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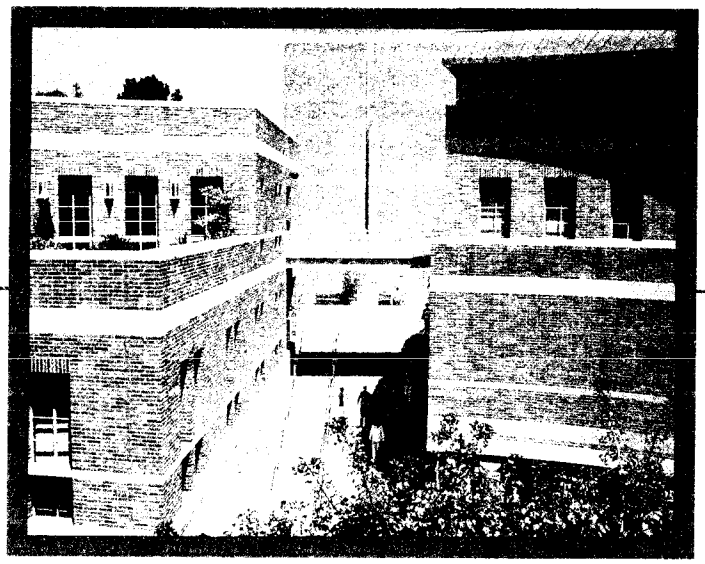

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Pricing of Options on Commodity Futures with Stochastic Term Structures of Convenience Yields and Interest Rates

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PRICING OF OPTIONS ON COMMODITY FUTURES WITH STOCHASTIC TERM STRUCTURES OF CONVENIENCE YIELDS AND INTEREST RATES

KRISTIAN R. MILTERSEN AND EDUARDO S. SCHWARTZ

ABSTRACT. We develop a model to value options on commodity futures in the presence of stochastic interest rates as well as stochastic convenience yields. In the development of the model we distinguish between forward and future convenience yields, a distinction that has not been recognized in the literature. Assuming normality of continuously compounded forward interest rates and convenience yields and log-normality of the spot price of the underlying commodity, we obtain closed form solutions generalizing the Black-Scholes/Merton's formulas. We provide numerical examples with realistic parameter values showing that even a short time lag between the maturity of a European call option and the underlying futures contract have a significant impact of the option price.

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1. Introduction

In a seminal paper Heath, Jarrow, and Morton (1992) develop a no-arbitrage model of the stochastic movements of the term structure of interest rates. The model takes as given the initial forward interest rate curve and derives the drift of the risk neutral forward interest rate process consistent with no arbitrage. The model can be used to value all types of interest rate derivatives. Reismann (1992), Cortazar and Schwartz (1994), Amin, Ng, and Pirrong (1995), and Carr and Jarrow (1995)[1] develop similar models for the term structure of commodity futures prices. These models take as given the initial term structure of commodity futures prices and derive its stochastic movement consistent with no arbitrage. The models can be used to value all types of commodity derivatives.

A different approach to the valuation of commodity derivatives is presented by Gibson and Schwartz (1990). They develop a two factor model where the first factor is the spot price of the commodity and the second factor is the instantaneous convenience yield. Schwartz (1997) extends this model by introducing a third stochastic factor, the instantaneous interest rate.

In this paper, we develop a model that generalizes and combines the two approaches by using all the information in the initial term structures of both interest rates and commodity futures prices. The model also fits into the general framework developed by Jarrow and Turnbull (1996). In addition, assuming normality of continuously compounded forward interest rates and convenience yields and log-normality of the spot price of the underlying commodity, we obtain closed form solutions for the pricing of options on futures prices as well as forward prices, which are in the spirit of Black and Scholes (1973) and Merton (1973). In the development of the model we distinguish between forward and future convenience yields, a distinction that has not been recognized in the existing literature.

An important aspect of building a stochastic model of the behavior of commodity prices is to consider mean-reversion. It is an empirically stylized fact that most commodity price processes are mean reverting, cf., e.g., Bessembinder et al. (1995). Standard no-arbitrage arguments completely determine the drift of the price processes under the equivalent martingale measure leaving no room for explicit modeling of mean reversion via the drift of the spot commodity price. However, the spot convenience yield process enters the drift of the spot commodity price under the equivalent martingale measure in such a way that a positive correlation between the spot commodity price and the spot convenience yield will have a mean reversion effect of the spot commodity price even under the equivalent martingale measure.

