The Use of Treasury Bill Futures in Strategic Asset Allocation Programs

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Asset allocation models are designed to improve investment results by varying the allocation of a portfolio between broad asset classes over time[3], in response to changing assessments of expected returns (and risk) on the different asset classes. In *tactical* asset allocation the objective function is defined over the one-period (typically one quarter) return on the portfolio. Tactical asset allocation models have been quite successful[4], and gained greatly in popularity following the Crash of 1987.

However, as Brennan, Schwartz and Lagnado(1996) have argued, the analysis of Merton (1971) implies that the tactical asset allocation strategy, when employed over more than one period, is logically flawed[5]. This is because it fails to take account of the possibility of future changes in the investment opportunity set. Yet it is the fact that the opportunity set *does* change over time which provides the basis for the tactical asset allocation strategy.

The importance of taking account of changes in the opportunity set may be illustrated with a simple example. Consider an investor who is investing over a 10 year horizon and has a utility function defined over wealth at the horizon. Suppose that the investor has the choice between investing in short term bonds or a single 10 year pure discount bond. If interest rates vary stochastically, the return on the 10 year bond will appear risky from the viewpoint of an investor who is concerned only with wealth at the end of one year. However, from the viewpoint

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[3] Portfolio risk for a large investor depends mainly on allocations to broad asset classes rather than the selection of individual securities. See Mulvey and Ziemba (1995).

[4] See, for example, Evnine and Henriksson (1987).

[5] Except in the case of a logarithmic utility function for which the analysis of Hakansson (1971) for example, shows that myopic behavior is optimal even in the face of stochastically evolving investment opportunities.

of the 10 year investor the 10 year bond is actually riskless. This is because when the price of the 10 year bond falls, future investment opportunities, in the shape of expected future interest rates, improve, and in fact, from the viewpoint of our 10 year investor who invests only in the 10 year bond[6], these two effects just offset each other. In other words, the 10 year bond, while risky in terms of its effect on wealth at the end of one year, has the additional desirable characteristic of allowing the investor to *hedge* against changes in future investment opportunities. This second aspect of long term bonds is entirely missed by tactical asset allocation models, which assume that the investor is concerned solely with wealth at the end of one period and therefore does not value the hedging characteristics of different financial instruments.

Brennan, Schwartz and Lagnado (1996) have coined the term *strategic asset allocation* to refer to dynamic asset allocation strategies that take account, not only of the time-variation in expected returns on different asset classes, but also of the investor's horizon. They show that investment strategies are significantly affected by the investor's time horizon, when the investment opportunity set is described by three state variables, the short term interest rate, the long term interest rate, and the dividend yield on the stock market portfolio. This paper extends that analysis by allowing the investor to take long and short positions in short term interest rate futures as well as in bonds, stock and cash. The significance of the short term interest rate futures contract is that it allows the investor to hedge against unanticipated changes in the short term rate of interest. A position in long term bonds allows the investor to hedge perfectly against changes in the long term rate and a position in stock allows the investor to hedge almost perfectly against

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[6] The effects are a little more subtle if the investor can divide his portfolio between the long term bond and short term bonds. See Brennan, Schwartz and Lagnado (1996).

changes in the dividend yield of the stock market portfolio because most of the change in the dividend yield is associated with the return in the stock market portfolio. Without this contract the investor is able to hedge against changes in only two of the three state variables. We demonstrate that this new investment opportunity, which allows the investor to hedge against changes in the future investment opportunity set associated with changes in the short term rate of interest, leads to a significant improvement in expected utility. In fact, under our assumptions and with historical parameter estimates, an investor with a 19 year horizon would be willing to pay as much as $3.04 per dollar of his initial wealth in order to have the right to trade in Treasury Bill futures as well as stock, bonds and cash.

II

Time-Variation in Expected Returns

While time-variation in the expected returns on short-term cash instruments is visible to the naked eye, and time variation in the expected returns on other fixed income instruments may reasonably be inferred by extension, the random walk model of stock price behavior has nevertheless continued to hold the ascendancy among academics until relatively recently[7]. This is despite the fact that it is more than 20 years since Lintner (1975) reported evidence that expected returns on stocks vary over time in a manner that is related to the short term interest rate; his results have been confirmed more recently by Keim and Stambaugh (1986), and others[8].

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[7] Bossaerts and Hillion (1994), and Pesaran and Timmerman (1995), argue that the best model for predicting monthly stock returns changes over time, possibly as the result of learning.

[8] See also Campbell (1987), and Pesaran and Timmerman (1995). Hodrick (1987) reports that the confidence level of the test statistic for rejecting the null of no effect of Treasury Bill

Therefore the first variable that we employ as a predictor of asset returns is the short term riskless interest rate. Of course, this variable not only predicts stock returns, but it is also the (forecast of the) return on cash, which is an important asset class in its own right. The second variable that we use as a predictor of expected stock returns is the dividend yield on a stock index portfolio, defined as the ratio of the past 12 months' dividends divided by the current stock price. This variable is more controversial. Fama and French (1988), and Campbell and Shiller (1988), have provided evidence of a relation between dividend yields and subsequent stock returns. Their findings have been criticized by Nelson and Kim (1993) as being the result of small sample bias, and although Hodrick (1992) concludes after a lengthy study that 'the estimates and the Monte Carlo studies support the conclusion that changes in dividend yields forecast significant expected changes in expected stock returns' (p383), Goetzmann and Jorion (1993) claim to show that his results do not take proper account of the fact that the regressor (the dividend yield) behaves like a lagged dependent variable because it depends on lagged returns. We nevertheless include it. The third variable that we include in our analysis is the yield on a long term bond, which is intended to proxy for the yield on a consol bond. In conjunction with the short term interest rate, this is equivalent to including a 'term premium' or slope of the yield curve as a predictor of stock returns. Campbell (1987), Fama and French (1989) and Keim and Stambaugh (1986) include such a variable, although Hodrick (1992) claims that it has no marginal explanatory power for

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returns on stock returns exceeds 0.999 for the period 1952 to 1987. Attempts to account for this empirical regularity include Fama (1981), Schwert (1981), Geske and Roll (1983), and Stulz (1986). Since the short term interest rate is closely related to the expected rate of inflation, stock returns are negatively related to the expected rate of inflation also. Kaul (1987) argues that this phenomenon is due not only to a negative relation between inflation and real activity but also to the counter-cyclical responses of the monetary authorities in the post-war era.

stock returns in the presence of the dividend yield and the Treasury Bill rate. Our primary reason for including the long term interest rate is its role as a predictor of changes in the short term interest rate which has previously been documented by Brennan and Schwartz (1982) and Fama (1976). In summary, the three variables that we use to predict asset returns are the short rate, r, the long rate, l, and the dividend yield on the stock portfolio, δ. These state variables are assumed to follow a joint continuous time Markov process.

Thus, denoting the instantaneous rate of return on the stock portfolio by dS/S, the state variables and the stock return are assumed to follow a joint stochastic process of the form:

$$\frac{dS}{S} = \mu_S dt + \sigma_S dz_S \tag{1}$$

$$dr = \mu_r dt + \sigma_r dz_r \tag{2}$$

$$dl = \mu_l dt + \sigma_l dz_l \tag{3}$$

$$d\delta = \mu_\delta dt + \sigma_\delta dz_\delta. \tag{4}$$

where the parameters μ_i, σ_i (I = r, l, δ, S) are at most functions of the state variables r, l, δ, and dz_i are increments to Wiener processes. The correlation coefficients between the increments to the Wiener processes are denoted by $\rho_{r\delta}$ etc.

The empirical data that are used in the analysis that follows are taken from the period January 1976 to December 1994. This sample period was chosen because data on interest rate

futures were available only from 1976. The specific data series are as follows:

Stock return - the monthly rate of return on the CRSP Value-Weighted Index.

Dividend Yield - the sum of the past 12 months' dividends on the CRSP Value-Weighted Index divided by the value of the index at the end of the previous month.

Short (term interest) rate - the yield on a one month Treasury Bill as of the beginning of each month, taken from the CRSP Government Bond Files[9].

Long (term interest) rate: the (continuously compounded) yield to maturity on the longest maturity, taxable, non-callable government bond excluding flower bonds, as of the beginning of each month. This was taken from the CRSP Government Bond Files.

Return on the long term bond - the monthly return including coupons on the bond used to compute the long rate.

In addition, data were obtained from Knight-Ridder on the International Monetary Market 3-month Treasury Bill futures contract. Monthly futures returns were computed using the month-end prices of the nearest to maturity contract that had more than one month to maturity when entered into[10].

III

The Portfolio Problem

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[9] We discovered an anomaly in this file concerning the 1 month rate for April 30, 1987. The value for the 1 month rate on this date was 2.507%; the yields on the bills maturing 1 week earlier and 1 week later averaged 4.815%. This average value was substituted for the observed value for the empirical analysis.

[10] The quotes were converted to prices, and 'returns' were computed.

