one continues investing in its R&D investment project and firm two abandons its R&D investment project, then the value of all future cash flows to firm one would be $V_1^{R\&D1}(Y_t, K_{1t}, t)$, whereas the value of all future cash flows to firm two would be zero. Naturally, $\hat{V}_i^{R\&D2}(Y_t, K_{1t}, K_{2t}, t) < V_i^{R\&D1}(Y_t, K_{it}, t)$, for both $i = 1$ and $i = 2$, i.e., the value to firm i of all future cash flows if it continues investing in R&D is lower if the other firm also continues investing in R&D than if the other firm abandons its R&D investment project, *ceteris paribus*. If $\hat{V}_i^{R\&D2}(Y_t, K_{1t}, K_{2t}, t) \geq 0$, for both $i = 1$ and $i = 2$, there is a unique Nash equilibrium in simple strategies. This equilibrium is that both firms continue investing. If, for one i, $\hat{V}_i^{R\&D2}(Y_t, K_{1t}, K_{2t}, t) \geq 0$ and $\hat{V}_j^{R\&D2}(Y_t, K_{1t}, K_{2t}, t) < 0$, for $j = 3 - i$, then the unique Nash equilibrium in simple strategies is that firm i continues investing in R&D, whereas firm $j = 3 - i$ abandons its R&D investment project. If $V_i^{R\&D1}(Y_t, K_{it}, t) < 0$, for both $i = 1$ and $i = 2$, then the unique Nash equilibrium in simple strategies is that both firms abandon their R&D investment projects. If, for one i, $V_i^{R\&D1}(Y_t, K_{it}, t) \geq 0$ and $V_j^{R\&D1}(Y_t, K_{jt}, t) < 0$, for $j = 3 - i$, then the unique Nash

equilibrium in simple strategies is that firm i continues investing in R&D, whereas firm $j = 3 - i$ abandons its R&D investment project. Finally, if $V_i^{R\&D1}(Y_t, K_{it}, t) \geq 0$ and $\hat{V}_i^{R\&D2}(Y_t, K_{1t}, K_{2t}, t) < 0$, for both $i = 1$ and $i = 2$, then there are multiple Nash equilibria in simple strategies. It is a Nash equilibrium in simple strategies that one of the firms continues investing in R&D and the other abandons its R&D investment project. The question is how the two firms should select a rule for which of the two firms should continue investing in R&D and which of them should abandon its R&D investment project. Any rule that given the information set available to the two firms unambiguously selects which of the two firms should continue investing in R&D and which of them should abandon its R&D investment project is a Nash equilibrium. In our numerical solution procedure presented in Section 3 we have implemented the rule that the firm with the highest value of continuing investing in R&D, given that the other firm abandons its R&D investment project, continues investing in R&D, and the other firm abandons its R&D investment project. In the zero-probability event that both firms have identical values of continuing investing in R&D, given that the other firm abandons its R&D investment project, we say that firm two continues investing in R&D and firm one abandons its R&D investment project.[21] That is, we have implemented the rule that it is firm one which continues investing in R&D and firm two which abandons its R&D investment project if and only if $V_1^{R\&D1}(Y_t, K_{1t}, t) > V_2^{R\&D1}(Y_t, K_{2t}, t)$. This Nash equilibrium among all Nash equilibria is the one that gives the highest *ex ante* values of the two firms' projects and therefore it should be the one Nash equilibrium that both firms would prefer to play. One economic motivation for this refinement criterion is a story related to mutual threads of war of attrition.[22] First of all we can think of the two firms as the two opponents of a potential war of attrition. Firm i's gain from winning this war of attrition by forcing firm $j = 3 - i$ to abandon its R&D investment project at date t is $V_i^{R\&D1}(Y_t, K_{it}, t)$. Both firms know their own and their opponent's potential gains from winning this war of attrition. Since the value of losing the potential war of attrition is zero, $V_i^{R\&D1}(Y_t, K_{it}, t)$ can also be interpreted as the maximum amount that firm i would be willing to spend in order to win this war of attrition. Faced with these facts we find it plausible that the firm with the lowest value of $V^{R\&D1}$ voluntarily abandons its R&D investment project without even starting the war of attrition. This implies that the firm with the highest value of $V^{R\&D1}$ can continue its R&D investment project as a monopolist without having to spend any money to win the war of attrition. It is also possible to think about this refinement in terms of mergers and acquisitions. In these cases the equilibrium outcome would be the same, i.e. the firm with the highest value of $V^{R\&D1}$ will continue its R&D investment project and the other firm would abandon its R&D project, but typically a Nash bargaining game would be involved and there would be a wealth transfer from the firm with the highest value of $V^{R\&D1}$ to the firm with the lowest value of $V^{R\&D1}$. In this case there could also be an increase in value to above the highest value of $V^{R\&D1}$ if the two R&D investment projects can be combined in some way as a result of a merger or an acquisition. Finally, there is also a Nash equilibrium in mixed strategies. This equilibrium is that firm $i = 3 - j$ continues investing with probability $\frac{V_j^{R\&D1}(Y_t,K_{jt},t)}{V_j^{R\&D1}(Y_t,K_{jt},t)-\hat{V}_j^{R\&D2}(Y_t,K_{1t},K_{2t},t)}$ and abandons with probability $1 - \frac{V_j^{R\&D1}(Y_t,K_{jt},t)}{V_j^{R\&D1}(Y_t,K_{jt},t)-\hat{V}_j^{R\&D2}(Y_t,K_{1t},K_{2t},t)}$, for both firms $i = 1$ and $i = 2$. Note that $V_i^{R\&D1}(Y_t, K_{it}, t) \geq 0$ and $\hat{V}_i^{R\&D2}(Y_t, K_{1t}, K_{2t}, t) < 0$, for both $i = 1$ and $i = 2$, such that the probabilities for continuing investing are positive and strictly less than one. This alternative equilibrium,

[21] It is not a zero-probability event if this happens at date zero, e.g., in the symmetric case where both duopolists are identical.
[22] Other articles that consider wars of attrition in duopoly situations are Ghemawat and Nalebuff (1985), Fudenberg and Tirole (1986), and Lambrecht (2001).

while theoretically appealing in the sense that it provides the same value to both projects at date t, does not provide the highest *ex-ante* value for the projects. Hence, we have found the Cournot-Nash type equilibrium investment decisions for date t. The date t values for each of the two projects corresponding to the outcome of this Cournot-Nash type investment game can then be assigned to $V_1^{R\&D2}$ and $V_2^{R\&D2}$, cf. equations (16) and (17) in Appendix B. This last step replaces equation (8) in the optimal stopping time problem. To make this method rigorous in continuous time we must derive two partial differential equations to solve for $\hat{V}_1^{R\&D2}$ and $\hat{V}_2^{R\&D2}$ assuming that both firms continue investing, and at each instant in time check that it is a Nash equilibrium for both firms to continue investing. As soon as one of the other three outcomes of the game becomes a Nash equilibrium the competitive interactions come to an end and the corresponding values from equation (12) can be substituted in as a boundary condition. Details can be found in Appendix B.

