Nominal Interest Rates and Loan Volume with Heterogeneous Beliefs

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Abstract

The large volume of loan transactions suggests widely divergent beliefs among borrowers and lenders; But most modern term structure theories make no prediction about volume because they assume homogeneous agents. Within these rational expectations, representative agent theories, loans are not only in zero **net** supply; they are also in zero **gross** supply.

Here, the shape of the average term structure, the bias in forward interest rates, and the volume of transactions are discussed in a heuristic setting with heterogeneity and incomplete rationality. A reconciliation of heterogeneity with continuous-time term structure theory is attempted, but the results are not encouraging. Empirical data are consistent with the importance of heterogeneous beliefs.

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I. Introduction

Beliefs about future real interest rates and future inflation, possibly modulated by risk premiums, underlie the maturity pattern of nominal yields observed at a given moment. As beliefs or preferences change over time, the term structure assumes various typical shapes, often upward-sloping, sometimes downward-sloping, occasionally non-monotonic.

Controversy surrounds the identification and relative importance of term structure determinants; even the term structure's average shape is debated. Beliefs are by nature transient, and time-dependent variables do not wander off to infinity or zero, so risk premiums are prime candidates for explaining systematic differences among yields of various maturities, i.e., the average shape. Not that risk premiums themselves are immutable. They too could fluctuate unpredictably or respond to diverse conditions (such as changing volatility), but unlike other determinants, risk premiums could also be stable and maturity dependent.

The term structure literature of an earlier era is represented by scholars such as Fisher [1930], Hicks [1939], and Lutz [1940-41], who imagined a loan market populated by heterogeneous agents with varying time preferences, risk tolerances, and divergent beliefs. In this tradition, Modigliani and Sutch [1966, 1967], explicitly related heterogeneity to the "preferred maturity habitat" of an individual borrower or lender, determined by an idiosyncratic lifetime pattern of consumption and investment. The term structure on a given date was merely a set of market-clearing prices; the volume of loan transactions depended on the extent of heterogeneity. Infra-marginal lenders (borrowers) expected future yields to be lower (higher); volume was a positive function of this disagreement.

These early term structure ideas were sumptuous in their generality, but the generality had a significant cost: virtually unmanageable complexity and a resulting failure to obtain a tractable analytic description of the term structure. Empirical implications were necessarily vague since borrower/lender heterogeneity could assume virtually any form.

Since the appearance of continuous-time methods in finance, term structure literature has embarked on a different tack. Recent theory is mathematically elegant, is usually based on rational expectations or the absence of arbitrage, and often provides closed-form algebraic depictions of the yield curve.

However, modern models have been purchased also at a high price: the troubling assumption of homogeneity across borrowers and lenders, commonly-held beliefs, and frequently, identical risk preferences. The well-known and wholly-admirable paper of Cox, Ingersoll, and Ross [1985, p. 387] is typical, "The economy is composed of identical individuals, each of whom seeks to maximize..." the expected utility of consumption flows over a finite period. Since the individuals are identical, their expectations (and other probability parameters) are identical, as are their utility functions and horizons.

A similar assumption is expressed in Vasicek [1977, p. 179] and Longstaff and Schwartz [1992, p. 1262]. In other papers, there is no explicit statement, but homogeneity of beliefs is implicitly assumed because a stochastic process is exogenously specified; See Richard [1978], Brennan and Schwartz [1979], Hull and White [1990], Heath, Jarrow, and Morton [1992]. The form of the process is common knowledge and its parameters are known by everyone.

Homogeneity is a particularly disquieting assumption for the loan market. The aggregate quantity of loans is always in zero net supply; there is a lender for every borrower. Consequently, if the volume of lending is non-zero, which it obviously is in reality, some agents must be behaving unlike some other agents. A collection of identical agents cannot depict this reality, for then the quantity of loans would also be in zero gross supply! Actual lenders and borrowers must have heterogeneous preferences or divergent beliefs or both.

Indeed, some writers have even seen heterogeneity as the fundamental *raison d'être* of markets; e.g., Chichilnisky [1994] says,

> People trade because they are different. Gains from trade...depend naturally on the diversity of the traders' preferences and endowments. The market owes its existence to the diversity of those who make up the economy, (p. 427.)

Harrison and Kreps [1978] base their analysis of speculative behavior squarely on a "key element...the existence of heterogeneous expectations within the community of investors," (p. 324.) Their investors are risk neutral, so to preclude infinitely large speculative positions, they also impose a ban on short selling. These assumptions allow them to construct examples in which investors pay more for an asset than they think it is really worth, intending to unload it later at an even higher price. Significant trading volume is an implication of this speculation. More recently, in research about equity markets and in macroeconomics, there has been much consideration of heterogenous agents; (see Bonfim [1996], Campbell, Grossman and Wang [1993], Constantinides and Duffie [1996], or Den Haan [1996] for interesting examples and further references.)

In contrast, most modern theories of the term structure are utterly silent about the volume of lending and the absolute quantity of outstanding loans. New loan originations, secondary market transactions of existing loans, and outstanding loan aggregates all change dramatically over time. Yet most recent theories do not even mention loan transactions.

Bond prices exist in a vacuum; analytic price functions are obtained, and prices vary intertemporally, but there is no trading.

Mentioning this in casual conversation often elicits a rejoinder that a phantom agent "represents" the amalgam of individuals. Allegedly, bond prices would be the same in

a market actually inhabited by the representative agent. (Cf. Rubinstein [1976]). Perhaps so, but there is something left unexplained, the large volume of loans observed in many markets[1]. A representative agent model cannot depict the **persistent** heterogeneity implied by this undeniable phenomenon.

An absence of trading volume is a hallmark of **all** theories based on rational expectations, not just the modern term structure theories. There is a deep sense in which information alone cannot elicit transacting, even if it is asymmetric among agents. Aumann [1976] states this idea in his usual succinct and graceful style,

> If two people have the same priors, and their posteriors for a given event are common knowledge, then these posteriors must be equal. This is so even though they may base their posteriors on quite different information, (p. 1236.)

By "common knowledge," Aumann means that each person knows the other's posterior, knows that the other knows his posterior, knows that the other knows that he knows the other's posterior,... Even though they have received "quite different information," they will eventually agree that it is unreasonable to disagree! After thinking over the other's initial opinion, the preliminary (common knowledge) posterior just after receiving information, each agent will revise his own posterior, observe his compatriot's revision, and revise again, etc. The only possible termination is perfect agreement.

Agreement eliminates information-motivated trading, entirely. As stated by Tirole [1982],

> ...rational and risk averse traders [with common priors] never trade solely on the basis of differences in information, (p. 1164.).

As a less-startling corollary, Tirole shows that price bubbles cannot exist in a fully dynamic rational expectations equilibrium.

Similarly, Milgrom and Stokey [1982] consider an initial complete-market equilibrium followed by the receipt of private information. They prove that every trader, including anyone who happens to receive information, is indifferent to further trading. Moreover, the information received from the observed market price change (which occurs with no volume) outweighs the private signal completely.

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[1]In most countries the market value of outstanding debt exceeds the capitalization of the stock market; Cf. Solnik [1996], Exhibit 6.1, p. 168 and Exhibit 8.1, p. 288, for a list of the major countries. In aggregate, just government-issued debt amounts to about $16.3 trillion while equity amounts to $12.6 trillion. Only a few countries, (Australia and the U.K.), have larger equity than debt markets. Also, trading volume is often higher in debt markets.

> "...Each agent's posterior beliefs given both price changes and private signal depend only on the price changes..." and "...the change in relative prices is a purely information phenomenon, i.e., the change is independent of traders' endowments, preferences, and prior beliefs." (p. 18).

Heterogeneous priors cannot be responsible for **persistent** trading volume, because learning something about the information possessed by others affects the posterior in a given period, the prior for the next period. This strongly suggests that trading volume will decline over time unless new and different agents with heterogeneous priors continually repopulate the market. Biais and Bossaerts [1995] present simulations which support this argument. Given a fixed set of agents, trading declines even when priors are extremely incompatible in the sense of not conforming to the Harsanyi [1967-68] specification[2]. It is hard to believe that new traders are in fact responsible for the absolute blizzard of originations and secondary trading observed daily in loan markets.

Thinking about "new" traders raises the question of whether heterogeneous beliefs or heterogenous preferences are the dominant cause of trading volume. In a single observation period, either type of heterogeneity would be observationally equivalent[3]. But it seems implausible that preferences fluctuate enough to explain the persistent, daily trading, representing a high turnover rate, actually observed in most financial markets. If heterogeneity of beliefs is indeed responsible, a major part of the future research agenda will be devoted to modelling how it arises and how it interacts with observable variables.

Some have suggested that incomplete rationality, (or "bounded" rationality,) or simply irrationality may be the underlying source of heterogenous beliefs and the explanation for the widely-observed trading in asset markets. For instance, Arthur [1994] argues that deductive reasoning has limited applicability to real-world problems,

> ...beyond a certain level of complexity human logical capacity ceases to cope - human rationality is bounded...(p. 406.)

Because of their computational limitations, agents must resort to more primitive methods, pattern recognition, rules of thumb, inductive reasoning. The "common knowledge"

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[2]Harsanyi specifies that each individual's prior **could** have been elicited from a single common "meta" prior updated with different information. Biais and Bossaerts give an example of a set of priors that could not have been produced in this manner.

[3]In a recent important contribution, Wang [1996] considers a loan market populated by two types of investors with distinct risk preferences, but with common probability beliefs. In a special case presented for its tractability, investors have either logarithmic or square root preferences. Wang shows that this leads to one class being lenders and the other class borrowers; total loan volume is derived endogenously.

assumption is abandoned and everyone must guess what everyone else is thinking and is likely to do. Such chaos is consistent with a persistently large volume of trading, though, of course, lots of trading does not prove that individuals are reacting to each other in this manner. It may prove possible, however, to model agents with bounded rationality and derive predictions about the interaction between the extent of heterogeneity and trading volume.

With bounded rationality, learning still occurs. Poor calculators though we humans are, eventually we should stumble toward better understanding, if only because individuals with weaker bounds are more prominently successful, (e.g., richer in financial markets), and can be mimicked. However, if the world is complicated enough, learning may take a long time, long enough so that even a trickle of newly-arriving agents with more dispersed priors is sufficient to sustain a significant volume of trading.

In the paper to follow, the next section (II) provides an intuitive discussion of heterogeneous borrowers and lenders, risk aversion, and loan volume. No claim is made that agents follow rational expectations. The purpose is to explore how heterogeneity can be manifest in real and nominal rates of return and in loan volume. Section III extends the intuitive discussion more formally by considering the influence of non-linearities in loan pricing, along with a preliminary investigation of preferences. Section IV presents an attempt to reconcile a modern continuous-time model with agent heterogeneity, using as an example the Cox-Ingersoll-Ross (CIR) model with stochastic inflation, while maintaining some semblance of agent rationality and the absence of obvious arbitrage. Section V presents a brief survey of associated empirical evidence without, however, being able to provide any definitive tests.

The focus throughout the paper is on the **average** shape of the nominal term structure, not on the particular form observed on a given date. This is tantamount to focussing on maturity-dependent risk premiums or, equivalently, on the biases in nominal forward interest rates as predictors of future spot yields properly adjusted for the statistical effects of non-linearities. More evanescent determinants of yield curves on particular dates, such as time-dependent consensus forecasts of future yields, are for the most part permitted to rest undisturbed.

II. A Heuristic Rationale for Maturity-Related Risk Premiums

II. A. The Market for Forward Loans.

Forward loans are the building blocks of term structure theory. A forward loan is a contract whose terms are negotiated today but whose cash flows begin at some future date. For example, one might wish to invest funds for one year starting two years from now. The interest rate on such a loan, the forward rate, can be set by mutual bargaining now between the borrower and lender, using a mechanism similar to the determination of a forward exchange rate or the price of a commodity for future delivery.

Alternatively, forward interest rates can be deduced from the prevailing term structure of zero-coupon bond yields. Lending for one year beginning two years hence is equivalent to lending now for three years and simultaneously borrowing the same amount for two years. The effective forward interest rate is thus determined by the current ("spot") yields for two- and three-year loans. Using continuous compounding, the forward interest rate is $F_{1,2}=3Y_3-2Y_2$, where Y_m denotes a continuously-compounded yield per annum for a m-year-to-maturity zero coupon bond[4].

In general, the forward interest rate negotiated now on a loan spanning k period, beginning after n periods, is related to <u>current</u> zero-coupon yields, (all continuously-compounded) by

$$F_{k,n} = [(n+k)Y_{n+k} - nY_n]/k. \tag{1}$$

A theory which explains the determination of forward rates for arbitrary k and n is a complete theory of the term structure of interest rates because <u>any</u> default-free bond can be constructed as a portfolio of forward loans. This section (II) develops a heuristic theory for a <u>given</u> but arbitrary k and n, i.e., a theory of the term structure's basic building block.

Agents can either borrow in lend forward at the rate $F_{k,n}$. If they borrow (lend) and do nothing else, they will receive (pay) cash at after n periods, when the "loan period" begins, and then pay (receive) cash after n+k periods, when the loan period ends. Such lenders know that they will have disposible funds after n periods but not require funds until after n+k periods while such borrowers know that they will receive funds after n+k

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[4]The present value PV of $100 after two years is $PV=100e^{-2Y_2}$; after three years, the loan payment would be PVe^{3Y_3}. The investment of $100 during the third year, but contracted now, has a continuously-compounded rate of return

$$F_{1,2}=\ln_e[(PV)e^{3Y_3}/100]=\ln_e[(PV)e^{3Y_3}/(PV)e^{2Y_2}]=3Y_3-2Y_2.$$

periods but require funds earlier, after n periods.

Other agents need not be certain about their demand or supply of disposible funds during the future loan period. If they borrow or lend in the forward market now, they can partially or fully reverse the transaction by lending or borrowing later. Any net **expected** reversal could be considered speculation; for instance, an agent with absolutely no intention of having an open loan during the future loan period might nonetheless borrow or lend forward, anticipating a completely offsetting quantity of lending or borrowing later. In reality, there may be some individuals who actually plan to borrow or lend during a given future loan period, but others with no spot loan demand can participate in forward lending activity. The latter correspond to speculators in futures markets, who always intend to close out their positions prior to expiration, while the former are akin to hedgers who intend to take or make delivery.

Currently-observed forward interest rates must be related somehow to aggregated **anticipations** about future spot yields to prevail when the loan period arrives. We know this by analogy to the situation under certainty; If the future spot yield were known for sure, it must equal the forward rate to preclude profitable and costless arbitrage.

Admitting uncertainty, consider the example of a hypothetical investor who expects to make a k-period loan beginning after n periods. This illustrative individual expects to receive cash after n periods and has already decided to spend it only after k additional periods. He faces a **timing** decision: either (a) negotiate a forward loan now and lock in the interest rate, $F_{k,n}$ or (b) wait. Waiting can last up to n periods, at which point lending for k periods earns the <u>then</u>-prevailing k-period spot yield, $Y_{k,t+n}$ (the current period is t.)

There is a fundamental difference between contracting forward and waiting. The forward rate is known now, in period t, but the future spot yield is uncertain; it will be known for sure only after n periods have passed. If the investor were indifferent to risk, we might be tempted to predict that he would lend at the forward rate if it exceeds what he currently (in t) expects the future spot yield to be; that is, if $F_{k,n} > E_t[\tilde{Y}_{k,t+n}]$; (the ~ indicates that $Y_{k,t+n}$ is uncertain at time t.) Conversely, he would wait and lend later in the spot market if he expects the future yield to be higher than the current forward rate[5]. If the investor were risk averse, one might be tempted to predict a decision to lend forward even at a forward rate somewhat lower than the spot rate he <u>expects</u> to prevail at the beginning of the future loan period.

Every market participant, whether risk averse or not, will make a timing decision about the forward loan based on a comparison of his own expectation about the future spot rate

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[5]Later, we shall see that the decision to contract now or wait may depend on a more complex condition than that algebraic difference between the forward rate and the expected future spot yield, even for a risk neutral agent.

with the observed forward rate. These simultaneous deliberations by **all** agents determine equilibrium in the forward market for k-period loans n periods hence.

If everyone were risk neutral, an equilibrium between the demand and supply of forward loans would be achieved when the forward interest rate equaled the market's consensus expected future spot yield in the following sense: Borrowers (Lenders) who expect the future yield to be lower (higher) than the currently-observed forward rate would not transact in the forward market, but would wait to borrow or lend in the future spot market. Thus, the market clearing forward interest rate would satisfy $F_{k,n}=E_t[\tilde{Y}_{k,t+n}]$ where the expectation is that of the marginal **forward** borrower and lender, the individual borrower (lender) with the lowest (highest) expected future yield among all those who engaged in a forward transaction.

With risk neutrality, speculators would attempt to undertake indefinitely large positions, individuals with low expected future spot yields lending in the forward market to those with high expected future spot yields. A speculative forward lender, for instance, would expect to profit by borrowing later at a spot yield that he now expects to be lower than the current market forward rate; vice versa for a speculative forward borrower.

A limit on such activity is imposed by the prospect of default. Remember that a forward lender creates a position by lending for a longer period, say n+k, and borrowing for a shorter period, n. After n periods, he must come up with cash. Even if he has no resources, he <u>will</u> have collateral after n periods, the previously-negotiated longer loan whose remaining term is now k. However, if his expectation about the spot yield turns out to be too low, the collateral will have insufficient market value; he will forced to default on the short maturing leg of his forward loan. The original counter-party to the forward loan will realize that default can occur and will, therefore, charge the speculator a higher rate on the borrowing side of the forward loan; (note that this charge will be imposed even with risk neutrality.) The charge for default decreases the implied net forward interest rate and renders the speculative position less attractive.

A symmetric consideration limits the quantity of speculative borrowing. Due to possible default, a higher interest rate will be imposed on the longer side of the forward loan, thereby increasing the forward rate[6].

With the more realistic assumption of risk averse agents, few agents would risk absolute

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[6]The governor represented by default can perhaps be more lucidly imagined by considering a speculative lender making a forward loan to a speculative borrower. Both are entering the forward contract because the forward rate, now observed by both, differs from the future spot yield each expects; the lender anticipates that the future spot yield will be lower than the forward rate while the borrower anticipates the opposite. One of them must be wrong. If he has no other resources, he will default.

ruin; but the possibility of default would still prompt counter-parties to charge default premiums that essentially personalize forward rates to particular individual speculators.

Whatever the degree and diversity of risk aversion among agents, the spot/forward timing decision has paramount importance. The forward rate's relation to the expected future spot yield is not directly influenced by aggregated expected lending and borrowing in the future loan period, i.e., by the "preferred habitat" of borrowers or lenders[7]. If all anticipated borrowers (or lenders) were to experience a proportional change in available funds, **but no change in beliefs**, both the forward and expected future spot rates would shift to a new level, but their difference would not necessarily be affected. For example, if many individuals who formerly expected to lend in the future loan period decided they needed to borrow instead, the expected future spot rate and the forward rate would both increase, leaving their difference unaltered so long as the abdicating group had representative beliefs. This implies that the shape of the average yield curve depends primarily on the forward versus spot timing decision.

II. B. Why should there be risk premiums?

If borrowers and lenders were sensitive to uncertainty in **nominal** rates of returns, both would deem it safer to lock in a forward interest rate rather than to wait for the future period and its then-prevailing spot yield. That future yield is, of course, uncertain when the forward rate is determined.

Since the future spot yield is risky, a risk averse lender would wait and lend in the spot market, rather than contract in the safer forward market, only if he expects the spot yield to be somewhat larger, i.e., if $E_t[\tilde{Y}_{k,t+n}] > F_{k,n}$; (the expectation here is the private personal belief of the risk averse lender.) This suggests that forward rates might be downward biased predictors of future yields whenever the attitudes of risk averse lenders predominate.

However, a risk averse borrower would have a different view. He too would consider the forward loan contract safer than waiting and borrowing at whatever might be the future spot yield. This implies that the borrower would be willing to pay a higher interest rate to lock in the forward loan, i.e., $E_t[\tilde{Y}_{k,t+n}] < F_{k,n}$ (given his expectation of future yields.) Thus, if borrower preferences dominate, forward rates would be biased high![8]

Given an ensemble of borrowers and lenders, each one expecting a different future spot nominal yield, risk aversion should increase the volume of forward loan contracts

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[7]Cf. Modigliani and Sutch [1966, 1967].

[8]Perhaps this accounts to some extent for the widespread assumption that forward rates are in fact unbiased. The risk aversions of borrowers and lenders do work in opposite directions on the bias. They could be exactly offsetting, a fortuitous outcome!

undertaken relative to the volume that would have been undertaken in the absence of risk aversion. With no risk aversion, an equilibrium forward rate would induce only about half the potential lenders and borrowers (weighted by their respective levels of loan demand) to contract forward. But with risk aversion, some additional lenders, those whose expected future yields slightly higher than the forward rate, could be coaxed into forward lending; similarly, some additional borrowers, those with slightly lower expected future yields, would supply those loans.

If risk aversion were widespread and market participants were concerned with nominal returns, most loans would be negotiated in the forward market. Most bond originations would be at long maturities. Few loans would be originated at short maturities; most short-maturity bonds would be aged bonds that originally had longer terms. The fact that short-maturity bonds are commonly originated in the loan markets of many countries hints that some assumption above is probably not valid. The most likely culprit is the assumption that all borrowers and lenders are interested in nominal returns.