The investor is assumed to be able to invest in three asset classes as well as to take positions in interest rate futures. The asset classes assumed to be available to the investor are cash with sure rate of return, r; stock, whose rate of return is given by equation (1); and consol bonds. The price of a consol bond, B(l), is inversely proportional to its yield, l. The total return on a consol bond is the sum of the yield and the price change; then a simple application of Ito's Lemma implies that the instantaneous total return on the consol bond is given by;

$$\frac{dB}{B} + l\,dt = \left(l - \frac{\mu_l}{l} + \frac{\sigma_l^2}{l^2} \right) dt - \frac{\sigma_l}{l} dz_l. \tag{5}$$

It is assumed that the futures price of the Treasury Bill depends only on the short term interest rate so that the proportional change in the futures price may be written as:

$$\frac{dF}{F} = \mu_F dt + f(r)\sigma_r dz_r. \tag{6}$$

Define x as the proportion of the investment portfolio that is invested in stock, y as the proportion that is invested in the consol bond, and let z denote the notional value of the futures position expressed as a proportion of wealth[11]. Then the stochastic process for wealth, W, is:

[11] We use the term 'notional value' to remind the reader that a futures position requires no initial investment. If Q is the notional value of a futures position then the number of futures contracts held is Q/F where F is the futures price.

$$
\begin{aligned}
\frac{dW}{W} &= \left[x\frac{dS}{S} + y\left(\frac{dB}{B} + l\,dt\right) + z\frac{dF}{F} + (1 - x - y)r\,dt \right] \\
&= \left[x(\mu_S - r) + y\left(l - r - \frac{\mu_l}{l} + \frac{\sigma_l^2}{l^2}\right) + z\mu_F + r \right] dt \\
&\quad + \left[x^2\sigma_S^2 + y^2\frac{\sigma_l^2}{l^2} + z^2 f^2 \sigma_r^2 - \frac{2xy\sigma_l\sigma_S\rho_{Sl}}{l} + 2xzf\sigma_S\sigma_r\rho_{rS} - \frac{2yzf\sigma_r\sigma_l\rho_{rl}}{l} \right]^{\frac{1}{2}} dz_W \\
&\equiv \mu_W\,dt + \sigma_W\,dz_W
\end{aligned}
\tag{7}
$$

The Bellman equation for the investor's stochastic optimal control problem is

$$
\max_{x,y,z} E[dV] = 0 \tag{8}
$$

Or,

$$
\begin{aligned}
\max_{x,y,z} [\, & V_W\mu_W W + V_r\mu_r + V_l\mu_l + V_\delta\mu_\delta - V_\tau \\
& + \frac{1}{2}V_{WW}\sigma^2{}_W W^2 + \frac{1}{2}V_{rr}\sigma^2{}_r + \frac{1}{2}V_{ll}\sigma^2{}_l + \frac{1}{2}V_{\delta\delta}\sigma^2{}_\delta \\
& + V_{W\delta}W\sigma_\delta\sigma_W\rho_{\delta W} + V_{rl}\sigma_r\sigma_l\rho_{rl} + V_{r\delta}\sigma_r\sigma_\delta\rho_{r\delta} \\
& + V_{Wr}W\sigma_r\sigma_W\rho_{rW} + V_{Wl}W\sigma_l\sigma_W\rho_{lW} + V_{l\delta}\sigma_l\sigma_\delta\rho_{l\delta}\,] = 0
\end{aligned}
\tag{9}
$$

The control problem (9) has four state variables including W. To reduce the number of state variables, we assume that utility is of the isoelastic form so that;

$$
V(r, l, \delta, W, 0) = \frac{1}{\gamma}W^\gamma, \qquad \text{for } \gamma < 1 \tag{10}
$$

Then it may be verified that $V(r, l, \delta, W, \tau)$ may be written as $\gamma^{-1}W^\gamma\, v(r, l, \delta, \tau)$, where:

$$v(r,l,\delta,0) = 1 \tag{11}$$

and,

$$\begin{aligned}
\underset{x,y,z}{\text{Max}} \Big[& \mu_W v + \frac{1}{\gamma}\mu_r v_r + \frac{1}{\gamma}\mu_l v_l + \frac{1}{\gamma}\mu_\delta v_\delta - \frac{1}{\gamma} v_\tau \\
& + \frac{1}{2}(\gamma - 1)\sigma_W{}^2 v + \frac{1}{2\gamma}\sigma_r{}^2 v_{rr} + \frac{1}{2\gamma}\sigma_l{}^2 v_{ll} + \frac{1}{2\gamma}\sigma_\delta{}^2 \\
& + \sigma_W \sigma_r \rho_{rW} v_r + \sigma_W \sigma_l \rho_{lW} v_l + \sigma_W \sigma_\delta \rho_{\delta W} v_\delta \\
& + \frac{1}{\gamma}\sigma_r \sigma_l \rho_{rl} v_{rl} + \frac{1}{\gamma}\sigma_r \sigma_\delta \rho_{r\delta} v_{r\delta} + \frac{1}{\gamma}\sigma_l \sigma_\delta \rho_{l\delta} v_{l\delta} \Big] = 0
\end{aligned} \tag{12}$$

where

$$\begin{aligned}
\sigma_W \sigma_r \rho_{rW} &\equiv \sigma_{Wr} = x\,\sigma_S \sigma_r \rho_{rS} - y\frac{\sigma_l}{l}\sigma_r \rho_{rl} + z\,f\,\sigma_r{}^2 \\
\sigma_W \sigma_l \rho_{lW} &\equiv \sigma_{Wl} = x\,\sigma_S \sigma_l \rho_{lS} - y\frac{\sigma_l{}^2}{l} + z\,f\,\sigma_r \sigma_l \rho_{rl} \\
\sigma_W \sigma_\delta \rho_{\delta W} &\equiv \sigma_{W\delta} = x\,\sigma_S \sigma_\delta \rho_{S\delta} - y\frac{\sigma_l}{l}\sigma_\delta \rho_{\delta l} + z\,f\,\sigma_r \sigma_\delta \rho_{r\delta}
\end{aligned} \tag{13}$$

Substituting for μ_W and collecting terms, we have finally:

$$
\begin{aligned}
\underset{x,y,z}{\mathrm{Max}}\{ & v[x(\mu_S - r) + y(l - r - \frac{\mu_l}{l} + \frac{\sigma_l^2}{l^2}) + z\mu_F + r \\
& + \frac{1}{2}(\gamma - 1)(x^2\sigma_S^2 + y^2\frac{\sigma_l^2}{l^2} + z^2 f^2 \sigma_r^2 - \frac{2xy\sigma_S\sigma_l\rho_{Sl}}{l} \\
& + 2xzf\sigma_{Sr} - \frac{2yzf\sigma_{rl}}{l})] \\
& + v_r\left[\frac{1}{\gamma}\mu_r + x\sigma_{Sr} - \frac{y}{l}\sigma_{rl} + zf\sigma_r^2\right] + v_l\left[\frac{1}{\gamma}\mu_l + x\sigma_{Sl} - \frac{y}{l}\sigma_l^2 + zf\sigma_{rl}\right] \\
& + v_\delta[\frac{1}{\gamma}\mu_\delta + x\sigma_{S\delta} - \frac{y}{l}\sigma_{l\delta} + zf\sigma_{r\delta}] - \frac{1}{\gamma}v_\tau \\
& + \frac{1}{\gamma}\left[\frac{1}{2}v_{rr}\sigma_r^2 + \frac{1}{2}v_{ll}\sigma_{ll}^2 + \frac{1}{2}v_{\delta\delta}\sigma_\delta^2 + v_{rl}\sigma_{rl} + v_{r\delta}\sigma_{r\delta} + v_{l\delta}\sigma_{l\delta}\right]\} = 0
\end{aligned}
\tag{14}
$$

The first order conditions for a maximum in (14) imply that the optimal controls, $x^* \equiv x^*(r, l, \delta, \tau)$, $y^* \equiv y^*(r, l, \delta, \tau)$, and $z^* = z^*(r, l, \delta,, \tau)$ are given by:

$$
\begin{bmatrix} \sigma_S^2 & -\frac{\sigma_{Sl}}{l} & \sigma_{Sr} \\ -\frac{\sigma_{sl}}{l} & \frac{\sigma_l^2}{l^2} & -\frac{\sigma_{rl}}{l} \\ \sigma_{Sr} & -\frac{\sigma_{rl}}{l} & \sigma_r^2 \end{bmatrix} \begin{bmatrix} x^* \\ y^* \\ z^* f \end{bmatrix} = \begin{bmatrix} rhs_1 \\ rhs_2 \\ rhs_3 \end{bmatrix} \tag{15}
$$

where:

$$rhs_1 = -\frac{1}{\gamma - 1}\left[\mu_S - r + \frac{v_r}{v}\sigma_{Sr} + \frac{v_l}{v}\sigma_{Sl} + \frac{v_\delta}{v}\sigma_{S\delta}\right]$$

$$rhs_2 = -\frac{1}{\gamma - 1}\left[l - r - \frac{\mu_l}{l} + \frac{\sigma_l^2}{l^2} - \frac{v_r}{v}\frac{\sigma_{rl}}{l} - \frac{v_l}{v}\frac{\sigma_l^2}{l} - \frac{v_\delta}{v}\frac{\sigma_{l\delta}}{l}\right]$$

$$rhs_3 = -\frac{1}{\gamma - 1}\left[\frac{\mu_F}{f} + \frac{v_r}{v}\sigma_r^2 + \frac{v_l}{v}\sigma_{rl} + \frac{v_\delta}{v}\sigma_{r\delta}\right]$$

To determine z^*, the optimal position in the futures, it is necessary to define μ_F, the drift of the futures price process, and f(r), the sensitivity of the change in the futures price to the change in the short term rate, r. One way to do this would be to use an equilibrium model of bond pricing such as the Cox-Ingersoll-Ross(1985) model or the Brennan-Schwartz (1982) model to determine the drift and sensitivity. However, both of these models require that a market risk aversion parameter be assessed, in addition to the parameters of the interest rate process. Therefore, for the sensitivity we adopted the simpler expedient of using the duration model[12]. Insofar as there is little maturity effect in interest rates out to the maturity of the underlying Treasury Bill, we expect that this will be quite accurate. Thus define I as the relevant interest rate. Then the price of a futures contract on a Bill maturing in T^* years, which is deliverable in T years, is given by $F(T, T^*) = \exp\{i(T - T^*)\}$. This implies that $dF/di = (T - T^*)F$. In our application $T - T^*$ is one quarter so that $f =- 0.25$, and we identify I with the short rate, r.

For, μ_F, the proportional drift in the futures price process, there are several possible modeling strategies. The first, which we refer to as *Model 1*, recognizes that μ_F is simply a risk premium associated with changes in the short rate, and assumes that the risk premium is constant

[12] See Ingersoll (1979).

over time; we take the risk premium as equal to zero, so that $\mu_F = 0$. *Model 2* assumes that $\mu_F = f\,\mu_r$; this is consistent with the assumption of the duration model that the term structure of interest rates is flat so that the yield underlying the futures price is the current short rate. However, unlike *Model 1,* it does not allow for the possibility that the futures price will reflect the expected change in the short rate (represented by the drift in equation (2)). A third possibility is to allow μ_F to be a linear function of r, l and δ, and to estimate the functional form as in equations (1) - (4) - we refer to this as *Model 3*[13].

In order to implement the model, it is necessary first to estimate the parameters of the stochastic process for the state variables. We consider that next.