The way to solve for the date zero Cournot-Nash type equilibrium R&D investment/abandonment strategies for the two firms is to successively solve for Nash equilibria in the game in Table 1 at each instant in time, starting with equations (13) and (14) at the entrance date into the monopoly phase and going backwards in time using equation (15) until date zero.

3. Numerical Solution Procedure

We solve the model numerically by applying a variation of the Longstaff-Schwartz method (Longstaff and Schwartz 2001). This is done by first simulating 100,000 discretizised[23] sample paths of the three governing state variables Y, K_1, and K_2. It is then easy to determine the two stopping times $\underline{\tau}$ and $\overline{\tau}$ for each sample path, cf. Figure 3. The value of all future profits to each of the two firms can easily be calculated at date $\overline{\tau}$ by equations (3) and (4) in both the case where the firm will be the only one on the market, i.e. a monopolist, and in the case where the firms will be sharing the market, i.e. duopolists. The reason that we have to calculate both profit values is that we do not know if and when one of the two firms will abandon its R&D investment project or if and when it may be hit by catastrophic events. The losing firm's value in the monopoly phase is, however, a little bit more problematic in that it involves the R&D investment/abandonment decision, cf. equations (5) and (9). The value of the losing firm, at any given date in the monopoly phase, of all future cash flows, if the firm continues investing in R&D, can be approximated by the following backward procedure similar in spirit to equation (7): regress the continuation value along each of the sample paths which are in the monopoly phase at the same date on a specific function of the current values along the corresponding sample paths of the two governing state variables, K_l and Y. The regression coefficients from this regression is used to approximate the value, at this date in the monopoly phase, of all future cash flows, if the firm continues investing in R&D, simply by applying the regression to the current values of the two governing state variables.[24] If the value of continuing investing in R&D is greater than the costs of investing in R&D for another quarter, the firm should continue investing in R&D, otherwise it should abandon its R&D investment project. This procedure gives us the value of all future cash flows for the losing firm back

[23]In our implementation we have used a discretization based on three-month intervals. That is, we have separated the twenty-year time period, $[0, T]$, into 80 quarters.

[24]The Longstaff-Schwartz method exploits the fact that the continuation value, i.e. the date t value of all future cash flows from the R&D investment project if the firm continues investing in R&D (including the option to abandon later on), is simply the date t conditional expectation of the value of the true future cash flows from the R&D investment project under an equivalent martingale measure, $\mathbb{Q}$. Since the current date t value of the two governing state variables is a sufficient statistic of all date t information, it follows that the conditional expectation under $\mathbb{Q}$ is some (unknown) function of the current realized values of the two governing state variables. It is this unknown function that we try to approximate/estimate at each instant in time. This is done by estimating ten coefficients in a cubic polynomial parameterization of the function.

to date $\underline{\tau}$ as well as the firm's R&D investment/abandonment decisions along each sample path for each quarter in the monopoly phase both in the case where both projects are alive and in the case where there is only one project alive. With the losing firm's R&D investment/abandonment decisions along each sample path for each quarter in the monopoly phase it is quite easy to calculate the winning firm's value of all future profits at date $\underline{\tau}$ by equations (10) and (11).

In the competitive R&D phase, the same method is applied to approximate the value of all future cash flows, given the current value of the (now three) governing state variables, K_1, K_2 and Y, cf. equation (15). However, as we saw in Section 2 the actual R&D investment/abandonment decision is somewhat more tricky in this situation as we, for each quarter, have to go through the game in Table 1 to find the Nash equilibrium R&D investment/abandonment decisions based on the values from equations (12) and (15).

When this backward procedure is carried out starting from date $T = 20$ and quarter by quarter going backwards through the duopoly phase, the monopoly phase, and the competitive R&D phase back to date zero, all the R&D investment/abandonment decisions for the two firms have been determined for each quarter and for each of the 100,000 sample paths, both in the case where there are still two projects alive and in the case where there is only one project alive. The actual values of all future cash flows from continuing the R&D investment project which are the results of the regressions that has been run are not used in the computation of the value of the projects, only the R&D investment/abandonment decisions they have caused are used.

In order to value the R&D investment projects for the two firms at date zero we implement a forward procedure that finds the value by evaluating the profit along each sample path taking into account the R&D investment/abandonment decisions derived by the backward procedure and averaging over all the sample paths.

4. Numerical Results

In this section we provide a numerical illustration of the model. Since we solve the problem using numerical simulations, we are able to characterize the solution in great detail. We first develop a base case and then we perform sensitivity analysis with respect to some of the key parameters of the model. We emphasize the differences between the competitive (duopoly) and monopoly solutions to the R&D investment project.

Table 2 contains the data for the base case. The inverse demand function parameters are the same as the ones used to construct Figure 2. Note that parameter a is a measure of the size of the market for the product and that parameter b is a measure of the depth of the market, i.e., a measure for how much the price would change for a given change in supplied quantity. The R&D investment cost parameters are the ones used to construct the sample paths in Figure 3. The other parameters are chosen to represent a typical drug development project (Schwartz 2001).

The results of the numerical simulations for the base case are shown in Table 3. As expected the value of the R&D investment project in the monopoly situation is higher than the sum of the values of the two R&D investment projects in the duopoly situation. The monopolist not only optimizes its production rate after completion of the R&D investment project in order to maximize its own per period production profits, but it also uses an optimal R&D investment/abandonment strategy during the period of R&D investments without having to consider competitive interactions. Note, however, that the present value of the revenues generated during the production period is 26% higher in the duopoly situation than in the monopoly situation even

Inverse demand function		
First parameter	a	15
Second parameter	b	0.1
Demand shocks		
Drift parameter	μ_y	0
Diffusion parameter	σ_y	20%
Initial value	Y_0	1
Estimated remaining R&D investment costs for firm one's project		
Initial value	K_{10}	100
Investment rate	I_1	10
Diffusion parameter of technical shocks	γ_1	20%
Drift parameter of input cost shocks	μ_{1k}	0
Diffusion parameter of input cost shocks	σ_{1k}	10%
Estimated remaining R&D investment costs for firm two's project		
Initial value	K_{20}	100
Investment rate	I_2	10
Diffusion parameter of technical shocks	γ_2	20%
Drift parameter of input cost shocks	μ_{2k}	0
Diffusion parameter of input cost shocks	σ_{2k}	10%
Other parameters		
Intensity of catastrophic events for firm one	λ_1	0.07
Intensity of catastrophic events for firm two	λ_2	0.07
Correlation between demand shocks and input cost shocks	ρ_{yk}	–0.1
Riskless interest rate	r	5%
Duration of the patent period	T	20
Numerical procedure parameters		
Number of simulated paths	N	100,000
Duration of each time step	Δt	0.25