II. C. Risk Aversion and Real Returns[9]

Some investors, such as life insurance companies, may focus on nominal returns because their liabilities are nominal, but many lenders and borrowers are much more interested in real returns. If we assume that real return-motivated agents dominate, the nominal term structure's average shape will be subject to the alternative considerations described in this section.

An *ex post* real return is defined as the nominal yield Y less the actual inflation rate, I, observed over the loan period, $r=Y-I$. An investor foregoes current consumption hoping for greater future consumption. Similarly, a borrower secures more consumption now. Although a borrower promises to pay back monetary units, he is anxious about the future consumption that must be foregone to arrange that payment.

Consider again the decision to either contract in the forward loan market, at a fixed nominal forward interest rate, or wait for the future nominal yield in the spot loan market. The *ex post* real rate of return earned on the forward loan will be $\tilde{r}_F = F_{k,n}-\tilde{I}_k$, where $\tilde{I}_k$ is the actual inflation over the future loan interval stretching over the k periods after t+n (the current period is t), and the ~ indicates that inflation is uncertain now, in period t. Although the forward interest rate is known at t and is therefore perfectly certain in nominal terms, it is risky in real terms because inflation will not become known before the end of the future loan period. (Table 1 presents a time line indicating when various quantities become known.)

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[9]This section draws heavily on the paper by Cornell [1978].

The corresponding real return obtained from the future spot yield is $\tilde{r}_Y = \tilde{Y}_{k,t+n} - \tilde{I}_k$. It is affected by the same risky inflation as the forward loan, but there is additional uncertainty arising from the nominal spot yield itself, which is not known when the decision is made (at t) to either contract forward or wait.

How would a risk averse lender or borrower compare these alternative real returns? Which has the greater risk? To gain insight about this, let us measure the volatilities of the two real returns by computing their statistical variances on the decision date t and decomposing these volatilities into their underlying constituents.

An *ex ante* real yield is the expected real return implied by the nominal spot yield on the date when the nominal spot yield becomes known, (in our framework, at period t+n); i.e., $y_{k,t+n} \equiv Y_{k,t+n} - E_{t+n}(\tilde{I}_k)$. It is the expected value of the *ex post* real return, $E_{t+n}[\tilde{r}_Y]$, but observed at t+n rather than at t[10]. The *ex ante* real yield might contain an embedded risk premium. Its variability over time could be caused by factors influencing riskless real returns, (such as demographics or savings propensities), or by factors relating to perceived risk, such as inflation volatility.

Actual inflation observed ultimately at t+n+k will be equal to inflation expected at the beginning of the future loan period, $E_{t+n}(\tilde{I}_k)$ plus $\tilde{\epsilon}$, a random prediction error. If expectations are being formed rationally, the prediction error should be completely unforecastable and thus uncorrelated with all other variables observed earlier.

Combining these terms, the observed *ex post* return from contracting at the spot yield can be expressed as

$$\tilde{r}_Y = \tilde{Y}_{k,t+n} - \tilde{I}_k = \tilde{y}_{k,t+n} + \tilde{E}_{t+n}(\tilde{I}_k) - [\tilde{E}_{t+n}(\tilde{I}_k) + \tilde{\epsilon}] = \tilde{y}_{k,t+n} - \tilde{\epsilon}, \tag{4}$$

i.e., the *ex post* real return from spot lending is simply the *ex ante* real yield less the unforecastable inflation prediction error.

The real return from nominal forward contracting (earlier, at t) depends also on inflation at the beginning of the future loan period,

$$\tilde{r}_F = F_{k,n} - \tilde{I}_k = F_{k,n} - [\tilde{E}_{t+n}(\tilde{I}_k) + \tilde{\epsilon}]. \tag{5}$$

Comparing the volatility of the two real returns at decision time t,

$$Var_t(\tilde{r}_F) - Var_t(\tilde{r}_Y) = Var_t[\tilde{E}_{t+n}(\tilde{I}_k)] - Var_t(\tilde{y}_{k,t+n}), \tag{6}$$

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[10]Note that $E_{t+n}(\tilde{I}_k)$ and $E_{t+n}(\tilde{r}_Y)$ are evaluated as of date t+n; the expectations are themselves random variables at the decision date t.

where "Var_t" denotes the statistical variance assessed on the decision date, t.

The result is remarkably simple: the relative risk of contracting in the forward loan market versus waiting for the spot loan market depends entirely on the volatility of expected inflation relative to the volatility of *ex ante* real yields. In most periods and in most countries, market folklore holds that expected inflation is more volatile than *ex ante* real yields. If this is correct, it is safer for both borrowers and lenders to wait and contract in the spot market rather than participate in the forward market[11].

Actually, the sign of expression (6) is an empirical question which is complicated by the fact that neither *ex ante* real yields nor expected inflation can be directly observed. We are obliged to rely on some indirect measure of their comparative volatilities. One approach would be to survey knowledgable market participants and ask for their inflation forecasts at the beginning of each loan period. To the extent that such responses are truthful, unbiased, and representative of the entire market, this would lead to a correct inference.

A second approach would be to deduce the information from public **observables**, from actual (i.e., *ex post*) inflation, nominal spot yields, and nominal forward rates. The variance of the real return $\tilde{r}_Y$ from a spot loan at the nominal yield $Y_{k,t+n}$ can be expressed in terms of observable quantities as

$$Var_t(\tilde{r}_Y) = Var_t(\tilde{Y}_{k,t+n}) + Var_t(\tilde{I}_k) - 2Cov_t(\tilde{I}_k, \tilde{Y}_{k,t+n}),$$

where "Cov_t" denotes the covariance between two random variables. These statistical variances and covariances must be measured at time t, when the decision is made to either contract in the forward market or postpone the loan decision until the later spot market. Of course, they might change over time.

Since the forward rate $F_{k,n}$ is known at decision time t, the volatility of the real return from forward lending is simply the volatility of the actual future inflation rate. Consequently, in terms of observables,

$$Var_t(\tilde{r}_F) - Var_t(\tilde{r}_Y) = Var_t(\tilde{I}_k) - [Var_t(\tilde{Y}_{k,t+n}) + Var_t(\tilde{I}_k) - 2Cov_t(\tilde{I}_k, \tilde{Y}_{k,t+n})].$$

The variances of actual inflation cancel; so

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[11]In a portfolio context, it could be asked whether total volatility really represents risk to a well-diversified agent. In general, it does not. However, the real interest rate and expected inflation are often pre-specified as non-diversifiable sources of risk whose volatilities do matter. Following Chen, Roll, and Ross [1986] these two variables have often been used in the equity asset pricing literature as non-diversifiable risk factors.

$$\mathrm{Var}_t(\tilde{r}_F)-\mathrm{Var}_t(\tilde{r}_Y) = 2\mathrm{Cov}_t(\tilde{I}_k, \tilde{Y}_{k,t+n})-\mathrm{Var}_t(\tilde{Y}_{k,t+n}). \tag{7}$$

Thus, the difference in volatilities between our unobservable objects of enquiry, the volatilities of expected inflation and *of ex ante* real yields, can be statistically estimated from the variance and covariance of the observables in expression (7).

The relative riskiness of spot and forward lending can also be assessed with these statistical estimates; it is more risky to contract in the forward market, compared to waiting and contracting in the future spot loan market, whenever the following rather curious condition is valid:

$$\mathrm{Cov}_t(\tilde{I}_k, \tilde{Y}_{k,t+n})/\mathrm{Var}_t(\tilde{Y}_{k,t+n}) > ½.$$

This is equivalent to

$$\rho_{Y,I} > ½[\sigma_Y/\sigma_I], \tag{8}$$

where $\rho_{Y,I}$ is the correlation between the nominal yield and the inflation rate and σ_j denotes the standard deviation of variable j, all parameters being assessed on decision date t.

According to this result, the real riskiness of undertaking a forward loan relative to waiting for the future spot market is determined by a combination of three factors, the volatility of the nominal spot yield, the volatility of inflation, and the correlation of inflation and nominal yields. If nominal spot yields were not correlated at all with inflation, the inequality in (8) could not hold; it would then be safer to contract in the forward loan market (for both lender and borrower.) Conversely, if nominal yields are highly correlated with inflation, the inequality in (8) might be valid; then it would be safer to wait and undertake loans in the spot market.

The actual relative risk is an empirical question which can be answered by estimating a regression of inflation on nominal yields,

$$\tilde{I}_k = a_t + b_t\ \tilde{Y}_{k,t+n}, \tag{9}$$

where a_t and b_t are the regression intercept and slope, respectively. If b_t is less (greater) than ½, it is safer to contract forward (wait for the spot market.) The econometric problem is that the coefficients in this regression are possibly time-varying because the relative volatilities of expected inflation and *ex ante* real yields need not be stationary. In the empirical section of this paper, this difficulty will be investigated. For now, we just remark that a simple linear regression should provide the right answer during any sub-period when the volatilities of expected inflation and real yields are stationary.

It may be interesting to ascertain the covariability of the *ex ante* real yield, y, and expected inflation E(I). Higher inflation has been associated with more variable inflation; See the strong cross-country connection documented Vogel [1974]. Hence, the risk

premium embedded in the *ex ante* real yield might rise with expected inflation, thereby producing positive covariability between the *ex ante* real yield and the expected inflation rate. Indeed, this might be a major reason why there is any intertemporal variation at all in *ex ante* real yields. However, as shown above, co-movement of expected inflation and the *ex ante* real yield does not influence the relative riskiness of forward and spot loan contracting, aside from of the direct impact of expected inflation volatility on *ex ante* real yield volatility.

II. D. The Volume of Forward Contracting and the Bias in Forward Rates with Risk Aversion When Expected Inflation is more Volatile than *Ex Ante* Real Yields.

If expected inflation is more variable than *ex ante* real interest yields, risk aversion has a negative impact on the volume of loans undertaken in the forward market. Risk averse lenders and borrowers will consider it safer to wait. Figure 1 illustrates this phenomenon. The left side of the Figure depicts the diversity of opinion, the cross-sectional distributions of future spot yields expected, on date t, by potential lenders and borrowers[12]. There would exist such a distribution whenever individuals disagree about the future, which might be at least some of the time! When there is no aversion to risk, the forward loan market reaches an equilibrium when the quantity demanded of forward loans by lenders whose expectations are less than the forward rate just matches the quantity supplied by borrowers whose expectations exceed the forward rate. The shaded regions, representing the volume of forward lending and borrowing, are equal in size.

When individuals are risk averse and expected inflation is more volatile than *ex ante* real yields, lender i can be enticed into the forward market only if $F_{k,n}=E_{i,t}[\tilde{Y}_{k,t+n}]+\phi_i$, where $\phi_i>0$ is that particular lender's risk premium added on to his personal expectation. Similarly, borrower j can be induced into the forward market only if he expects the interest cost to be lower, $F_{k,n}=E_{j,t}[\tilde{Y}_{k,t+n}]-\phi_j$, with $\phi_j>0$.

The right side of Figure 1 depicts the situation with risk aversion. It plots the cross-sectional distributions of expected future yields plus or minus risk premiums for lenders and borrowers respectively. Since risk premiums are strictly positive, the lenders' distribution has been shifted to the right and the borrowers' distribution to the left. The

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[12]Technically, the cross-sectional distributions in Figure 1 plot the quantities of financial resources of individuals possessing a given expected spot yield; the distributions do not depict the number of individuals with each expectation. Individuals within each distribution, though denoted "lender" and "borrower" in the Figure, are actually any agents who might potentially transact in the forward market, not just those who know for sure that they will be lenders or borrowers during the future loan period. Consequently, the same individual could be in both populations, as a forward lender if the forward rate is high and a borrower if it is low, relative to their personal expected spot rate.

equilibrium forward rate still determines the quantity of loans undertaken in the forward market. The volume of forward lending is less than it would have been without risk premiums; the shaded regions are unambiguously smaller[13].

If risk aversion is large enough or if inflation volatility is large relative to the volatility of *ex ante* real interest rates, we might expect that only a small volume of lending would occur in the forward market and most lending would be postponed until the spot market. Few long-term fixed-rate bonds would be originated. Long-term bonds would have to bear floating rates or else most borrowers and lenders would simply roll over short-term commitments[14].

When the market is populated entirely by risk averse private borrowers and lenders, the forward interest rate will not necessarily be a biased predictor of the future spot yield. We see from the figure that the forward rate overpredicts the yield expected by the marginal lender while it underpredicts the yield expected by the marginal borrower. If lenders as a group are more risk averse than borrowers, we might conclude that forward rates would be biased high, and vice versa if borrowers are more risk averse. We have no way of ascertaining whether there is such an asymmetric difference[15].

However, when a sovereign government adopts a policy of originating long-term fixed-rate bonds, it is behaving like a large risk neutral borrower in the sense that it auctions nominal instruments at whatever price the market will bear. To successfully sell its bonds to risk averse private lenders, it might very well be obliged to offer biased-high embedded forward rates and thus high yields. Moreover, the larger the volume of government borrowing relative to the natural size of the private market, the higher yields must be to entice cautious lenders into what they perceive (correctly) as risky positions. This implies that the yield curve will be upward sloped on average.

Remember that a long-term bond is equivalent to a short-term bond plus a sequence of

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[13]The non-shaded portions of the distributions indicate the <u>potential</u> volume of forward loans that might have been undertaken; these are either loans that will later be undertaken in the spot market or potential forward loans by speculators that will not be undertaken at all.

[14]A similar figure could illustrate the more unusual situation when expected inflation is <u>less</u> volatile than the *ex ante* real yield. The risk premiums would change signs; lenders (borrowers) would have negative (positive) values of ϕ. The distributions would be shifted to the left (right) for lenders (borrowers) and, of course, there would be a correspondingly greater volume of contracting in the forward loan market.

[15]Exactly this asymmetry, "a constitutional weakness" on the lending side of the market, was employed by Hicks [1939, pp. 146-147] to explain the forward rate's supposed upward bias.

forward loans. A thirty-year Treasury, for instance, has an embedded one-year fixed rate forward loan to begin in every one of the next 29 years! A private lender contemplating each separate forward contract might find it very risky indeed in real terms; yet he would be extending that forward loan as a component of the portfolio represented by the Treasury. Given the range of possible inflation rates 29 years in the future and the concommitant real risks, only nominal return-oriented lenders could find such a forward rate palatable at the rather modest historical premium indicated by the average term structure. It would be interesting to investigate the real vs. nominal orientation of typical lenders to the Treasury as a function of maturity; one might anticipate that real (nominal) return-oriented lenders would populate the short (long) end.

The forward market equilibrium is illustrated in Figure 2 for a market segment dominated by real return-oriented agents except for the government. Lenders as a group are more risk averse than borrowers, partly because the government is a large and relatively risk neutral borrower, so the lender distribution is shifted more to the right by risk premiums than the borrower distribution is shifted to the left. (The figure assumes that expected inflation is more volatile than *ex ante* real yields.) If borrowers and lenders have roughly similar views about future spot yields in that their cross-sectional means are approximately the same, then the forward rate will be biased high relative to their mean expectation.

For each forward rate embedded in a long-maturity, a risk averse lender will demand a risk premium. Thus, a Treasury yield, a weighted average of the forward rates, will imply a total interest expense that exceeds, by the weighted average of the risk premiums, the expected costs of rolling over short maturities. If risk premiums increase with the time until each forward loan begins, the yield curve will be upward-sloped on average and borrowing costs will increase with maturity. We should expect this result <u>if</u> the government borrows at long maturities at fixed nominal rates and <u>if</u> the desired quantity of private lending decreases with maturity.

The riskiness of forward lending should increase with the waiting time until the future loan period begins, whenever the volatility of expected inflation increases more than the volatility of *ex ante* future real yields; i.e., for a given decision period t, when $Var_t[\tilde{E}_{t+n}(\tilde{I}_k)]-Var_t(\tilde{y}_{k,t+n})$ increases with n. Average yield curves from many countries suggest just such a pattern. However, they suggest also that the <u>marginal</u> increase in relative risk <u>decreases</u> with n. The average yield curve is more steeply sloped upward near the short end and it flattens for maturities beyond, usually, about five years.

III. Non-Linearity in the Timing Decision to Either Contract Currently in the Forward Market or Wait to Contract in the Spot Market, the Discrete Case.

In the previous section, the decision to borrow or lend by contracting currently in the forward market was contrasted to waiting, through a comparison of the respective returns. Expectations and volatilities of interest rates, nominal and real, were the metrics of decision. However, interest rates are related non-linearly to wealth levels and wealth may itself be related non-linearly to the ultimate determinant of decision, the utility of consumption. This has important empirical implications. It probably affects the observed bias between forward interest rates and subsequent spot yields.

III. A. Risk Neutrality: Non-Linearity Induced by Jensen's Inequality

Risk neutral agents are motivated by expected real consumption. In discrete time, this implies that an agent who has decided to become a lender focuses on maximizing expected consumption at the end of the loan period while borrower decisions are driven by the maximization of expected consumption at the beginning of the loan period. Of course, in a more rigorous and less heuristic setting, the decision to either borrow or lend is endogenous, but for clarity we shall postpone that rigor[16].

A risk neutral lender makes a decision at date t to contract in the forward market, rather than wait to lend in the spot market, if his personally anticipated real terminal wealth is larger contracting forward; i.e., if $E_t[e^{\tilde{r}_F}] > E_t[e^{\tilde{r}_Y}]$, where $\tilde{r}_Y$ and $\tilde{r}_F$, are respectively, the *ex post* real returns from lending spot and forward, respectively; (See equations (4) and (5))[17].

Since the unexpected inflation component ($\tilde{\epsilon}$) of these real returns is unrelated to any other variable, it cancels from both sides of the above inequality, which is thereby reducible to

$$e^{F_{k,n}} E_t[e^{-\tilde{E}_{t+n}(\tilde{I}_k)}] \;>\; E_t[e^{\tilde{y}_{k,t+n}}].$$

As a reminder, $F_{k,n}$ is the currently (in t) observed forward rate for a future loan spanning

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[16]The intuition of this approach can be illustrated as follows: Suppose an agent receives incomes I_1 and I_2 at the beginning and end of the period, respectively; Consumption is denoted C_1 and C_2. The budget constraint is $C_2 - I_2 = e^r(I_1 - C_1)$, where r is the real return on the loan. For a lender, $I_1 > C_1$, and terminal consumption increases with e^r. For a borrower, $I_2 > C_2$, and $C_1 - I_1 = e^{-r}(I_2 - C_2)$, hence initial consumption increases with e^{-r}.

[17]This implicitly assumes that the amount of the future loan is determined exogenously.

k periods beginning at t+n, I_k is the actual inflation rate over the same loan period, and $y_{k,t+n}$ is the real interest rate[18] at the beginning of the loan period; the ~ denotes uncertainty at t.

Applying Jensen's inequality and the law of iterated expectations, there exists a non-negative constant[19], $\psi_E \geq 0$, such that the expectation on the left side of the inequality can be written as $E_t[e^{-\tilde{R}_{t+n}(\tilde{I}_k)}] \equiv [e^{-E_t(\tilde{I}_k)}][e^{\psi_E}]$. Similarly, $E_t[e^{\tilde{y}_{k,t+n}}] \equiv [e^{E_t(\tilde{y}_{k,t+n})}][e^{\psi_y}]$ with $\psi_y \geq 0$. Substituting into the inequality,

$$[e^{F_{k,n}}][e^{-E_t(\tilde{I}_k)}][e^{\psi_E}] > [e^{E_t(\tilde{y}_{k,t+n})}][e^{\psi_y}],$$

using the definition of the real *ex post* yield, $\tilde{y}_{k,t+n} \equiv \tilde{Y}_{k,t+n} - \tilde{I}_k$, and rearranging, we obtain,

$$F_{k,n} > E_t[\tilde{Y}_{k,t+n}] + (\psi_y - \psi_E). \tag{10}$$

The second term on the right side of this expression implies that a risk-neutral lenders's timing decision will not be determined solely by whether the forward rate is greater than his personally anticipated future nominal spot yield. For example, if expected inflation has larger volatility than real interest rates, the term $\psi_y - \psi_E$ will usually be negative. This implies that some lenders may choose to contract forward even though they expect future nominal yields to be higher than the forward rate (and, of course, **all** lenders who expect future yields to be lower than the forward rate will wish to contract forward.) This somewhat non-intuitive result is induced by the fact that expected terminal real wealth from forward lending increases with the volatility of expected inflation (!) as a consequence of Jensen's inequality.

In contrast to a lender, a risk neutral borrower seeks to maximize the real consumption value of a discount bond sold at the beginning of the future loan period. He will choose to contract in the forward market if he expects the real price of a discount bond sold forward to exceed the price of a bond sold later in the spot market, if $E_t[e^{-\tilde{f}_F}] > E_t[e^{-\tilde{f}_Y}]$.

Going through the same steps, the borrower's inequality can be reduced to

$$F_{k,n} < E_t[\tilde{Y}_{k,t+n}] - (\psi_y - \psi_E). \tag{11}$$

Comparing (10) and (11), the personal decision criteria for the risk neutral borrower and lender, respectively, we see that Jensen's inequality operates on the forward/spot comparison like a **negative** risk premium. The larger the volatility of expected inflation

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[18]By definition, $y_{k,t+n} \equiv Y_{k,t+n} - E_{t+n}(\tilde{I}_k)$, where Y is the nominal interest rate on a default-free loan at t+n.