IV

Estimation of the Stochastic Process

Following Brennan, Schwartz and Lagnado (1996), we assume that the expected returns on stocks and bonds, and the drifts of the dividend yield and short rate, are linear functions of the three state variables, r, l, and δ, while the volatility of each state variable is assumed to be proportional to its current level, and the volatility of the stock rate of return is taken as constant. This implies from equation (5) that the drift of the long rate is a non-linear function of the state variables, being equal to the product of l and a linear function of the state variables. This specification implies that the joint stochastic process may be written as:

[13] We continue to assume that the proportional innovation in the futures price is perfectly correlated with innovations in the short rate. See Shiller (1979) for a similar finding.

$$\frac{dS}{S} = (a_{S1} + a_{S2}\delta + a_{S3}r + a_{S4}l)dt + \sigma_S dz_S \tag{16}$$

$$dr = (a_{r1} + a_{r2}\delta + a_{r3}r + a_{r4}l)dt + r\sigma_r dz_r \tag{17}$$

$$dl = l(a_{l1} + a_{l2}\delta + a_{l3}r + a_{l4}l)dt + l\sigma_l dz_l \tag{18}$$

$$d\delta = (a_{\delta 1} + a_{\delta 2}\delta + a_{\delta 3}r + a_{\delta 4}l)dt + \delta\sigma_\delta dz_\delta \tag{19}$$

The dividend yield is defined as the sum of the past 12 months' dividends divided by the current level of the stock index, S. The specification (19) must therefore be regarded as an approximation since the stochastic process for lagged dividends is not modeled explicitly; to have done so would have introduced a fourth state variable into the analysis which would have considerably increased the difficulty of solving the control problem. However, we expect that the stochastic increment to the dividend yield will have a strong negative correlation with the return on the stock, since most of the stock return is accounted for by price changes.

The joint stochastic process was estimated on monthly data for the period January 1976 to December 1994, using a discrete approximation to the continuous process. The time-paths of r, l and δ during the sample period are shown in Figure 1. It is apparent that there is the most variability in the short rate, followed by the long rate, while the dividend yield fluctuates between 2% and 6%.

The system of equations (16)-(19) was estimated by non-linear seemingly unrelated regression using TSP. Table 1 reports the regression estimates. As previous investigators have

found, the expected return on common stocks is negatively related to the current level of the short rate and positively related to the level of the dividend yield, but is not significantly related to the long rate (and therefore to the slope of the yield curve). As Brennan and Schwartz (1982) have found, the change in the short rate is negatively related to its current level and positively but not significantly related to the level of the long rate. The change in the long rate is negatively related to its current level and positively related to the short rate at conventional levels of significance[14]. The change in the dividend yield is negatively related to its current level, so that it shows mean reversion; in addition, it is positively related to the short rate[15]. Table 2 reports the correlations between the innovations in the state variables. As anticipated, the innovation, or unexpected change, in the dividend yield is very highly negatively correlated with the innovation in stock returns; it is also positively correlated with the innovation in the long rate, because the innovation in the long rate is negatively correlated with the innovation in stock returns. The innovations in the long and short rates have a correlation of 0.36, while the correlation between the stock return and the innovation in the short rate is only -0.02.

While the focus of this paper is on the optimal investment policy assuming that the parameters of the stochastic process are known, we shall also perform an out of sample experiment in which the policy used for the second half of the sample period is computed using

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[14] This implies that the risk premium on long-term bonds, like that on stocks, is *negatively* related to the level of the short rate.

[15] This corresponds to the negative coefficient of the stock return on the short rate, since a negative stock return is associated with an *increase* in the dividend yield.

parameter estimates derived from the first half. Therefore, Tables 3 and 4 report the parameter estimates obtained for the two halves of the sample period separately[16].

IV

Empirical Analysis of the Role of Futures

The stochastic optimal control problem was solved[17] using the parameter estimates reported in Table 1 and a value of the risk aversion parameter, γ, of -5.0. This high value for the risk aversion parameter was chosen because we take the estimated parameters of the stochastic process as known, and with a low risk aversion parameter there is a tendency for the model to take unreasonably aggressive positions[18]. The holdings of bonds, stock and cash were restricted to between zero and 100%, and the absolute value of the nominal value of the futures position was restricted to be less than ten times wealth. The proportional drift of the futures price was first computed according to *Model 1*.

Asset Proportions

The stochastic optimal control problem yields the optimal portfolio position x^*, y^*, z^*, as functions of the state variables r, l,δ and time to maturity, τ. Then assuming that the horizon was December 31 1994, the strategy functions were combined with the values of the state variables

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[16] To conserve space we do not report the correlations of the innovations for the subperiods.

[17] For a discussion of the solution procedure see Brennan, Schwartz and Lagnado (1996).

[18] For evidence on the importance of parameter errors in mean-variance analysis see Chopra dn Ziemba (1993). We are currently working on incorporating estimation risk into the analysis.

for each month of the sample period to yield times series of the optimal portfolio positions in the different asset classes. Figure 2 shows the optimal positions in bonds, stock and futures[19] - the optimal cash position follows as a residual. There is considerable month to month variability in the portfolio positions - it is noticeable that the futures position tends to be negatively related to the bond position; this is what we should expect, since under *Model 1* the Treasury Bill future provides costless insurance against changes in the short rate and the short rate and the long rate are positively correlated[20].

Figure 3 and Table 5 relate the portfolio positions to the state variables. In Figure 3 the time series of the holdings of bonds and stock are plotted along with their expected returns computed from equations (5), (16) and (18) using the state variables and the parameter estimates in Table 1. The Figure shows that there is positive relation between the expected returns on both bonds and stock and the allocation to those asset classes. Table 5 reports the results of regressions of portfolio proportions[21] on the expected excess returns on bonds and stock. The stock and bond allocations are strongly positively related to the own expected returns and negatively related to the expected returns on the other asset class. The futures position is positively related to the expected returns on stock, and strongly negatively related to the expected

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[19] The proportional futures position is scaled by a factor of 5 for visual clarity.

[20] It has been suggested to us that perhaps the T-Bill futures contract improves welfare by relaxing the constraint on bond short sales. The fact that the futures position tends to be short when the bond position is long implies that this is not the main economic benefit of introducing the futures contract.

In terms of volatility, an allocation to futures of 100% of wealth is roughly equivalent to a 100% investment in stock.

[21] We include in the regressions only those observations for which the portfolio allocation to the asset class was not at its upper or lower limit.

returns on bonds. Most significantly, while the allocations to stock and futures are not significantly time dependent, the futures allocation is positively related to the time to the horizon. The nominal futures position expressed as a proportion of wealth increases by about 6% for every year to maturity. Table 6 reports the results of regressing the futures position on the stock and bond allocations as well as time to maturity, using all the observations. The futures position is strongly negatively related to the bond allocation, and positively related to the stock allocation; the former is consistent with the correlation of - 0.36 between bond returns and the innovation in the short rate. The positive correlation with the stock allocation is harder to explain; it maybe that this is an artifact of the constraint on the sum of the bond and stock allocations. The futures position increases with the time to the horizon.

For comparison, the optimal strategy without futures contracts was also computed. Figure 4 shows the time series allocations to bonds and stock with and without futures. The differences are often large: there is a tendency for the allocation to bonds to be larger when the opportunity set includes futures, but the effect of futures trading on the stock position is more ambiguous.

Investment Returns and the Value of Investment Opportunities

Figure 5 plots the time paths of wealth per dollar invested under the optimal strategies with and without futures, as well as an all-stock strategy. The figure also shows the certainty equivalent of wealth under the strategies. This is defined as the sure amount at the horizon that the investor would exchange for his current wealth *and the opportunity to invest up to the horizon.* Thus the difference between current wealth and the certainty equivalent represents the value of the remaining investment opportunities up to the horizon. The certainty equivalent, CE,

is defined by

$$\frac{1}{\gamma}(CE)^{\gamma} = V(W, r, l, \delta, \tau)$$

$$= \frac{1}{\gamma} W^{\gamma} v(r, l, \delta, \tau)$$

Hence

$$CE(W, r, l, \delta, \tau) = W\,[v(r, l, \delta, \tau)]^{\frac{1}{\gamma}}$$

It is readily verified that since $E[dV] = 0$, $E[d(CE^{\gamma})] = 0$. Moreover, $CE(W, r, l, \delta, 0) = W$. Thus, under the optimal strategy the certainty equivalent raised to the power γ follows a random walk, and its terminal value is equal to the terminal wealth realized under the optimal strategy. Therefore, the accumulated volatility of the certainty equivalent over the life of the strategy is an appropriate measure of the risk of the strategy[22].

Not surprisingly, both asset allocation strategies substantially outperform the all-stock strategy. The introduction of futures leads to a substantial decrease in risk; as can be seen from the figure, the volatility of the certainty equivalent is much less when futures contracts are introduced[23]. The initial certainty equivalent for the with futures strategy is 10.90 versus only

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[22] Consider the volatilities of wealth and the certainty equivalent of wealth under a strategy of investing in a pure discount bond with maturity equal to the horizon. Under this riskless strategy the volatility of wealth will be positive while the volatility of the certainty equivalent will be zero.

[23] The volatility of the monthly proportional change in the certainty equivalent is 0.064 without futures and 0.052 with futures.

7.86 for the without futures strategy; thus, the *ex-ante* value of the futures opportunity is $3.04 per dollar of initial wealth when the horizon is 19 years[24]. The fact that the final certainty equivalent (and therefore final wealth) is higher under the without futures strategy must be attributed to sampling variability.

The above results are obtained in sample and therefore overstate the advantages of the asset allocation strategies. Therefore, for comparison we show in Figure 6 the results of computing the optimal strategy from *Model 1* using the parameter values reported in Table 3 for the first half of the sample period, and then implementing this strategy for the second half of the sample period without further updating of the parameter values. We do this both with and without trading in futures contracts. The first thing to note is the downward trend in the certainty equivalents. This reflects the fact that the realized investment opportunities are not as good as the *ex-ante* assessment of them which treats the parameter values as known. In addition, the certainty equivalent of the with futures strategy is above that of the without futures strategy , not only at inception[25], but throughout. While the final wealth under the pure stock strategy exceeds that under the asset allocation strategies, it should be noted that, not only are the parameters of the asset allocation strategies not updated, but also the pure stock strategy is much more risky[26].