In Section 2 we establish the differences between forward and future convenience yields and state the terminology of the model. In Section 3 we develop the model and in Section 4 we specialize it to the Gaussian case and obtain closed form solutions for options on commodity futures as well as commodity forwards. In Section 5 we provide various special cases and in Section 6 we provide a numerical example. Finally, Section 7 concludes.

2. Preliminaries

The basic elements we work with in this paper are zero-coupon bond prices, $P(t,T)$, for all maturities, $T \geq t$, the spot price of the underlying commodity, S_t, forward prices of the commodity, $F(t,T)$, and futures prices of the commodity, $G(t,T)$, for all maturities, $T \geq t$, at any date $t \geq 0$. Note that since, in this model, we assume stochastic interest rates we will have to distinguish between forward and futures prices.

[1] Carr and Jarrow (1995) use a binomial approach and allow for stochastic interest rates.

To start, assume the primitives in the paper by Schwartz (1997). That is, we have a filtered probability space, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, and three adapted stochastic processes fulfilling sufficient integrability conditions such that the expectations used in the analysis are well defined. The three processes are the spot price of the underlying commodity, S, the spot convenience yield, δ, and the spot interest rate, r. Let $E[\cdot|\mathcal{F}_t]$ denote the conditional expectation under an equivalent martingale measure conditional on the date t information, $\mathcal{F}_t$. Using standard arguments, we have[2]

$$P(t,T) = E\big[e^{-\int_t^T r_s ds} \mid \mathcal{F}_t\big],$$

$$S_t = E\big[e^{-\int_t^T r_s ds} e^{\int_t^T \delta_s ds} S_T \mid \mathcal{F}_t\big], \tag{1}$$

$$F(t,T) = \frac{E\big[e^{-\int_t^T r_s ds} S_T \mid \mathcal{F}_t\big]}{P(t,T)}, \tag{2}$$

and

$$G(t,T) = E\big[S_T \mid \mathcal{F}_t\big], \tag{3}$$

for any given date t and future date $T \geq t$.

Using the characteristics of the spot price from Equation (1) we have

$$\begin{aligned}
S_t &= E\big[e^{-\int_t^T r_s ds} e^{\int_t^T \delta_s ds} S_T \mid \mathcal{F}_t\big] \\
&= E\big[e^{\int_t^T \delta_s ds} \mid \mathcal{F}_t\big] E\big[e^{-\int_t^T r_s ds} S_T \mid \mathcal{F}_t\big] + \mathrm{Cov}\big(e^{\int_t^T \delta_s ds}, e^{-\int_t^T r_s ds} S_T \mid \mathcal{F}_t\big) \\
&= E\big[e^{\int_t^T \delta_s ds} \mid \mathcal{F}_t\big] P(t,T) F(t,T) + \mathrm{Cov}\big(e^{\int_t^T \delta_s ds}, e^{-\int_t^T r_s ds} S_T \mid \mathcal{F}_t\big) \\
&= E\big[e^{\int_t^T \delta_s ds} \mid \mathcal{F}_t\big] E\big[e^{-\int_t^T r_s ds} \mid \mathcal{F}_t\big] E\big[S_T \mid \mathcal{F}_t\big] + E\big[e^{\int_t^T \delta_s ds} \mid \mathcal{F}_t\big] \mathrm{Cov}\big(e^{-\int_t^T r_s ds}, S_T \mid \mathcal{F}_t\big) \\
&\quad + \mathrm{Cov}\big(e^{\int_t^T \delta_s ds}, e^{-\int_t^T r_s ds} S_T \mid \mathcal{F}_t\big) \\
&= E\big[e^{\int_t^T \delta_s ds} \mid \mathcal{F}_t\big] P(t,T) G(t,T) + E\big[e^{\int_t^T \delta_s ds} \mid \mathcal{F}_t\big] \mathrm{Cov}\big(e^{-\int_t^T r_s ds}, S_T \mid \mathcal{F}_t\big) \\
&\quad + \mathrm{Cov}\big(e^{\int_t^T \delta_s ds}, e^{-\int_t^T r_s ds} S_T \mid \mathcal{F}_t\big),
\end{aligned} \tag{4}$$

where $\mathrm{Cov}(X,Y|\mathcal{F}_t)$ denotes the conditional covariance between the stochastic variables X and Y, i.e.,

$$\mathrm{Cov}(X,Y|\mathcal{F}_t) = E[XY|\mathcal{F}_t] - E[X|\mathcal{F}_t]E[Y|\mathcal{F}_t].$$

