Dynamic Choice and Risk Aversion

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Abstract

This paper studies the dynamic asset allocation between a riskless and a risky asset with stochastic volatility. The optimal portfolio weight is derived explicitly. I show that intuitions about risk aversion from static choice problems may not apply in a dynamic choice setting. For example, a more risk-averse investor may hold more risky assets; the equilibrium risk premium may decrease with the risk aversion of the representative agent; an infinitely risk averse agent may hold positive amounts of the risky asset; and a risk-averse investor may short (sometimes infinite amount of) a risky asset with strictly positive risk premium. I argue that these counter-intuitive results are due to rebalancing and can be understood using the concept of equivalent myopic measure. The results suggest that it may be not appropriate to use stock holdings as a proxy for risk aversion and that the dynamic nature of asset allocation may be another source of market non-participation.
Dynamic Choice and Risk Aversion

This paper studies the dynamic asset allocation between a riskless and a risky asset with stochastic volatility. The optimal portfolio weight is derived explicitly. I show that intuitions about risk aversion from static choice problems may not apply in a dynamic choice setting. For example, a more risk-averse investor may hold more risky assets; the equilibrium risk premium may decrease with the risk aversion of the representative agent; an infinitely risk averse agent may hold positive amounts of the risky asset; and a risk-averse investor may short (sometimes infinite amount of) a risky asset with strictly positive risk premium. I argue that these counter-intuitive results are due to rebalancing and can be understood using the concept of equivalent myopic measure. The results suggest that it may be not appropriate to use stock holdings as a proxy for risk aversion and that the dynamic nature of asset allocation may be another source of market non-participation.

1 Introduction

In this paper I study a dynamic asset allocation problem when stock returns display stochastic volatility. I derive the optimal portfolio weight explicitly. The optimal portfolio weight is used to study market timing and the equilibrium restrictions on the asset price dynamics. The optimal portfolio weight also highlights the qualitative differences between dynamic and static portfolio choices. For example, in the dynamic setting, a risk averse agent may hold an infinite amount of the risky asset even though the risk premium is finite; a risk averse agent may short the risky asset and a more risk averse agent may hold more risky assets even if the risk premium is strictly positive. These differences defy intuitions about risk aversion in static choice problems, which is all the more puzzling considering dynamic choice problems in multiple periods can be reduced to a one-period static choice problems by dynamic programming principle. I show that the asset returns have a different distribution, which is termed equivalent myopic distribution (EMD), in the reduced one-period problem. The apparent counter-intuitive results are due to the dependence of the equivalent myopic distribution on the risk aversion of the agent.

The paper uses the framework of Merton (1969, 1971). I assume there are two assets, a riskless asset and a risky one. The riskless asset has a constant return. The return of the risky asset follows a continuous time diffusion process; the instantaneous variance of the
risky asset follows a diffusion process (so the asset returns have stochastic volatility). The risk premium of the risky asset has constant elasticity of volatility (CEV); that is, the risk premium is a power function of the volatility.\(^1\) For analytical tractability and I assume that the variance process is a power function of a square-root process because I believe that the qualitative behavior of the optimal portfolio weight does not depend on the detail of the volatility process. The agent rebalances continuously to maximize a power utility\(^2\) over the end of period wealth. Under the above assumptions, the optimal portfolio weight is derived explicitly as a function of the instantaneous volatility, the (constant) interest rate, the investment horizon, the constant relative risk aversion coefficient and the parameters of the volatility process.

The optimal portfolio weight in general depends on the volatility, which implies that the agent times the volatility. The dependence is determined by the instantaneous market price of risk (IMPR), which is the ratio of the instantaneous risk premium over the instantaneous variance. If the IMPR is increasing in volatility, the return for taking risk is high at high volatilities, so the agent holds more risky assets when the volatility is high. On the other hand, if the IMPR is decreasing in volatility, the return for taking risk is low at high volatilities, the agent holds less risky asset when the volatility is low.

However, market clearing of the risky asset and zero net supply of the riskless asset require the risky asset portfolio of representative agent to be 1 and therefore a constant. This implies that for the agent in the model to be the representative agent, the portfolio weight cannot depend on the volatility. This is only true when IMPR is constant. In this case, the equilibrium risk premium can be computed. Interestingly, I show that the equilibrium risk premium in the dynamic economy can decrease with the risk aversion of the representative agent. This is in contrast to the mean-variance theory, in which the equilibrium risk premium is proportional to the risk aversion of the representative agent.

The optimal portfolio weight is a monotonic function of the investment horizon. If the instantaneous Sharpe ratio (ISR) (which is the ratio of instantaneous risk premium over the instantaneous volatility) is decreasing in volatility, the optimal portfolio weight

\(^1\)Merton (1980) pointed out that the risk premium is crucial for portfolio choice and is hard to estimate empirically. He proposes that the risk premium may depend on powers of volatility with the power to be either 0, or 1, or 2.

\(^2\)Since the interest rate is constant, one can also get explicit solutions for constant absolute risk averse (CARA) and constant hyperbolic risk averse (HARA) investors.
of a conservative\textsuperscript{3} investor is decreasing in the horizon, from the myopic portfolio weight\textsuperscript{4} at short horizons to a finite value at infinite horizons; the optimal portfolio weight for aggressive investors is increasing in the horizon, from the myopic portfolio weight and may reach positive infinity at a finite horizon. This is because the aggressive investors are always willing to take advantages of any opportunities and the advantage of the dynamic choice over the multiple-periods may be considered by them to be so huge that they would take an infinite position. If the ISR is increasing in volatility, the optimal portfolio weight of a conservative investor is increasing from the myopic portfolio weight at short horizon to a finite value at infinite horizons; the optimal portfolio weight for an aggressive investor is decreasing from the myopic portfolio weight, and surprisingly, may reach negative infinity at a finite horizon. This is the consequence of rebalancing and is only true in the dynamic choice problem with multiple periods.

The most striking feature of the optimal portfolio weight is its dependence on risk aversion. If the ISR is decreasing in volatility, the portfolio weight is decreasing in risk aversion and reaches $+\infty$ for risk aversion coefficient in $(0,1)$. If the ISR is increasing in volatility, the portfolio weight can be either decreasing in risk aversion and reaches $+\infty$ at 0 if the dynamic hedging effect is small, or increasing and reaches $-\infty$ for risk aversion coefficient in $(0,1)$ if the dynamic hedging effect is large. In the last case, a more risk averse investor may hold more of the risky asset and a risk averse investor may even short the risky asset despite a strictly positive risk premium. These results are even more surprising considering that dynamic choice problems can be and are often reduced to static problems using dynamic programming principle. The puzzle is resolved by using the concept of equivalent myopic distribution or equivalent myopic measure (EMM). The equivalent myopic distribution (EMD) is the distribution of the asset return in the reduced static problem. The EMD depends on the risk aversion of the agent—it is different for different investors (of different risk aversions) because of the endogeneity of rebalancing (different investors will rebalance differently). This is demonstrated readily when asset returns follow a diffusion process, in which case, a change to an equivalent

\textsuperscript{3}Throughout the paper, I will refer to investors with constant relative risk aversion (CRRA) larger than 1 as conservative investors and those with constant relative risk aversion less than 1 but larger than 0 as aggressive investors. Note that aggressive investors are still risk averse.

\textsuperscript{4}The myopic portfolio weight of the risky asset is equal to the ratio of risk premium over variance divided by the relative risk aversion coefficient.
measure amounts to a change of expected return (drift).

Although there have been many studies on dynamic portfolio choice since Mossin (1967), Samuelson (1968) and Merton (1968), it seems that few have attempted to understand the qualitative differences between dynamic and static choice and the role of rebalancing. The counter-intuitive results regarding risk aversion is a general phenomenon in dynamic choice problems. For example, they exist in models with predictability, such as Kim and Omberg (1996), Brennan, Schwartz, and Lagnado (1997), Barberis (1999), Brennan and Xia (1999), Campbell and Viceira (1999), Liu (1999), and Skiadas and Schroder (1999). However, in these models, the risk premium can be negative, so the counter-intuitive results can be branded as pathological problems associated with negative risk premium. Instead, in the stochastic volatility model studied in this paper, the risk premium is always positive and the short rate is constant, so the counter-intuitive result can only come from the effects of dynamic rebalancing.

Recently, there have several studies on portfolio problems with stochastic volatility, such as Liu (1998) and Chacko and Viceira (1999). Liu (1998) assumes that the risk premium is proportional to the variance while Chacko and Viceira (1999) assume that the risk premium is independent of variance respectively. In this paper, I consider a more general risk premium. More importantly, neither Liu (1998) nor Chacko and Viceira (1997) studies the role of rebalancing, which is studied in this paper. Longstaff (2000) studies portfolio choice where assets have stochastic volatility under logarithmic utility and his focus is on liquidity. Liu, Longstaff, and Pan (2001) study the effects of jumps in both stock return and volatility. Finally, Ang and Bekaert (1998) and Das and Uppal (1999) study portfolio problems which can be viewed as alternative specifications of the stochastic volatility.

The paper is organized as follows. In section 2, I will review two important properties of static portfolio choice. In section 3, I use binomial models to give intuition on dynamic choice and equivalent myopic measures. In section 4, I specify the asset price dynamics and the utility function of the agent. In section 5, I derive the optimal portfolio weight and discuss its dependence on the volatility, investment horizon, risk aversion, and the equilibrium implications. I also fit a stochastic volatility model to US market data and study the portfolio weight with the estimated parameters. In section 6, I study EMD for the stochastic volatility model and use it to explain the counter-intuitive results of
dynamic choice problems. I conclude in section 7. I leave the calculation details to the appendix.

2 Static Choice and Risk Aversion

In this section, I will discuss two important properties of static choice which highlight the difference between the static and dynamic choice. I will call them the participation theorem and the calibration theorem respectively.

Theorem 2.1 (Participation Theorem) If the risk premium $E[r_e]$ of an asset is positive (negative), then a risk averse agent will hold a positive (negative) amount of the asset.