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[24] When the portfolio proportions are unconstrained the value rises to $5.20. The fact that the value of the futures contract is higher when borrowing and short sales are allowed demonstrates that the primary benefit of the contract is not to circumvent these restrictions, but to improve hedging opportunities.

[25] The certainty equivalent of the with futures strategy must be at least as great of the without futures strategy when the wealth is the same for the two strategies .

[26] The standard deviation of the monthly return of the stock strategy is 4.4%, as compared with 1.5% and 1.6% for the two asset allocation strategies.

To this point we have relied on the assumption of *Model 1* that the futures price process has zero drift. We consider next the other two models of the futures price process.

Alternative Models of Futures Price Behavior

Model 2 assumes that the proportional drift in the futures price is $\mu_F = f\,\mu_r$ where f is -0.25 for the T-Bill futures contract and μ_r is the drift in the short rate. *Model 3* requires that the conditional drift of the futures price be estimated by regressing the proportional change in the futures price on r, l, and δ. Table 7 reports the regression estimates for the whole sample period and two subperiods. It is evident that there is some degree of predictability in the futures price based mainly on the long rate.

Figure 7 shows the wealth and certainty equivalents yielded by the asset allocation strategies based on the three models of futures prices for the whole sample period, with parameters estimated in sample. The certainty equivalent for *Model 1* which assumes no predictability in the futures price is, not surprisingly, initially below the certainty equivalents for the other two models which do assume predictability. *Model 2,* which assumes that the futures price change is as predictable as the changes in the short rate itself, has the highest initial value of the certainty equivalent, but this declines rapidly, as the realized predictability turns out to be less than that assumed. Moreover, the spurious predictability of this model causes it to take large bets on futures, with the result that the return and the certainty equivalent are much more volatile than for the other two models, and for these models there is no downward trend in the certainty equivalent. *Model 1*, which assumes no predictability in the futures price yields the highest final wealth, and the difference between it and *Model 3* is statistically significant at the 5% level.

Figure 8 depicts the out-of-sample performance of the asset allocation strategies based on

the three models of futures price behavior. The parameters are estimated over the first half of the sample period and the strategies are implemented over the second half. For all three models the certainty equivalents show a declining trend, reflecting the failure of the predictive models to perform as well out of sample; however, the rate of decline is greatest for *Model 2* with its unrealistic assumption that the futures price will move one for one with the short term rate. The performance of *Models 1 and 3* is almost indistinguishable.

V

Conclusion

In this paper we have shown that the opportunity to trade in short term interest rate futures can be valuable for an investor who has a long term horizon. There appear to be three reasons for this. First, the futures contract allows the investor to hedge part of the risk of the long term bond, because innovations in the short rate are correlated with innovations in the long rate. Secondly, the short rate is an important determinant of the investor's opportunity set, both because it is the return on an asset class, and because it is an important predictor of stock returns. Therefore the futures contract allows the investor to hedge against adverse shifts in the investment opportunity set. Finally, insofar as the changes in the futures price are predictable, trading in this security allows the investor to place favorable bets on these price changes. We have compared three models of futures price changes. These all assume that the proportional innovation in the price is given by the (negative of the)duration of the underlying bill times the innovation in the short rate, but differ in their assumptions about the proportional drift. *Model 1* assumes that the drift is zero; *Model 2* assumes that it is equal to the (negative of the)duration of

the underlying bill times the drift of the short rate. *Model 3* assumes that the drift can be written as a linear function of the three state variables, r,l and δ. We find that asset allocation strategies based on *Models 1 and 3* outperform those based on *Model 2* which assumes too much predictability in futures price changes.

Most of the analysis was conducted using in-sample parameter estimates of the stochastic process governing asset returns since our concern has been with comparing the theoretical value of the futures contract with a known stochastic process for returns. The tendency for the certainty equivalent of wealth to decline over time when the strategies are implemented out of sample points to the need to take account of estimation risk in constructing asset allocation strategies. This is the subject of ongoing research.

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	Constant	δ	r	l	σ
dS/S	-0.037 (2.25)	1.904 (3.95)	-0.451 (2.69)	0.025 (0.11)	0.041
dr	-0.008 (2.48)	0.234 (2.54)	-0.124 (3.36)	0.074 (1.60)	0.114
dl	0.033 (1.99)	-0.284 (0.59)	0.328 (2.28)	-0.520 (2.33)	0.038
dδ	0.002 (2.56)	-0.072 (3.18)	0.021 (2.72)	-0.004 (0.42)	0.044

(t-statistics in parentheses)
Log Likelihood: 1670

The Estimated Stochastic Process for the State Variables and the Stock Return
January 1976 - December 1994

Table 1

	Stock return	r	l	δ
Stock return	1.0			
r	-0.022	1.0		
l	-0.358	0.362	1.0	
δ	-0.944	0.068	0.326	1.0

Correlations of State Variables and Stock Return Innovations
January 1976 - December 1994

Table 2

	Constant	δ	r	l	σ
dS/S	-0.080 (2.60)	2.322 (3.21)	-0.574 (2.77)	0.328 (1.27)	0.039
dr	-0.0003 (0.05)	0.046 (0.29)	-0.112 (1.92)	0.078 (1.27)	0.109
dl	0.076 (2.31)	-1.154 (1.52)	0.474 (2.24)	-0.588 (2.21)	0.039
dδ	0.005 (3.10)	-0.106 (2.93)	0.032 (3.06)	-0.024 (1.86)	0.041

(t-statistics in parentheses)
Log Likelihood: 854
The Estimated Stochastic Process for the State Variables and the Stock Return
for the First Half of the Sample Period:
January 1976 - June 1985

Table 3

	Constant	δ	r	l	σ
dS/S	-0.071 (1.89)	6.602 (4.47)	-0.759 (2.32)	-1.329 (2.13)	0.040
dr	-0.008 (1.70)	0.451 (2.14)	-0.153 (3.10)	0.003 (0.04)	0.118
dl	0.081 (2.41)	-0.321 (0.25)	0.225 (0.78)	-1.049 (1.86)	0.035
dδ	0.003 (1.92)	-0.232 (4.16)	0.028 (2.16)	0.047 (2.03)	0.044

(t-statistics in parentheses)
Log Likelihood: 835
The Estimated Stochastic Process for the State Variables and the Stock Return
for the Second Half of the Sample Period:
July 1985 - December 1994

Table 4

	Stock Allocation	Bond Allocation	Futures Position
R^2	0.83	0.78	0.44
NOBS	165	141	227
Intercept	0.373 (18.84)	0.235 (8.39)	-0.438 (2.64)
Time to Maturity	0.000 (0.08)	-.0004 (1.46)	0.0048 (3.69)
Expected Excess Return on Stock	4.730 (26.97)	-3.497 (16.63)	6.081 (8.47)
Expected Excess Return on Bond	-3.213 (20.27)	5.795 (20.61)	-9.985 (9.78)

Regressions of Asset Allocations on Time to Horizon,
and Expected Excess Returns on Stock and Bonds.

Table 5

Intercept	Stock Allocation	Bond Allocation	Time to Horizon	R^2	Nobs
-0.685 (2.80)	1.007 (3.44)	-1.413 (4.81)	0.0061 (4.65)	0.41	227

Regression of Futures Position on Stock and Bond Allocations,
and Time to Horizon

Table 6

	Intercept	δ	r	l	R^2	Nobs
January 1976 - December 1994	-0.002 (2.95)	0.040 (1.67)	-0.023 (0.027)	0.027 (2.50)	0.05	227
January 1976 - June 1985	-0.006 (2.93)	0.116 (2.46)	-0.031 (2.35)	0.034 (2.08)	0.07	114
July 1985 - December 1994	-0.000 (0.23)	-0.068 (2.23)	0.002 (0.22)	0.032 (2.37)	0.06	113

Regression of Proportional Change in Futures Price ($\Delta F/F$)
on State Variables, δ, r, and l.

Table 7

Figure Legends

Figure 1: Monthly time series of stock dividend yield (d), short term interest rate (r), and long term interest rate (l).

Figure 2: Monthly asset allocations to bonds and stock, and futures position derived from *Model 1*, using in sample parameter values from Tables 1 and 2. Futures position divided by 5.

Figure 3: Monthly time series of allocations to stocks and bonds derived from *Model 1,* using in sample parameter values from Tables 1 and 2, and expected returns on stocks and bonds calculated using the coefficients in Table 1.

Figure 4: A comparison of the allocation to stocks and bonds when the opportunity set includes futures contracts and does not. The allocation is derived using *Model 1* when futures contracts are included. Based on in sample parameter estimates reported in Tables 1 and 2.

Figure 5: In sample certainty equivalents and wealth for asset allocation strategies with futures (ce1, w1) and without futures contracts (ce0, w0) and for an all stock strategy (ws). The allocation is derived using *Model 1* when futures contracts are included. Based on in sample parameter estimates reported in Tables 1 and 2.

Figure 6: Out of sample certainty equivalents and wealth for asset allocation strategies with futures (ce1, w1) and without futures contracts (ce0, w0) and for an all stock strategy (ws). The allocation is derived using *Model 1* when futures contracts are included. Based on out of sample parameter estimates reported in Table 3.

Figure 7: In sample certainty equivalents and wealth for asset allocation strategies with futures based on the three models of futures price changes. *Model 1* (ce1, w1), *Model 2* (ce2, w2), *Model 3* (ce3, w3). Based on out of sample parameter estimates reported in Tables 1 and 2.

Figure 8: Out of sample certainty equivalents and wealth for asset allocation strategies with futures based on the three models of futures price changes. *Model 1* (ce1, w1), *Model 2* (ce2, w2), *Model 3* (ce3, w3). Based on out of sample parameter estimates reported in Table 3.