[19]The constant ψ_E is non-negative since e^{-x} is a convex function of x. If x is a random variable with positive variability, ψ_E is strictly positive. For example, if x is normally distributed with standard deviation σ_x, then $\psi_x = \frac{1}{2}\sigma_x^2$.

relative to the volatility of real interest rates, the lower the forward rate acceptable to the lender and the greater the forward rate acceptable to the borrower (both relative to their respective personally-anticipated future spot nominal yields.) Thus, under risk neutrality, increasing inflation volatility, *ceteris paribus*, acting through Jensen's inequality, increases the volume of forward contracting.

III. B. Discrete-time Risk Aversion and the Bias in Forward Rates.

For expositional simplicity, we investigate risk aversion by examining a special class of utility functions defined with respect to a payment W, which can be either the real payoff (for a lender) or real cost (for a borrower) of a default-free loan over a given discrete loan period, from t+n to t+n+k[20]; specifically,

$$U(W)=W^{1-\lambda}/(1-\lambda), \text{ with } \lambda \geq 0.$$

There are two interesting special cases: for $\lambda=0$, the agent is risk neutral with respect to the real payment; for $\lambda=1$, $U(W)=\ln(W)$, the commonly-used logarithmic utility function defined over the real payment; (a non-negative λ excludes risk loving agents[21].) This real payment occurs at different times for a lender and a borrower to coincide with increasing consumption. For the lender, consumption increases with larger payoffs at the terminal date, t+n+k. For the borrower, consumption increases with larger proceeds at the initial loan period date, t+n. Timing decisions with respect to either contracting in the forward market or waiting to contract in the spot market are made by both parties at an earlier date, t.

A lender with this utility function will decide to contract in forward loan market, rather than wait for the spot loan market, if

$$E_t\{[e^{\bar{r}}F]^{(1-\lambda)}/(1-\lambda)\} > E_t\{[e^{\bar{r}}Y]^{(1-\lambda)}/(1-\lambda)\}. \tag{12}$$

Similarly, a borrower will contract in the forward market if

$$E_t\{[e^{-\bar{r}}F]^{(1-\lambda)}/(1-\lambda)\} > E_t\{[e^{-\bar{r}}Y]^{(1-\lambda)}/(1-\lambda)\}. \tag{13}$$

For the logarithmic case, $\lambda=1$, the lender's (borrower's) inequality simplifies readily to

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[20]A time line is given in Table 1.

[21]Since $-WU''/U'=\lambda$, this utility function is characterized by constant relative risk aversion.

$$F_{k,n} > (<) \; E_t[\tilde{Y}_{k,t+n}]. \tag{14}$$

Logarithmic lenders (borrowers) will transact in the forward market if the nominal forward interest rate is greater (less) than their individually-anticipated future spot nominal yield. Thus obtains a version of the "pure" expectations hypothesis in the sense that the marginal decision-maker, either lender or borrower, possesses an expectation about future nominal yields just equal to the market-determined forward rate. In this particular case of logarithmic utility, the non-linearity induced by Jensen's inequality is precisely offset by risk aversion; agents then base their loan timing decisions entirely on expected nominal interest rates.

For non-logarithmic cases, $\lambda \neq 1$, the inequalities reduce to

$$F_{k,n} > E_t[\tilde{Y}_{k,t+n}] \pm [\psi_y(\lambda)-\psi_E(\lambda)]/(1-\lambda), \tag{15}$$

with the second term on the right being preceded by a plus sign for lenders and a minus sign for borrowers. The non-negative constants $\psi_y(\lambda)$ and $\psi_E(\lambda)$ depend on the risk aversion parameter λ. Typically, these constants are increasing in the absolute value of $(1-\lambda)$; for example, if the real interest rate or the expected inflation rate were normally-distributed, $\psi_y(\lambda)$ or $\psi_E(\lambda)$ would increase linearly with $(1-\lambda)^2$.

Logarithmic utility separates two regions in which the marginal agent employs either biased low or biased high forward rates. In the likely circumstance that expected inflation is more volatile than real interest rates; (or, at least, agents believe this a the decision time t), the numerator in the second term on the right side of (15) would ordinarily be negative. If risk aversion is higher than logarithmic, $\lambda > 1$, a lender would contract in the forward loan market only if forward rates are somewhat higher than his anticipated future spot yield. Conversely, when risk aversion is relatively low, and lower than logarithmic, $\lambda < 1$ and a lender might contract forward even if he expects the future yield to be higher than the currently-observed forward interest rate.

A borrower's perspective is a mirror image. At higher levels of risk aversion than logarithmic, he can be coaxed into forward borrowing only if he expects future yields to be somewhat higher than the current forward rate; while with lower risk aversion, he might agree to borrow in the forward market even if its expected interest expense were higher.

For $\lambda > 1$, the term $[\psi_y(\lambda)-\psi_E(\lambda)]/(1-\lambda)$ in (15) is identical to the risk premium ϕ discussed heuristically in Section II and portrayed in Figure 1. When expected inflation is more volatile than real interest rates, $\psi_y(\lambda) < \psi_E(\lambda)$, and a positive ϕ is added to each lender's expected yield to obtain the cross-sectional distribution which the forward rate divides into forward and spot lenders, while a positive ϕ is **subtracted** from each borrower's expected yield to obtain the corresponding cross-sectional distribution which the forward rate separates into spot and forward lenders. Equilibrium in the forward loan market is

determined by that particular forward interest rate for which the volume of lending and borrowing are equal; Cf. Figure 1.

However, we have now discovered something beyond heuristics about the determinants of ϕ. It is positive for lenders and negative for borrowers only when risk aversion is relatively large, exceeding logarithmic, i.e., when $\lambda > 1$. For lower risk aversion, $\lambda \leq 1$, the influence of Jensen's inequality is as large or larger than the influence of risk aversion. This is critical for the volume of lending undertaken in the forward market. As we have seen previously, with risk neutrality, ($\lambda = 0$), Jensen's inequality actually serves to increase the volume of forward transacting in response to increasing variability in expected inflation, *ceteris paribus*. But an increasing degree of risk aversion offsets this propensity to transact forward; as aversion to risk increases, lenders and borrowers will wait in larger numbers for the spot market to arrive.

These statements are reversed if real interest rates are more volatile than expected inflation rates. In such circumstances, forward lending would experience the smallest volume when agents were risk neutral. Forward loan transactions volume would then increase with risk aversion.

IV. Heterogeneity, Loan Volume, and the Forward Rate Bias in Continuous Time.

Continuous-time models of the term structure are more mathematically elegant than discrete-time models; they have the potential to provide a more rigorous and complete theory without some of the simplifying assumptions that discrete models require for clarity.

For instance, in the preceding discrete analysis, a troubling assumption was that agents had already completed the borrowing/lending decision prior to the timing decision, (either forward or spot.) This is obviously unrealistic. Moreover, in discrete time, it is tedious to consider the interaction of multiple loan periods; yet we ought to permit the possibility that decisions to contract forward or spot over one loan period may be influenced by the opportunities to finance such a position by contracting spot or forward over a different period. Such possibilities raise the specter of intertemporal riskless arbitrage, which any respectable theory should rule out, at least if it's obvious. Examining discrete loan periods one at a time cannot guarantee the absence of arbitrage.

In a continuous-time framework, agents are assumed to hold beliefs about the intertemporal evolution of particular quantities such as the instantaneous short rate of interest, commodity prices, expected instantaneous inflation, instantaneous short-rate volatility, etc. In most models, these beliefs are homogeneous across agents and, in fact, are true. This permits each agent to construct and examine **truly** risk-free positions, which must earn the risk-free return (instantaneously) if there is no pure arbitrage cash flow available.

Heterogeneity is an immense complication. When beliefs are not the same, by necessity some are incorrect. An agent may **think** he is examining a risk-free position, only to learn with hindsight and perhaps with regret, that it really wasn't riskless after all. This implies that the usual reliance on arbitrage restrictions to obtain analytic valuation results is no longer available. How far can we go without the consummate aid of the no-arbitrage method?

In continuous time, it is convenient to work with instantaneous forward rates, i.e., a forward rate for a future loan period of infinitesimal length,

$$f_{n,t} \equiv \lim_{k\to 0}\{[(n+k)Y_{n+k,t} - nY_{n,t}]/k\}.$$

Since $P_{n,t}=e^{-nY_{n,t}}$, $f_{n,t}=-\partial \ln P_{n,t}/\partial n=-(\partial P_{n,t}/\partial n)/P_{n,t}$ where $P_{n,t}$ is the price at time t of a default-free zero-coupon bond with term-to-maturity n and $Y_{n,t}$ is its yield. The forward/spot contracting decision at period t is based on comparing the instantaneous forward rate observed at t with the instantaneous yield, $Y_{0,t+n}$, to be observed at t+n.

When there is a single state variable, many continuous-time term structure models imply that a default-free, zero-coupon bond's price follows a geometric Brownian motion of the

form[22]

$$dP = \alpha P dt + \beta P dz$$

where the drift and volatility parameters, α and β respectively, can be functions of time and of other underlying parameters. For a particular bond with term n, $-\ln(P_n)=nY_n$, and Ito's lemma can be applied to the log transformation to obtain

$$d[-\ln(P_n)] = [(½)\beta_n^2-\alpha_n]dt - \beta_n dz.$$

Since

$$f_n = \lim_{\Delta n \to 0}\{[-\ln(P_{n+\Delta n})+\ln(P_n)]/\Delta n\}$$

the forward rate's stochastic process satisfies

$$df_n = [½(\partial\beta_n^2/\partial n)-(\partial\alpha_n/\partial n)]dt - (\partial\beta_n/\partial n)dz. \tag{16}$$

This expression, or an analogous expression generalized to multiple factor processes, can be used to deduce the relation between forward rates and expected future spot yields. It will be possible to compare not only their relative expected values and thus the bias in forward rates, if any, but also to ascertain how the forward rate and the expectation of its companion future yield vary dynamically. This represents an improvement over the discrete-time approach, which has traditionally been limited to investigating just the bias.

IV. A. Forward and Spot Rates, a Continuous-Time Example with the Single-Factor CIR Model.

As a preliminary illustration, we investigate the forward/spot timing problem in the context of the familiar Cox-Ingersoll-Ross (CIR) [1985] single factor model[23]. The single factor version of the CIR model involves **real** interest rates only, though some have applied it empirically to nominal rates as well; e.g., see Brown and Dybvig [1986]. The CIR model is driven by a specification of the dynamics of the instantaneous real yield, (r in their notation), which follows a mean-reverting square root autoregressive process

$$dr = \kappa(\theta-r)dt+\sigma\sqrt{r}\,dz$$

where dz is a Wiener process and κ, θ, and σ are constants, σ being the scale factor for

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[22]If a variable has no time subscript, it is observed at t.

[23]The CIR model is used here for expositional purposes because of its familiarity to many readers. Other similar models would have provided similar insights.

volatility, θ the long-term mean of r, and κ the speed of adjustment of r to its long-term mean.

CIR obtain the price of a default-free zero-coupon bond with term-to-maturity n and show that its dynamics are described by the stochastic differential equation

$$dP_n/P_n = r[1-\lambda B_n]dt - B_n\sigma\sqrt{r}dz, \tag{17}$$

where λ is a risk parameter and B_n depends on λ, the term-to-maturity n, and the parameters κ and σ of the interest rate process; B_n does not depend on the interest rate r nor on its long-term mean θ[24]. The risk parameter is related to the covariability of the instantaneous interest rate and real wealth; if λ is negative, expected returns on bonds exceed r and increase with term-to-maturity.

Substituting in (16), the dynamics of the forward rate in the CIR model are

$$df_n = [r(B_n\sigma^2+\lambda)(\partial B_n/\partial n)]dt + [\sigma\sqrt{r}(\partial B_n/\partial n)]dz. \tag{18}$$

In general, both the drift and the volatility of the forward rate depend on the passage of time. For $n>0$, a non-zero value for the drift implies a bias in the forward rate for any agent whose expectations are rational based on his own information set. This follows from the observation that a rational agent cannot expect a change in his own expectation of the future instantaneous yield. His expected yield must have a zero drift. Thus, a downward (upward) biased forward rate is implied by a positive (negative) drift in (18).

In the CIR model, every agent is the same; their expected future spot yield is given by

$$E_t(\tilde{y}_{0,t+n}) \equiv E_t(r_{t+n}) = r_te^{-\kappa n} + \theta(1-e^{-\kappa n}), \tag{19}$$

The expectation is a weighted average of the current instantaneous interest rate, r_t, and its long-term mean, θ; the weights depend on the speed of adjustment, κ, and the time, n, until the future spot rate is observed. A lower-case "y" is used for the future yield since the single-factor CIR model is only for real interest rates. Subscripts are added to r to denote its value at different points in time.

Expression (19) implies that the expected real yield of a CIR agent follows its own dynamic process. Applying Ito's lemma and recognizing that $\partial t=\partial(-n)$, the drift of the

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[24]The exact expression for B_n, given in their equation (23), p. 393, is

$$B_n=2(e^{\gamma n}-1)/[(\kappa+\gamma+\lambda)(e^{\gamma n}-1)+2\gamma],$$

where $\gamma\equiv[(\kappa+\lambda)^2+2\sigma^2]^{1/2}$. Their notation has been simplified slightly here.

expectation is given by

$$\kappa(\theta-r_t)[\partial E_t(r_{t+n})/\partial r_t]+\partial E_t(r_{t+n})/\partial(-n)=\kappa(\theta-r_t)e^{-\kappa n}+(r_t-\theta)\kappa e^{-\kappa n}=0.$$

Consequently,

$$dE_t(y_{t+n}) = [0]dt + [e^{-\kappa n}\sigma\sqrt{r_t}]dz. \tag{20}$$

As required by the CIR agent's rationality, the drift in the expected future spot yield is zero at all times. As the instantaneous loan period approaches, ($n\to 0$), the volatility of the expected spot yield converges to that of the instantaneous interest rate. However, for positive term-to-maturity ($n>0$) and mean reversion ($\kappa>0$), the volatility of the expectation is always less than its terminal value; it increases monotonically with the passage of time. This is intuitive because the longer until maturity, the closer the expected spot rate cleaves to its long-term mean θ.

With the instantaneous loan period approaching, it can be verified that $\lim_{n\to 0}B_n=0$ and $\lim_{n\to 0}(\partial B_n/\partial n)=1$. Thus, from (18), the volatility of the forward rate converges to the volatility of the instantaneous spot rate, (i.e., to $\sigma\sqrt{r}$), while the drift of the forward rate converges to $r\lambda$[25]. It might seem curious that the drift of the forward rate does not converge to the drift of the spot rate since the two rates must be equal at $n=0$. But because the forward rate is generally biased, its drift must correct the bias asymptotically; essentially, the forward rate is tracking $E_t(r_{t+n})$ with a vanishing bias which depends on λ, while r_t is converging toward θ. In the CIR framework, negative λ is the standard case. Consequently, the forward rate drift is asymptotically negative (as $n\to 0$) and the forward rate bias is asymptotically positive.

The forward rate bias under the CIR model can be computed directly and related to both volatility and risk. Figure 3 depicts several cases. Panel A is for an instantaneous risk parameter λ of zero. The panel shows the expected path of the instantaneous rate, r, and four different yield curves (each one labelled "y"), two for increasing rates and two for decreasing rates, each with two levels of volatility, $\sigma=5\%$, 15% per annum, plus corresponding forward rate curves, labelled "f".

With $\lambda=0$, it can be verified from (18) that the forward rate has a positive drift at all

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[25]The CIR price of a discount bond is $P_n=A_ne^{-B_nr}$ where A_n is a rather complicated function of all the parameters; see Cox, Ingersoll, and Ross [1985, p. 393]. The forward rate can also be expressed as $f_n=r(\partial B_n/\partial n)-(\partial A_n/\partial n)/A_n=r(\partial B_n/\partial n)+\kappa\theta B_n$. Applying Ito's lemma then gives an alternative form of the forward rate drift, $\kappa(\theta-r)(\partial B_n/\partial n)+\partial f_n/\partial t$. Though tedious, it can be shown by direct calculation that $\lim_{n\to 0}(\partial f_n/\partial t)=r(\lambda+\kappa)-\kappa\theta$, again verifying that the forward rate drift converges to $r\lambda$ with approaching maturity. For $n>0$, the drift can also be written as $-r[\kappa(\partial B_n/\partial n)+\partial^2B_n/\partial n^2]=r(\partial B_n/\partial n)[w\lambda+(1-w)(\gamma-\kappa)]$ where the weight $w\equiv 2\gamma/[2\gamma+(\gamma+\kappa+\lambda)(e^{\gamma n}-1)]$.

times prior to maturity. Consequently, as the figure shows, the forward rate curve lies everywhere below the path of expected future spot rates. Forward rates are biased downward. The two levels of volatility illustrate the influence of Jensen's inequality; the forward rate is only slightly biased for $\sigma=5\%$, but the bias is as much as 100 basis points when $\sigma=15\%$. Remember, this occurs even though the price of risk is zero.

It is interesting to compare this result with the discrete-time case discussed in Section III. There, it was shown that risk neutral borrowers can be induced to contract forward only if the forward rate is biased downward, provided that inflation is non-stochastic.[26] Risk neutral lenders, however, will contract only when the forward rate is biased upward. By happenstance apparently, the CIR equilibrium with a zero price of risk ($\lambda=0$) produces the phenomenon favored by borrowers, a downward-biased forward rate.

Panel B of Figure 3 holds volatility constant at $\sigma=10\%$/annum and alters the instantaneous price of risk, λ. In the CIR model, $\lambda<0$ corresponds to a positive risk premium for bond returns which increases with maturity. Intuitively, this would be the standard case. As the figure shows, the downward bias in forward rates increases with algebraically larger values of λ. Consequently, a larger bond return premium offsets Jensen's inequality; if the price of risk is large enough (i.e., λ is negative enough), forward rates can be biased high estimates of future spot rates. As we have already discovered, the forward rate drift approaches $r\lambda$ with approaching maturity; so the bias eventually becomes unambiguously positive (negative) for negative (positive) λ. The intuition for this result is the decreasing influence of Jensen's inequality with the passage of time. At the instant of maturity, it has disappeared entirely.

Finally, panel C depicts a special case of very high volatility ($\sigma=50\%$/annum), for which the CIR model implies a humped yield curve. In this circumstance, forward rates are grossly downward biased when risk is nil because of Jensen's inequality. However, for a large enough price of risk, (negative λ), forward rates can be biased upward over short horizons and simultaneously biased downward over longer horizons. The price of risk and Jensen's inequality are fighting over the bias; the latter wins early while the former wins later. There is even one horizon, about six years in the illustration, where the two effects are exactly offsetting and the bias is negligible.

Agents comparing forward rates with spot rates expected to prevail in the future loan period can calculate directly the stochastic relation at t between the forward rate and the expectation. In fact, a bias function with zero volatility can be constructed: a forward loan quantity $Q \equiv e^{-rn}/(\partial B_n/\partial n)$ has exactly the same stochastic component as $dE_t(r_{t+n})$. The intertemporal change in this bias function is therefore deterministic. From (18) and (20),

$$Qdf_n - dE_t(r_{t+n}) = [Qr(B_n\sigma^2+\lambda)(\partial B_n/\partial n)-0]dt + [Q\sigma\sqrt{r}(\partial B_n/\partial n)-e^{-rn}\sigma\sqrt{r}]dz,$$

or,

$$Qdf_n - dE_t(r_{t+n}) = [r(B_n\sigma^2+\lambda)e^{-rn}]dt. \tag{21}$$

[26]Cf. equations (10) and (11) with $\psi_E=0$, (i.e., no inflation volatility) and $\psi_y>0$.

The (non-stochastic) drift in the bias function decreases with increasing risk, (increasingly negative values of λ), and with term-to-maturity and increases with short rate volatility. The bias' drift converges to $r\lambda$ as maturity nears[27].

It is somewhat contradictory to investigate heterogeneity within the context of a model such as CIR. Different agents, each attempting to estimate the CIR parameters, could derive diverse values for at least some of them. But this violates a fundamental assumption of the CIR model. Even under the doubtful assumption that everyone believes in a unique stochastic factor, r, agents would not agree on hedge ratios across bonds or among forward rates, to the extent that their parameter beliefs varied.

Nonetheless, applying the CIR model empirically necessarily involves statistical estimation, thereby implicitly admitting the possibility of heterogeneity. Brown and Schaefer [1994] use British Government Index-Linked "gilts," whose yields are approximately real and thus come close to the spirit of the single-factor CIR model. Table 6 of their paper presents CIR parameters estimated for different years, along with standard errors for each year. There is considerable intertemporal variability in the point estimates. For example, an estimate of $\kappa+\lambda$ varied from -.168 to +.094 from 1984 to 1985. The standard errors also are not trivial; some are even larger than the point estimates. Brown and Schaefer conclude,

> ...the [CIR] theory relies...on the parameters...remaining stable and, at least over annual subperiods, we are able to firmly reject stability. [Also] we obtain three (largely) independent but strongly inconsistent estimates of the degree of mean reversion in real rates, [parameter κ], (p. 38.)

In other words, in reality there is even heterogeneity across different estimation methods, not to mention what there must be across different investigators. Using constructed *ex post* real returns for short-term U.S. data, (maturities less than twelve months), Gibbons and Ramaswamy [1993] also report intertemporal instability in some of the parameters.

Despite a possible internal inconsistency, we might gain a modest insight by flying in the face of reality, assuming common beliefs and stationarity about some of the CIR parameters, while allowing a limited degree of heterogeneous belief about others.