TABLE 2. Base case parameter values.

though on average the price per unit sold is lower in the duopoly situation than in the monopoly situation. This is due to the fact that, on average, the total number of units produced is 34% higher in the duopoly situation than in the monopoly situation. There are three reasons, all working in the same direction, for this higher production in the duopoly situation. Firstly, there is a higher probability that a product is developed in the duopoly situation than in the monopoly situation since (i) catastrophic events are diversified over the two duopolists and (ii) the R&D investment costs for the winning duopoly firm are on average smaller than the R&D investment costs for the monopoly firm because the winning duopoly firm typically has the estimated remaining R&D investment cost process which hits zero first. Secondly, on average the time to completion for the first product is shorter in the duopoly situation than in the monopoly situation so there is a longer time period to produce before the patents expire. Thirdly, in those cases where both duopolists

	Duopoly situation	Monopoly situation
Value of R&D investment project(s)	6.9	9.6
Present value of revenues	62.6	49.5
Present value of R&D investment costs	55.7	39.9
Total number of units produced	10.8	8.0
Present value of revenue per unit produced	5.8	6.2
Present value of R&D investment costs per unit produced	5.2	5.0
Probability of catastrophic events for each firm	24.2%	34.4%
Probability of economic abandonment for each firm	52.6%	32.2%
Total probability of not completing for each firm	76.7%	66.6%
Probability that at least one firm completes	42.1%	33.4%
Probability that both firms complete	4.9%	n/a

TABLE 3. Results of base case analysis using the base case parameter values from Table 2.

complete their R&D investment projects, the production rate is higher after the second firm completes its R&D investment project (see figure 2). The flip side of the benefits of having more units produced in the duopoly situation is that the total present value of the R&D investment costs is higher in the duopoly situation than in the monopoly situation. Even on a per unit basis the present value of the R&D investment costs is higher in the duopoly situation than in the monopoly situation, since there is a duplication of the R&D investment costs.

Now let us turn to the analysis of the probabilities of abandonment in the two situations.[25] The probability of abandonment of the R&D investment project because it is no longer profitable to continue investing (economic abandonment) for each of the two duopolists is much larger (52.6%) than it is for the monopolist (32.2%), since the expected future cash flows to each of the two duopolists are smaller. A consequence of this is that on average the life time of an R&D investment project initiated by a duopolist is shorter than the life time of an R&D investment project initiated by a monopolist, and therefore the probability of catastrophic events is smaller for a given R&D investment project if it is initiated by a duopolist (24.2%) than if it is initiated by a monopolist (34.4%). The outcome of these two opposing effects is that, on a per project basis, the probability of completion of the R&D investment project is always higher if the R&D investment project is initiated by a monopolist than if it is initiated by a duopolist. If we take into account that there are two duopolists, each with an R&D investment project; however, the probability that at least one of the two duopolists will complete is always higher (42.1%) than the probability that the monopolist will complete (33.4%). Finally, in our base case, the probability that both duopolists complete is only 4.9%. That is, the conditional probability that the second duopolist completes, conditional on the first completing, is only 21.0%, which is lower than the unconditional probability that a given duopolist will complete (23.3%).

In summary, if we only take into account the value of the R&D investment projects, the monopoly situation is the superior alternative for society. But, if society's objective criteria include the number of units produced, the price per unit produced, the time to complete the R&D investment project, and the

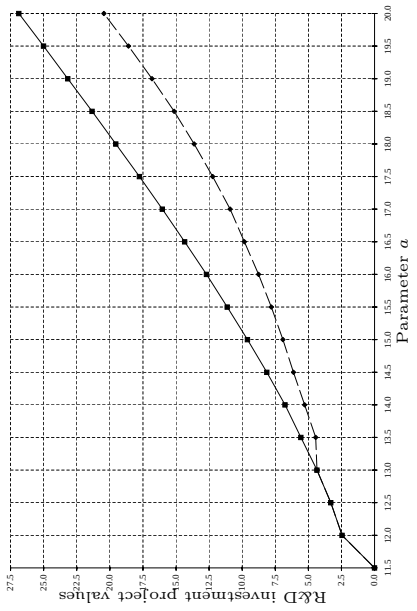
[25]Note that these are risk neutral probabilities since we have only developed the stochastic processes under an equivalent martingale measure, $\mathbb{Q}$.

probability of successful completion, the duopoly situation may very well be the superior alternative. Note that these benefits of the duopoly situation came about not only because of the lower price charged when both duopolists are producing, but also because of the fact that the chance of completion is higher and the expected time to completion of the first R&D investment project is shorter. These results are robust to all the numerical simulations we have performed.

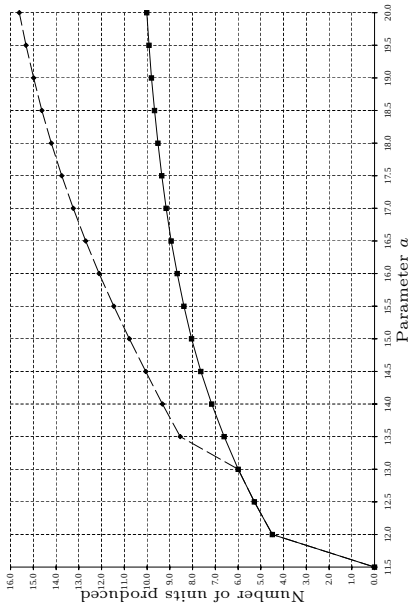
To get to a deeper understanding of the insights of the model, we now perturb some of the key parameters of the model and analyze their impact, for both the duopoly and monopoly situations, on valuation, number of units produced, and probabilities of completion. We first analyze the sensitivity to the size of the market. The parameter a of the inverse demand function captures the number of units demanded for a given price or the price that can be charged for a given quantity produced.[26] We vary a from 11.5 up to 20 in Figure 4. When $a = 11.5$, the market is so small that it is not optimal to initiate the R&D investment project even in the monopoly situation, so that the value of the R&D investment project is zero, the number of units produced is zero, and the probability of (economic) abandonment is 100%. When $a = 12$, 12.5, or 13, it is optimal for the monopolist to initiate the R&D investment project, but the market is still not large enough to make it optimal for both of the duopolists to initiate their R&D investment projects. However, according to the rules of the R&D investment game given by our choice of refinement criterion in the competitive R&D phase, cf. Table 1 in Section 2, in this situation one of the firms' (firm two's) R&D investment projects is initiated and the other firm's (firm one's) R&D investment project is immediately abandoned. Thus, the duopoly and monopoly situations are identical when $a = 12$, 12.5, or 13. When a is 13.5 or higher, both duopolists initiate their R&D investment projects at date zero and the differences between the duopoly and monopoly situations start arising. The scenario when $a = 15$ corresponds to the base case discussed in detail above, cf. Table 3.