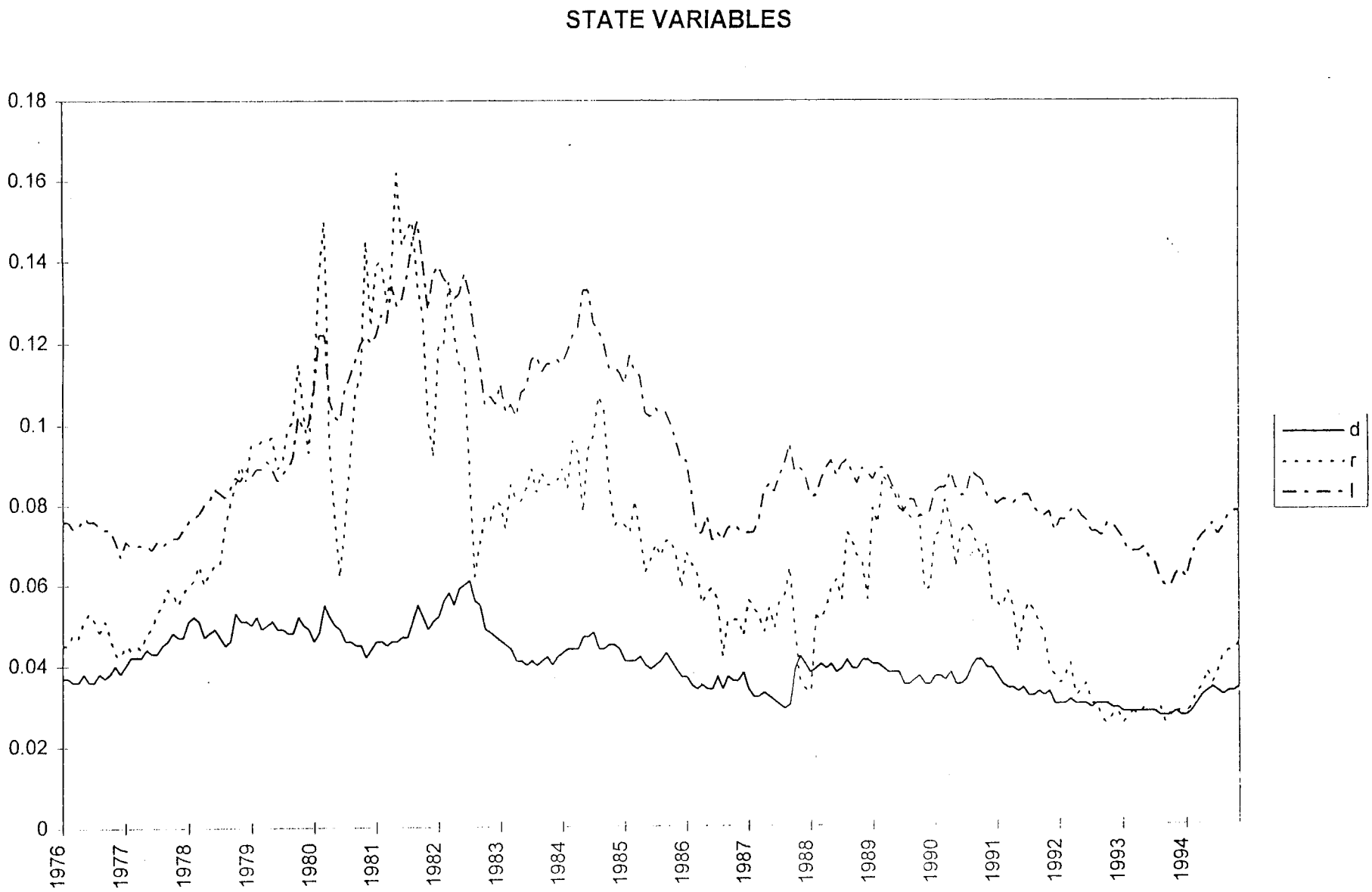
STATE VARIABLES
0.18
0.16
0.14
0.12
0.1
0.08
0.06
0.04
0.02
0
1976
1977
1978
1979
1980
1981
1982
1983
1984
1985
1986
1987
1988
1989
1990
1991
1992
1993
1994
d
r
l

Figure 1

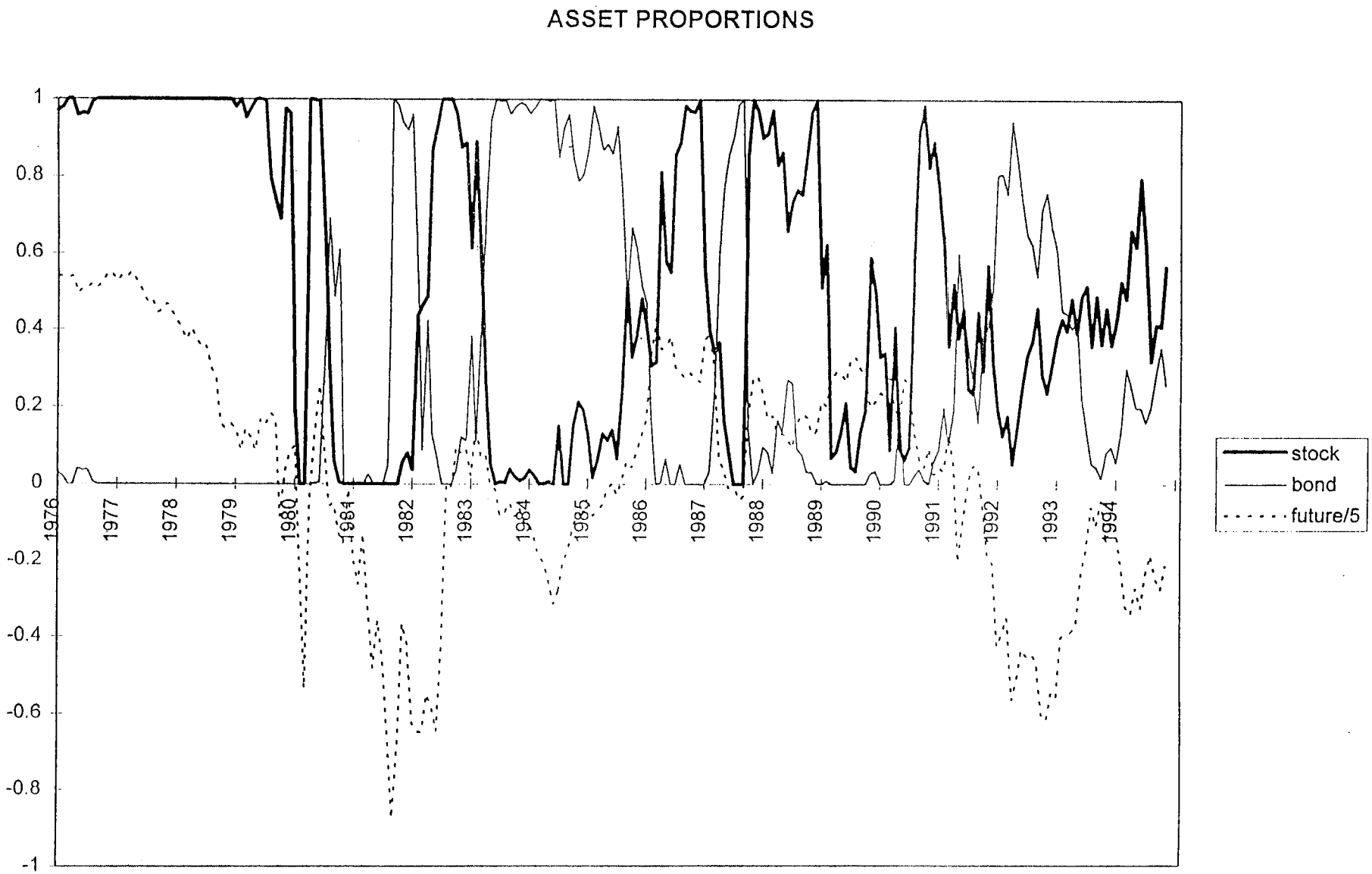
ASSET PROPORTIONS
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
1976
1977
1978
1979
1980
1981
1982
1983
1984
1985
1986
1987
1988
1989
1990
1991
1992
1993
1994
stock
bond
future/5

Figure 2

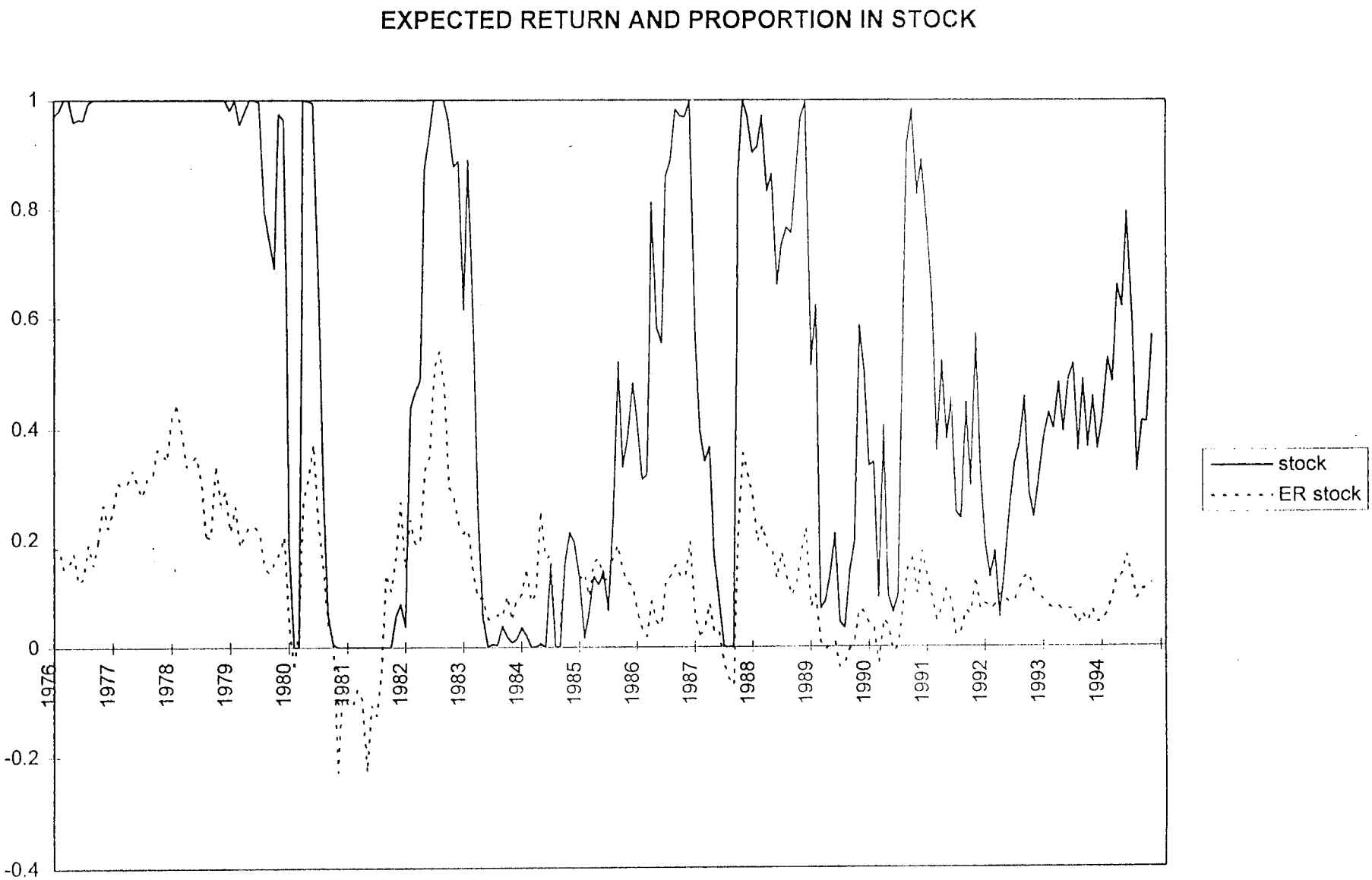
EXPECTED RETURN AND PROPORTION IN STOCK
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
1976
1977
1978
1979
1980
1981
1982
1983
1984
1985
1986
1987
1988
1989
1990
1991
1992
1993
1994
stock
ER stock