The theorem and the proof are given in Huang and Litzenberger (1988). The proof is simple. Suppose $E[r_e] > 0$, the marginal utility at $\phi = 0$ is $E[U'(r_f + \phi r_e)]|_{\phi=0} = U'(r_f)E[r_e] > 0$, this implies that $\phi^* > 0$ because the utility function $U$ is concave (the agent is assumed to be risk averse) in wealth and therefore the expected utility function is concave in $\phi$.

The participation theorem states that agents should always take advantage of the excess returns of risky assets. An application to the US stock market implies that all investors should hold positive amount of stock because the stock risk premium is positive. The fact that a significant proportion of US population does not hold stocks is the so-called non-participation puzzle, which is being actively studied in the literature.

Theorem 2.2 (Calibration Theorem) If the risk premium $E[r_e]$ of an asset is positive (negative), the optimal portfolio weight decreases (increases) with the risk aversion of the agent.

The calibration theorem is a variation of Arrow’s theorem on insurance premium. The proof is given in the Appendix. The theorem is very intuitive, it implies that a less risk averse agent holds more of the risky asset. This theorem has both theoretical as well as practical importance. The optimal portfolio weight as well as many other economic variables, such as the pricing kernel in the consumption based asset pricing model, depends
on the risk aversion $\gamma$ of agents. But the risk aversion of agents is not easily measured. One common method to calibrate $\gamma$, which is used in academic studies as well as in practice, is to infer it from the portfolio weight of the agents. The Calibration Theorem provides the theoretical foundation for this method.

Both the participation and the calibration theorems provide fundamental intuition for expected utility theory as a theory of risk and have economic importance. However, I will show that both theorems are violated in the dynamic choice models. Since the pattern of violation depends on whether the relative risk aversion is greater or smaller than 1, I provide the following definition for convenience.

**Definition 2.1 (Conservative and Aggressive Agents)** A conservative (aggressive) agent is a risk averse agent with constant relative risk aversion $\gamma$ greater than (smaller than) 1.

Note that the utility function of constant relative risk aversion is $\frac{w^{1-\gamma}}{1-\gamma}$. For conservative agents ($\gamma > 1$) is bounded from above by 0 and is unbounded from below when the return approaches zero. The agent with this utility function suffers huge loss of utility when the return is close to zero, so conservative agents cares relatively more about the variance than the expected return. The utility function for aggressive agents ($\gamma < 1$) is bounded from below by 0 and is unbounded from above. The agent with this utility function does not suffer much loss of utility when the return is close to zero and benefit more from consecutive large returns, so aggressive agents care relatively more about the expected return than the variance of returns.
3 Binomial Models

In this section, I solve the optimal static portfolio weight for a one-period binomial tree model and the optimal dynamic portfolio weight for a two-period binomial tree model. The dynamic portfolio weight has qualitatively different properties than the static portfolio weight because of rebalancing. The concept of equivalent myopic measure is used to understand the effects of the rebalancing and the behavior of dynamic choice.

3.1 One-Period Binomial Tree Model

The static portfolio weight in a one-period binomial tree model will help us study and understand the two-period dynamic choice model later.

**Definition 3.1 (One-Period Binomial Model)** A one-period binomial tree model has two assets. The riskfree asset has gross return of 1 and the risky asset has excess return $r^e$ of $u > 0$ with probability $p$ and $d < 0$ with probability $q = 1 - p$.

This is the standard one-period binomial tree model with riskfree return 1. The absence of arbitrage means that $u > 0$ implies $d < 0$.

**Definition 3.2 (One-Period Agent)** The objective of the one-period agent is to choose the portfolio weight $\phi$ to maximize the utility function over the end-of-period wealth

$$U = E \left[ \frac{1}{1-\gamma} (1 + \phi r^e)^{1-\gamma} \right].$$

The expected utility can be explicitly expressed as

$$U = \frac{p(1 + \phi u)^{1-\gamma} + q(1 + \phi d)^{1-\gamma}}{1 - \gamma}.$$

The first order condition is given by

$$up(1 + \phi^* u)^{-\gamma} + dq(1 + \phi^* d)^{-\gamma} = 0,$$

which lead to the following optimal portfolio weight of the risky asset,

$$\phi^* = \frac{A - 1}{u - Ad}.$$
where $A = \left( \frac{pu}{-qd} \right)^{\frac{1}{2}}$. Note that $A > 1$ ($A < 1$) if the risk premium $E[r_e] = pu + qd$ is positive (negative). The properties of the optimal portfolio weight is summarized in the following proposition.

**Proposition 3.1 (Optimal One-Period Portfolio Weight)** The optimal portfolio weight $\phi^*$ of the one-period agent has the following properties: a) when the risk premium is positive (negative), $\phi^*$ is positive (negative) and is a decreasing (increasing) function of the risk aversion $\gamma$; b) when $\gamma \to \infty$, $\phi^* \to 0$; c) when $\gamma \to 0$, $\phi \to -\frac{1}{d}$ ($\phi \to -\frac{1}{u}$) if the risk premium is positive (negative).

Property a) confirms the participation and the calibration theorem. Property b) is quite intuitive, an infinitely risk averse agent does not like any variation in his terminal wealth and any position (long or short) in the risky asset will lead to variations. Property c), which says that an agent which is close to risk neutral will hold a finite and bounded position, is a little puzzling, because a risk-neutral agent ($\gamma = 0$) will take an infinite position, This is due to the fact that marginal utility approaches $-\infty$ when the wealth approach $0$, the wealth has to be positive. This implies that portfolio weight is bounded

$$-\frac{1}{u} < \phi < \frac{1}{-d},$$

as long as $\gamma \neq 0$. The first inequality implies the investor will not short the risky asset of more than $\frac{1}{u}$ of the wealth and second inequality implies that the investor will not hold the risky asset more than $\frac{1}{-d}$ fraction of the wealth. Note that these bounds do not depend on the probability $p$.\(^{5}\) The portfolio weight of the least risk averse agent (one with $\gamma$ close to zero) does not depend on the probability $p$: it is $\frac{1}{-d}$ if the risk premium is positive and $-\frac{1}{u}$ if the risk premium is negative. The portfolio weights of the agents with finite risk aversion does depend on $p$.

Reducing $u$ or $|d|$ increases the magnitude of the bounds of the portfolio weight. In diffusion models, $u$ and $-d$ goes to zero\(^{6}\), the bounds become $+\infty$ and $-\infty$ and the portfolio weight can become arbitrarily large or arbitrarily small (arbitrarily negative).

\(^{5}\)See Liu, Longstaff and Pan (2000) for a similar discussion related to dynamic choice in a jump-diffusion model.

\(^{6}\)Dixit and Pindyck (1994) provide a nice illustration that diffusion processes are binomial tree models with infinitesimal $u$ and $d$. 

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3.2 Two Period Binomial Tree Model

We now study the dynamic asset allocation in a two-period binomial tree model.

Definition 3.3 (Two-Period Binomial Models) A two-period binomial tree model has two assets. The riskfree asset has gross return 1 in both periods. In the first period, the risky asset has excess return of \( u \) with probability \( p \) and \( d \) with probability \( q = 1 - p \). In the second period, a) if the first period return is \( u \), the risky asset has excess return of \( u_1 \) with probability \( p \) and \( d \) with probability \( q = 1 - p \); b) if the first period return is \( d \), the risky asset has excess return of \( u \) with probability \( p \) and \( d_1 \) with probability \( q = 1 - p \).

If \( u_1 = u \) and \( d_1 = d \), the asset is called an IID asset; if \( u_1 = u \) and \( d_1 > d \), the asset is called a reversal asset; if \( u_1 > u \) and \( d = d_1 \), the asset is called a momentum asset.

The two-period model defined above is almost the same as the standard two-period binomial tree model that is used in option pricing, with the exception that the excess returns in the second periods are \((u_1, d, u, d_1)\) instead of \((u, d, u, d)\). By varying \( u_1 \) or \( d_1 \), we introduce the correlations into asset returns. For example, if \( u_1 = u \) and \( d_1 = d \), the return of the risky asset is IID and thus the term IID asset; if \( u_1 = u \) and \( d_1 > d \), the return of the risky asset is negatively correlated and the return of a stock has reversal effects and thus the term reversal asset; if \( u_1 > u \) and \( d = d_1 \), the return of the risky asset is positively correlated\(^7\) and the return of a stock has momentum effects and thus the term momentum asset. Clearly, both momentum and reversal assets dominate the IID asset (in the sense that the payoff of the former is greater than or equal to that of the latter state by state).

Definition 3.4 (Buy-and-Hold Two-Period Agent) The objective of the buy-and-hold two-period agent is to choose the portfolio weight \( \phi_h \) at time 0 (the beginning of the first period) to maximize the utility function over the end of the second period wealth

\[
U = E_0 \left[ \frac{1}{1 - \gamma} (1 + \phi_h ((1 + r_1^c)(1 + r_2^c) - 1))^{1 - \gamma} \right].
\]

The optimal portfolio problem of a buy-and-hold agent is very similar to the one-period problem and its properties are summarized in the following proposition.

\(^7\)There are other ways of introducing correlation, for example, one can choose \( u_1 = u \) and \( d_1 < d \). However, this may lead to negative risk premium even if the risk premium are positive in the IID case.
Proposition 3.2 (Optimal Buy-and-Hold Portfolio Weight) The optimal portfolio weight $\phi$ of the buy-and-hold two-period agent has following properties. a) when the risk premium is positive (negative), the $\phi^*_0$ is positive (negative) and is a decreasing (increasing) function of the risk aversion $\gamma$; b) when $\gamma \to \infty$, $\phi^*_0 \to 0$; c) when $\gamma \to 0$, $\phi^*_0 \to -\frac{1}{d+d_1+d_1}$ ($\phi^*_0 \to -\frac{1}{u+u_1+u_1}$) if the risk premium is positive (negative); d) furthermore, the optimal portfolio weights for both the momentum and reversal asset are greater than the IID asset.