Figure 4(a) shows that as the size of the market increases, the values of the R&D investment projects in both the duopoly and the monopoly situations increase, but the R&D investment project values in the monopoly situation (solid line) increase more than in the duopoly situation (dashed line) such that the absolute difference between the monopoly and the duopoly situations also increases. Figure 4(b) shows that the number of units produced also increases with the market size. The increase is, however, stronger for the duopoly situation (dashed line) than for the monopoly situation (solid line). The three reasons for a larger production in the duopoly situation than in the monopoly situation mentioned above get stronger the larger the size of the market. This is because as the demand increases, the better the projects are, and therefore the later will one of the duopolists eventually abandon for economic reasons. This implies that the diversification effects and the effect that one of the duopolists completes on average earlier are increased, and at the same time the probability that both of the duopolists complete increases. In Figure 4(d) we present the present values of the R&D investment costs per unit produced and the present value of revenues per units produced for both the duopoly and monopoly situations. As the size of the market increases, the price per unit sold also increases, and then also the present value of the revenue per unit produced will increase in both the monopoly (dashed-dotted line) and the duopoly (dotted line) situations. As the present value of revenues per unit produced increases, the firms are willing to incur more R&D investment costs per unit

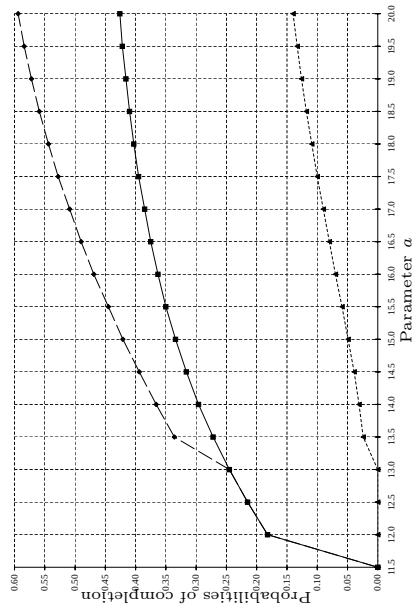
[26]Our specification of the demand function is such that the optimal quantity produced both in the monopoly and duopoly situations is independent of a, such that there is only a price effect when the size of the market is changed. Note, however, that there is an indirect effect on the quantity produced since, as the price increases, on average more R&D investment projects are completed, which implies a higher average number of units produced.



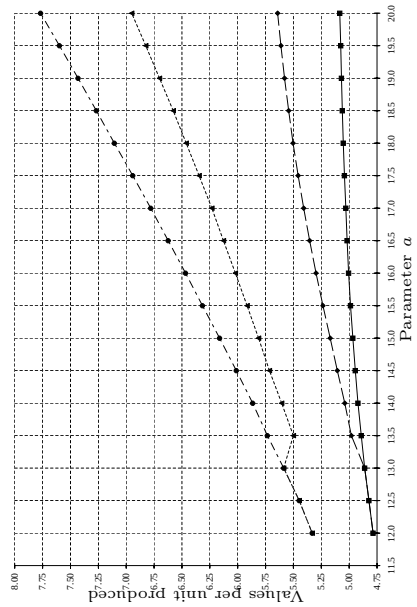
(a) Values of R&D investment projects in the monopoly (solid line) and the duopoly (dashed line) situations as a function of market size, a.



(b) Units produced in the monopoly (solid line) and the duopoly (dashed line) situations as a function of market size, a.



(c) Probabilities of completion in the monopoly situation (solid line), and probabilities that at least one of the duopolists completes (dashed line), and that both of the duopolists (dotted line) complete in the duopoly situation as a function of market size, a.



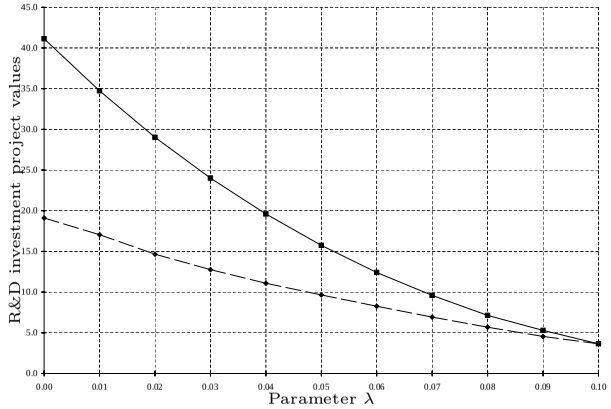
(d) Present value of costs (solid line) and revenues (dashed-dotted line) in the monopoly situation and present value of costs (dashed line) and revenues (dotted line) in the duopoly situation per unit produced as a function of market size, a.

FIGURE 4. Sensitivity analysis with respect to the market size, a.

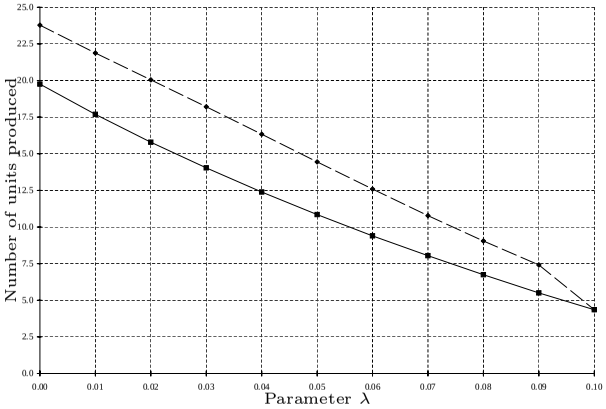
produced in order to obtain these per unit revenues. Hence, also the present value of R&D investment costs per unit produced increases in both the monopoly (solid line) and the duopoly (dashed line) situations. The two inner curves correspond to the duopoly situation and the two outer curves correspond to the monopoly situation. The difference between the two inner curves and the difference between the two outer curves are the R&D investment project values per unit produced to the duopolist and monopolist, respectively. The probabilities of completion are depicted in Figure 4(c). As we can see, both the probability that at least one of the duopolists completes (dashed line) and the probability that both of the duopolists complete (dotted line) increase with market size in absolute terms as well as relative to the monopoly situation (solid line).