Figure 3a

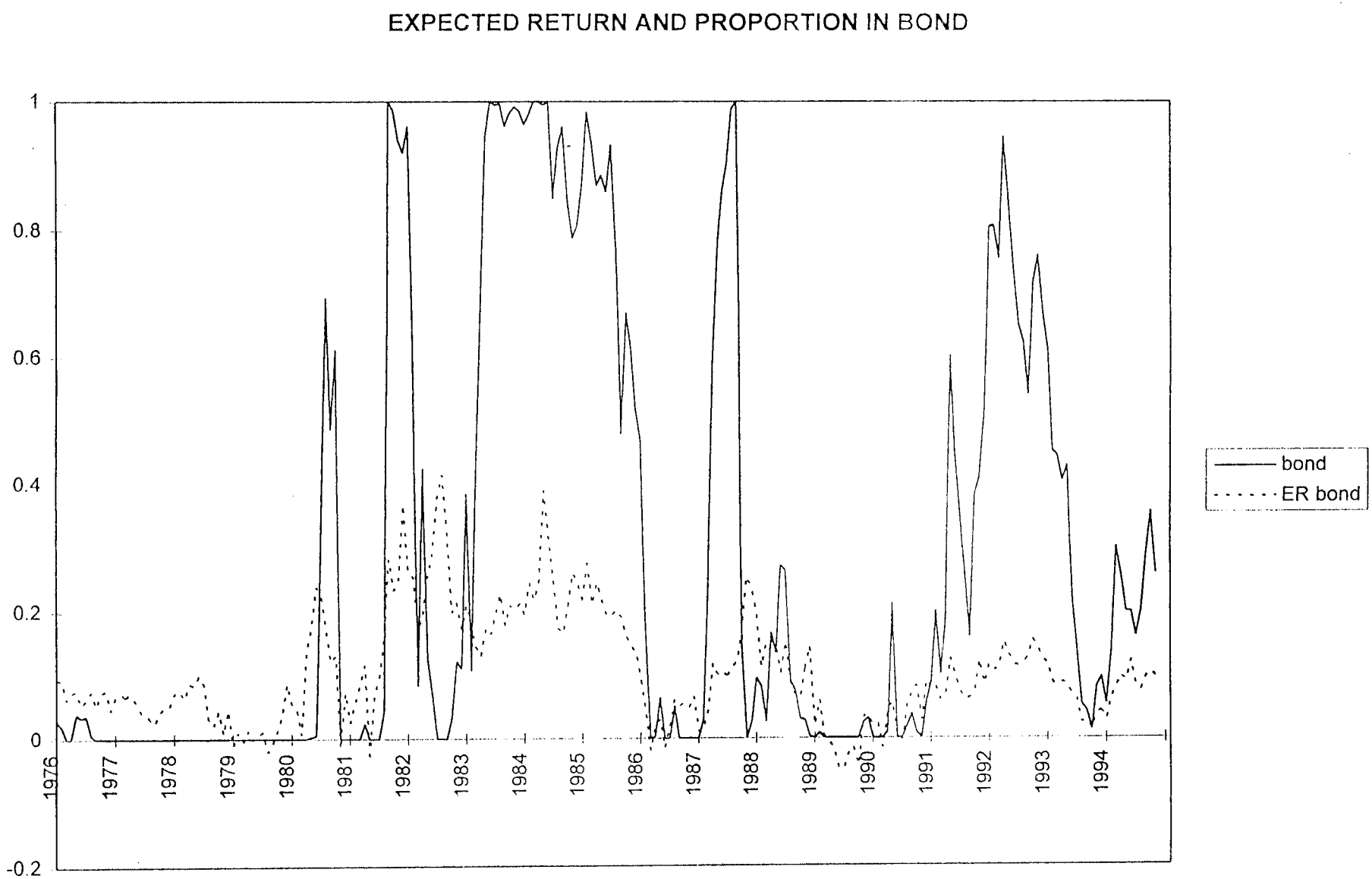
EXPECTED RETURN AND PROPORTION IN BOND
1
0.8
0.6
0.4
0.2
0
-0.2
1976
1977
1978
1979
1980
1981
1982
1983
1984
1985
1986
1987
1988
1989
1990
1991
1992
1993
1994
bond
ER bond

Figure 3b

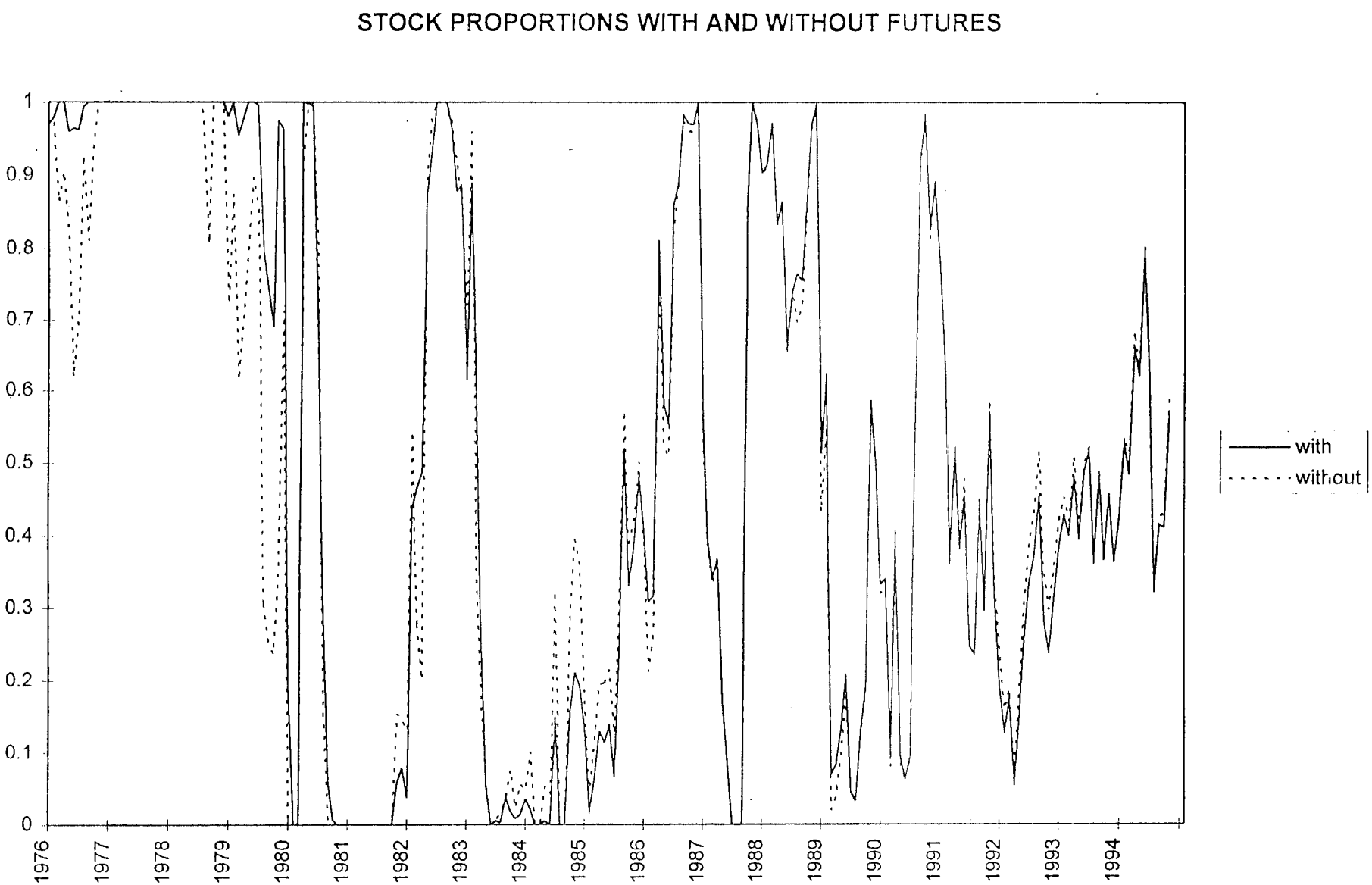
STOCK PROPORTIONS WITH AND WITHOUT FUTURES
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1976
1977
1978
1979
1980
1981
1982
1983
1984
1985
1986
1987
1988
1989
1990
1991
1992
1993
1994
with
without