Equation (4) implies that the forward price of the commodity can be written as

$$F(t,T) = \frac{S_t - \mathrm{Cov}\big(e^{\int_t^T \delta_s ds}, e^{-\int_t^T r_s ds} S_T \mid \mathcal{F}_t\big)}{P(t,T) E\big[e^{\int_t^T \delta_s ds} \mid \mathcal{F}_t\big]}$$

[2]All equations between stochastic variables throughout the paper are to be understood as almost surely equations under the given probability measure.

and that the futures price of the commodity can be written as

$$
\begin{aligned}
G(t,T) &= \frac{S_t - \operatorname{Cov}\big(e^{\int_t^T \delta_s ds}, e^{-\int_t^T r_s ds} S_T \mid \mathcal{F}_t\big) - E\big[e^{\int_t^T \delta_s ds} \mid \mathcal{F}_t\big] \operatorname{Cov}\big(e^{-\int_t^T r_s ds}, S_T \mid \mathcal{F}_t\big)}{P(t,T) E\big[e^{\int_t^T \delta_s ds} \mid \mathcal{F}_t\big]} \\
&= F(t,T) - \frac{\operatorname{Cov}\big(e^{-\int_t^T r_s ds}, S_T \mid \mathcal{F}_t\big)}{P(t,T)} \qquad (5) \\
&= F(t,T) - \frac{S_t}{P(t,T)} \operatorname{Cov}\big(e^{-\int_t^T r_s ds}, \frac{S_T}{S_t} \mid \mathcal{F}_t\big).
\end{aligned}
$$

Equation (5) establishes the general relation between futures and forward prices.

Similar to the Heath-Jarrow-Morton approach, we prefer to work with continuously compounded forward interest rates, $f(t,s)$, that is, we *define* the forward interest rate, $f(t,s)$, such that the zero-coupon bond price is

$$
P(t,T) = E\big[e^{-\int_t^T r_s ds} \mid \mathcal{F}_t\big] = e^{-\int_t^T f(t,s) ds}. \qquad (6)
$$

Moreover, we would like to use the same approach for the forward prices of the commodity, hence, we *define* the continuously compounded forward convenience yields, $\delta(t,s)$, such that the forward price is

$$
F(t,T) = \frac{S_t}{P(t,T)} e^{-\int_t^T \delta(t,s) ds} = S_t e^{\int_t^T (f(t,s) - \delta(t,s)) ds}.
$$

That is,

$$
e^{-\int_t^T \delta(t,s) ds} = \frac{1 - \operatorname{Cov}\big(e^{\int_t^T \delta_s ds}, e^{-\int_t^T r_s ds} \frac{S_T}{S_t} \mid \mathcal{F}_t\big)}{E\big[e^{\int_t^T \delta_s ds} \mid \mathcal{F}_t\big]}. \qquad (7)
$$

We call $\delta(t,\cdot)$ the term structure of forward convenience yields, just like $f(t,\cdot)$ is called the term structure of forward interest rates. Note that if the spot convenience yield, δ_s, is deterministic for all s, then $\delta(t,s) = \delta_s$, for all t and s such that $t \leq s$.

Similarly, we will use this approach for the futures prices of the commodity, hence, we *define* the continuously compounded future convenience yields, $\epsilon(t,s)$, such that the futures price is

$$
G(t,T) = \frac{S_t}{P(t,T)} e^{-\int_t^T \epsilon(t,s) ds} = S_t e^{\int_t^T (f(t,s) - \epsilon(t,s)) ds}.
$$

That is,

$$
e^{-\int_t^T \epsilon(t,s) ds} = \frac{1 - \operatorname{Cov}\big(e^{\int_t^T \delta_s ds}, e^{-\int_t^T r_s ds} \frac{S_T}{S_t} \mid \mathcal{F}_t\big)}{E[e^{\int_t^T \delta_s ds} | \mathcal{F}_t]} - \operatorname{Cov}\big(e^{-\int_t^T r_s ds}, \frac{S_T}{S_t} \mid \mathcal{F}_t\big). \qquad (8)
$$

We call $\epsilon(t,\cdot)$ the term structure of future convenience yields. Note that if the spot convenience yield, δ_s, is deterministic for all s, then the future convenience yield will still reflect the correlation between the spot price of the underlying commodity and the spot interest rate, hence, it is not necessarily the case that $\epsilon(t,s) = \delta_s$, for all t and s such that $t \leq s$.