The intensity of catastrophic events, λ, measures the probability per unit of time that the value of the R&D investment projects will vanish due to unforeseen external circumstances and thereby quantifies how challenging the R&D investment projects are. An increase in the value of λ clearly decreases the value of the R&D investment projects both in the duopoly and monopoly situations, but it also increases the advantage that diversification has on the R&D investment project values in the duopoly situation compared to the monopoly situation. In Figure 5 we vary λ for both firms ($\lambda = \lambda_1 = \lambda_2$) from zero (no catastrophic events) up to 0.10. When $\lambda = 0.10$, we get the same scenario as explained above where the R&D investment project is so marginal that only one of the two duopolists would optimally initiate investment in the R&D investment project, and therefore the results for the monopoly and duopoly situations are identical in this scenario. For all other values of λ in Figure 5 we get that both duopolists optimally initiate their investment in the R&D investment project at date zero. Recall that in the base case $\lambda = 0.07$. Keep in mind that for this analysis we maintain the inverse demand function fixed, so that all the results are due to the effect that the catastrophic events have on the completion of the R&D investment projects. Figure 5(a) shows that the value of the R&D investment projects in the monopoly situation (solid line) is more than twice what it is in the duopoly situation (dashed line) for $\lambda = 0$; this is mostly because the duopolists incur much higher R&D investment costs without getting the benefits of the diversification of the catastrophic events (since there are no catastrophic events). But when λ increases, the value of the R&D investment project in the monopoly situation (solid line) decreases much faster than the value of the R&D investment projects in the duopoly situation (dashed line) mainly due to the increased diversification of catastrophic events in the duopoly situation. In Figure 5(b) we see that the number of units produced decreases as λ increases in both situations. However, the relative difference between the number of units produced in the monopoly situation (solid line) and in the duopoly situation (dashed line) is smaller than the relative difference between the values of the R&D investment projects because the R&D investment duplication effect does not directly influence the number of units produced.

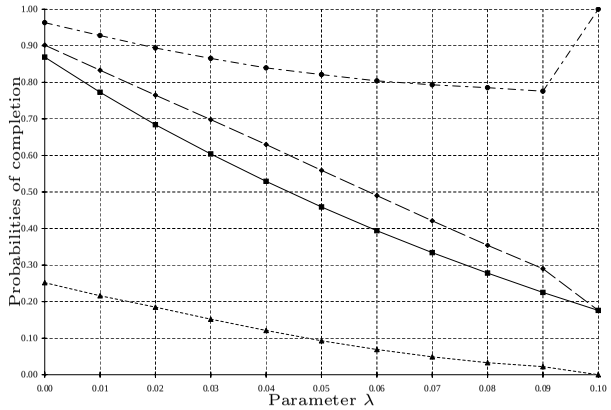
Figure 5(d) illustrates the effect of λ on the present value of revenues and the present value of R&D investment costs per unit produced for both the monopoly and the duopoly situations. The present value of revenues per unit produced increases in both the monopoly (dashed-dotted line) and the duopoly (dotted line) situations because as λ increases, the R&D investment projects become more and more marginal and the threshold level of the demand shocks for abandonment becomes higher, *ceteris paribus*. Hence, as λ increases the average price conditional on production occurring also increases. R&D investment costs per unit produced also increase both in the monopoly (solid line) and the duopoly (dashed line) situations because since the revenues per unit produced are higher, the firm is willing to incur higher R&D investment costs per unit produced, but they even go up faster than the revenues per unit produced so that the project



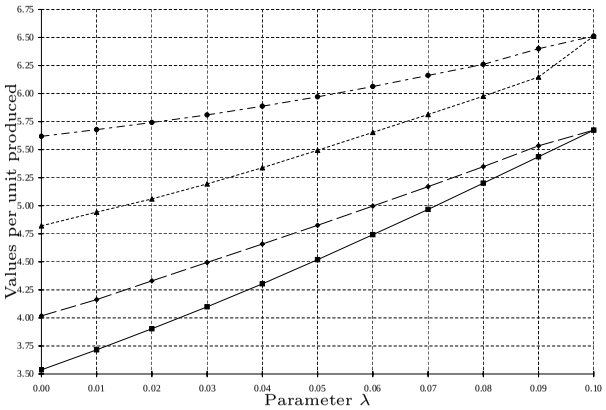
(a) Values of R&D investment projects in the monopoly (solid line) and the duopoly (dashed line) situations as a function of intensity of catastrophic events, λ.



(b) Units produced in the monopoly (solid line) and the duopoly (dashed line) situations as a function of intensity of catastrophic events, λ.



(c) Probabilities of completion in the monopoly situation (solid line), probabilities that at least one of the duopolists completes (dashed line), that both of the duopolists (dotted line) complete, and that the second duopolist completes conditional on that the first one completes (dashed-dotted line) in the duopoly situation as a function of intensity of catastrophic events, λ.



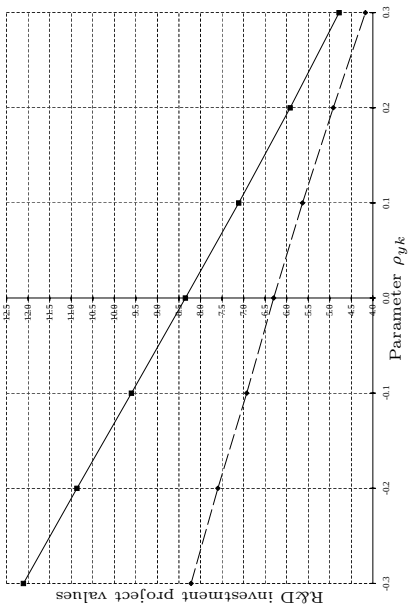
(d) Present value of costs (solid line) and revenues (dashed-dotted line) in the monopoly situation and present value of costs (dashed line) and revenues (dotted line) in the duopoly situation per unit produced as a function of intensity of catastrophic events, λ.

FIGURE 5. Sensitivity analysis with respect to the intensity of catastrophic events, $\lambda = \lambda_1 = \lambda_2$.