Figure 4c

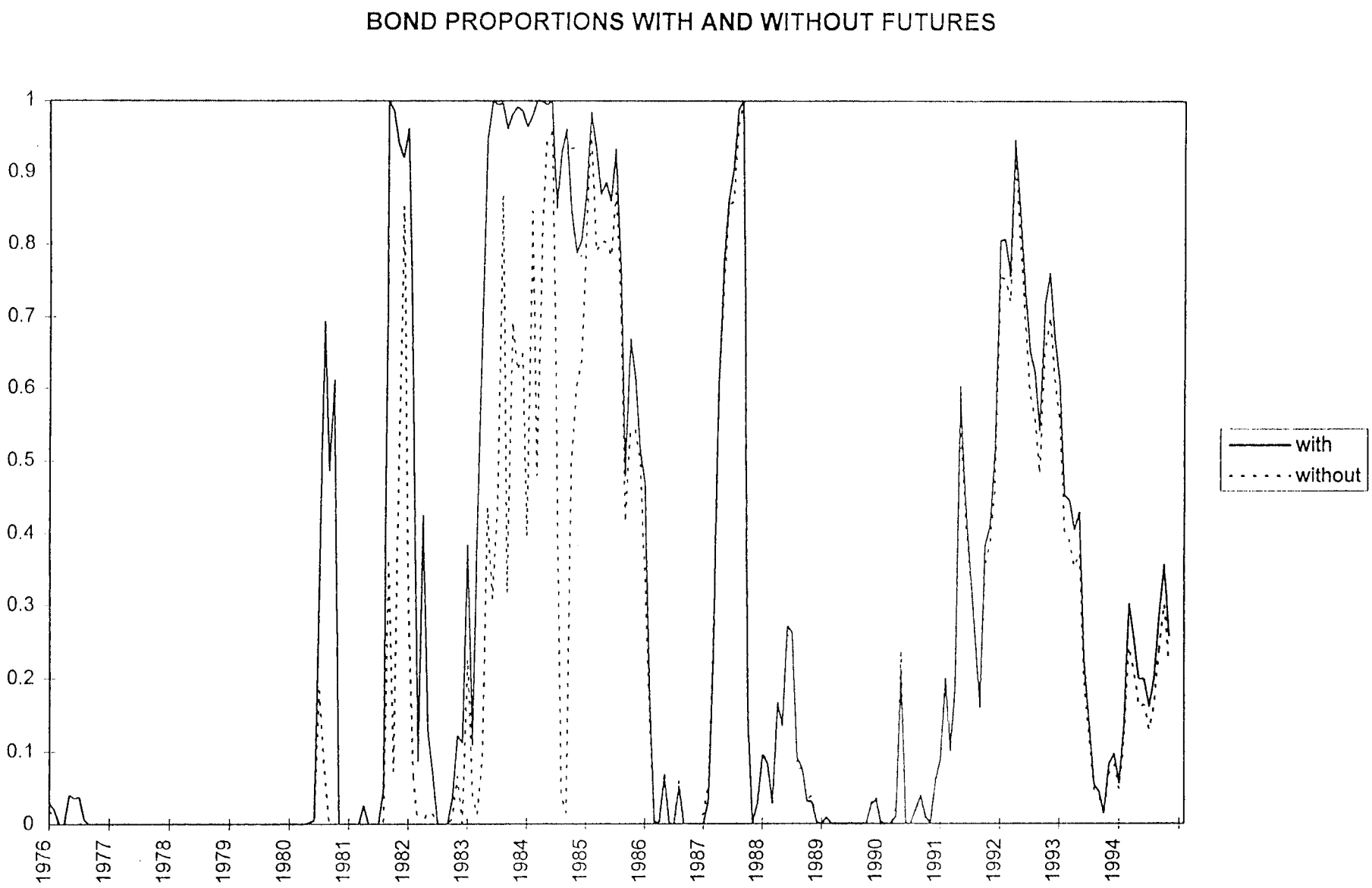
BOND PROPORTIONS WITH AND WITHOUT FUTURES
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1976
1977
1978
1979
1980
1981
1982
1983
1984
1985
1986
1987
1988
1989
1990
1991
1992
1993
1994
with
without

Figure 4d

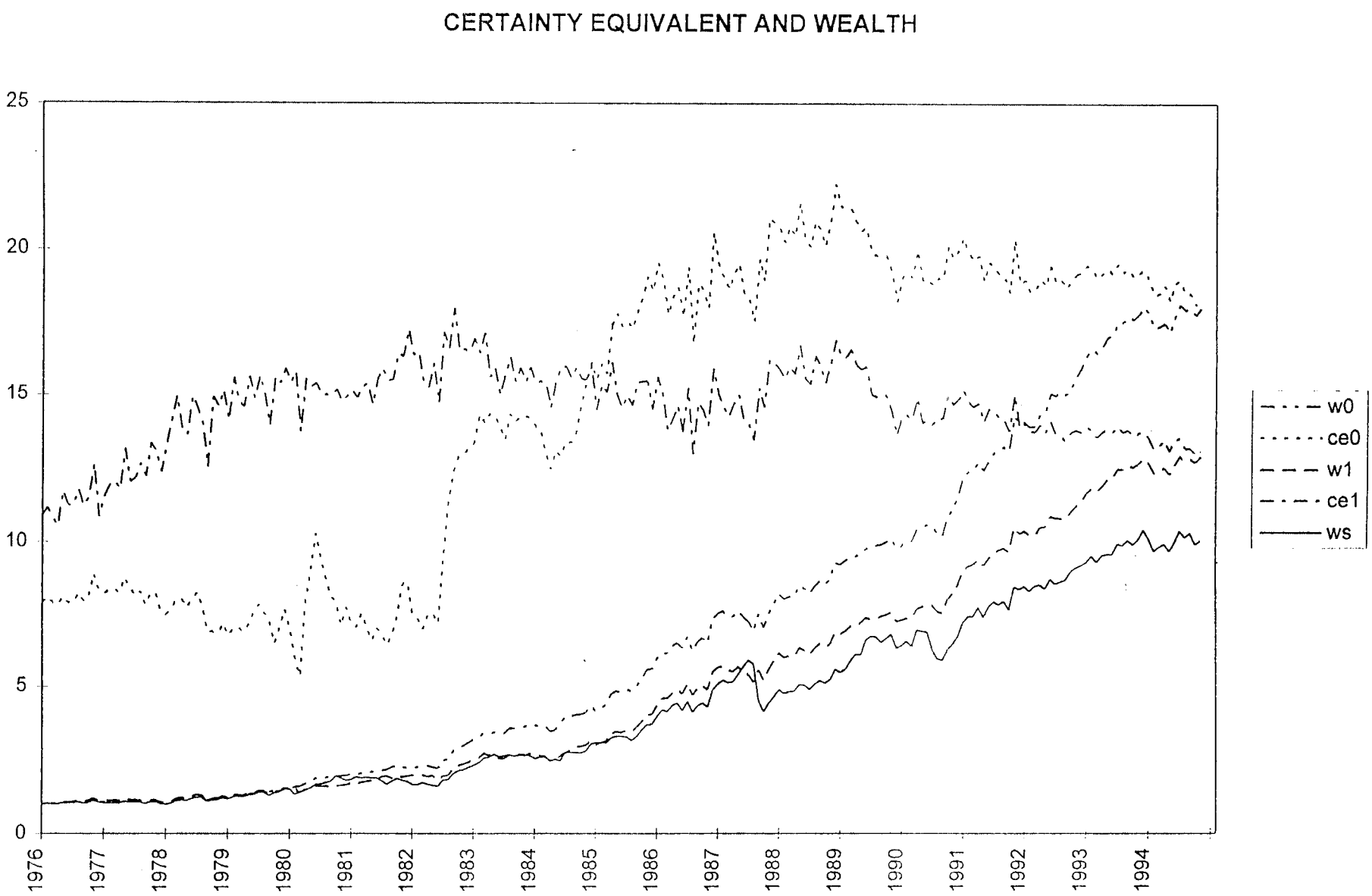
CERTAINTY EQUIVALENT AND WEALTH
25
20
15
10
5
0
1976
1977
1978
1979
1980
1981
1982
1983
1984
1985
1986
1987
1988
1989
1990
1991
1992
1993
1994
w0
ce0
w1
ce1
ws

Figure 5

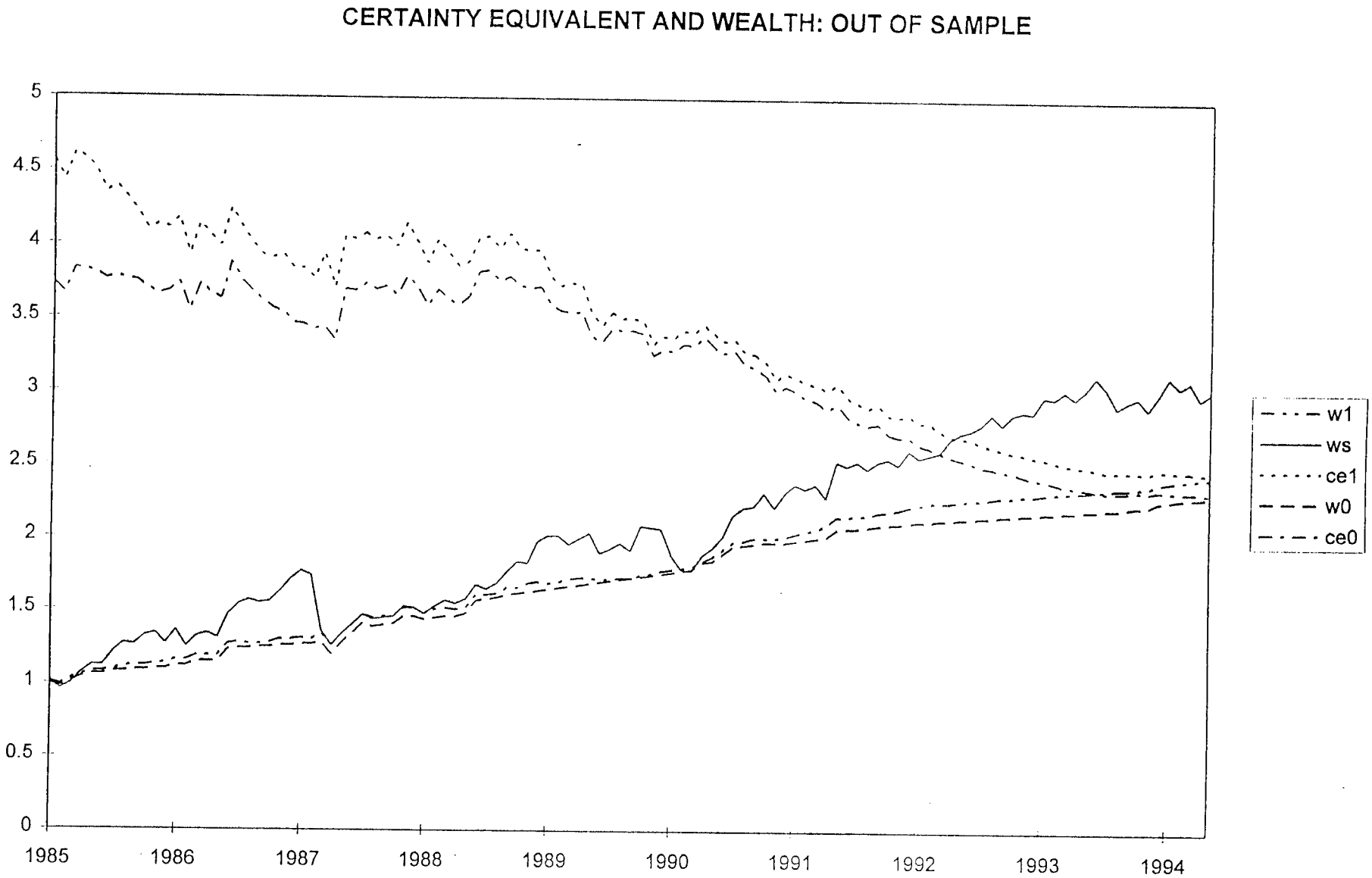
CERTAINTY EQUIVALENT AND WEALTH: OUT OF SAMPLE
5
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
1985
1986
1987
1988
1989
1990
1991
1992
1993
1994
w1
ws
ce1
w0
ce0