The buy-and-hold strategy is a static portfolio choice problem therefore both the participation theorem and the calibration theorem apply. Therefore property a) and property b) hold because they are general property of the static choice. Properties c) is similarly to that of the one-period case, the bounds on the portfolio are changed because the maximum return and minimum returns are changed. Property d) is the consequence of the dominance of the momentum and reversal assets over the IID asset. We now turn to dynamic portfolio weight.

Definition 3.5 (Dynamic Two-Period Agent) The objective of the dynamic two-period agent is to choose the portfolio weight $\phi_0$ at time 0 (the beginning of the first period) and $\phi_1$ at time 1 (the beginning of the second period) to maximize the utility function over the end of the second period wealth

$$U = E_0 \left[ \frac{1}{1-\gamma} \left( 1 + \phi_0 r_1^e \right) \left( 1 + \phi_1 r_2^e \right)^{1-\gamma} \right].$$

The agent with the above objective can solve the optimization problem using dynamic programming principle. The optimal portfolio weight for the second period conditioned on the realization of the first period excess return is solved first; then the optimal portfolio weight for the first period is solved, assuming the agent will choose the optimal portfolio weight in the second period. Graphically, the dynamic programming principle can be represented as follows

$$\max U = \max_{\phi_0, \phi_1} E_0 \left[ \left( 1 + \phi_0 r_1^e \right)^{1-\gamma} \max_{\phi_1} E_1 \left[ \frac{1}{1-\gamma} \left( 1 + \phi_1 r_2^e \right)^{1-\gamma} \right] \right].$$

Denote the optimal portfolio weight for the second period by $\phi_1^*$, we get

$$\max U = \max_{\phi_0} E_0 \left[ \frac{1}{1-\gamma} \left( 1 + \phi_0 r_1^e \right)^{1-\gamma} E_1 \left[ \left( 1 + \phi_1^* r_2^e \right)^{1-\gamma} \right] \right]$$

$$= \max_{\phi_0} E_0 \left[ \frac{1}{1-\gamma} \left( 1 + \phi_0 r_1^e \right)^{1-\gamma} \xi_1 \right],$$

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where $\xi$ is defined by

$$\xi_1 = E_1 \left[ (1 + \phi_1^* r_2^*)^{1-\gamma} \right].$$

(5)

Since the portfolio problem at time 1 is clearly static ($\phi_1^*$ is a static portfolio weight condition on the information of time 1), only $\phi_0^*$ is dynamic. The properties of the dynamic portfolio weight $\phi_0^*$ is summarized in the following proposition.

**Proposition 3.3 (Optimal Dynamic Portfolio Weight)** The optimal portfolio weight $\phi_0^*$ of the two-period agent has the following properties: a) even when the risk premia are positive, $\phi_0^*$ can be negative and may not be decreasing in risk aversion $\gamma$; b) when $\gamma \to \infty$, $\phi_0^* \to 0$; c) conservative (aggressive) agents will hold less (more) the momentum assets than the IID assets; if the momentum effect is strong ($u_1 >> u$), conservative agents will short the momentum asset even though its risk premia are strictly positive; d) aggressive (conservative) agents will hold less (more) reversal assets than IID assets; if the reversal effect is strong enough ($d_1$ is close enough to 0), aggressive agents will short the reversal asset even though its risk premia are strictly positive.

The effect of rebalancing is demonstrated clearly by studying the portfolio weight at the beginning of the second period in the reversal asset case. In this case, the second return conditioned on that the first period excess return is $d$ is better than if the first period excess return is $u$, because $d_1 > d$. The dynamic agent takes advantage of this by holding more in the $d$ state than the $u$ state; it is easily shown that $\phi_1^{*d}$, the second period portfolio weight condition on that the first period return is $d$ is greater than $\phi_1^{*u}$, the second period portfolio weight condition on that the first period return is $u$. However, the buy-and-hold agent does not adjust the portfolio at the beginning of the second period, therefore, the portfolio weights will be $\frac{\phi_2^*(1+d)}{1+\phi_2^*d}$ and $\frac{\phi_2^*(1+u)}{1+\phi_2^*u}$ respectively. This is not efficient, because the portfolio weight in the risky asset is lower (that is, $\frac{\phi_2^*(1+d)}{1+\phi_2^*d} < \frac{\phi_2^*(1+u)}{1+\phi_2^*u}$) when the return is better.

Another striking difference is that the buy-and-hold portfolio weight $\phi_0^*$ of both the momentum asset and reversal assets are greater than that of the IID asset, as shown in proposition (3.2). This is intuitive because the two-period return of the former is better than that of the latter state by state. However, the dynamic portfolio weight $\phi_0^*$ can be either greater than or smaller than that of the IID asset.
To understand the intuition, consider the reversal asset for example. A conservative agent suffers large utility loss when there is a large negative return, which is less likely than IID case, so the portfolio weight is larger. An aggressive agent benefit from large utility gains when there is a large positive return. The way to take advantage of the superior second period return in the $d$ state is to hold less risky assets than the IID asset at the beginning of the first period so that the wealth is higher if the first period return is $d$. In fact, if the advantage is large enough ($d_1$ is close enough to 0 and $\gamma$ is close enough to 0), the agent will short the assets at the beginning of the first period, so that effectively, the reversal asset is changed into a momentum asset.

Now consider the momentum asset. An aggressive agent receives a large utility gain when there is a large positive return, which is more likely than the IID case; so the portfolio weight is larger. A conservative agent takes advantage of the momentum by reducing the holding of the risky asset. The utility loss in the $u$ state from doing so more than offsets from the utility gain in the $d$ state because the agent suffers a large utility loss when there is a large negative return. In fact, if the advantage is large enough ($u_1$ is large enough and $\gamma$ is large enough), the agent will short the assets at the beginning of the first period. Effectively the momentum asset is changed into a reversal asset.

Still, a third difference exists between the dynamic and buy-and-hold agents in the bounds on the portfolio weight. The bounds on the dynamic portfolio weight are $\frac{1}{d}$ and $-\frac{1}{u}$, which are the same as the one-period agent discussed above. However, the bounds on the buy-and-hold portfolio weight are $\frac{1}{-(d+d_1+dd_1)}$ and $-\frac{1}{u+u_1+uu_1}$, which is easily understood since they are the bounds that keep the wealth positive.

The above intuition will be formalized next.

### 3.3 Equivalent Myopic Measure

The fact that property a) (violation of both participation theorem and calibration theorem) and properties c) and d) are different from their one-period counterparts is surprising, because the two-period dynamic problem can be reduced to a one period problem by equation (4). To make the one-period problem (4) look like a standard one-period static choice problem (1), we need to absorb the random variable $\xi_1$. This is accomplished by a change of measure.
Definition 3.6 The equivalent myopic measure (EMM) $Q$ is defined by the following Radon-Nykodym derivative

$$\frac{dQ}{dP} = \frac{\xi_1}{E_0[\xi_1]}.$$ 

The distribution of $r^*_1$ under EMM will be termed equivalent myopic distribution, or equivalent distribution for short. Similarly, the probability and risk premium under EMM will be called equivalent myopic probability (or equivalent probability) and equivalent myopic risk premium (or equivalent risk premium) and denoted $\hat{p}^e$ and $E^Q[r^e]$ respectively. The original measure $P$ will be called the physical measure and $I$ will use the terms such as physical distribution, physical probability and physical risk premium to distinguish them from their equivalent counterparts.

Using the definition of EMM, the following proposition is obvious.

**Proposition 3.4** The dynamic choice problem over the two-period is reduced to an equivalent one-period static problem

$$\max_{\phi_0, \phi_1} U = E_0 \left[ \frac{1}{1 - \gamma}((1 + \phi_0 r^e_0)(1 + \phi_1 r^e_2))^{1 - \gamma} \right] = \max_{\phi_0} E^Q_0 \left[ \frac{1}{1 - \gamma}(1 + \phi_0 r^e_0)^{1 - \gamma} \right]. \quad (6)$$

Therefore solving $\phi_0$ for the two-period binomial model under the physical measure is equivalent to solving $\phi_0$ for a one-period binomial model under EMM, provided that the equivalent myopic measure instead of the physical measure is used. The EMM summarizes the effects on the portfolio weight due to future period returns.

It is easy to verify that the equivalent myopic probability $\hat{p}^e$ is equal to $\frac{p_{\xi^e_1}}{p_{\xi^e_1} + q_{\xi^e_1}}$, and the equivalent myopic risk premium $E^Q[r^e]$ is given by $\frac{up_{\xi^e_1}}{p_{\xi^e_1} + q_{\xi^e_1}} + \frac{dq_{\xi^e_1}}{p_{\xi^e_1} + q_{\xi^e_1}}$.

**Proposition 3.5** Suppose the risk premium is positive. If the risky asset is a momentum asset, the equivalent probability $\hat{p}^e$ is greater (smaller) than the physical probability $p$ and the equivalent risk premium $E^Q_0[r^e_1]$ is greater (smaller) than the physical risk premium $E_0[r^e_1]$ for aggressive (conservative) agents; if the risky asset is a reversal asset, the equivalent probability $\hat{p}^e$ is smaller (greater) than the physical probability $p$ and the equivalent risk premium $E^Q_0[r^e_1]$ is smaller (greater) than the physical risk premium $E_0[r^e_1]$ for aggressive (conservative) agents.

The proof is given in the Appendix. The proposition can be understood intuitively as follows. Consider the case of reversal, a conservative agent (with $\gamma > 1$) is less likely to
suffers large utility loss from large negative returns because of the reversal, so the agent will view the $u$-state probability higher than $p$. An aggressive agent (with $\gamma < 1$) is less likely to achieve the higher utility because of the reversal, so the agent will view the $u$-state probability less than $p$.