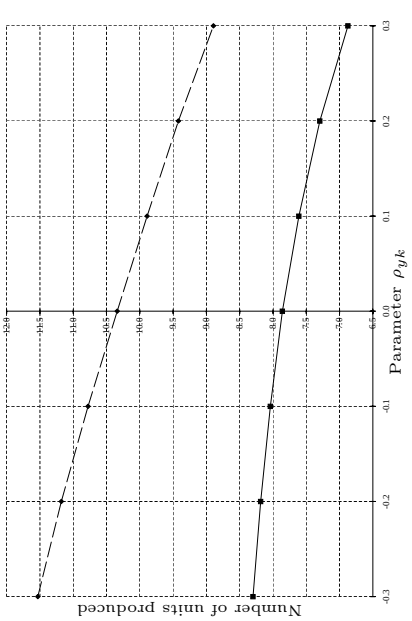
values per unit produced go down. In Figure 5(c) we observe that the probabilities of completion decrease as λ increases. As a measure of the catastrophic events diversification effect we compute the conditional probability that a monopolist would complete, given that at least one of the duopolists completes (dashed-dotted line). Since the conditional probabilities decrease, the advantage of diversification increases. This continues until the last value of λ, where the monopoly and duopoly situations are identical so in this case there is no diversification effect. Even for $\lambda = 0$ there is a small diversification effect due to the fact that on average the surviving duopolist would be the one with the lowest realized R&D investment costs; i.e. the duopolist with the estimated remaining R&D investment cost process that hits zero first.

Figure 6 analyzes the effects of changing the correlation between demand shocks and the input cost shocks. As mentioned earlier this correlation will depend on the type of product being considered. We change the correlation between –0.3 and +0.3. The base case has $\rho_{yk} = -0.1$. Clearly, the more negative the correlation is, the better the R&D investment projects are since this implies a higher probability than when R&D investment costs turn out to be low, demand is high and vice versa. De facto this gives a higher volatility of the cash flows. A simple option argument implies that this results in higher values for the R&D investment projects, higher number of units produced, and higher probabilities of completion in both the monopoly and duopoly situations. In addition, a more negative correlation implies that on average higher demand shocks for the cases that are completed and therefore a higher on average revenue per unit produced both in the monopoly (dashed-dotted line) and the duopoly (dotted line) situations, cf. Figure 6(d).

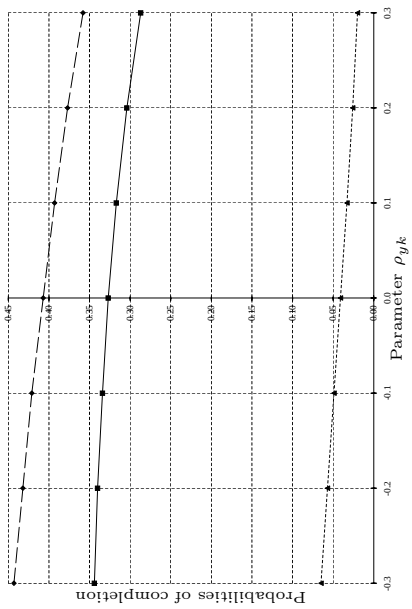
Up to now we have concentrated on analyzing the effect of competition on patent-protected R&D investment projects for the case where both duopolists are identical. Now we look at a particular case of asymmetry which highlights a new angle of competition between the duopolists. In Figure 7 we change the level of technical shocks for firm two, γ_2, from 0.1 to 0.3 while keeping constant firm one's technical shocks at $\gamma_1 = 0.2$ as well as all other parameters. Figure 7 reports the valuation results of this exercise. First note the value of the R&D investment project for a monopolist when its technical shocks change (dotted line). As expected, higher uncertainty increases the value of the R&D investment project. This result follows from standard real option arguments. The corresponding duopoly firm value (solid line) grows faster with uncertainty than does the value of the corresponding monopoly firm in percentage terms. For the lowest uncertainty in the figure, $\gamma_2 = 0.1$, it is not optimal for firm two to initiate the R&D investment project and therefore its value is zero. Most interesting, the value of firm one's project (dashed line) decreases dramatically. Recall that for this firm *all* parameters have been kept constant. We can see here that changes in the other firm's technical uncertainty have a big impact on this firm's value, and all of this impact is due to the competitive interactions between the firms. At $\gamma_2 = 0.1$, because firm two does not invest, firm one becomes a monopolist. Hence, its value (dashed line) will be the same as that of the pure monopolist (dotted line) for $\gamma_2 = 0.2$. In summary, as the uncertainty of firm two's project increases, the value of firm two's project increases, the value of firm one's project decreases, but the aggregate value of both firms' projects still increases (dashed-dotted line) except for $\gamma_2 = 0.1$, where firm two does not initiate investment. This example illustrates the importance of taking into account competitive interactions in real option valuation. Even without any changes in the parameters of a particular firm, its value can vary because of changes in the competitive environment such as changes of a parameter in a competing firm.



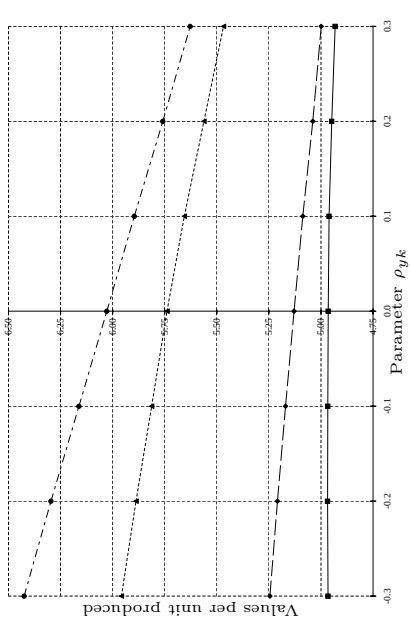
(a) Values of R&D investment projects in the monopoly (solid line) and the duopoly (dashed line) situations as a function of correlation coefficient, ρ_{yk}.



(b) Units produced in the monopoly (solid line) and the duopoly (dashed line) situations as a function of correlation coefficient, ρ_{yk}.



(c) Probabilities of completion in the monopoly situation (solid line), and probabilities that at least one of the duopolists completes (dashed line), and that both of the duopolists (dotted line) complete in the duopoly situation as a function of correlation coefficient, ρ_{yk}.



(d) Present value of costs (solid line) and revenues (dashed-dotted line) in the monopoly situation and present value of costs (dashed line) and revenues (dotted line) in the duopoly situation per unit produced as a function of correlation coefficient, ρ_{yk}.

FIGURE 6. Sensitivity analysis with respect to the correlation coefficient, ρ_{yk}.