The simplest approach would exploit the CIR feature that, θ, the long-term mean of r,

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[27]This result generalizes readily to any single factor interest rate model for which the expected future spot rate is some function G(r) and the discount bond price is some other function P=F(r). Then $Q=-P\{F'(r)/[\partial G'(r)/\partial n]\}$ is a scaling factor which multiplies the forward rate to produce a non-stochastic bias function.

does not appear explicitly in the processes for P and f[28]. The crucial function B_n, which appears in the stochastic dynamics of both prices and forward rates, (Cf. equations 17 and 18), depends on three of the model's four parameters, κ, λ, and σ, in addition to n, the time-until-maturity, (that we can safely assume would be known by everyone.) What happens if everyone agrees on all features and parameters of the single-factor CIR model except θ? Does this lead to a positive volume of loan originations, some agents lending to others? How does loan volume change over time as agents improve their estimates of θ?

If all parameters except θ were fixed and common knowledge, agents would agree completely about the dynamics of prices (17) and forward rates (18), and expectations of future spot yields (20). None of these stochastic differential equations depends on θ. Hedging strategies would appear perfectly sensible to everyone; e.g., the construction of a riskless position from any two bonds[29] would be identical across agents. Agents would not agree on the drift term of the instantaneous short rate, $\kappa(\theta\text{-}r)$, but they would also not be able to tell from a finite sample who was right.

The levels of forward rates would appear puzzling. Agent i believes that the long-run mean interest rate is θ_i and therefore expects to observe a market forward rate

$$f_{n,i} = r(\partial B_n/\partial n) + \kappa B_n \theta_i. \tag{22}$$

which follows from the alternative expression for the forward rate given in footnote 25 after indexing the only parameter about which agents can disagree. If the observed forward rate were actually lower (higher), the agent would be tempted to borrow (lend) in the forward loan market, anticipating the opportunity to cover by reversing the loan at a profit later in the spot market. Since (22) is linear in θ_i, it represents a linear excess supply indicator function for lending; a positive volume of forward lending by agent i when the market forward rate f_n exceeds $f_{n,i}$ and a positive volume of forward borrowing when $f_n < f_{n,i}$.

Although we can deduce the direction of each agent's action, whether he will be a forward borrower or a lender, we cannot determine the quantity of forward transacting; (22) indicates only the direction, not the amount. At this point, we can only say that loan volume would not be zero. Intuitively, the quantity of loan transactions at a given moment would have to depend on at least four elements: (1) the cross-sectional diversity

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[28]The dynamics of prices and forward rates depend on θ, but only indirectly through the influence of r.

[29]Reminder: since there is only one stochastic factor, all bond returns are perfectly correlated instantaneously. A long position $\$(B_m/B_n)$ in a bond of term n and a short position of \$1 in a bond of term m has zero return volatility and certain return r.

in θ_i, (2) the confidence of each agent in his belief about θ_i[30], (3) the degree of risk, and (4) the credit quality (or wealth) of each agent. Volume would increase with disagreement and confidence and decrease with risk. The impact of credit quality would depend on its cross-sectional relation, if any, with confidence and risk.

It is apparent that we are not even close to an analytic theory that combines the elements described intuitively in the previous paragraph, and yet those elements arose in a very simple situation which relied on an assumption of considerable agreement: a single stochastic factor driving the term structure according to a specific mean-reverting square root process with known volatility, known speed of adjustment, and known and uniform risk! Actually, we don't know any of these things.

More disquieting still, agents undoubtedly learn at least something over time as they observe the profits and losses on positions taken earlier. The degree of risk might very well be stable, but confidence in a particular belief about θ_i cannot withstand substantial and persistent excursions of r away from θ_i particularly when they are accompanied by significant financial losses. Revised beliefs must follow along with an alteration in cross-agent dispersion.

IV. B. The CIR Model with Nominal Interest Rates, Uncertain Inflation, and Stochastic Expected Inflation.

An annoying feature of single-factor continuous-time term structure models with homogeneous agents is their implication of perfect correlation between any two bonds. Actually, the trouble is deeper. Perfect correlation is also present between bond returns and changes in forward rates, yields, the instantaneous short rate, and even in expected future short rates. This is evident in the single-factor CIR model by noting that the same stochastic constituent, dz, appears in the dynamics of every variable; (Cf. (17), (18), and (20)).

Perfect correlation is not a characteristic of actual bond returns, forward rates, or yields, but this in itself is not a sufficient reason to reject such models. A better reason is that heterogeneity is hard to imagine in such a context. How can agents agree perfectly on constructing a perfect hedge yet disagree about whether to borrow or lend? The results are contrived and unconvincing. Perhaps a multiple-factor model specified with plausible economic variables such as nominal interest rates, inflation, and expected inflation, will provide a venue for examining heterogeneity more easily and be more empirically realistic

[30]Harris and Raviv [1993] presume that "...speculative trading is most likely when the beliefs of the traders are most diffuse" and "...the posterior beliefs of a speculator are most sensitive to the next signal when his or her current beliefs are most diffuse." (p. 490.)

at the same time.

In the penultimate section (7) of their paper, CIR present a three-factor model intended to capture the simultaneous movements of real interest rates, expected inflation rates, and unexpected inflation. The real interest rate process remains the same as in their single factor model,

$$dr = \kappa_r(\theta_r - r)dt + \sigma_r \sqrt{r}\, dz_r. \tag{23}$$

They propose two alternative versions of the process describing expected inflation, but for illustrative purposes we shall employ just the simpler one, their "model 2,"

$$d\hat{e} = \kappa_{\hat{e}}(\theta_{\hat{e}} - \hat{e})dt + \sigma_{\hat{e}} \sqrt{\hat{e}}\, dz_{\hat{e}}, \tag{24}$$

where $\hat{e}$ denotes the **instantaneous** expected inflation rate[31]. Like the real interest rate, the instantaneous expected inflation can drift away from its long-term level $\theta_{\hat{e}}$, and it reverts with its own speed of adjustment $\kappa_{\hat{e}}$.

Finally, the commodity price level π follows a geometric diffusion whereby the instantaneous **actual** inflation rate is given by

$$d\pi/\pi = \hat{e}dt + \sigma_\pi \sqrt{\hat{e}}\, dz_\pi. \tag{25}$$

Hence, $\sigma_\pi \sqrt{\hat{e}}\, dz_\pi$ is the instantaneous **unexpected** rate of inflation.

CIR make the simplifying assumption that inflation has no real effects; consequently, there is no correlation between movements in inflation, either expected or unexpected, and movements in real interest rates. They do permit correlation between actual inflation and percentage changes in expected inflation, $\rho \equiv \mathrm{Cov}[dz_\pi, dz_{\hat{e}}]$.

From this specification, CIR derive the real price of a nominal bond, i.e., a discount bond at time t which pays a real value of $\$1/\pi_{t+n}$ for sure at maturity, t+n. At t, the nominal value of this bond is simply its real price multiplied by the current price level, π_t. Their pricing function implies an instantaneous **nominal** forward interest rate, F_n, with term-to-maturity n,

$$F_n = r(\partial B_n/\partial n) + \hat{e}(\partial C_n/\partial n) + \kappa_r\theta_r B_n + \kappa_{\hat{e}}\theta_{\hat{e}} C_n, \tag{26}$$

where B_n, the same function that appeared in the CIR single-factor model, depends only on maturity, the parameters κ_r and σ_r of the real interest rate process (23), plus the price

[31]Note that $\hat{e}$ differs from the inflation rate expected for a **fixed** future infinitesimal period; $\hat{e}$ is the rate anticipated over the next instant. Consequently, it can be expected to change (and have a non-zero drift) without violating rationality. Subscripts have been added to parameters where they contribute to clarity.

of risk λ. The new function C_n depends only on maturity, the parameters of the expected and unexpected inflation processes $\kappa_{\hat{e}}$, $\sigma_{\hat{e}}$, σ_π, and their correlation, ρ. The form of C_n is similar to that of B_n[32]. Notice that (26) contains only two state variables, r and $\hat{e}$, the instantaneous real interest rate and the instantaneous expected inflation rate, while the basic CIR setup includes a third factor, the commodity price level, π. Although π does not appear explicitly in (26) because F_n is a nominal forward rate, F_n does depend on the stochastic properties of unexpected inflation as well as those of expected inflation.

As the forward rate nears maturity, it is straightforward to show from (26) that

$$\lim_{n\to 0} F_n = r + \hat{e}(1-\sigma_\pi^2). \tag{27}$$

Perhaps surprisingly, the forward rate does not converge to the Fisherian definition of the nominal yield; this would have been $r+\hat{e}$, the instantaneous real rate plus the instantaneous expected inflation rate.

In continuous time, the classic Fisher equation does not hold; the nominal interest rate must equal the real rate plus expected inflation adjusted downward by an amount that increases with the volatility of unexpected inflation. The downward adjustment is required by Jensen's inequality to equilibrate the real return on a nominal bond with the return on a real bond. It is reminiscent of the discrete-time results in Section III.A and III.B above; e.g., expected terminal real wealth from forward lending increases with the volatility of expected inflation, implying that the nominal forward rate acceptable to a risk-neutral lender decreases with increasing inflation volatility; (Cf. inequality (10)).

Applying Ito's lemma to (26), we obtain the following expression describing the dynamics of the instantaneous nominal forward rate,

$$\begin{aligned} dF_n = {} & \{-r[\partial B_n/\partial n + \partial^2 B_n/\partial n^2] - \hat{e}[\partial C_n/\partial n + \partial^2 C_n/\partial n^2]\}dt \\ & + (\partial B_n/\partial n)\sigma_r \sqrt{r}\, dz_r + (\partial C_n/\partial n)\sigma_{\hat{e}} \sqrt{\hat{e}}\, dz_{\hat{e}}. \end{aligned} \tag{28}$$

The instantaneous nominal spot yield, R, conforms to the Fisherian relation modified by Jensen's inequality,

$$R = r + \hat{e}(1-\sigma_\pi^2),$$

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[32]It is given in CIR's equation (56), p. 403,

$$C_n = [2(e^{\xi n}-1)(1-\sigma_\pi^2)]/[(\xi+\kappa_{\hat{e}}+\rho\sigma_\pi\sigma_{\hat{e}})(e^{\xi n}-1)+2\xi],$$

where $\xi \equiv [(\kappa_{\hat{e}}+\rho\sigma_\pi\sigma_{\hat{e}})^2+2\sigma_{\hat{e}}^2(1-\sigma_\pi^2)]^{1/2}$.

at any moment. Consequently, we can compute its expectation directly from the assumed stochastic processes for r and ê as,

$$E_t(R_{t+n}) = \theta_r + (r-\theta_r)e^{-\kappa_r n} + (1-\sigma_\pi^2)[\theta_\hat{e} + (\hat{e}-\theta_\hat{e})e^{-\kappa_\hat{e} n}], \tag{29}$$

and the stochastic dynamics of this expectation conform to

$$d[E_t(R_{t+n})] = 0dt + e^{-\kappa_r n}\sigma_r \sqrt{r}\,dz_r + (1-\sigma_\pi^2)e^{-\kappa_\hat{e} n}\sigma_\hat{e} \sqrt{\hat{e}}\,dz_\hat{e}, \tag{30}$$

whose drift is, of course, zero.

Comparing (30) to (28), the forward rate and the expectation of its corresponding expected future nominal spot rate are not perfectly correlated because the coefficients multiplying the two stochastic Weiner process, (dz_r and $dz_\hat{e}$), are not equal in the two equations. Consequently, it is usually impossible, using only a single forward rate, to form a bias function QdF_n-$d[E_t(R_{t+n})]$ without some random component[33]; (recall that this was possible when there is a single stochastic state variable.) Only at the instant of maturity do the forward rate and the expected future spot rate become perfectly correlated.

But a non-stochastic bias function could be formed as a linear combination of any two different forward rates; i.e., it is possible to mimic perfectly the stochastic process of the expected future spot nominal yield by a combination of two observable forward rates. For notational simplicity, define $B'_j \equiv \partial B_j/\partial j$. Consider the stochastic process of the combination, $Q_j dF_j + Q_k dF_k$, $(j \neq k)$, where

$$Q_j = [C'_k e^{-\kappa_r n} - B'_k(1-\sigma_\pi^2)e^{-\kappa_\hat{e} n}]/(B'_j C'_k - B'_k C'_j)$$

and

$$Q_k = [-C'_j e^{-\kappa_r n} + B'_j(1-\sigma_\pi^2)e^{-\kappa_\hat{e} n}]/(B'_j C'_k - B'_k C'_j).$$

Unlike the expected future nominal yield, $d[E_t(R_{t+n})]$, the drift of the combination $Q_j dF_j + Q_k dF_k$ will not necessarily be zero, but the stochastic component of the combination will be identical to the sum of the two stochastic elements of (30). Subtracting the drift from the observed changes in this combination of forward rates provides a perfect mimicking portfolio for the expected future spot nominal yield.

But for agents to agree on the same non-stochastic bias portfolio, every parameter which

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[33]There is a trivial and unlikely exception; if, by some fortuitous miracle, we had the following condition,

$$(\partial C_n/\partial n) = (\partial B_n/\partial n)e^{(\kappa_r - \kappa_\hat{e})n}(1-\sigma_\pi^2),$$

a riskless bias function could be constructed with a single forward rate.

appears in the expressions for Q_j and Q_k would have to be common knowledge. We observe by inspection, therefore, that they could only agree to disagree about θ_r and θ_δ, the long-term means of the real interest rate and the expected inflation rate, respectively. Consequently, pushing the analysis a step farther has not gained us very much. Disagreement about any other parameter essentially renders the model internally inconsistent. Agents would not then agree on hedge ratios. They would not agree on what constitutes a riskless hedge. They would necessarily write down divergent partial differential equations for bond prices and therefore produce incompatible characterizations of "equilibrium" prices. Trading volume would depend, of course, on how fervently and confidently each agent believed in his own pricing calculation.

The drift term in (28) is not necessarily zero, which implies that forward rates are biased. Figure 4 illustrates the direction and extent of the bias for some particular parameter values. In Panel A, nominal yields are expected to remain unchanged. Increasing either the volatility of real interest rates or the volatility of expected inflation decreases (algebraically) the forward rate bias. Because the price of risk (λ is -6.5% in this example), the forward rate can be biased high if the volatilities are relatively low. However, as either volatility increases, the positive bias decreases and eventually turns negative as Jensen's inequality becomes dominant.

Panel B illustrates that the volatility of unexpected inflation also decreases the forward rate's bias. The figure plots the bias (instead of the rate level) because unexpected inflation volatility also influences the expectation of future nominal spot rates; (Cf. (29)). The top curve in 4-B conforms to a flat expectation, the same as in panel A, but the bottom curve with higher unexpected inflation volatility implies an expectation of slightly decreasing expected future nominal spot yields. Higher unexpected inflation volatility shifts downward both the forward rate and the expected future spot rate and it also decreases the bias.

The CIR nominal rate model with stochastic inflation is not in accordance with the intuitive discussion of Section II. Recall the intuition that the direction of forward rate bias depends on whether expected inflation is more or less volatile than the real interest rate. A risk averse lender, for instance, considers it risky to contract nominally in the forward market if expected inflation is volatile; it is safer to wait for the spot market, provided that the real interest rate is relatively less volatile. In the CIR model, however, everyone is instantaneously risk averse, so the volatilities of both real interest rates and expected inflation act in the same direction.

This CIR implication exemplifies the homogeneity issue of continuous-time term structure models. If there were some heterogeneity; for example, if the government were a relatively risk-neutral long-maturity nominal borrower while private lenders were quite risk averse, one would expect nominal yields to increase with maturity and rate of increase to rise with the volatility of expected inflation, *ceteris paribus*. In other words, the forward rate bias would increase with inflation volatility. Yet the CIR model prescribes a

decrease! The CIR result can be blamed on homogeneity, of preferences. Since CIR borrowers and lenders face identical risks, the forward rate's bias is determined by Jensen's inequality. Consequently, the assumption of homogeneity may have been responsible for the model bypassing an obvious and important determinant of actual nominal yield curves.

V. Empirical Evidence.

With heterogeneity among borrowers and lenders, none of whom necessarily follow completely rational strategies, a heuristic discrete-time argument was advanced in Section II that:

> (1) Contracting in the forward loan market is riskier than waiting to contract in the spot market when the intertemporal variation in expected inflation exceeds the variation in *ex ante* real yields. ("Risk" is defined as the volatility of the real return.) In this circumstance, a regression of *ex post* inflation rates on beginning-of-period spot nominal yields will have a slope coefficient greater than one-half, provided that beliefs are relatively stable throughout the interval of empirical observation.

If private individuals are risk averse and forward contracting is riskier because expected inflation is more volatile than *ex ante* real yields, then,

> (2) both lenders and borrowers can be persuaded to contract in the forward loan market only by being offered preferable rates, higher forward rates for lenders and lower rates for borrowers.
>
> (3) The volume of private forward lending/borrowing will be smaller the greater the degree of risk aversion and/or the greater the volatility of expected inflation, *ceteris paribus*.
>
> (4) A sovereign government issuing long-term fixed-rate bonds will have larger borrowing costs on average than it would incur by rolling over shorter-maturity fixed-rate loans (or by issuing floating rate loans.)

In addition, if the government is a large fixed-rate borrower at long maturities,

> (5) Forward rates will be biased high estimates of future spot yields; the bias represents a risk premium which will probably increase with the time until the future loan period.
>
> (6) If (5) is true, the yield curve will be sloped upward[34] on average.

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[34]This does not imply that the yield curve will be sloped upward on any particular calendar date. It can be downward sloped, even though it contains risk premiums that increase with maturity, whenever short term rates are expected to decline precipitously enough; (e.g., in Germany during 1993.) On average, however, the slope of the yield

Section III showed that these same conclusions are valid when private borrowers have logarithmic preferences while the government borrower is less risk averse. However, if private risk aversion is less acute than logarithmic, forward rates could conceivably be biased downward by Jensen's inequality, (again assuming that expected inflation if more volatile than real interest rates.)

Section IV's excursion into continuous-time models of the term structure, using the Cox-Ingersoll-Ross (CIR) model [1985] with stochastic inflation as an example, revealed two interesting features:

First, it is difficult to embed a significant degree of borrower/lender heterogeneity in such a model because its pricing equations are derived by constructing riskless positions from a combination of securities. If agents disagree about parameters of the underlying stochastic processes, they will almost surely disagree about whether a particular portfolio position is truly risk free. Only disagreement about the long-term levels of real interest and expected inflation makes sense within the CIR framework.

Second, assuming homogeneity, the CIR model implies that the bias in nominal forward rates decreases with the volatilities of both the real interest rate and the expected inflation rate. This implication represents an empirical lever for ascertaining whether the CIR model or a model based on incomplete rationality and heterogeneity is more consistent with reality.

V. A. The Average Nominal Term Structure.

An important datum for all models is the shape of the average term structure of nominal yields. If it is upward sloping, perhaps expected inflation varies more than *ex ante* real interest rates and lenders are more risk averse than borrowers, or perhaps homogeneous agents in the CIR world are very risk averse.

The empirical problem with this approach is to know how long a period is required before the average observed yield curve can be regarded as truly representative. Surely, it would require averaging over several business cycles. In the U.S., yield curve data are available since the mid-1920's, now a total of almost seventy years. What do they reveal?

Ibbotson Associates [1995] tabulates total rates of nominal return for U.S. Treasury bonds of long and short maturities, over the calendar period 1926-94 inclusive. On average over this entire sample period, long-term bonds returned 4.8 percent per annum while short-term bonds returned 3.7 percent. There were long sub-periods when the returns were higher on short-term bonds, particularly when the market was surprised by increasing yields. But

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curve at a given point will equal the marginal increase in risk premium with respect to maturity.

such episodes were at least partially offset by other occasions when the surprise went in the opposite direction and yields declined[35]. Over the entire period, the U.S. Treasury could have saved about 110 b.p. per year per marginal dollar shifted from long to short maturities. This is probably an underestimate of the Treasury's potential cost savings in the future because yields increased modestly over the entire 1926-94 period, thereby inducing a small unexpected *ex post* decrease in realized long-term bond returns.

Data for the Federal Republic of Germany are available over a shorter time period, but they tell pretty much the same story. From June, 1973 through October, 1995, the average new issue yield on the ten-year Bund was 7.79 percent per annum. There were no short-term new issues by German government entities over much of this period, but the EuroMark one-month deposit rate averaged 6.48 percent. The difference of 131 basis points might even be understated slightly because of greater default risk in EuroMark deposits, the liabilities of private borrowers.

A plot of these German yields and the corresponding long/short yield difference is provided in Figure 5. Notice that the 10-year Bund yield was higher than the EuroMark yield for long periods, e.g., 1974-79, 1982-89, 1994-95. The Bund yield was higher in 67 percent of the months. As is typical for yield curve fluctuations, short-term rates were more volatile than long-term rates; the standard deviation of EuroMark yields was 2.51, approximately twice the standard deviation of Bund yields, 1.26. If successive monthly observations of the yield difference can be regarded as statistically independent, the t-statistic of the yield difference was 11.4, thereby indicating a high degree of confidence in the proposition that long-term yields are systematically higher than short-term yields.