The striking properties of the EMM is that it depends on the risk aversion of the agent, this is due to rebalancing and the fact that agents with different risk aversion will rebalance differently and achieve different level of utility. Because of this, participation and calibration theorems do no apply; even for stocks with strictly positive risk premia, a more risk averse agent may hold more risky assets; more surprisingly, a risk averse agent may short risky assets. Consider the example of $d_1 > d$ and $u_1 = u$ (the reversal asset). From Proposition (3.5), we know that $p^\xi < p$ ($p^\xi > p$) if $\gamma < 1$ ($\gamma > 1$). For aggressive (conservative) agents, the reversal asset is worse (better) than IID asset, thus aggressive (conservative) agents will hold less (more) than IID assets. One can also verify that when $d_1$ is close to 0, the equivalent risk premium $E^Q[r_e]$ can become negative for some $\gamma < 1$ even though the physical risk premium is positive. When this happens, the agent will short the risky asset ($\phi^*_0 < 0$).

4 Formulation

In this section I will specify the stochastic volatility model and the utility function of the agent.

4.1 Asset Price Dynamics

The price of the risky asset satisfies the following equation

$$dP_t = P_t(r + \lambda V_t^{\frac{1+\beta}{2}}) + P_t \sqrt{V_t} dB_t.$$  \hspace{1cm} (7)

To focus on stochastic volatility, I will assume that the short rate $r$ is constant.\footnote{Liu (1997) studies a portfolio choice problem in which both the short rate and the volatility of stock returns are stochastic.} The risk premium $\lambda V_t^{\frac{1+\beta}{2}}$, where $\lambda$ and $\beta$ are both constants, has constant elasticity with respect to volatility (or variance) (CEV). Without loss of generality, I will assume that $\lambda \geq 0$. This
means that the risk premium is positive, which is economically sensible. The solution also applies for \( \lambda < 0 \). Since the risk premium depends on the volatility \( V_t \) of the return process, all of the risk of the return are priced. Therefore the above model is a model for market portfolio and it may not be an appropriate model for an individual asset. Merton (1980) pointed out that it is difficult to estimate the risk premium. He proposes three possible forms for the risk premium, \( \lambda (\beta = -1), \lambda V_t^{1/2} (\beta = 0) \), and \( \lambda V_t (\beta = 1) \). Subsequent empirical studies have yield conflicting results on the relation between volatility and risk premium; some have found positive relationships, some have found negative relationships, while still others have found no relationship. Because of these conflicting results, I will study portfolio choice for the whole CEV class of risk premia (that is, for all values of \( \beta \)).

As will be seen later, the dependence of the portfolio weight on the volatility is determined by the IMPR and the risk aversion and horizon dependence is determined by the ISR. In the above specification of the asset price dynamics, note that IMPR is \( \lambda V^{\beta - 1/2} \) and the ISR is \( \lambda V^\beta \). I also note that only the risk premium for the asset return risk is relevant; the portfolio weight does not depend on the risk premium for volatility risk. This is in contrast to option pricing literature, where the risk premium is one of the most important factors in determining option prices when the market is incomplete (i.e., when the volatility risk is not priced by the market).

### 4.2 Volatility Process

Because the qualitative behavior of the portfolio weight does not depend on the specification of the dynamics for the variance process \( V_t \), I will choose the variance process such that the portfolio weight can be derived explicitly. To this end, I assume an instantaneous variance process

\[
V_t = X_t^{\beta/3}
\]

Because portfolio choice depends critically on risk premium as well as volatility, there is an important difference between portfolio problems with predictability and stochastic volatility. In the case of predictability models, the volatility is implicitly assumed to be constant while predictability specifies the risk premium. In stochastic volatility models, the focus is often on the volatility and risk-premium is largely left unspecified. As Merton (1980) pointed out, this is fine if the researcher is only interested in derivative pricing. For portfolio choice problems, the risk premium plays a critical role.
where $X_t$ is a square-root process,

$$dX_t = (k - KX_t)dt + \sigma \sqrt{X_t} dB_t^V.$$ 

Using Ito’s lemma, one can show that the above assumption for $V_t$ is equivalent to directly assuming that the variance process $V_t$ satisfies the following equation

$$dV_t = \frac{1}{\beta} V_t^{1-\beta} \left\{ \left( k + \frac{\sigma^2(1-\beta)}{2\beta} - KV_t^\beta \right) dt + \sigma V_t^{\frac{\beta}{2}} dB_t^V \right\}, \beta \neq 0. \quad (9)$$

Since $V_t$ is a deterministic function of a square-root process with long-term mean $\frac{k}{K}$, $V_t$ is well-defined as long as $\frac{k}{K} \geq 0$. I will assume that $\sigma > 0$ for definiteness.\(^{10}\) I will assume that the correlation between the Brownian motions $dB_t$ and $dB_t^V$ is $\rho$ which is a constant. Note that the correlation between the “shock” $\sqrt{V_t} dB_t$ to the stock return and “shock” $\frac{\sigma}{\beta} V_t^{1-\beta/2} dB_t^V$ to the variance process $V_t$ is $\text{sign}(\beta)\rho$, that is, the correlation between the two shocks is $\rho$ when $\beta > 0$ and $-\rho$ when $\beta < 0$.

A few comments are in order. First, when $\beta = 1$, the return dynamics (7, 8) is the same as that of the Heston model (1993). Heston (1993) did not explicitly specify the risk premium of the stock return since his primary interest is option pricing. The risk premium of the form $\lambda V_t$ is proposed by Bates (1997) and used in Bakshi, Cao, and Chen (1997). This form of risk premium is motivated by CAPM and is originally suggested for any stochastic volatility model (not just the Heston model) by Merton (1980). The portfolio selection in this model is studied in Liu (1998). Second, when $\beta = -1$, the return dynamics (7, 8) is the same as proposed by by Chacko and Viceira (1999). For this dynamics, they study the portfolio choice problem when the agent has recursive utility defined over intermediate consumption.

### 4.3 Utility of the Agent

The agent’s objective is to allocate funds between a riskless asset with constant return $r$ and the risky asset to maximize the end-of-period wealth

$$\max E_0 \left[ \frac{W_t^1}{1 - \gamma} \right], \quad (10)$$

\(^{10}\)There is no loss of generality since $-\sigma$ should lead the same observable because we can change $B_t^V$ to $-B_t^V$.  

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where $W_T$ is the end of period wealth of a self-financing trading strategy $\phi_t$

$$dW_t = W_t \left( r + \lambda \phi_t V_t^{1+\delta} \right) dt + \phi_t \sqrt{V_t} dB_t. \quad (11)$$

The utility over the end of the period wealth (instead of over intermediate consumption) is chosen for three reasons. First, maximizing end of the period wealth controlling for risk is the objective of many investors, such as fund managers. Second, I would like to study the effects of investment horizon on portfolio choice. The concept of investment horizon is blurred when there is intermediate consumption. Third, and most importantly, the main focus of the paper is on the effects of investors' risk aversion on portfolio choice. With power utility over intermediate consumption, it is difficult to distinguish risk aversion from elasticity of intertemporal substitution.

5 Optimal Dynamic Strategy

The optimization problem (10) can be solved using the method of Liu (1999). The optimal portfolio weight is given in the following theorem.

**Theorem 5.1** The optimal portfolio weight is given by

$$\phi_t^* = V_t^{\frac{\gamma - 1}{2}} \left( \frac{1}{K - \frac{1-\gamma}{\gamma} \lambda \rho \sigma + \xi \coth(\xi \tau / 2)} \right) \gamma \cot(\eta \tau / 2), \quad (12)$$

$$= V_t^{\frac{\gamma - 1}{2}} \frac{\lambda (K + \xi \coth(\xi \tau / 2))}{\gamma (K + \xi \coth(\xi \tau / 2)) - (1 - \gamma) \lambda \rho \sigma} \quad (13)$$

$$= V_t^{\frac{\gamma - 1}{2}} \frac{\lambda \cot(\eta \tau / 2)}{\gamma (K + \xi \coth(\xi \tau / 2)) - (1 - \gamma) \lambda \rho \sigma} \quad (14)$$

$$= V_t^{\frac{\gamma - 1}{2}} \frac{\lambda (K + \eta \cot(\eta \tau / 2))}{\gamma (K + \eta \cot(\eta \tau / 2)) - (1 - \gamma) \lambda \rho \sigma} \quad (15)$$

with $\tau = T - t$, $\xi = \sqrt{K^2 - \frac{1}{\gamma} (2K \lambda \rho \sigma + \lambda^2 \sigma^2)}$, and $\eta = -i \xi$.

The proof is given in the Appendix. When $\gamma \geq 1$, the parameter $\xi$ is real for all $\tau \geq 0$. When $\gamma < \gamma_{\text{min}} \equiv 1 - \frac{K^2}{(K + \lambda \rho \sigma)^2 + \lambda^2 \sigma^2 (1 - \rho^2)}$, $\xi$ is purely imaginary. In this case, $K^2 - \frac{1-\gamma}{\gamma} (2K \lambda \rho \sigma + \lambda^2 \sigma^2) < 0$ and $\eta = -i \xi$ is real. The above expressions are valid even when $\xi$ or $\eta$ are purely imaginary. The portfolio weight is given in both $\xi$ and $\eta$ because at least one of them is real and it is convenient to work with real variables. Note also that
the optimal dynamic portfolio weight does not depend on the long-run mean \( \frac{\kappa}{\lambda} \) of the variance process \( V \).

### 5.1 Volatility Timing

**Proposition 5.1 (Volatility Timing)** The optimal portfolio weight is decreasing in volatility if IMPR is decreasing in volatility (\( \beta < 1 \)); and increasing if IMPR is increasing (\( \beta > 1 \)). The ratio of myopic component and intertemporal hedging component is independent of the volatility.

The proof is obvious using Theorem 5.1. Proposition 5.1 implies that the agent times the volatility as long as \( \beta \neq 1 \). The intuition is clear. When IMPR is increasing in volatility (\( \beta > 1 \)), the return for bearing risk increases with volatility, therefore the agent will hold more risky assets when the volatility is high. When IMPR is decreasing in volatility (\( \beta < 1 \)), the return for bearing risk decreases with volatility, therefore the agent will hold less risky assets when the volatility is high.