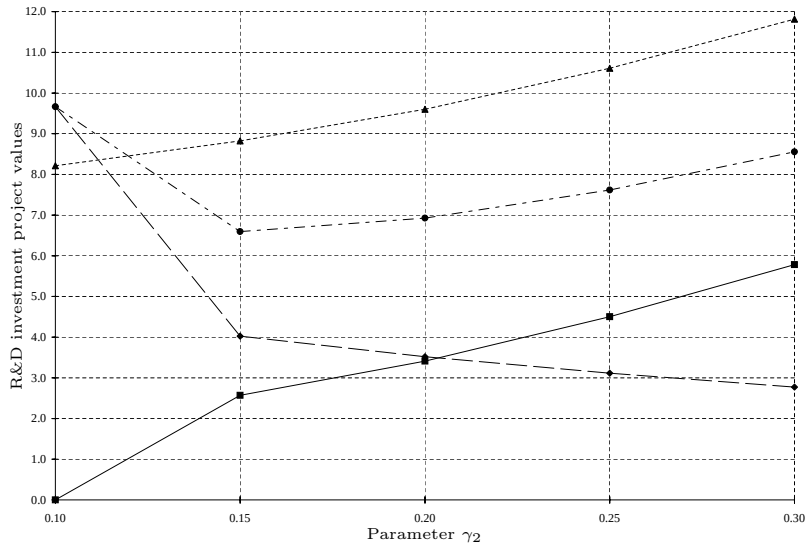


FIGURE 7. Sensitivity analysis with respect to the level of technical shocks for firm two, γ_2. Value of the R&D investment project in the monopoly situation (dotted line), value of firm one's R&D investment project (dashed line), value of firm two's R&D investment project (solid line), and value of both firms' R&D investment projects (dashed-dotted line) in the duopoly situation as a function of the level of technical shocks for firm two, γ_2.

5. Conclusion

In this article we have developed a model to analyze patent-protected R&D investment projects when there is competition in the development and marketing of the resulting product (in our case a medical drug). The competitive interactions that occur substantially complicate the solution of the problem since each of the duopolists has to take into account not only the factors that affect its own decisions, but also the factors that affect the decisions of the other duopolist. The real options framework utilized to deal with investments under uncertainty is extended to incorporate the game theoretic concepts required to deal with these interactions.

Implementation of the model shows that competition in R&D not only increases production and reduces prices, but also shortens the time of developing the product and increases the probability of a successful development. These benefits to society are countered by increased total investment costs in R&D and lower aggregate value of the R&D investment projects.

Some extensions of the model would be easy to implement. For example, the current implementation of the model assumes that both patents expire at the same date and that net cash flows revert to zero. It would be trivial to add a terminal value for the R&D investment projects. It would be somewhat more difficult to implement different expiration dates for the patents in our numerical procedure, though strictly in our formulation of the model this would be of no benefit since generic drugs would be introduced as soon as the

first patent expires. A more challenging task would be to increase the number of competitors, but we believe that the qualitative nature of the results would not change.

In some sense our choice of refinement criterion in the case of multiple Nash equilibria in the game played by the duopolists favors the duopoly situation. The refinement criterion we use is that when it is not optimal for both duopolists to continue investing, but it would be optimal for a monopolist to continue investing, then the duopolist with the highest value of continuing to invest will proceed investing while the other will abandon. This reasonable rule is not just *any* Nash equilibrium, it is *the* Nash equilibrium that gives the highest *ex-ante* value for the duopolist's R&D investment projects and we motivated this choice of refinement criterion by an economic argument of mutual threats of war of attrition. It is, however, not the only possible Nash equilibrium; mixed strategies among other equilibria would also be possible (cf. Huisman and Kort 1999, Huisman, Thijssen, and Kort 2001). It is also possible to think in terms of mergers and acquisitions but we have not pursued any of these two last refinements further in this article.

A more interesting and complex extension would be to add asymmetric information to the model. In the development of the model we assumed that all participants know the value of the state variables at each date and that they use these values to make their decisions. It is somewhat unrealistic to suppose that a duopolist has as good an estimate of the remaining R&D investment costs to complete the other duopolist's project as it has of its own project. More realistically, it estimates the competitor's cost with some error.

The model developed here could potentially help in the formulation of a public policy with respect to the encouragement of investments in R&D. At the very least it contributes to our understanding of these important issues in economic development.

Appendix A. The Partial Differential Equation for the Value of the Losing Firm's Project

The partial differential equation (PDE) corresponding to the intuitive derivation in equation (7) can be derived by Itô's lemma and (other) standard arguments from stochastic control theory. Basically, the PDE is derived by applying Itô's lemma to $V_l^{M2l}(Y_t + dY_t, K_{lt} + dK_{lt}, t + dt)$, eliminating the martingale terms by taking the expectation under $\mathbb{Q}$, and differentiating on both sides of equation (7) with respect to time, i.e. 'dividing' by dt. Starting with the boundary condition from equation (6)

$$V_l^{M2l}(y, 0, t) = V_l^{D2}(y, t),$$

for $y \geq 0$ and $t \in [0, T]$, the function V_l^{M2l} must fulfill the PDE,

$$\begin{aligned}
&\frac{1}{2}\sigma_y^2 y^2 \frac{\partial^2}{\partial y^2} V_l^{M2l}(y,k,t) + \sigma_y \sigma_{lk} \rho_{yk} y k \frac{\partial^2}{\partial y \partial k} V_l^{M2l}(y,k,t) \\
&\qquad + \frac{1}{2}(\gamma_l^2 I_l k + \sigma_{lk}^2 k^2) \frac{\partial^2}{\partial k^2} V_l^{M2l}(y,k,t) + \mu_y y \frac{\partial}{\partial y} V_l^{M2l}(y,k,t) + (\mu_{lk} k - I_l) \frac{\partial}{\partial k} V_l^{M2l}(y,k,t) \\
&\qquad\qquad + \frac{\partial}{\partial t} V_l^{M2l}(y,k,t) - I_l - (r + \lambda_l) V_l^{M2l}(y,k,t) = 0,
\end{aligned}$$

for all $y \geq 0$, $k \geq 0$, and $t \in [0, T]$, and the free boundary condition

$$V_l^{M2l}(y, k, t) \geq 0.$$

That is, the R&D investment continuation region is $\{(y,k,t) \in \mathbb{R}^2 \times [0,T] : V_l^{M2l}(y,k,t) > 0\}$. $V_l^{M2l}(y,k,t) = 0$ is the condition for when the optimal economic abandonment decision should be taken.