To make the German data consistent with the U.S. data reported by Ibbotson, approximate holding-period total returns were calculated from the two German series; (a total return comprises both coupons and price appreciation or depreciation.) For the EuroMark series, this is straightforward because the beginning-of-month yield is the one-month holding period return over the subsequent month. For the Bund, the new issue was assumed to have been purchased and held for one month. It was then sold and the proceeds were reinvested in the next new issue. This involves an approximation because there are no available secondary market prices for each old bond being sold; instead, it's price was assumed to have been fixed such that its secondary market yield matches that of the new

[35]An example of the former occasion was the decade of the 1960's. From the beginning of 1961 through the end of 1969, long-term U.S. Treasury bonds returned 0.2% per annum while short-term bonds returned 4.0% per annum. An example of the latter occasion was the decade of the 1980's during which long-term bonds returned 14.6% while short-term bonds returned 8.6%.

issue[36].

Accumulating these one-month holding-period returns over the entire sample period, June, 1973 through October, 1995, one Deutchesmark invested at the beginning of June, 1973 would have grown to DM4.049 in the one-month EuroMark market. In the ten-year Bund market, it would have grown to DM7.032. Given a particular amount of national indebtedness outstanding now (at the end of this past October), the German government could have spent 74% more in 1973 if it had financed those 1973 expenditures with one-month maturities rather than with ten-year Bunds!

On a per annum basis, the compounded holding-period returns were 6.45 percent for one-month EuroMarks and 9.09 percent for ten-year Bunds. The difference, 264 basis points, is higher than the average new-issue yield difference of 131 basis points mainly because of the average decline in long-term yields over the sample period[37].

V. B. Estimating the Relative Volatilities of Expected Inflation of Real Interest Rates.

An indirect method of measuring the relative volatilities of expected inflation and *ex ante* real yields estimates the slope coefficient, b, in regression (9). Fama [1975] reported such a regression using U.S. Treasury spot nominal yields and actual inflation measured by changes in the U.S. consumer price index (CPI) over the period January, 1953 through July, 1971. He did not use earlier data because, he argued, the CPI was not a reliable indicator of inflation prior to 1953 and the short-term Treasury market was rigged by government price fixing for a number of years in the 1940's and early 1950's. His regressions were calculated assuming constant coefficients over the sample period.

Using one-month Treasury Bill yields, his estimated value of b was .98 with a standard error of only .0003. This implies very strongly that expected inflation was more volatile than *ex ante* real yields at one-month horizons during that specific era in the U.S. For longer-term bills, the results were similar. The two- and three-month horizons had estimated slope coefficients of .96 and .92. Even longer horizons were available after 1958; for the sample period March, 1959 through July, 1971, the coefficients were 1.03, .97, and 1.01 for four, five and six-month horizons, respectively. All were statistically

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[36]The technique for obtaining a holding-period return from a time series of yields was derived by Shiller [1990].

[37]In June, 1973, the 10-year Bund yield was 9.75%; in October, 1995, it was 6.52%. To the extent that this overall decline was unanticipated, the resulting holding-period return is a biased high estimate of future long-term borrowing costs. Notice that the effect is virtually nil for one-month EuroMarks; the holding-period return was 6.45 per annum, on average, while the mean new-issue yield was 6.48 percent.

significantly higher than 1/2.

Fama's regression at the one-month horizon were extended here to cover the entire period of data now available for U.S. Treasury bills, from January, 1926 through August, 1995 (the latest month available.)

Over those seven decades, the regression slope coefficient was only .582. Although the coefficient was algebraically larger than 1/2, it was not significantly larger. But remember that a simple regression assumes stationarity. Such was manifestly not the case throughout U.S. history. During the period studied by Fama, the coefficient was indeed much larger, .98 as he reported. But over the 1926-52 period, it was actually negative, -1.16. A plot of *ex post* real U.S. returns, Figure 6, show a substantially larger variability in *ex post* real returns prior to 1953. This may have been the source of Fama's suspicion about the pre-1953 data; either the observed nominal interest rate or the measured inflation rate may have been so noisy that estimation of (9) is unreliable.

Since the end of Fama's sample period, (July, 1971), the coefficient has been consistently greater than 1/2. From January, 1952 through the most recent month, it was .778 (standard error of .048). From October, 1979 through December, 1989, which many economists believe was a period of high real interest rates, possibly induced by a change in Federal Reserve monetary policy in October, 1979, the coefficient was .806 (standard error of .115). As Figure 6 shows, this last period was indeed characterized by a relatively high <u>level</u> of *ex post* real rates on one-month Treasury bills; this is consistent with a relatively large embedded risk premium in the nominal yield.

The clear evidence of non-stationarity over available U.S. history impels us to contemplate a time-varying relative risk estimator. There are many "high-tech" methods, but Occam's Razor motivates a simple one, at least as a first attempt: viz., estimate the parameters from <u>each</u> noisy observation. Equations (6) and (7) imply that the difference between variances, expected inflation less *ex ante* real yields, can be estimated for each month t in the sample as

$$Y_{1,t-1}[2I_t - Y_{1,t-1}], \tag{31}$$

where I_t is the observed inflation rate over the month and $Y_{1,t-1}$ is the one-month Treasury Bill nominal yield at the beginning of the month. The estimator in (31) is biased upward by the amount $E(Y_1)[2E(I)-E(Y_1)]$, but for monthly returns this bias likely to be negligible.

Figure 7 plots the estimator in (31) over the available U.S. sample, Panel A for the entire period and Panel B for some interesting sub-periods. The month-by-month value of the estimator is presented along with its twelve-month moving average. Figure 7-C confirms that the estimator has a substantial amount of intertemporal dependence. An ARIMA model detects first-order autoregressive <u>and</u> moving average components.

The results are striking and sensible, even with data prior to 1953. In the early part of

the great depression, the estimator was consistently negative, thereby implying a larger volatility in real interest rates than in expected inflation; certainly a reasonable result for that episode of U.S. history. In Fama's 1953-71 sample period, the estimator was either close to zero or positive, the latter revealing greater volatility in expected inflation than in real rates; immediately following Fama's sample period, the estimator became larger, (over the period from 1971-79.) Then, following the 1979 change in Federal Reserve monetary strategy, the estimator fell sharply and even became modestly negative. This accords with historical folklore suggesting increasing **variability** in *ex ante* real rates during that period.

Notice too that from the 1933 through the mid-1940s the estimator was very small and indistinguishable from zero for the most part; i.e., there was virtually no difference between the volatilities of inflationary expectations and real rates. Since observed inflation had little volatility during this sub-period, unless these data were riddled with mis-measurement, as Fama indeed argued, real interest rates too must have had low volatility.

Turning back to Germany, Figure 8 plots *ex post* real returns from one-month EuroMark loans over the post-1972 era. The striking feature of this plot is the pronounced seasonal; real returns turn out to be very high many Junes and negative many Januaries. Figure 9 shows that this seasonal can be traced to the German CPI, not to the EuroMark yield. An examination of the data reveals that the German CPI takes a big jump from December to January while it displays a very small increase, or frequently even a decrease, from June to July. Surely this cannot represent the true inflation process in Germany.

Using these data, despite some misgivings, the estimated slope coefficient from regression (9) over the 1973-95 period for Germany is almost exactly 1/2; it is .499 and is, of course, not significantly different from 1/2. If we take this result at face value, the volatilities of expected inflation and *ex ante* real yields must have been about the same in Germany over the past 22 odd years. But should we rely on these data which obviously contain considerable measurement error of a seasonal variety, not to mention other measurement error of which we remain ignorant?

In an effort to glean something meaningful from these data, the inflation series was smoothed with a 12-month moving average, to remove the annual seasonal. The smoothed series is plotted also in Figure 9. The Figure reveals a clear trend in the *ex post* average real yield over the past 22 years in Germany; the gap between the EuroMark yield and subsequent inflation has widened substantially since 1978. Although it is not quite econometric cricket to run a regression with smoothed series, the simple slope coefficient of smoothed inflation on nominal yield exceeds 1/2 substantially.

Figure 10 plots the estimator (31) for the available German sample period; this portrays the month-by-month estimated value of the difference in volatilities between expected inflation and real interest rates. During roughly the first half of the sample, the estimator was consistently positive. It then decreased to near zero and decreased further to a

negative level in the second half of the sample. This pattern coincides with known historical events in Germany, particularly the real uncertainty occasioned by German reunification and the substantial increase in Government expenditures for rehabilitation of the former East Germany.

V. C. Estimated Risk Premiums Based on Survey Expectations.

By surveying market participants, expectations about future inflation or spot yields can be elicited directly. To the extent that these are representative, they offer useful data for estimating risk premiums.

Gendreau [1995] analyses the risk premiums imputed from a survey which asked knowledgable individuals to give forecasts of the spot yield on a U.S. Treasury ten-year bond three and twelve months ahead[38]. The yield forecasts are subtracted from the corresponding forward rate computed from the currently-observed yield curve to obtain two risk premiums; $Prem_{.25}=F_{10,.25}-E_t(\tilde{Y}_{10,t+.25})$ and $Prem_1=F_{10,1}-E_t(\tilde{Y}_{10,t+1})$, in our notation, where "Prem" denotes the computed risk premium and the subscripts are in units of one year. Note that the loan periods of these two calculations are different. $Prem_{.25}$ corresponds to a ten-year forward loan to begin in three months while $Prem_1$ corresponds to a ten-year forward loan to begin in one year.

Over the data availability period for the survey forecasts, November 1990 through October 1995, these implied risk premiums fluctuated widely, from a high of 65 basis points to a low of -51 basis points. Their means were positive, 7 and 12 basis points for the $Prem_{.25}$ and $Prem_1$ respectively. As Gendreau notes, however, their small size on average does not imply that they are unimportant. During some periods, they were economically significant.

Gendreau also reports that time variations in the imputed risk premiums are associated with changes in other risk-related variables. In particular, the implied volatility from Treasury bond options, the credit spread between low- and high-rated corporate bonds, and the immediately previous month's increase in the ten-year Treasury spot yield all have statistically significant positive affects on the two imputed risk premiums. This is strong evidence that $Prem_{.25}$ and $Prem_1$ are indeed measuring meaningful time varying risk premiums in Treasury bond forward rates.

Perhaps most important, a portfolio strategy of investing 100% in ten-year Treasuries when the risk premium is positive and only 75% in ten-year Treasuries (and 25% in short-term

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[38]The forecasts are gathered by Consensus Economics, Inc., from a variety of individuals and entities such as firms, trade associations, and universities. See Gendreau [1995] for details.

Treasury bills) when the premium is negative was compared to a constant policy of investing 75%/25% in the ten-year and bills, respectively. The portfolio strategy earned an extra 78 bp (per annum) using $Prem_{.25}$ and 134 bp using $Prem_1$. Statistical significance was only marginal for $Prem_{.25}$ but was at the 1% level for $Prem_1$.

A sovereign government would be in a position opposite to the portfolio investor. If the government would borrow at long-maturity when the risk premium is negative and at least partly at short maturity when the premium is positive, it should be able to generate a cost savings of a similar magnitude. Of course, this implies that the government has access to accurate consensus expectations from surveys or develops its own accurate expectations internally.

For Germany, consensus survey expectations for the ten-year Bund yield were collected by the same agency for a brief period, from November, 1986 through the end of 1991. Again, respondents were asked for two forecasts, one three months into the future and another twelve months ahead. This allows the calculation of imputed risk premiums by subtracting their mean forecasts from current forward rates. The resulting risk premiums are plotted in Figure 11. The mean values were 10 and 14 basis points for $Prem_{.25}$ and $Prem_1$, respectively. Although small, both are statistically significantly positive[39].

V. D. The Forward Premium, the relative volatilities of expected inflation and real rates, and the relative level of U.S. Federal Debt: A Cursory Examination.

This section will investigate whether Federal debt is related to a direct measure of the risk premium embedded in forward rates. Our measure of the risk premium will be the difference between the forward rate and the subsequently observed spot interest rate which covers the same calendar period. To obtain as many sample points as possible, we use the one-month forward rate for a single month in advance and the spot one-month Treasury Bill rate at the beginning of the next month. This difference is plotted for its total available sample period in Figure 12. Clearly, it is a very noisy measure; an impressively high level of explanatory power is unlikely.

For a more limited sub-sample of U.S. History, the relative outstanding amounts of Federal and Private debt are available. Figure 13 shows these series over their available sample period, Panel A giving the individual series (inflation adjusted) and Panel B providing the time series of the ratio of Federal to Private debt. All Federal debt in the United States was fixed rate during this sample period, but there is no information available on the proportion of private debt that was index-linked or floating. Consequently, the ratio of Federal to Private total debt may be a noisy estimator of the

[39]Assuming that the time series observations are independent of each other.

theoretical predictor, the proportion of all fixed rate debt borrowed by the government[40].

The risk measure will be the slope coefficient b_t from equation (9) estimated over the 36 months prior to the month in which the *ex post* forward rate bias is observed. A rolling 36-month estimator was chosen in an effort to reduce estimation error. Recall that b_t increases with increasing volatility of expected inflation relative to real interest rate volatility. Of course, a 36-month rolling estimator of b_t is closely related to a moving average of the direct volatility estimator pictured in Figure 7. For the regression estimator, however, there is the advantage of having its associated standard error, which can be usefully employed in a weighted regression. Figure 14 graphs the estimated b_t over the sample period of available aggregate debt series.

Table 2 contains the results of several regression models using these data. The Federal/Private debt ratio has the right sign but it is only marginally significant, perhaps not a surprising result given the noise in both the dependent and explanatory variables. The adjusted R-squares indicate a very modest degree of explanatory power. The coefficient for relative riskiness is statistically significant but it has the wrong sign! During this sample period, the higher the estimated volatility of expected inflation relative to the volatility of real interest rates, the lower the average forward rate premium over the subsequent spot nominal interest rate. This is indeed an empirical puzzle which requires further investigation.

V. E. Summary of Empirical Data

Both the U.S. and German data provide adequate evidence of time variation in the relative volatilities of expected inflation and real interest rates. Moreover, there is at least some evidence of intertemporal variation in risk premiums. It would be interesting to ascertain whether these episodes coincide with alterations in the volume of private nominal loan contracting. We should not necessarily expect to find a correlation between the forward rate bias (or the risk premium) and the relative volatility of expected inflation and real rates. We should expect, however, to find such a correlation in periods when the government nominal borrowing represents a large fraction of all borrowing, i.e., when private borrowing would otherwise have been small at nominal fixed rates and the proportion of private borrowing at floating rates would have been large.

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[40]It would be more desirable, of course, to also account for the maturity composition of Federal debt. In those maturity ranges where Federal debt represents a larger fraction of all debt, one would expect the risk premiums to be larger. Similarly, when the overall maturity composition of Federal debt is shortened, as it has been during the first term of the Clinton administration, one would expect a flattening of the yield curve, *ceteris paribus*. The lack of these more refined data might also reduce empirical explanatory power.

It would also be interesting to calculate the volume/risk premium connection in countries where governments issue some indexed bonds. For example, Woodward [1990] presents a time series of *ex ante* real interest rates derived from British government index-linked bonds and a time series of expected inflation rates derived by subtracting the real interest rate from the nominal yield on a non-indexed government bond. He notes that the resulting estimate "...will overstate the expectation [of inflation] because it will include the [risk] premium," (p. 377.) In the United Kingdom, one might anticipate that the risk premium would depend both on the volatility of expected inflation relative to real yields and on the proportion of government borrowing in fixed versus indexed form.

Recently, Kandel, Ofer and Sarig [1996] calculated risk premiums by using Israeli nominal and indexed bonds. They found a statistically significant relation between the level of the risk premium and the *ex ante* volatility of inflation, as estimated from the dispersion of relative prices within the general consumer price index; (See their Section V.) Presumably, the significance of this relation should increase with the magnitude and proportion of government borrowing in the nominal form.

The preliminary investigation with U.S. data in the preceding section raised more questions than it answered. Although the forward rate premium over the subsequent spot rate increased with the ratio of Federal to Private debt, the effect was not very significant. Moreover, the effect of the estimated relative volatilities of expected inflation to real interest rates was significant with the wrong sign.

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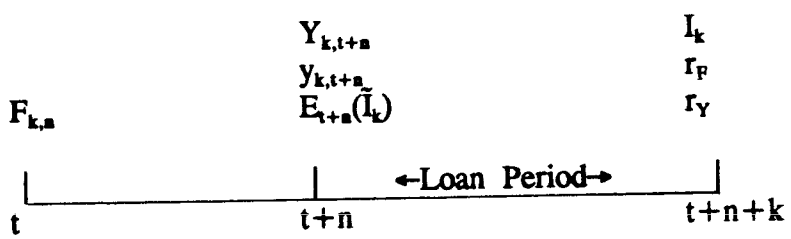
Table 1

Time Line and Glossary

The time line below indicates when interest rates and inflation become known for certain and when investor anticipations are formed. The symbols are defined below the time line. All rates are continuously-compounded.



t	The current period; when a decision is made either (1) to contract for a loan in the forward market or (2) to defer contracting until the later spot loan market.
$F_{k,n}$	The forward nominal interest rate on a loan for k periods, contracted in period t, to commence after period t+n.
$Y_{k,t+n}$	The spot nominal yield on a loan for k periods, contracted at t+n, (and beginning immediately thereafter.)
I_k	The actual inflation rate during the loan period, from t+n to t+n+k.
r_F	The realized real return, (*ex post*), on the forward loan, $r_F = F_{k,n} - I_k$.
r_Y	The realized real return on the spot loan, $r_Y = Y_{k,t+n} - I_k$.
$y_{k,t+n}$	The *ex ante* real yield on a k-period spot loan as of t+n, $y_{k,t+n} \equiv Y_{k,t+n} - E_{t+n}(\tilde{I}_k) = E_{t+n}(\tilde{r}_Y)$, where $E_{t+n}(.)$ denotes an expectation formed at period t+n. The ~ denotes uncertainty.

Table 2

Forward Rate Prediction Error (Estimated Risk Premium)
Regressed on the Relative Level of U.S. Federal Debt
and an Estimate of Expected Inflation Less Real Interest Volatilities
January, 1970 through December, 1992 (monthly observations)

Federal/Private Debt Ratio	Relative Volatility b_t	Adjusted R-Square	Durbin-Watson
Ordinary Least Squares			
0.633	-.380	0.0528	1.32
(1.71)	(-3.57)		
Generalized (Weighted) Least Squares[41]			
0.675	-.359	0.0389	1.75
(1.91)	(-2.97)		
Cochrane-Orcutt OLS (For Autocorrelated Disturbances)			
0.623	-.400	0.162	1.94
(1.19)	(-2.72)		

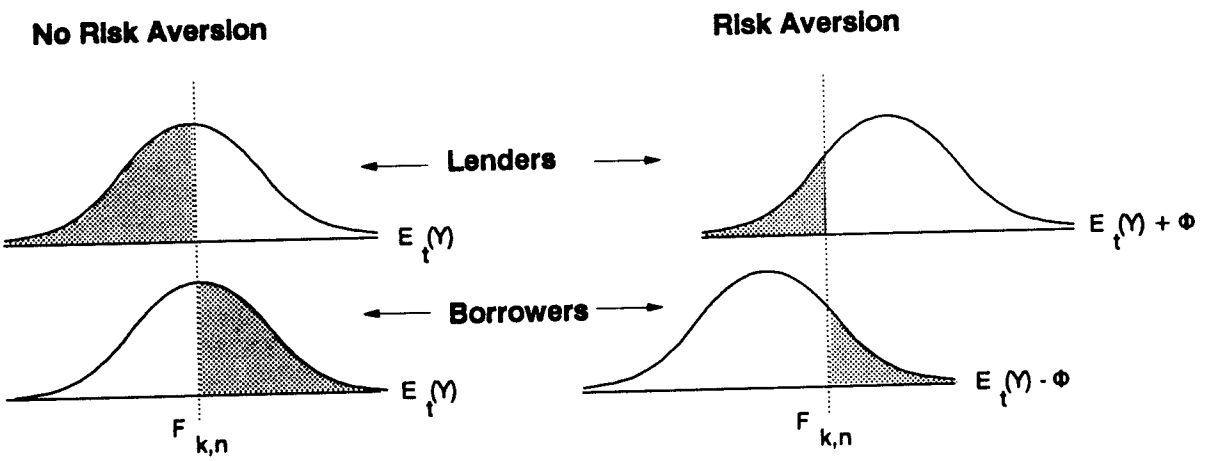
ML Estimate of First-Order Autocorrelation of disturbances: 0.339

T-Statistics are in parentheses.

[41]The weights are the reciprocals of the standard errors of b_t.