### 5.2 Equilibrium

Since the asset price process is exogenously specified, one frequently asked question is that whether the asset price dynamics arises as equilibrium price of an economy. Because there is a strictly positive pricing kernel for the price dynamics, the answer is yes, as showed by Constantinides (1992).

A more interesting question to ask is that, “suppose the agent is the representative agent, can the price process be the equilibrium price process?” The answer is no in general. If the agent times the markets, the market clearing condition, which requires the agent to hold entire market portfolio, will be violated. However, for the special case of \( \beta = 1 \), that is, when IMPR is a constant, the portfolio weight is constant and the price process can be the equilibrium price process with the agent be the representative agent. This agrees with the general results of He and Leland (1993), who show that IMPR is completely determined by the risk aversion of the representative agent. In this paper, the relative risk aversion of the representative agent is a constant, so IMPR has to be a constant for the price process to be an equilibrium price.
For the case of $\beta = 1$, the equilibrium risk premium parameter $\lambda_e$ can be solved from the following equation (market clearing condition)

$$\frac{\lambda_e(K + \xi_e \coth(\xi_e \tau/2))}{\gamma(K + \xi_e \coth(\xi_e \tau/2)) - (1 - \gamma)\lambda_e \rho \sigma} = 1$$

where $\xi_e = \sqrt{K^2 - \frac{1 - \gamma}{\gamma} (2K \lambda_e \rho \sigma + \lambda_e^2 \sigma^2)}$. For example, suppose the investor is the representative agent with infinite horizon $\tau = +\infty$, we can solve $\lambda_e$ explicitly

$$\lambda_e = \gamma - K \rho \sigma + \rho \sqrt{\left(\frac{K}{\sigma}\right)^2 + \gamma(\gamma - 1)}.$$

When the dynamic hedging effect is the largest, that is, when $K = 0$ or $\sigma = \infty$ and $\rho = -1$, the equilibrium risk premium parameter further reduces to

$$\lambda_e = \gamma - \sqrt{\gamma(\gamma - 1)}. \quad (16)$$

In this case, the equilibrium risk premium is decreasing with the risk aversion $\gamma$. This is the direct consequence of the portfolio weight may be decreasing in $\gamma$, which will be discussed later. Note that this is in direct contrast with the mean-variance equilibrium risk premium, which is proportional to $\gamma$. Another interest feature of equation (16) is that it is only defined for $\gamma \geq 1$. When $\gamma < 1$, the agent views the investment opportunity characterized by the asset price dynamics to be so good that he will take an infinity position for any $\lambda$, so there is no equilibrium risk premium.

I should remark that the optimal portfolio weight does depend on the volatility when there is intermediate consumption, even for $\beta = 1$. This implies that that the asset price dynamics specified in section (4) cannot be the equilibrium price process if the representative agent has CRRA utility over intermediate consumption. One can also show that, along the same line, if markets clear, the asset price dynamics in many predictability models cannot be the equilibrium asset price dynamics if the representative agent has CRRA utility over intermediate consumption. This has interesting implications for recent empirical studies, such as Campbell (1993, 1996), Hodrick, Ng, and Sengmueller (1999), Lettau and Ludvigson (2000a, 2000b), and Chen (2000), that test predictability models by imposing the condition that the representative agent has essentially CRRA utility.
The equilibrium risk premium coefficient $\lambda_e$ as a function of risk aversion $\gamma$. The correlation coefficient is $\rho = -1$. $K = 1$ for the solid lines and $K = 0$ for the dashed-dotted line. The representative agent is assumed to have an infinite investment horizon.

Figure 1: Equilibrium Risk Premium Coefficient as a Function of Risk Aversion
5.3 Investment Horizon

When asset returns display stochastic volatility, the optimal portfolio weight depends on the investment horizon, contrary to the case when asset returns are distributed independently over time. At very short horizon \( (\tau \to 0) \), the portfolio weight is given by

\[
\phi = \frac{\lambda}{\gamma} \left( 1 - \tau \rho \sigma \left( 1 - \frac{1}{\gamma} \right) \lambda \right)
\]  

(17)

For \( \gamma > 1 \), the portfolio weight and utility function are well defined for all horizons. At very long horizon \( (\tau = \infty) \), the portfolio weight is given by

\[
\phi = \frac{1}{1 + \gamma (\frac{1}{\rho^2} - 1)} \left( \frac{\lambda}{\rho^2} + \frac{K - \sqrt{K^2 + (1 - \frac{1}{\gamma}) (2K \rho + \sigma \lambda) \sigma \lambda}}{\rho \sigma} \right)
\]  

(18)

The following proposition summarizes the horizon dependence of the portfolio weight.

**Proposition 5.2 (Horizon Dependence)** For conservative investors \( (\gamma > 1) \), the optimal portfolio weight is increasing in investment horizon if the Sharpe ratio is increasing with volatility \( (\beta > 0 \text{ and } \rho < 0) \); the optimal portfolio weight is decreasing in investment horizon if the Sharpe ratio is decreasing with volatility \( (\beta < 0) \).

For aggressive but still risk-averse investors \( (0 < \gamma < 1) \), the optimal portfolio weight is decreasing in investment horizon if the Sharpe ratio is increasing with volatility \( (\beta > 0 \text{ and } \rho < 0) \) and may (if \( 2K \rho + \lambda \sigma < 0 \)) go to negative infinity at a finite horizon if \( \gamma \) is too close to 0; the optimal portfolio weight is increasing in investment horizon if the Sharpe ratio is decreasing with volatility \( (\beta < 0 \text{ and } \rho > 0) \) and always goes to infinity when \( \gamma \) is close to 0.

In all cases, the effect (the magnitude) of intertemporal hedging demand is increasing with investment horizon.

Note that the condition for dependence in volatility is different from the condition for horizon dependence. These two conditions are two different measures of risk premium required to compensate risk. While I do not derive the portfolio weight for risk premium specification more general than CEV, I expect, qualitatively, that if the risk is well compensated, the investor may view high volatility as a good thing and thus hold more stocks when the volatility is high and vice versa. The constant ratio of the myopic component to
This graph summarizes the possible investment horizon dependence of the optimal portfolio weight when the correlation coefficient \( \rho < 0 \). The solid line represents agents with \( \gamma > 1 \), the dashed-dotted line \( \gamma_{\text{min}} < \gamma < 1 \) the dashed line \( 0 < \gamma < \gamma_{\text{min}} \). The dashed line approaches \(-\infty\) at a finite horizon.

Figure 2: The Optimal Portfolio Weight as a Function of Investment Horizon

The intertemporal hedging component is due to the specification of the CEV risk premium and variance processes. In general, this is not true in the Stein-Stein model, as shown by Liu (1998), for example.

Note that \( \cot(\eta \tau/2) \) varies monotonically from \(+\infty\) to \(-\infty\) when \( \tau \) changes from 0 to \( 2\pi/\eta \). The denominator in equation (15) approaches 0 when \( \tau \to \tau_{\text{max}} \equiv -\frac{2}{\eta} \arctan \left( \frac{\eta}{K - \frac{1}{2} \lambda \rho} \right) \). In this case, the portfolio weight reaches infinity at finite \( \tau \), and the investor will take an infinite position (short if \( \rho < 0 \) or long if \( \rho < 0 \)) in the risky asset. That is, a risk averse agent may short stock (sometimes infinite amount), even though the risk premium is finite.
This graph summarizes the possible investment horizon dependence of the optimal portfolio weight when the correlation coefficient $\rho > 0$. The solid line represents agents with $\gamma > 1$, the dashed-dotted line $\gamma_{\min} < \gamma < 1$, and the dashed line $0 < \gamma < \gamma_{\min}$. The dashed line approaches $+\infty$ at a finite horizon.

Figure 3: The Optimal Portfolio Weight as a Function of Investment Horizon
and strictly positive, cannot be true in static choice problems.

When $\beta = 1$, the risk premium $\lambda V_t$ is proportional to the variance $V_t$, and the result in the above proposition reduces to the result of Liu (1998), which shows that the intertemporal hedging demand is positive and the portfolio weight is increasing in investment horizon. When $\beta = -1$, the risk premium $\lambda V_t$ is independent of the variance $V_t$, and the result in the above proposition is similar to the result of Chacko and Viceira (1999) which shows that the intertemporal hedging demand is negative and the portfolio weight is decreasing in investment horizon. The two-period example suggests that the qualitative behavior of the portfolio weight is determined by the Sharpe ratio. The Sharpe ratio is increasing with variance in Liu (1998) while it is decreasing in Chacko and Viceira, this explains why the portfolio behavior is qualitatively different in the two papers.

5.4 Risk Aversion

The most interesting features of the optimal portfolio weight are its dependence on the risk aversion, which is summarized in the following proposition.

Proposition 5.3 (Risk Aversion Dependence) If the Sharpe ratio is decreasing in volatility ($\beta < 0$ and $\rho > 0$), the optimal portfolio weight is decreasing in $\gamma$ and always reaches $+\infty$ for non-zero investment horizon $\tau$ when $\gamma$ is small enough.

If the Sharpe ratio is decreasing in volatility ($\beta > 0$ and $\rho < 0$) and $2K\rho + \lambda \sigma \leq 0$, the optimal portfolio weight is decreasing in $\gamma$.

If the Sharpe ratio is decreasing in volatility ($\beta > 0$ and $\rho < 0$) and $2K\rho + \lambda \sigma > 0$, the portfolio is increasing for small $\gamma$, reaches $-\infty$ for non-zero investment horizon $\tau$, and decreasing for large $\gamma$. So the portfolio weight is non-monotonic in $\gamma$.

The proof is given in the appendix. This proposition says that a more conservative (bigger $\gamma$) investor may hold less stock, in a simple system, in which there are only two assets and the short rate is constant and there is no pathological features, such as negative risk premium. This is rather counter-intuitive, especially considering that stock holding is often used as a proxy of the risk aversion of the agent.
This graph shows that the optimal portfolio weight is always decreasing in risk aversion when the correlation coefficient $\rho > 0$.