Figure 6

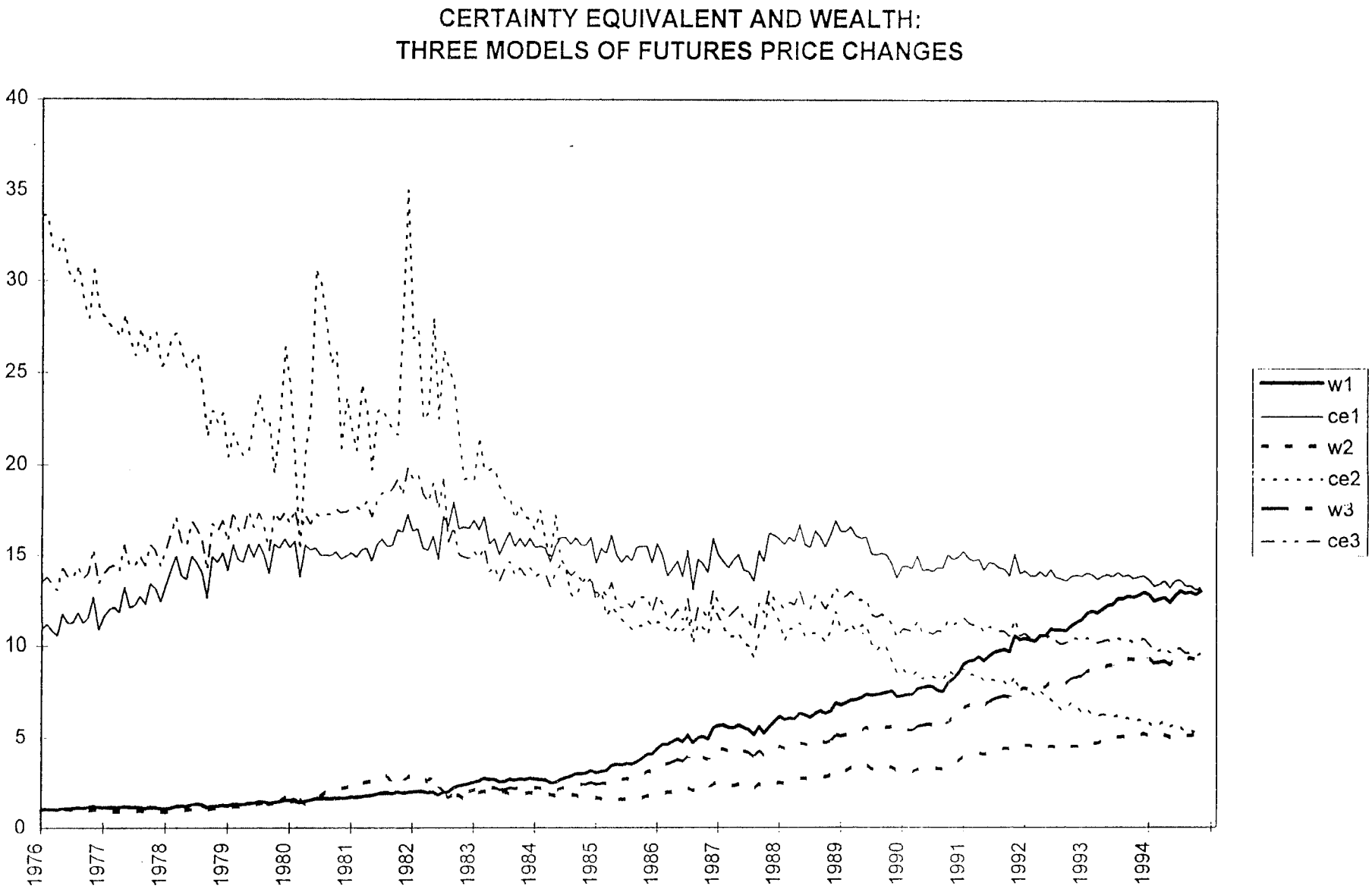
CERTAINTY EQUIVALENT AND WEALTH:
THREE MODELS OF FUTURES PRICE CHANGES
40
35
30
25
20
15
10
5
0
1976
1977
1978
1979
1980
1981
1982
1983
1984
1985
1986
1987
1988
1989
1990
1991
1992
1993
1994
w1
ce1
w2
ce2
w3
ce3

Figure 7

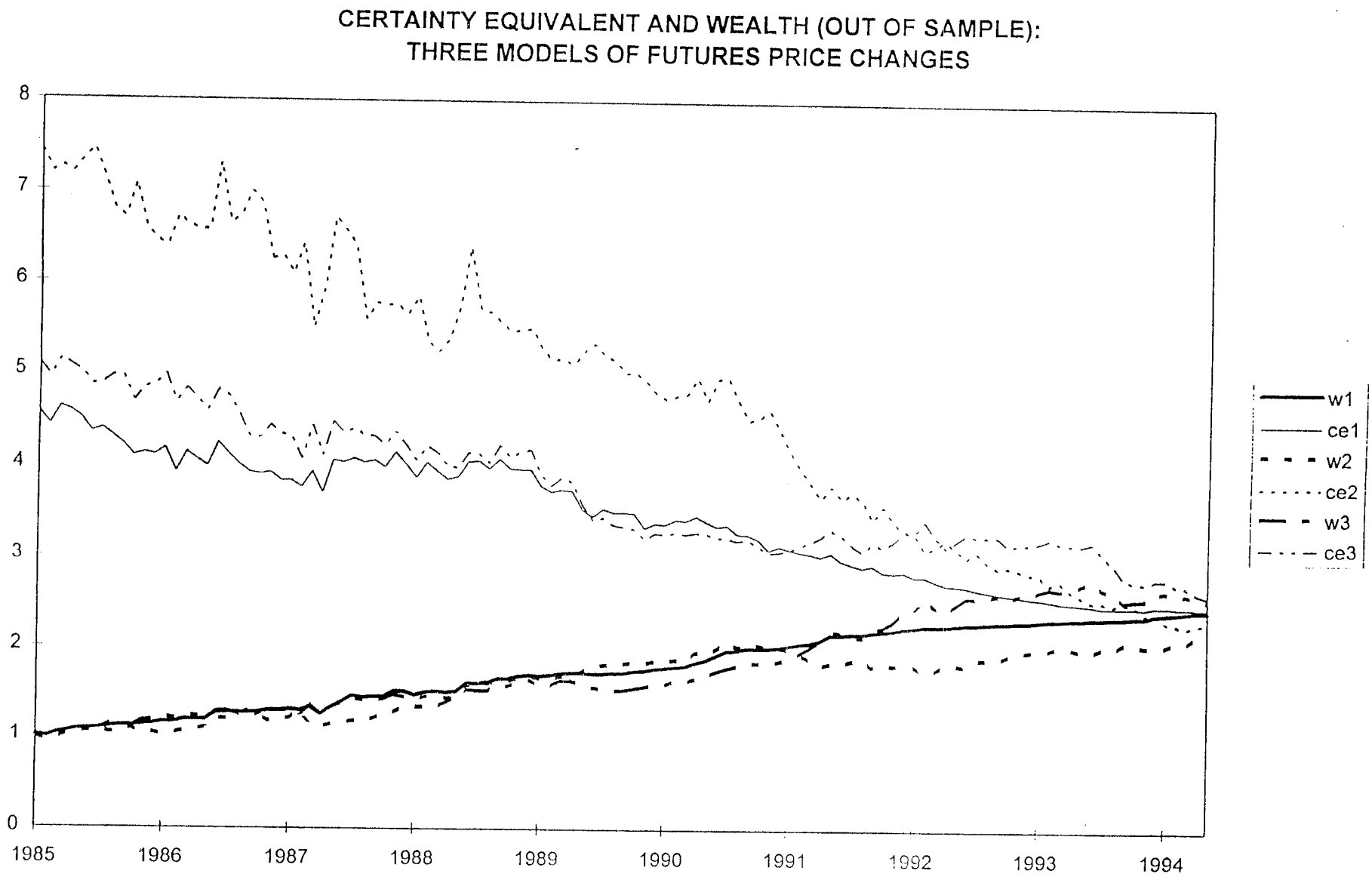
CERTAINTY EQUIVALENT AND WEALTH (OUT OF SAMPLE):
THREE MODELS OF FUTURES PRICE CHANGES
8
7
6
5
4
3
2
1
0
1985
1986
1987
1988
1989
1990
1991
1992
1993
1994
w1
ce1
w2
ce2
w3
ce3

Figure 8