Figure 1

Forward Loan Market Equilibrium

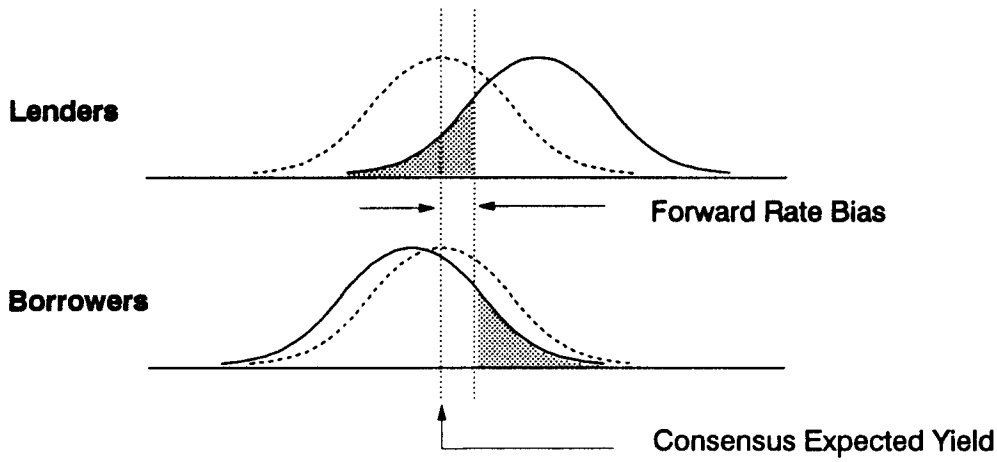


Shaded Areas Indicate Forward Loan Contracting Volume

Figure 2

Forward Loan Market Equilibrium

With Asymmetric Risk Aversion



- - - - - - - - - - - - - - Distribution of expected future spot yields

———————— Distribution of expected future spot yields plus (for lenders) or minus (for borrowers) risk premiums

Shaded Areas Indicate Loan Volume

Figure 3, Panel A

CIR Single-Factor Model (Zero Price of Risk)

Jensen's Inequality Effect on Forward Rate Bias

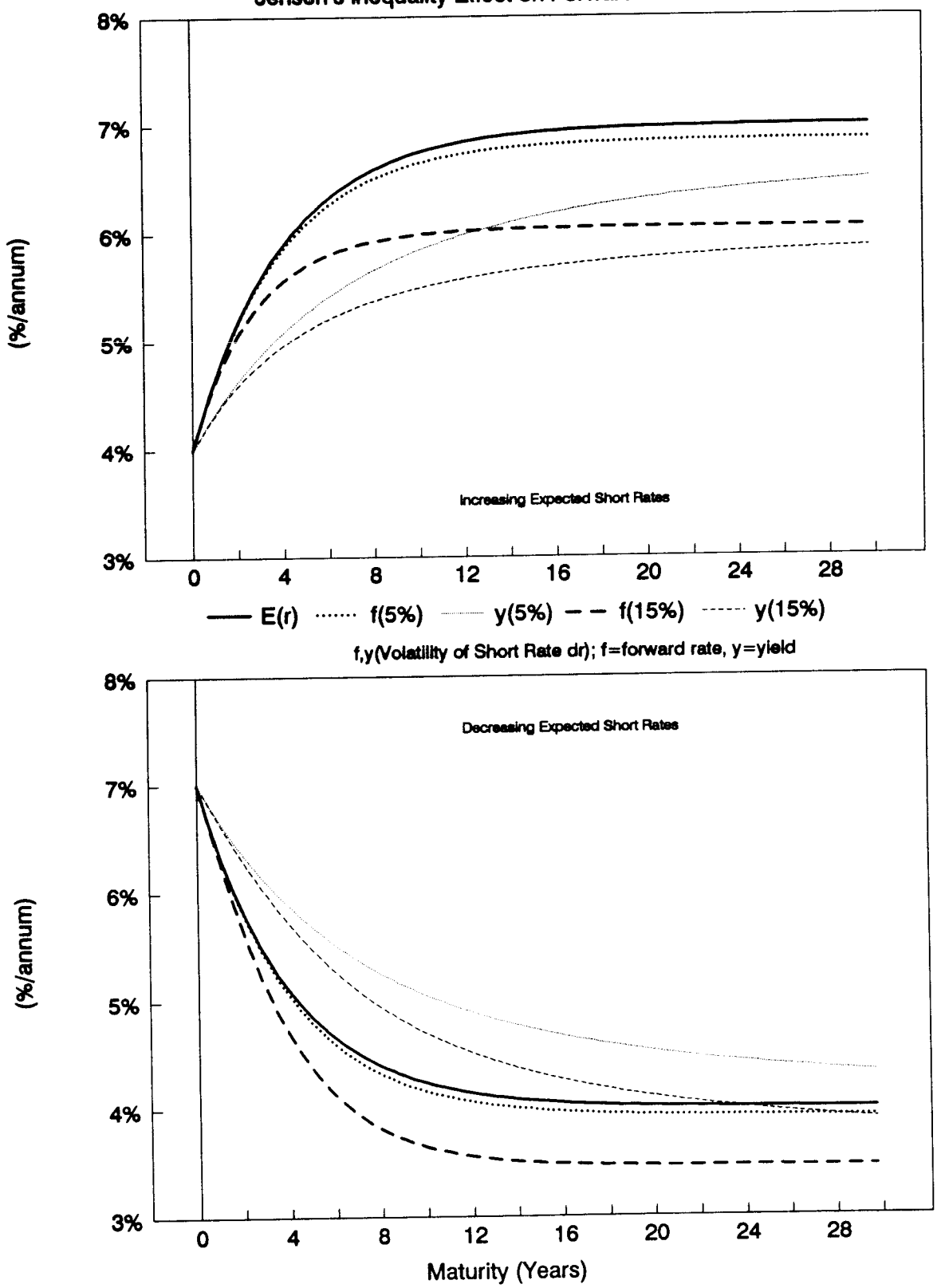


Figure 3, Panel B

CIR Single-Factor Model

Effect of Price of Risk on Forward Rate Bias

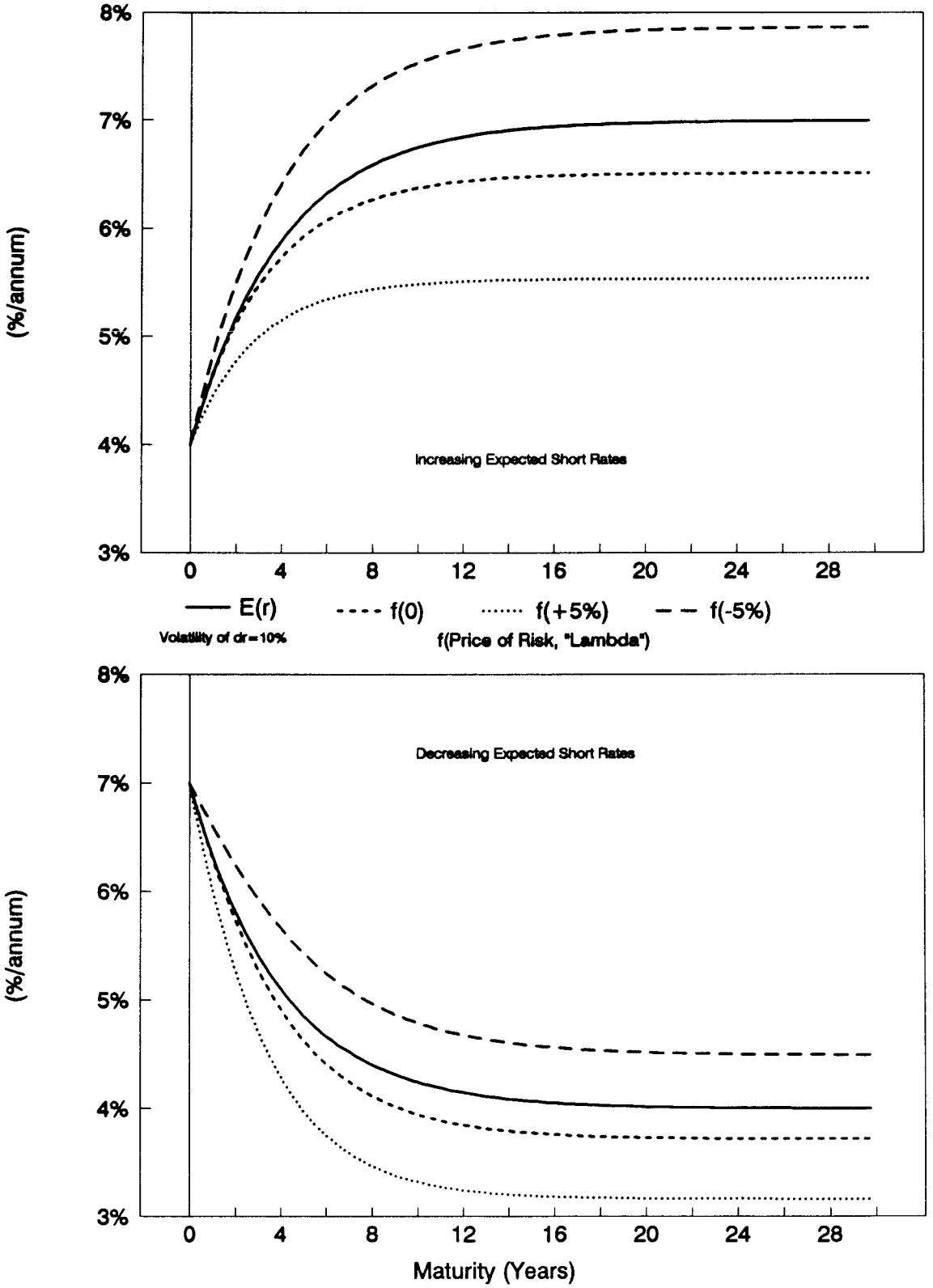


Figure 3, Panel C

CIR Single-Factor Model (High Volatility)

Humped Yield and Forward Rate Curves

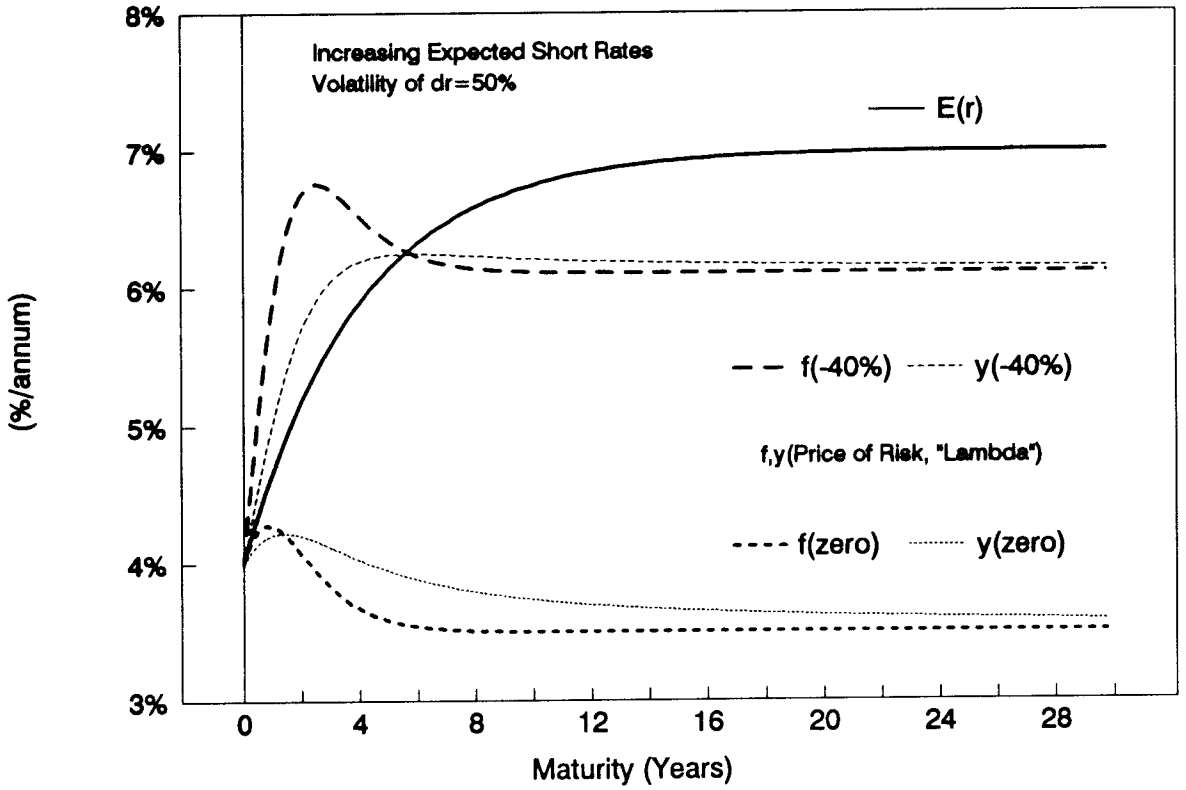
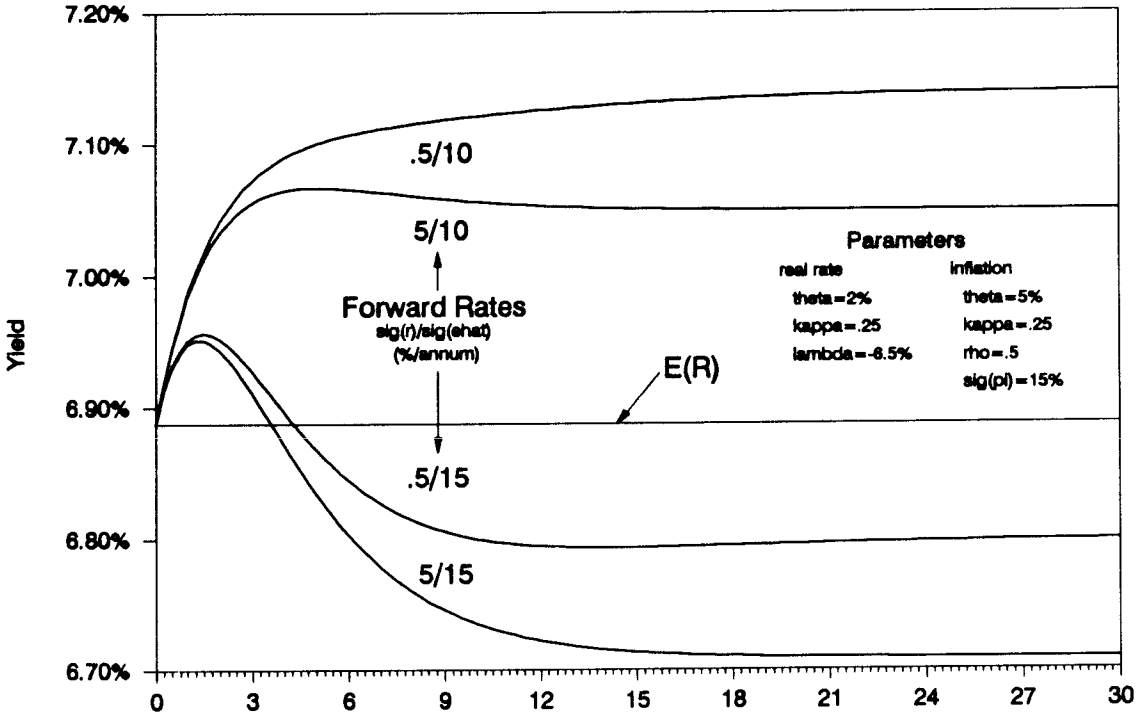


Figure 4

CIR Nominal Forward Rate Bias

Panel A: Effect of real rate and expected inflation volatilities



Panel B: Effect of Unexpected Inflation Volatility

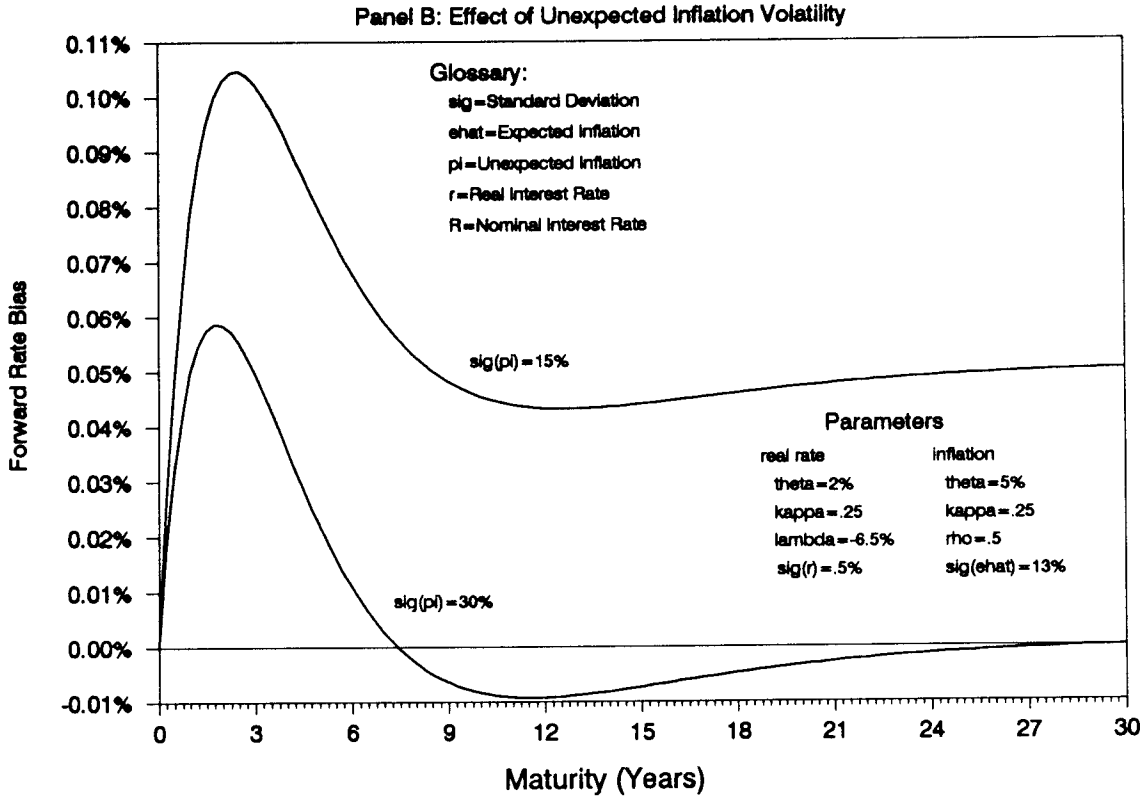


Figure 5

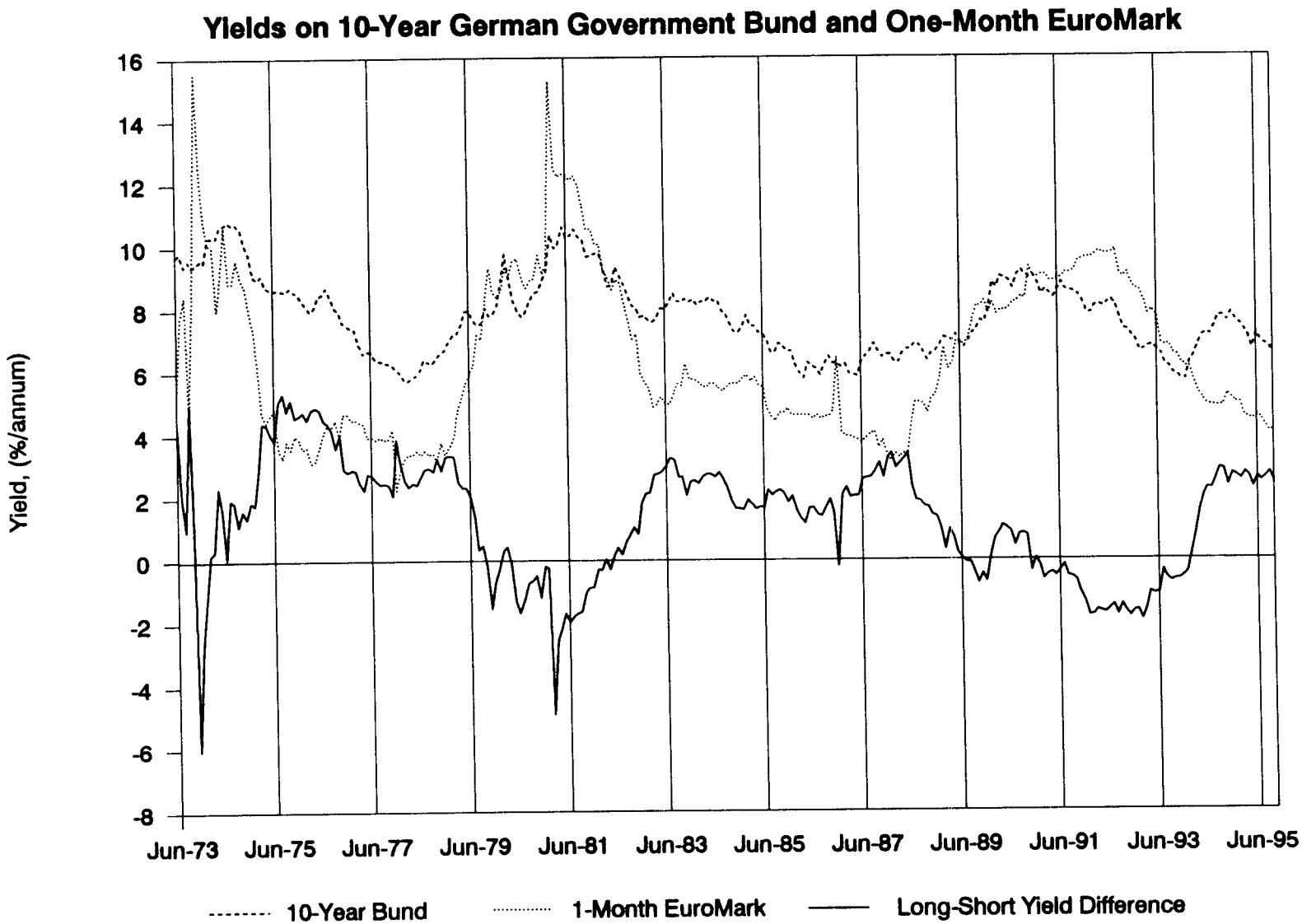
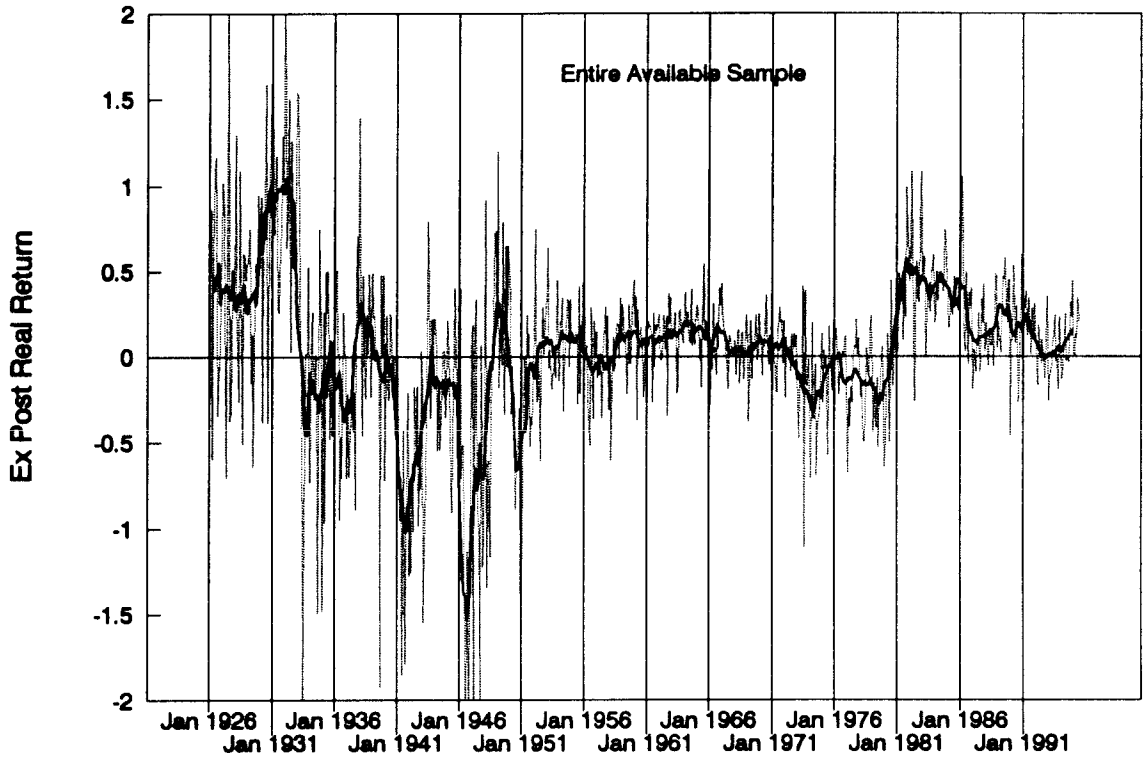