Figure 4: The Optimal Portfolio Weight as a Function of Risk Aversion
This graph summarizes the possible risk aversion dependence of the optimal portfolio weight when the correlation coefficient $\rho < 0$ and the investment horizon $\tau$ is finite but non-zero. The solid line shows that the optimal portfolio weight is increasing for low risk aversion $\gamma$ if the dynamic hedging effect is large ($2K\rho + \lambda \sigma > 0$); the dashed-dotted line shows that the optimal portfolio weight is always decreasing if the dynamic hedging effect is small ($2K\rho + \lambda \sigma < 0$).

Figure 5: The Optimal Portfolio Weight as a Function of Risk Aversion
This graph summarizes risk aversion dependence of the optimal portfolio weight when the correlation coefficient $\rho = -1$ and the investment horizon $\tau$ is infinite. The solid line shows that the optimal portfolio weight monotonically increases to a strict positive constant as $\gamma \to \infty$ if the dynamic hedging effect is large ($2K\rho + \lambda \sigma > 0$); the dashed-dotted line shows that the optimal portfolio weight is monotonically decreases to 0 as $\gamma \to \infty$ if the dynamic hedging effect is small ($2K\rho + \lambda \sigma < 0$).

Figure 6: The Optimal Portfolio Weight as a Function of Risk Aversion
Another interesting example is that an infinitely risk averse agent may hold a strictly positive amount of risky asset. When $\tau \to \infty$, $K = 0$, and $\rho = -1$,

$$\phi = V_t^{\frac{\gamma - 1}{2}} \frac{\lambda}{\gamma - \sqrt{\gamma(\gamma - 1)}},$$

which is increasing in risk aversion. For $\beta = 1$, this is equivalent to the fact that the equilibrium risk premium is decreasing in $\gamma$. When $\gamma \to \infty$, the above portfolio weight approaches to $V_t^{\frac{\gamma - 1}{2}} \frac{\lambda}{2} > 0$, which is positive. I should remark that if either $\tau$ is finite or $\rho < -1$, $\phi \to 0$ when $\gamma \to \infty$.

5.5 Calibration Exercise

The stochastic volatility model with $\beta = 1$ can be estimated rather easily. Data used to estimate the data generating process $(V_t, P_t)$ of the stochastic volatility model are the yields of the zero-coupon bond of one-year maturity and the stock returns. The constant short rate $r$ is computed from the mean yield of the zero-coupon bond. The volatility $V_t$ of the stock return is not observed. Nevertheless, the parameters of the model can be identified using the moments of the stock returns.$^{11}$ In Liu (1998) the unconditional moments of stock returns up to third power are computed in closed form. The parameters are estimated using GMM. The estimation results are given in Table 5.5.

Note that $2K\rho + \lambda\sigma < 0$, so there is no counter-intuitive behavior with the market data.

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$^{11}$Pan (1998) and Wu (1998) estimate a similar model using stock return data as well as option data.
This graph plots investment horizon dependence of the optimal portfolio weight using the empirically estimated parameters in Table 5.5. The solid line plots the portfolio weight for the agent with $\gamma = 4$ and the dashed-dotted line plots the portfolio weight for the agent with $\gamma = 0.9$.

Figure 7: The Optimal Portfolio Weight as a Function of Investment Horizon
This graph plots investment horizon dependence of the optimal portfolio weight using the empirically estimated parameters in Table 5.5. The investment horizon is 2 years.

Figure 8: The Optimal Portfolio Weight as a Function of Risk Aversion
6 Equivalent Myopic Distribution

One of the most intuitive results in static choice models is that a more risk averse agent will hold less risky assets. This is apparently not true in dynamic portfolio choice problems. Another important and intuitive result is that agents will hold positive amount of risky asset if the risk premium is positive. Since the stock returns have positive risk premium, this result leads to the so-call market non-participation puzzle, which refers to the fact that many people do not hold stocks. However, proposition (5.3) shows that, with multiple period economy with portfolio rebalancing, an investor may not hold positive amount of risky assets even if the risk premium is strictly positive.

These counter-intuitive results have not been studied before, probably because many well-known multi-period models\textsuperscript{12} are effectively single-period models, so intuitions from static choice models apply. These counter-intuitive results do exist in dynamic choice with predictability models, see for example, Kim and Omerberg (1996), Brennan, Schwartz and Lagnado (1997), Brennan and Xia (1999), Campbell and Viceira (1999), Liu (1999), and Xia (2000); however, in these models, the risk premium can be negative, so it is not clear whether these counter-intuitive results are consequence of dynamic choice or the pathological effects of negative risk premium. The stochastic volatility model studied in this paper has positive risk premia\textsuperscript{13} and a constant short rate, so these counter-intuitive results can only be due to dynamic rebalancing.

These counter-intuitive results can be understood using the EMM. As illustrated in the two-period binomial example, when we reduce a multi-period dynamic asset allocation problem to a one-period static problem, the risky asset has a different distribution under the reduced problem.

**Definition 6.1 (Equivalent Myopic Measure: Stochastic Volatility Model)** The equivalent myopic measure of a dynamic asset allocation problem is defined by the following Radon-Nykodym derivative

\[
\frac{dQ}{dP} = \frac{\xi_T}{E_0[\xi_T]},
\]

\textsuperscript{12}For example, there are two important classes of models: first, any model with buy-and-hold strategy; and second, models with portfolio rebalancing but asset returns are independently distributed over time. In these models, portfolio choice problems are equivalent to single-period portfolio choice models.

\textsuperscript{13}With mild restriction on parameters and a strictly positive initial condition, the variance process thus the risk premium is strictly positive.
where
\[ \xi_T = \exp \left( (1 - \gamma) \int_0^T (r + \phi^*(\mu - r)dt + \phi^*dB_t) \right). \]

The distribution under EMM will be termed equivalent myopic distribution, or equivalent distribution for short. Similarly, the risk premium under EMM will be called equivalent myopic risk premium (or equivalent risk premium). The original measure \( P \) will be called the physical measure.

Obviously, \( \xi_T \) is the continuous-time analog of \( \xi_1 \) in the two-period model and it summarizes the effects of future opportunities as far as the dynamic choice is concerned. The equivalent distribution in general depends on the the state variable, the risk aversion (because different agent rebalances differently), and the investment horizon (because the effects of future opportunities are different for different horizons).

For the case of continuous-time diffusion models, only drift is changed when changing to an equivalent measure. So the equivalent myopic distribution is completely determined by the equivalent expected return or equivalent myopic risk premium (EMRP).

**Proposition 6.1 (EMRP and Dynamic Portfolio Weight)** The equivalent risk premium for the dynamic choice problem (10) is given by
\[
V_t^{\frac{\hat{a} + 1}{2}} \frac{1}{\lambda} \left( 1 + \frac{1}{K - \frac{1-\gamma}{\gamma^2} \lambda \rho \sigma + \xi \coth(\xi_T/2)} \right) \frac{(1 - \gamma)\lambda \rho \sigma}{\gamma} \tag{19}
\]
or equivalently
\[
V_t^{\frac{\hat{a} + 1}{2}} \frac{1}{\lambda} \left( 1 + \frac{1}{K - \frac{1-\gamma}{\gamma^2} \lambda \rho \sigma + \eta \cot(\eta \tau/2)} \right) \frac{(1 - \gamma)\lambda \rho \sigma}{\gamma}. \tag{20}
\]

The optimal dynamic portfolio weight in the physical measure is the optimal myopic portfolio weight (mean-variance portfolio weight) in the EMM.

The proof is straightforward. EMRP can be readily computed using the definition, and dividing the above EMRP by the variance \( V_t \) (note that the variance does not change when changing to an equivalent measure for a diffusion process) gives rise to the myopic portfolio weight, which is equal to the dynamic portfolio weight.

From equation (19) or (20), it is obvious that EMRP depends on risk aversion. EMRP is equal to risk premium when either \( \rho = 0 \) or \( \gamma = 1 \), which is expected, since in both
cases the dynamic choice problems reduce to static choice problems. It can be shown that EMRP is decreasing with $\gamma$ if $\rho > 0$ and increasing if $\rho < 0$. So when $\rho > 0$, the optimal portfolio weight $\frac{EMRP}{\gamma V_t}$ is decreasing in $\gamma$, since both $EMRP$ and $\frac{1}{\gamma}$ are. However, when $\rho < 0$, the optimal portfolio weight $\frac{EMRP}{\gamma V_t}$ may increase with $\gamma$. 
7 Conclusion

In this paper, I derive the closed-form portfolio weight for a dynamic asset allocation problem in which the risky asset displays stochastic volatility. I show that there are qualitative differences between dynamic and static choice problems. For example, both participation theorem and calibration theorems of static choice models are violated in the dynamic choice models. I argue that the counter-intuitive results in the dynamic setting are due to rebalancing and can be understood using the concept of the equivalent myopic measure. The dynamic problem under the physical measure can be reduced to a static problem under EMM, and violations of the participation and calibration theorems are due to the fact that the distribution under EMM depends on the risk aversion. My results show that dynamic rebalancing may be another source of market non-participation; they also imply that stock holdings may not be good proxies for risk aversion. The results and methods in the paper can be also used to study equilibrium implications of exogenously specified asset price dynamics.
Table 1: Estimates of Stochastic Volatility

<table>
<thead>
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<th></th>
<th>$\theta = \frac{k}{K}$</th>
<th>$K$</th>
<th>$\sigma$</th>
<th>$\lambda$</th>
<th>$\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Estimate</td>
<td>0.0195</td>
<td>3.4871</td>
<td>0.3503</td>
<td>3.8240</td>
<td>-0.7561</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>0.001420</td>
<td>1.3416</td>
<td>0.2750</td>
<td>0.9676</td>
<td>0.4064</td>
</tr>
</tbody>
</table>

The volatility process $V$ and the stock price $P$ satisfy

$$dV = (k - KV)dt + \sigma \sqrt{V} dB^V,$$

$$d\ln P = \left(r + \left(\lambda - \frac{1}{2}\right)V\right)dt + \sqrt{V} dB.$$