Appendix B. The Partial Differential Equation for the Value of Both Firms' Projects in the Competitive R&D Phase

The Nash equilibria in the normal form representation of the R&D investment game from Table 1 determine the functions $V_1^{R\&D2}$ and $V_2^{R\&D2}$, cf. equation (15), as the following:

(16)

$$V_1^{R\&D2}(Y_t, K_{1t}, K_{2t}, t) = \begin{cases} \hat{V}_1^{R\&D2}(Y_t, K_{1t}, K_{2t}, t) & \text{if } \hat{V}_1^{R\&D2}(Y_t, K_{1t}, K_{2t}, t) \geq 0 \text{ and} \\ & \quad \hat{V}_2^{R\&D2}(Y_t, K_{1t}, K_{2t}, t) \geq 0, \\ V_1^{R\&D1}(Y_t, K_{1t}, t) & \text{if } (\hat{V}_1^{R\&D2}(Y_t, K_{1t}, K_{2t}, t) \geq 0 \text{ and} \\ & \quad \hat{V}_2^{R\&D2}(Y_t, K_{1t}, K_{2t}, t) < 0) \text{ or} \\ & \quad (V_1^{R\&D1}(Y_t, K_{1t}, t) \geq 0 \text{ and} \\ & \quad V_2^{R\&D1}(Y_t, K_{2t}, t) < 0) \text{ or} \\ & \quad (V_1^{R\&D1}(Y_t, K_{1t}, t) > V_2^{R\&D1}(Y_t, K_{2t}, t) \geq 0 \text{ and} \\ & \quad \hat{V}_2^{R\&D2}(Y_t, K_{1t}, K_{2t}, t) < 0), \\ 0 & \text{if } V_1^{R\&D1}(Y_t, K_{1t}, t) < 0 \text{ or} \\ & \quad (\hat{V}_1^{R\&D2}(Y_t, K_{1t}, K_{2t}, t) < 0 \text{ and} \\ & \quad \hat{V}_2^{R\&D2}(Y_t, K_{1t}, K_{2t}, t) \geq 0) \text{ or} \\ & \quad (V_2^{R\&D1}(Y_t, K_{2t}, t) \geq V_1^{R\&D1}(Y_t, K_{1t}, t) \geq 0 \text{ and} \\ & \quad \hat{V}_1^{R\&D2}(Y_t, K_{1t}, K_{2t}, t) < 0), \end{cases}$$

and

(17)

$$V_2^{R\&D2}(Y_t,K_{1t},K_{2t},t)=\begin{cases}\hat{V}_2^{R\&D2}(Y_t,K_{1t},K_{2t},t) & \text{if } \hat{V}_1^{R\&D2}(Y_t,K_{1t},K_{2t},t)\geq 0 \text{ and}\\ & \quad \hat{V}_2^{R\&D2}(Y_t,K_{1t},K_{2t},t)\geq 0,\\ V_2^{R\&D1}(Y_t,K_{2t},t) & \text{if } (\hat{V}_1^{R\&D2}(Y_t,K_{1t},K_{2t},t)<0 \text{ and}\\ & \quad \hat{V}_2^{R\&D2}(Y_t,K_{1t},K_{2t},t)\geq 0) \text{ or}\\ & \quad (V_1^{R\&D1}(Y_t,K_{1t},t)<0 \text{ and}\\ & \quad V_2^{R\&D1}(Y_t,K_{2t},t)\geq 0) \text{ or}\\ & \quad (V_2^{R\&D1}(Y_t,K_{2t},t)\geq V_1^{R\&D1}(Y_t,K_{1t},t)\geq 0 \text{ and}\\ & \quad \hat{V}_1^{R\&D2}(Y_t,K_{1t},K_{2t},t)<0),\\ 0 & \text{if } V_2^{R\&D1}(Y_t,K_{1t},t)<0 \text{ or}\\ & \quad (\hat{V}_1^{R\&D2}(Y_t,K_{1t},K_{2t},t)\geq 0 \text{ and}\\ & \quad \hat{V}_2^{R\&D2}(Y_t,K_{1t},K_{2t},t)<0) \text{ or}\\ & \quad (V_1^{R\&D1}(Y_t,K_{1t},t)>V_2^{R\&D1}(Y_t,K_{2t},t)\geq 0 \text{ and}\\ & \quad \hat{V}_2^{R\&D2}(Y_t,K_{1t},K_{2t},t)<0).\end{cases}$$

Equations (16) and (17) are similar in spirit to equation (8) for the optimal stopping time problem.

The same methodology as sketched in Appendix A can be used to derive the PDEs corresponding to the intuitive derivation in equation (15). Starting with the boundary conditions from equations (13) and (14), the functions $V_1^{R\&D2}$ and $V_2^{R\&D2}$ must fulfill the PDE

$$\begin{aligned}&\frac{1}{2}\sigma_y^2y^2\frac{\partial^2}{\partial y^2}V_i^{R\&D2}(y,k_1,k_2,t)+\sigma_y\sigma_{1k}\rho_{yk}yk_1\frac{\partial^2}{\partial y\partial k_1}V_i^{R\&D2}(y,k_1,k_2,t)\\ &\quad+\sigma_y\sigma_{2k}\rho_{yk}yk_2\frac{\partial^2}{\partial y\partial k_2}V_i^{R\&D2}(y,k_1,k_2,t)+\frac{1}{2}(\gamma_1^2I_1k_1+\sigma_{1k}^2k_1^2)\frac{\partial^2}{\partial k_1^2}V_i^{R\&D2}(y,k_1,k_2,t)\\ &\quad+\sigma_{1k}\sigma_{2k}k_1k_2\frac{\partial^2}{\partial k_1\partial k_2}V_i^{R\&D2}(y,k_1,k_2,t)+\frac{1}{2}(\gamma_2^2I_2k_2+\sigma_{2k}^2k_2^2)\frac{\partial^2}{\partial k_2^2}V_i^{R\&D2}(y,k_1,k_2,t)\\ &\quad+\mu_yy\frac{\partial}{\partial y}V_i^{R\&D2}(y,k_1,k_2,t)+(\mu_{1k}k_1-I_1)\frac{\partial}{\partial k_1}V_i^{R\&D2}(y,k_1,k_2,t)\\ &\quad+(\mu_{2k}k_2-I_2)\frac{\partial}{\partial k_2}V_i^{R\&D2}(y,k_1,k_2,t)+\frac{\partial}{\partial t}V_i^{R\&D2}(y,k_1,k_2,t)\\ &\quad+\lambda_jV_i^{R\&D1}(y,k_i,t)-I_i-(r+\lambda_1+\lambda_2)V_i^{R\&D2}(y,k_1,k_2,t)=0,\end{aligned}$$

for $i\in\{1,2\}$ and the free boundary conditions

$$V_1^{R\&D2}(y,k_1,k_2,t)\geq 0$$

and

$$V_2^{R\&D2}(y,k_1,k_2,t)\geq 0.$$

That is, the R&D investment continuation region for both firms, cf. equations (16) and (17), is $\{(y, k_1, k_2, t) \in \mathbb{R}^3 \times [0, T] : V_1^{R\&D2}(y, k_1, k_2, t) > 0 \text{ and } V_2^{R\&D2}(y, k_1, k_2, t) > 0\}$. $V_1^{R\&D2}(y, k_1, k_2, t) = 0$ or $V_2^{R\&D2}(y, k_1, k_2, t) = 0$ is the condition for when the optimal economic abandonment decision should be taken for (at least) one of the two firms. The boundary condition that gives the value to each of the two projects when the R&D investment continuation region is left can be derived from equations (16) and (17).

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