Yields on 10-Year German Government Bund and One-Month EuroMark
Yield, (%/annum)
16
14
12
10
8
6
4
2
0
-2
-4
-6
-8
Jun-73
Jun-75
Jun-77
Jun-79
Jun-81
Jun-83
Jun-85
Jun-87
Jun-89
Jun-91
Jun-93
Jun-95
10-Year Bund
1-Month EuroMark
Long-Short Yield Difference

Figure 6

U.S. Ex Post Real Returns

From One-Month Treasury Bills



Monthly Observation —— 12-Month Moving Average

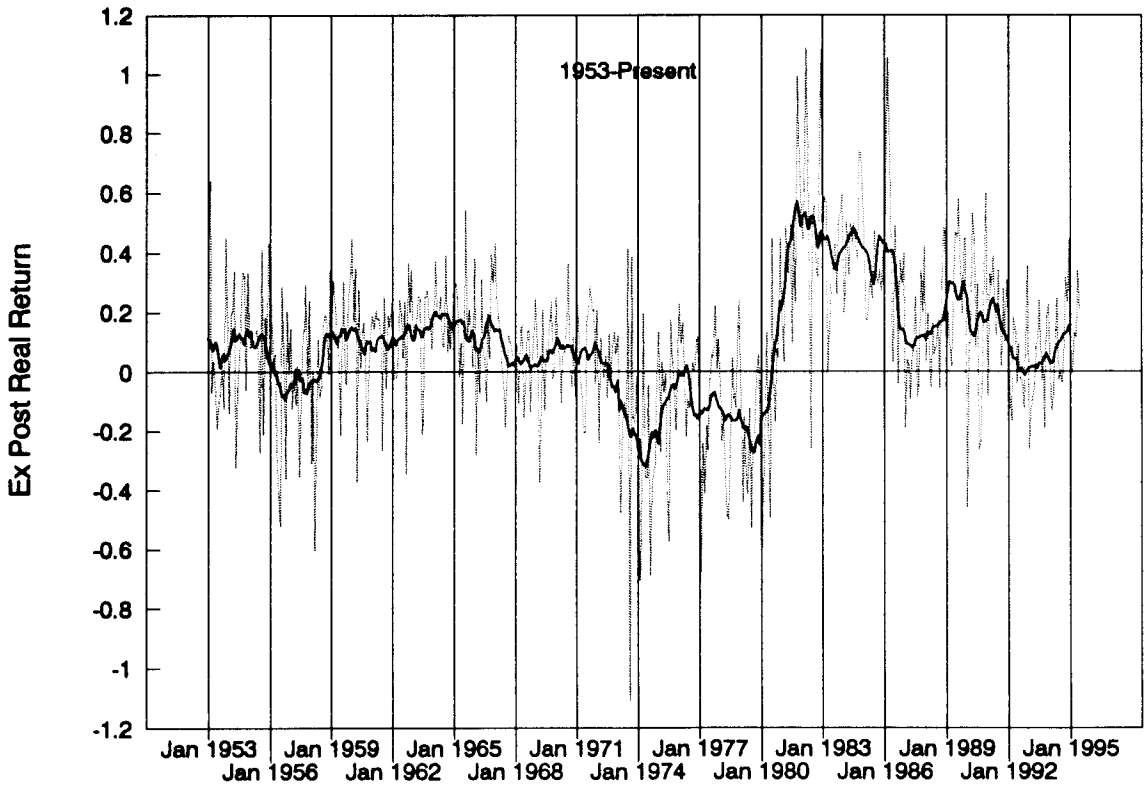


Figure 7, Panel A

Volatility Difference

U.S. Expected Inflation Less Real Interest Rate

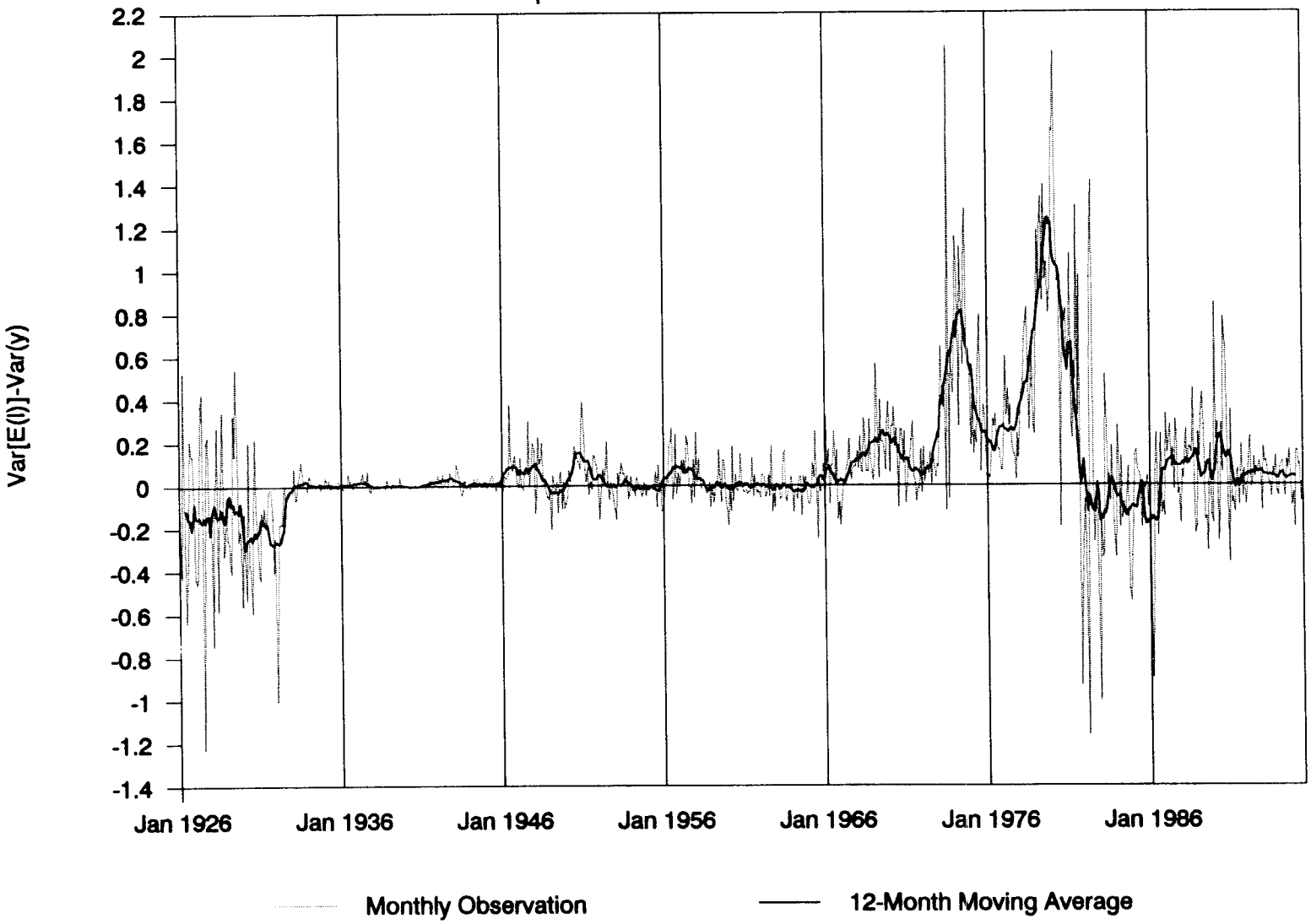


Figure 7, Panel B

Volatility Difference

U.S. Expected Inflation Less Real Interest Rate, Sub-Periods

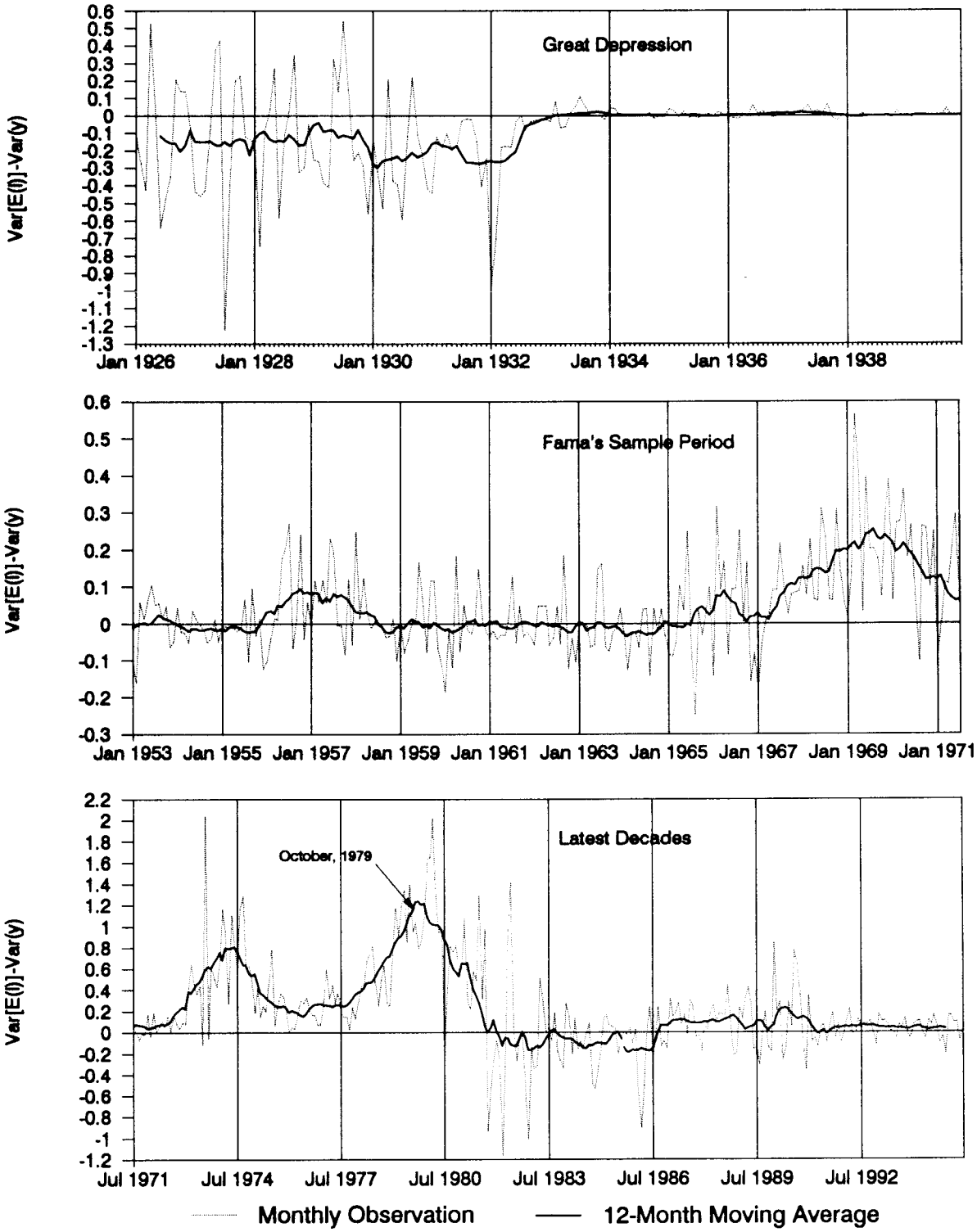
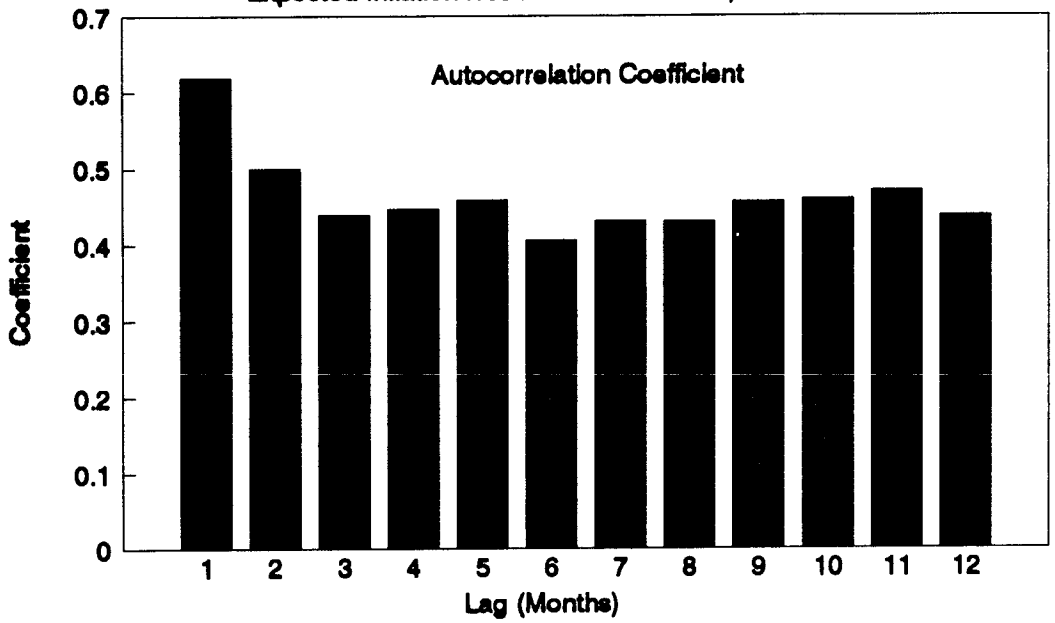
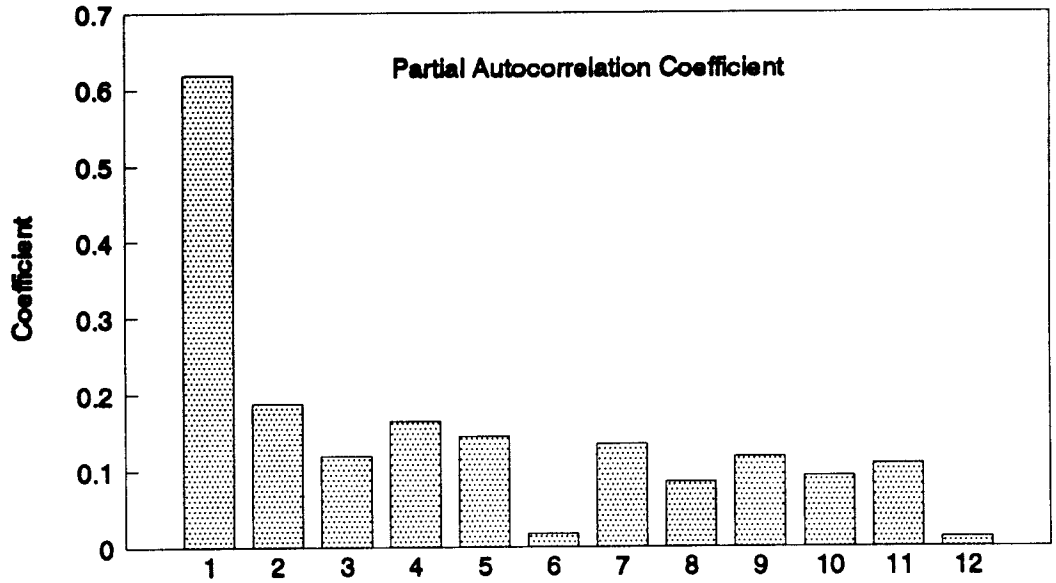


Figure 7, Panel C

Autocorrelation Structure of Volatility Difference

Expected Inflation less Real Interest Rate, 1926-95





ARIMA Model

	AR(1)	AR(2)	MA(1)	MA(2)
Coefficient:	1.17	-.189	.770	.0559
T-Statistic:	(11.5)	(-1.93)	(7.25)	(.678)