The correlation between $B^V$ and $B$ is $\rho$. The volatility process $V$ is not observed. The constant short rate $r$ is the mean yield of the zero-coupon bond with one year maturity. The $(V_t, P_t)$ process is estimated from the data on the stock return $R_t = \ln(P_t/P_{t-1})$ using GMM.
References


8 Appendix

8.1 Proof of Theorem 2.2

Proof. The proof is a variation of the proof of Arrow's theorem on insurance premium. Suppose utility function $U$ is more concave than utility function $V$ and both are increasing. Then there exists a concave function $G$, such that $U = G(V)$. Suppose the optimal portfolio weight for $V$ is $\phi^*$, and the risk premium is positive so that $\phi^* > 0$, so $E[V'(r_f + \phi^* r_e) r_e] = 0$, then

$$E[U'(r_f + \phi^* r_e) r_e] = E[G'(r_f + \phi^* r_e) V'(r_f + \phi^* r_e) r_e]$$

$$= E[(G'(r_f + \phi^* r_e) - G'(r_f)) V'(r_f + \phi^* r_e) r_e 1_{\{r_e \geq 0\}}]$$

$$+ E[(G'(r_f + \phi^* r_e) - G'(r_f)) V'(r_f + \phi^* r_e) r_e 1_{\{r_e < 0\}}]$$

$$\geq E[(G'(r_f) - G'(r_f)) V'(r_f + \phi^* r_e) r_e 1_{\{r_e \geq 0\}}]$$

$$+ E[(G'(r_f) - G'(r_f)) V'(r_f + \phi^* r_e) r_e 1_{\{r_e < 0\}}] = 0.$$

This implies that the optimal portfolio weight for $U$ is smaller than $\phi^*$.

For the case of CRRA preference, we can explicitly show that the derivative of the optimal portfolio weight with respect to the relative risk aversion coefficient $\gamma$ is negative. The first order condition for $\phi^*$ is

$$E[(r_f + \phi^* r_e)^{-\gamma} r_e] = 0.$$

Taking derivative with respect to $\gamma$ of the above equation, we have

$$\frac{\partial \phi^*}{\partial \gamma} = \frac{E[(r_f + \phi^* r_e)^{-\gamma} r_e \ln(r_f + \phi^* r_e)]}{\gamma E[(r_f + \phi^* r_e)^{-\gamma-1} r_e^2]}$$

$$= \frac{E[(r_f + \phi^* r_e)^{-\gamma} r_e \ln(1 + \phi^* \frac{r_e}{r_f})]}{\gamma E[(r_f + \phi^* r_e)^{-\gamma-1} r_e^2]}$$

$$= \frac{\text{cov}((r_f + \phi^* r_e)^{-\gamma} r_e, \ln(1 + \phi^* \frac{r_e}{r_f}))}{\gamma E[((r_f + \phi^* r_e)^{-\gamma-1} r_e^2)},$$

where I used the first order condition for the second equality and third equality. Since $\ln(r_f + \phi^* r_e)$ is strictly increasing if $\phi^* > 0$, it follows that

$$E[(r_f + \phi^* r_e)^{-\gamma} r_e \ln(1 + \phi^* \frac{r_e}{r_f})]$$

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\[ E[(r_f + \phi^* r_e)^{-\gamma} r_e \ln(1 + \phi^* r_e) 1_{\{r_e \geq 0\}}] + E[(r_f + \phi^* r_e)^{-\gamma} r_e \ln(1 + \phi^* r_e) 1_{\{r_e < 0\}}] \geq E[(r_f + \phi^* r_e)^{-\gamma} r_e (\ln 1) 1_{\{r_e \geq 0\}}] + E[(r_f + \phi^* r_e)^{-\gamma} r_e (\ln 1) 1_{\{r_e < 0\}}] = 0. \]

So we have $\frac{\partial \phi^*}{\partial \gamma} < 0$. Note that $\ln(r_f + \phi^* r_e)$ is strictly decreasing if $\phi^* < 0$ and we will have $\frac{\partial \phi^*}{\partial \gamma} > 0$.

### 8.2 Proof of Proposition 3.5

Consider reversal asset case (the momentum asset case can be dealt with similarly). From the definition of $\xi_1$, we have

\[ \xi_1 = E\left[ (1 + \phi_1^* r_e)^{1-\gamma} \right], \]

where $\phi_1^*$ is the optimal portfolio weight, one can show that

\[ \xi_1^u = p(1 + \phi_1^* u)^{1-\gamma} + q(1 + \phi_1^* d)^{1-\gamma} = \left( \frac{u - d}{u - Ad} \right)^{1-\gamma} (pA_1^{1-\gamma} + q) = \left( \frac{u - Ad}{u - d} \right)^\gamma \frac{u - d}{u} q, \]

where $A = \left( \frac{pq}{-qd} \right)^{1/\gamma}$. The following properties $\xi_1^u$ are not needed in the proof but can be helpful in understanding special cases: 1) $\xi_1^u = 1$ if $\gamma = 1$ or $A = 1$; 2) when $\gamma \to 0$, $\xi_1^u \to \frac{u - d}{u - d} p$; 3) when $\gamma \to 0$, $\xi_1^u \to \frac{u - d}{u - d} q$ if $A > 1$ (if $A < 1$); 4) when $\gamma \to \infty$, $\xi_1^u \to \left( \frac{pq}{-qd} \right)^{-\gamma} \frac{u - d}{u} q$. Similarly, one can show that

\[ \xi_1^d = p(1 + \phi_1^* u)^{1-\gamma} + q(1 + \phi_1^* d_1)^{1-\gamma} = \left( \frac{u - d_1}{u - Ad_1} \right)^{1-\gamma} (pA_1^{1-\gamma} + q) = \left( \frac{u - Ad_1}{u - d_1} \right)^\gamma \frac{u - d_1}{u} q, \]

where $A = \left( \frac{pq}{-qd} \right)^{1/\gamma}$. To prove that $\xi_1^u > \xi_1^d$ for $\gamma > 1$ ($\gamma < 1$), note that

\[ \xi_1^u = \left( \frac{u - Ad}{u - d} \right)^\gamma \frac{u - d}{u} q = \left( 1 + (p/q)^{1/\gamma} x^{1-\gamma} \right)^\gamma (1 + x)^{1-\gamma} q, \]

where $x = \frac{d}{u}$. Therefore,

\[ \frac{\partial \ln \xi_1^u}{\partial (-d)} = \frac{\partial \ln \xi_1^u}{\partial x} \frac{1}{u} = qu(1 - \gamma) \frac{1 - (p/q)^{1/\gamma} x^{1-\gamma}}{(1 + x)(1 + (p/q)^{1/\gamma} x^{1-\gamma})}, \]

which is positive (negative) if $\gamma > 1$ ($\gamma < 1$), noting that $1 - (p/q)^{1/\gamma} x^{1-\gamma} < 0$ because the risk premium is assumed to be positive.
8.3 Proof of Theorem 5.1

The price process can be expressed in terms of the state variable $X_t$

$$dP_t = P_t(r + \lambda X_t^{1/2} + (\lambda X_t^{1/2} + 1)) + P_t X_t^{\frac{1}{2}} dB_t.$$  \hspace{1cm} (21)

The wealth process $W_t$ satisfies the following equation

$$dW_t = W_t(r + \phi_t \lambda X_t^{1/2} + 1) dt + W_t \phi_t X_t^{1/2} dB_t$$

The optimization problem (10) is studied in Liu (1998). We will sketch the procedure here. The derived utility function is the utility of when the optimal allocation $\phi^*_t$ is followed

$$J(W, V, t) = E_0 \left[ \frac{W_t^1}{1 - \gamma} \right].$$

The Bellman-Hamilton-Jacobi equation (HJB) implies that $J$ satisfies the following equation

$$\max_{\phi} \left[ \dot{J} + \frac{1}{2} W^2 \phi^2 X^{1/2} J_{WW} + W[r + \phi \lambda X^{1/2} + 1] J_W 
+W \phi \sigma X^{1/2} J_{WX} + \frac{1}{2} \sigma X^{1/2} J_{XX} + (k - KX) J_X \right] = 0,$$

$$J(T, W, X) = \frac{W_{1-\gamma}}{1-\gamma},$$ \hspace{1cm} (22)

where $\dot{J}$, $J_W$ and $J_X$ denote the derivatives of $J$ with respect to $t$, $W$ and $X$ respectively.

The first order condition gives

$$\phi^* = -\frac{J_W}{W J_{WW}} X^{-\frac{1}{2} + \frac{1}{2}} \left( \lambda + \sigma \rho \frac{\partial \ln J_W}{\partial X} \right).$$

We now have

$$\dot{J} - \frac{1}{2} X \frac{J_W^2}{J_{WW}} \left( \lambda + \sigma \rho \frac{\partial \ln J_W}{\partial X} \right)^2 + W J_W r
+ \frac{1}{2} \sigma^2 X J_{XX} + (k - KX) J_X = 0.$$

Guess that the value function $J$ has the form

$$J(W, X, t) = \frac{W_{1-\gamma}}{1-\gamma} e^{(c(t) + d(t)X)},$$

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the above PDE reduces to
\[ \dot{c} + \dot{d}X + \frac{1 - \gamma}{2 \gamma^2} X (\lambda + \gamma \sigma d(t))^2 + \frac{1 - \gamma}{\gamma} r \]
\[ + \frac{\gamma}{2} \sigma^2 X d^2 + (k - KX)d = 0. \]

The functions \(c(t)\) and \(d(t)\) satisfy the following ordinary differential equation (ODE)
\[ \dot{c} + kd + \frac{1 - \gamma}{\gamma} r = 0, \]
\[ \dot{d} + \left( -K + \frac{1 - \gamma}{\gamma} \lambda \sigma \rho \right) d + \frac{\sigma^2}{2} \left( 1 + (\gamma - 1)(1 - \rho^2) \right) d^2 + \frac{1 - \gamma}{2 \gamma^2} \lambda^2 = 0, \]
\[ c(T) = d(T) = 0. \]

Solving the above ODE, we obtain
\[ c = \frac{2k}{\sigma^2(\rho^2 + \gamma(1 - \rho^2))} \ln \left( \frac{2\delta e^{(\xi + K)\gamma/2}}{\left( (K - (1 - \gamma)/\gamma \lambda \rho \sigma) + \delta \right) \left( \exp(\xi) - 1 \right) + 2\delta} \right) + \frac{1 - \gamma}{\gamma} r_\tau \]
\[ d = \frac{1}{((K - (1 - \gamma)/\gamma \lambda \rho \sigma) + \delta) \left( \exp(\xi) - 1 \right) + 2\delta} \]
(23)

with \(\tau = T - t\), \(\delta = \frac{1 - \gamma}{2 \gamma^2} \lambda^2\), and
\[ \xi = \sqrt{(K - (1 - \gamma)/\gamma \lambda \rho \sigma)^2 + 2\delta(\rho^2 + \gamma(1 - \rho^2))} \sigma^2 = \sqrt{K^2 - \frac{1 - \gamma}{\gamma} (2K \lambda \rho \sigma + \lambda^2 \sigma^2)}. \]

The readers are referred to Liu(1999) for calculation detail.

Note that the function \(c(t)\) is not used in \(\phi^*\). The optimal weight \(\phi^*\) is given by
\[ \phi^*_t = X_t^{-\frac{\gamma}{2} + \frac{1}{2}} \left( \frac{1}{\gamma} \lambda + \rho \sigma d(t) \right) \]
\[ = V_t^{-\frac{1}{2} + \frac{1}{2}} \left( \frac{1}{\gamma} \lambda + \rho \sigma d(t) \right). \]
(24)

When \(\gamma \geq 1\), the parameter \(\xi\) is real and it is easy to verify that the function \(c(t)\) and \(d(t)\) is well-defined over \([0, T]\). When \(\gamma < 1\), it is possible that \(\xi\) will be imaginary. This happens when In this case, \(d(t)\) is still real but can reaches infinite at finite \(t < T\). When \((K - (1 - \gamma)/\gamma \lambda \rho \sigma)^2 < -2\delta(\rho^2 + \gamma(1 - \rho^2))\sigma^2\), \(\xi = i\eta\) is purely imaginary and so \(\eta\) is real. In this case, one can easily show that
\[ d = -\frac{2}{K - \frac{1 - \gamma}{\gamma} \lambda \rho \sigma + \eta \cot(\eta \gamma/2)} \delta. \]
(25)
8.4 Proof of Proposition 5.3

\[
\phi^* = V_t^{a-1} \frac{\lambda(K + \xi \coth(\xi\tau/2))}{\gamma(K + \xi \coth(\xi\tau/2)) - (1 - \gamma)\lambda\rho\sigma} \tag{26}
\]

\[
= V_t^{a-1} g. \tag{27}
\]

First consider \( \rho > 0 \). Let \( \Delta = 2K\rho + \lambda\sigma, a = K/\sqrt{\Delta\lambda\sigma}, b = \frac{\lambda\rho\sigma}{\sqrt{\Delta}} \). then \( \xi = \sqrt{\Delta}\sqrt{a^2 + 1 - \frac{1}{\gamma}} \) and \( \gamma = \frac{1}{1+a^2-z^2}, z = \frac{\xi}{\sqrt{\Delta\lambda\sigma}} \) and \( \nu = \sqrt{\Delta\lambda\sigma}\tau \), so

\[
g = \frac{K + \xi \coth(\xi\tau/2)}{\gamma(K + \xi \coth(\xi\tau/2)) - (1 - \gamma)\lambda\rho\sigma} = \frac{(a + z \coth(z\nu/2))(1 + (a^2 - z^2))}{a + z \coth(z\nu/2) - b(a^2 - z^2)} = \frac{(a + f)(1 + a^2 - z^2)}{h},
\]

where \( h = a + z \coth(z\nu/2) - b(a^2 - z^2) \), \( f = z \coth(z\nu/2) \). It is straightforward to show that

\[
\frac{\partial g}{\partial z} = \frac{(f'(1 + a^2 - z^2) + (a + f)(-2z))h - (f' + 2bz)(a + f)(1 + a^2 - z^2)}{h^2} < 0. \tag{28}
\]

Now consider the case \( \rho < 0 \) and \( 2K\rho + \lambda\sigma < 0 \). It is easy to verify that \( \phi^* \) is decreasing in \( \gamma \) when \( \gamma > 1 \) using Equation 12,

\[
\phi^* = V_t^{a-1} \frac{1}{\gamma} \left( 1 + \frac{1}{K - \frac{1}{\gamma}\lambda\rho\sigma + \xi \coth(\xi\tau/2)} \right). \tag{29}
\]

Note that

\[
\frac{\partial \xi}{\partial \gamma} = \frac{(2K\rho + \lambda\sigma)\lambda\sigma}{2\xi\gamma^2} < 0.
\]

Note that \( \xi \coth(\xi\tau/2) \) is increasing in \( \xi \) and therefore is decreasing in \( \gamma \). Hence \( \frac{1}{K - \frac{1}{\gamma}\lambda\rho\sigma + \xi \coth(\xi\tau/2)} \) is increasing in \( \gamma \) (noting that \( \lambda\rho\sigma < 0 \)). Because \( \frac{1}{\gamma} \) is decreasing in \( \gamma \) and is negative when \( \gamma > 1 \), it follows that the term in the bracket of the above equation is decreasing in \( \gamma \). Since \( \frac{1}{\gamma} \) and the bracket term are both positive and decreasing in \( \gamma \), their product is also decreasing, which implies that \( \phi^* \) is.
Now consider the case of $\gamma < 1$, $\rho < 0$, and $2K\rho + \lambda\sigma < 0$. We have
\begin{equation}
g = \frac{K + \xi \coth(\xi \tau/2)}{\gamma(K + \xi \coth(\xi \tau/2)) - (1 - \gamma)\lambda\rho\sigma} = \frac{\gamma(1 + x \coth(x\mu/2))}{1 + f} = \frac{1 + f}{\gamma(1 + f - C) + C}
\end{equation}
where $x = \frac{\xi}{K}$, and $C = \frac{-\lambda\sigma}{K}$, and $f(x) = x \coth(x\mu/2)$.

\[
\frac{\partial}{\partial x} g = \frac{f'(\gamma(1 + f - C) + C) - (1 + f)(\gamma f' + (1 + f - C)\gamma_x)}{(\gamma(1 + f - C) + C)^2} = \frac{f'(-\gamma C + C) - (1 + f)(1 + f - C)\gamma_x}{(\gamma(1 + f - C) + C)^2} = \frac{f'(-\gamma C + C) - (1 + f)(1 + f - C)\gamma_x}{(\gamma(1 + f - C) + C)^2}
\]

Note that
\[
x = \sqrt{1 + (1/\gamma - 1)C(2 - C/\rho^2)}
\]
and
\[
\gamma^{-1}_x = \frac{1}{2x} \gamma^2 C(2 - C/\rho^2).
\]
so
\[
\frac{f'(-\gamma C + C) - (1 + f)(1 + f - C)\gamma_x}{(\gamma(1 + f - C) + C)^2} = \frac{f'(x^2 - 1)(C + \frac{x^2 - 1}{2 - \frac{C}{\rho^2}}) + (1 + f)(1 + f - C)2x}{(\gamma(1 + f - C) + C)^2}
\]
For $x \geq 1$, then $f'(x^2 - 1)(C + \frac{x^2 - 1}{2 - \frac{C}{\rho^2}}) = f'(C(x^2 - 1) + \frac{(x^2 - 1)^2}{2 - \frac{C}{\rho^2}}) > 0$ (noting that $f'(x) > 0$ \forall x). Since $f(x) \geq x$, we know that $(1 + f(x))(1 + f(x) - C)2x \geq (1 + x)(1 + x - C)2x \geq 0$. We used the fact that $x \geq x_{\text{min}} = \sqrt{(C - 1)^2 + (1/\rho^2 - 1)C^2}$ so that $x \geq C - 1$.

Now we consider $x_{\text{min}} < x < 0$. Note that since $1 - x^2 - C(2 - C/\rho^2)$ for all $x \geq x_{\text{min}}$, we know that $f'(x^2 - 1)(C + \frac{x^2 - 1}{2 - \frac{C}{\rho^2}}) \geq 0$. Note that $f'(x) \geq 1$, we know $f'(x^2 - 1)(C + \frac{x^2 - 1}{2 - \frac{C}{\rho^2}}) \geq (x^2 - 1)(C + \frac{x^2 - 1}{2 - \frac{C}{\rho^2}})$.

So we have
\[
f'(x^2 - 1)(C(2 - \frac{C}{\rho^2} + x^2 - 1) + (1 + f)(1 + f - C)2x(2 - \frac{C}{\rho^2})
\]

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\[ \geq (x^2 - 1)(C(2 - \frac{C}{\rho^2}) + x^2 - 1) + (1 + x)(1 + x - C)2x(2 - \frac{C}{\rho^2}) \]
\[ = (1 - x^2)^2 - (1 - x^2)C(2 - \frac{C}{\rho^2}) + 2x((1 + x)^2 - (1 + x)C)(2 - \frac{C}{\rho^2}) \]
\[ = (1 - x^2)^2 + (2 - \frac{C}{\rho^2})(1 + x^2)(2x - C) \]
\[ = (1 + x)^2((1 - x)^2 + (2 - \frac{C}{\rho^2})(2x - C)) \]
\[ = (1 + x)^2(2 - 2x + x^2 - 1 + (2 - \frac{C}{\rho^2})(2x - C)) \]
\[ = (1 + x)^2(2(1 - x) + (2 - \frac{C}{\rho^2})2x) \geq 0. \]

When \( \rho < 0 \) and \( 2K\rho + \lambda\sigma > 0 \), one can easily show that the portfolio weight also reaches \(-\infty\) for small enough \( \gamma \) as long as \( \tau > 0 \). So the portfolio weight cannot be decreasing in \( \gamma \) for any strictly positive \( \tau \).