Adjusted R-Square: 0.453

Figure 8

Ex Post Real Return from One-Month EuroMark

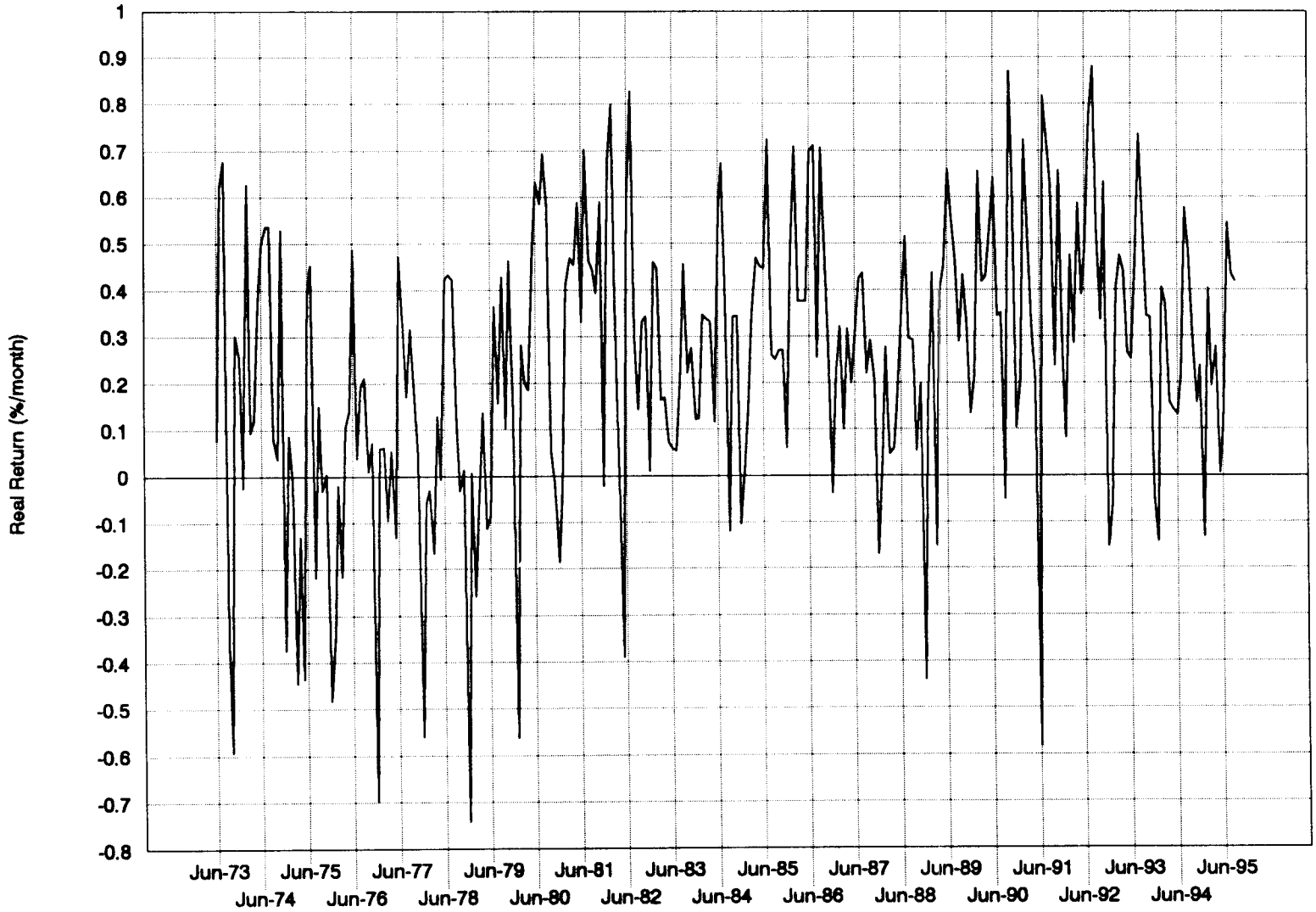


Figure 9

One-Month EuroMark and Subsequent Month's Inflation

with 12-Month Moving Average of Inflation

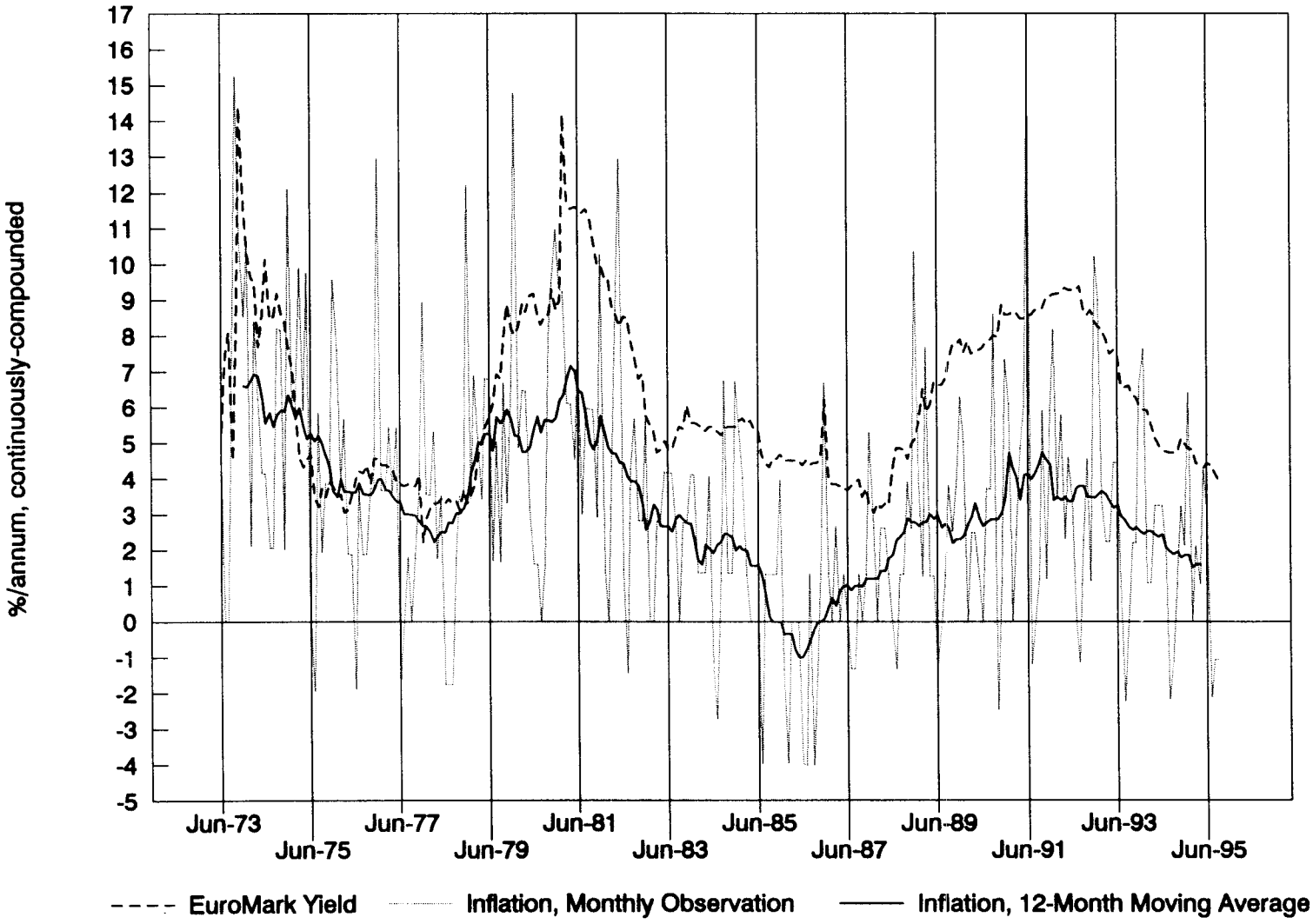


Figure 10

Volatility Difference

German Expected Inflation Less Real Interest Rate

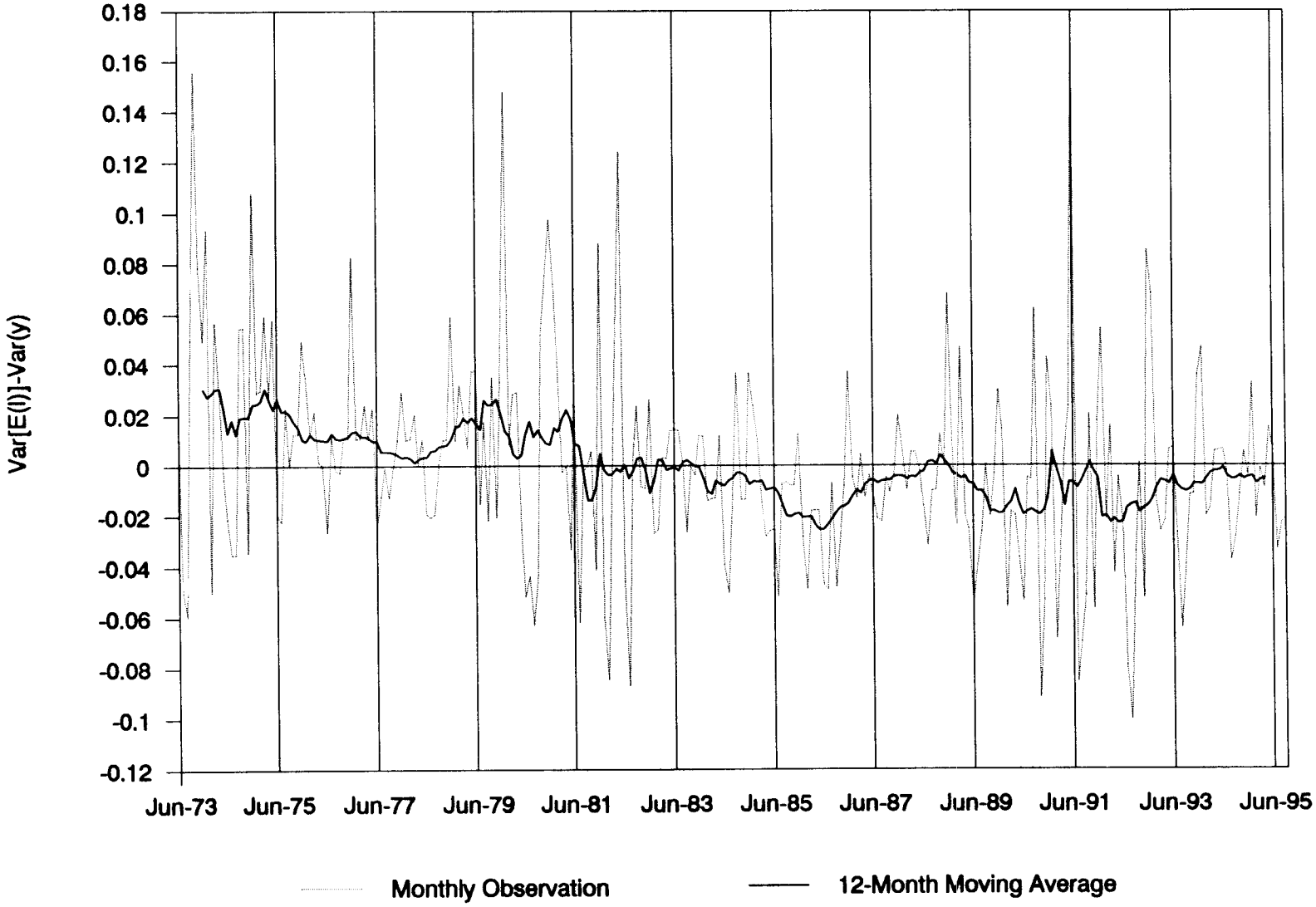


Figure 11

Imputed Risk Premiums, Ten-Year Bund Forward Rates

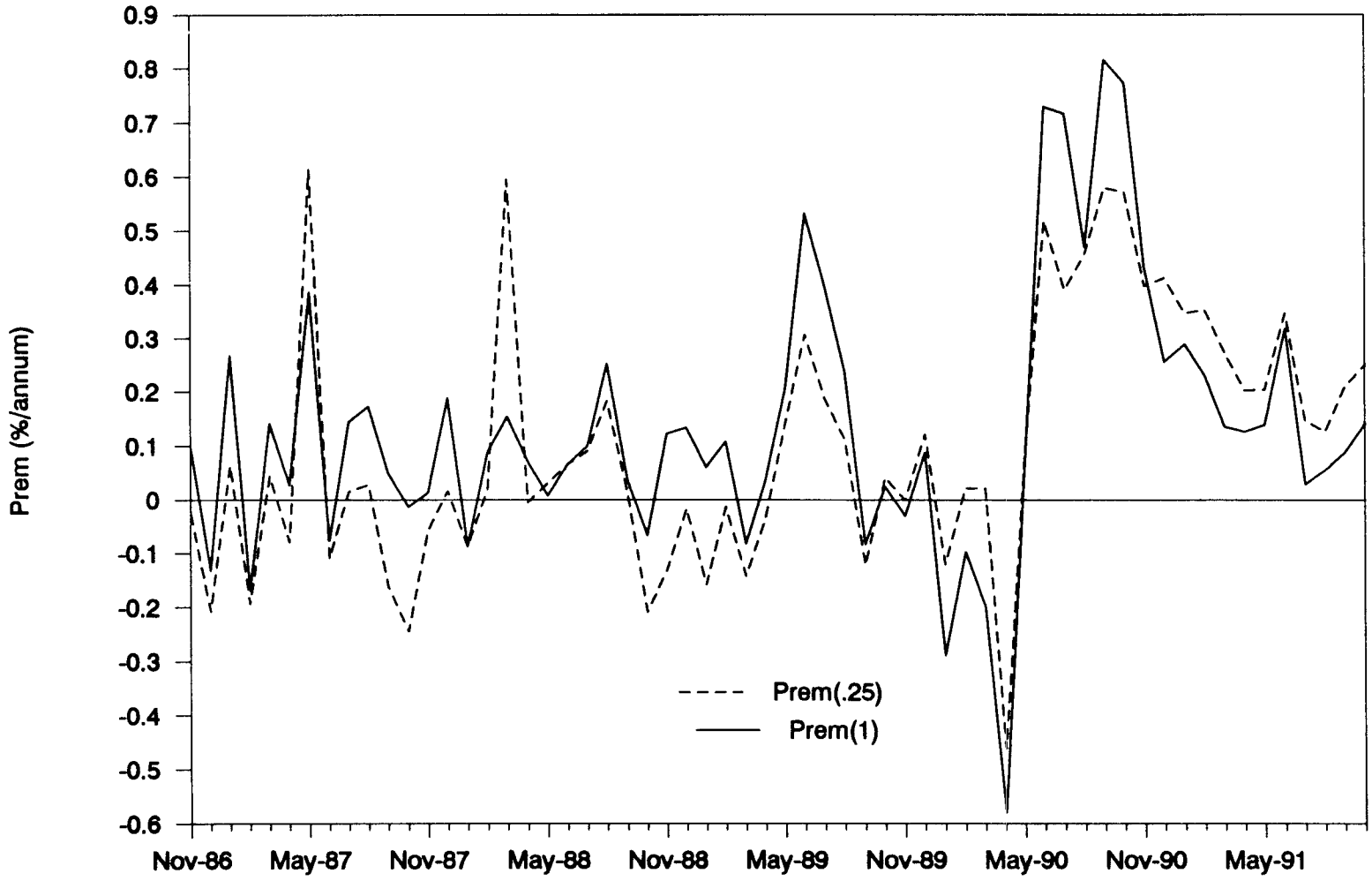


Figure 12

Forward Rate Ex Post Bias

US T-Bills, One-Month-Ahead Forecast, from CRSP "Fama" File

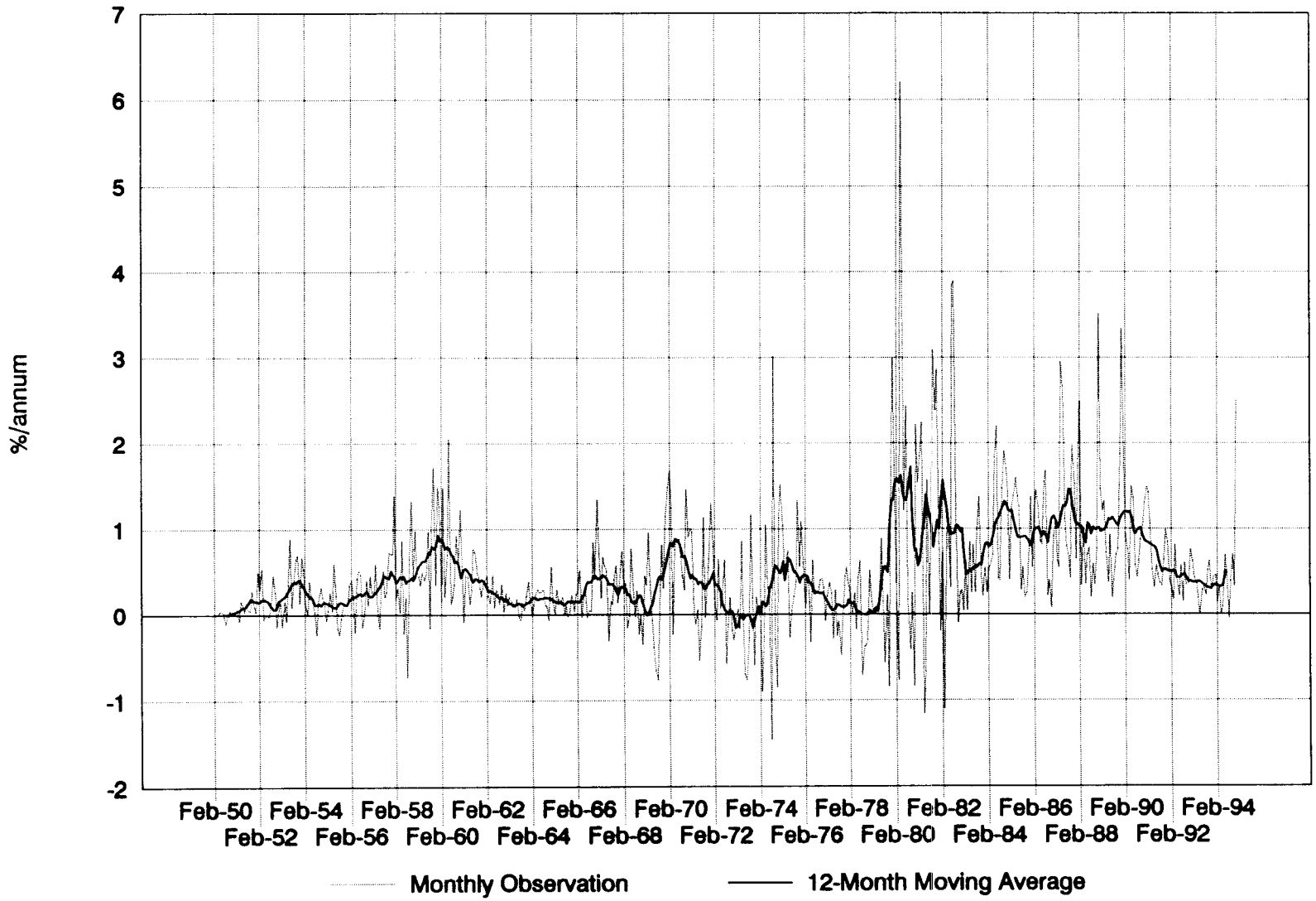


Figure 13, Panel A

Federal and Private Outstanding Debt

USA, 1987 Constant Dollars

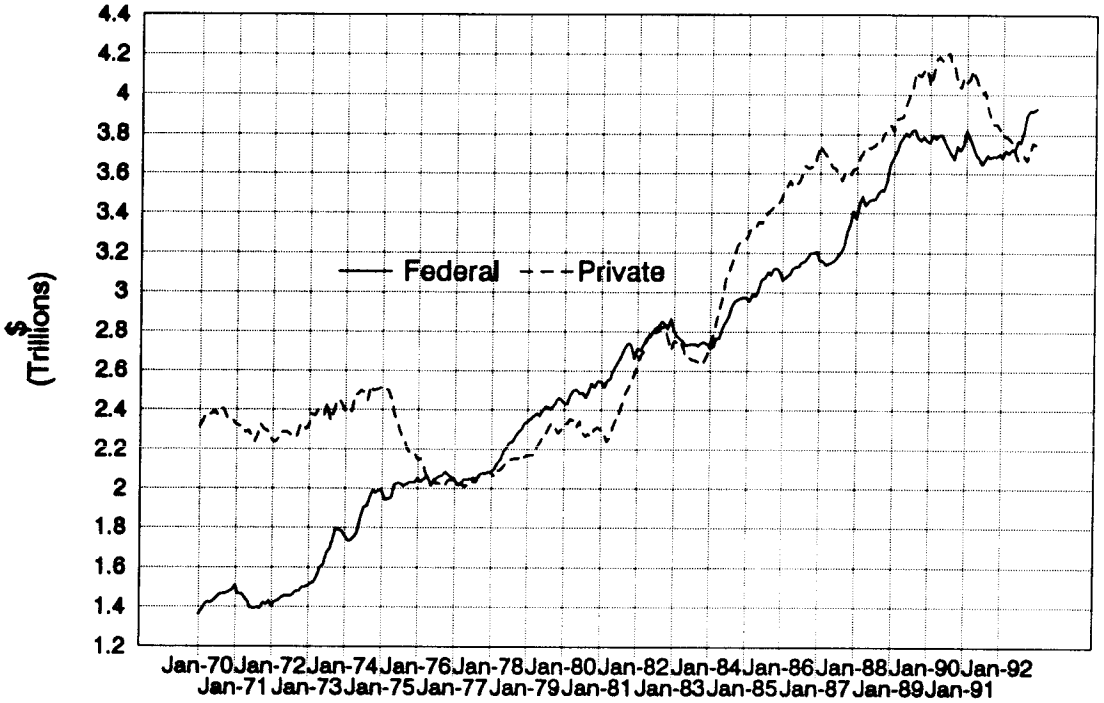


Figure 13, Panel B

Debt Outstanding, Federal/Private

USA, 1970-1992

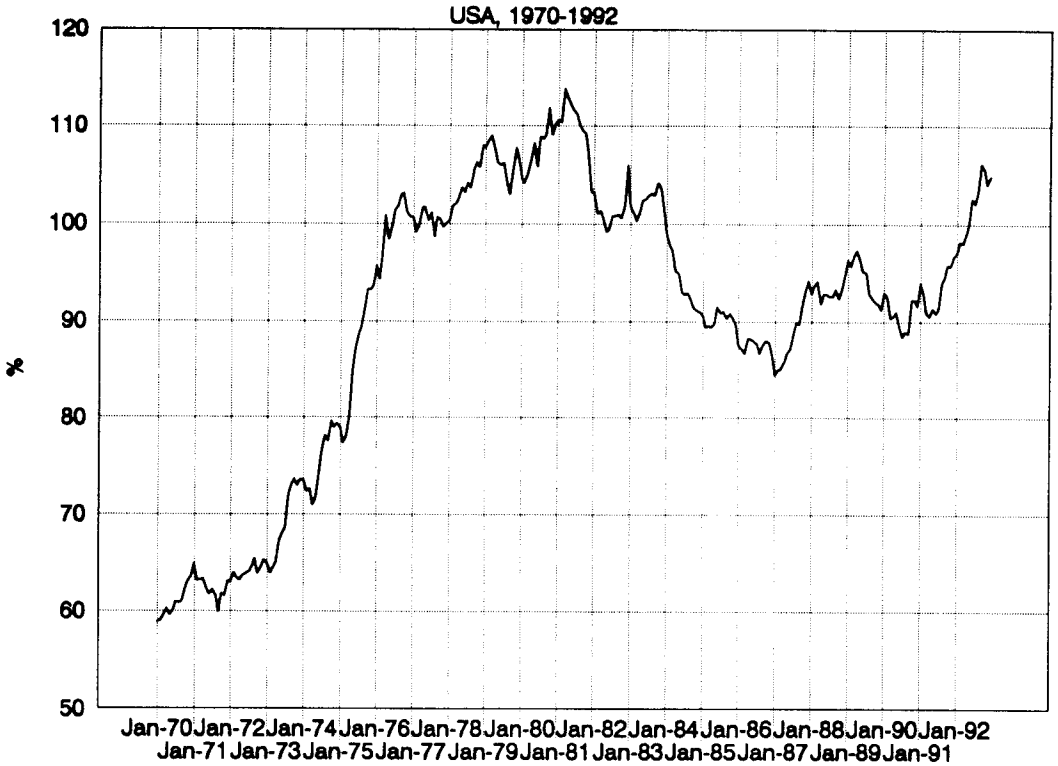


Figure 14

Estimated Volatility Difference

from 36-month rolling regressions