Differentiating, in Equation (6), with respect to T, dividing by $P(t,T)$, and taking the limit $T \downarrow t$ establishes the connection between the forward interest rate and the spot interest rate

$$
f(t,t) = r_t, \qquad (9)
$$

for all t. A similar task on Equations (7) and (8) gives the connection between the future convenience yield, the forward convenience yield, and the spot convenience yield

$$\delta(t,t) = \epsilon(t,t) = \delta_t, \tag{10}$$

for all t.

To ease the notation, we introduce

$$Y(t,T) = e^{\int_t^T (f(t,s)-\epsilon(t,s))ds}. \tag{11}$$

That is, the futures price of the commodity can be written as

$$G(t,T) = S_t Y(t,T). \tag{12}$$

Note that, since $Y(t,t) = 1$, $G(t,t) = S_t$ as expected given no-arbitrage restrictions.

3. The Model

The observables of the model are zero-coupon bond prices at date 0, $P(0,T)$, for all maturities, $T > 0$, the spot price of the underlying commodity, S_0, forward prices of the commodity, $F(0,T)$, and futures prices of the commodity, $G(0,T)$, for all maturities, $T > 0$.

Our stochastic model of future price movements consists of three processes, the spot price of the underlying commodity, the term structure of forward interest rates, and the term structure of future convenience yields. Since our objective is pricing of derivative securities written on the futures prices we are only concerned with the stochastic behavior of these three processes under an equivalent martingale measure. As it turns out in the Heath-Jarrow-Morton analysis it is most convenient to model the price fluctuations of the zero coupon prices by explicitly writing up the stochastic differential equation (SDE) for the continuously compounded forward interest rates, f. That is,

$$f(t,s) = f(0,s) + \int_0^t \mu_f(u,s)du + \int_0^t \sigma_f(u,s) \cdot dW_u, \tag{13}$$

where W is a standard d-dimensional Wiener process.[3] The same is true for the price fluctuations of the futures prices of the commodity, hence, we will explicitly write up the SDE for the continuously compounded future convenience yields, ϵ. That is,

$$\epsilon(t,s) = \epsilon(0,s) + \int_0^t \mu_\epsilon(u,s)du + \int_0^t \sigma_\epsilon(u,s) \cdot dW_u. \tag{14}$$

Finally, the spot price of the underlying commodity is modeled explicitly as

$$S_t = S_0 + \int_0^t S_u \mu_S(u)du + \int_0^t S_u \sigma_S(u) \cdot dW_u. \tag{15}$$

Possible correlation between the three processes comes via the specification of the diffusion terms (the σ's), since it is the same vector Wiener process, W, that is used in all three SDEs. So far the drift terms (the μ's) and the diffusion terms (the σ's) are not specified further, however, they must fulfill certain regularity conditions such that strong solutions of the stated SDEs exist. For example, they can be bounded previsible stochastic processes. If we further impose no-arbitrage restrictions on our model we can derive restrictions

[3] "$\cdot$" denotes the standard Euclidean inner product of $\mathbb{R}^d$ and the corresponding norm is defined as $||x||^2 = x \cdot x$ for any $x \in \mathbb{R}^d$.

on the stated SDEs under an equivalent martingale measure which will completely determine the drift terms (the μ's).

Standard no-arbitrage restrictions imply that the drift of the spot commodity price process is determined as

$$\mu_S(t) = r_t - \delta_t$$

under an equivalent martingale measure, cf., e.g., Equation (1). Hence, using the connection between the spot and the forward/future rates from Equations (9) and (10) we derive

$$\mu_S(t) = f(t,t) - \epsilon(t,t).$$

Similarly, we have from the Heath-Jarrow-Morton analysis the no-arbitrage restriction for the drift of the forward interest rate process is given by

$$\mu_f(t,s) = \sigma_f(t,s) \cdot \Big(\int_t^s \sigma_f(t,v)dv \Big)$$

under an equivalent martingale measure.

We are now aiming to get a similar restriction on the drift of the future convenience yield process, μ_ϵ. Itô's lemma on the zero-coupon bond prices from Equation (6) using the SDE of the forward interest rates from Equation (13) yields the dynamics (or the SDE) of the zero-coupon bond prices

$$\begin{aligned} P(t,T) = P(0,T) &+ \int_0^t P(u,T)\Big(f(u,u) - \int_u^T \mu_f(u,s)ds + \frac{1}{2}\Big\| \int_u^T \sigma_f(u,s)ds \Big\|^2 \Big) du \\ &- \int_0^t P(u,T)\Big(\int_u^T \sigma_f(u,s)ds \Big) \cdot dW_u, \end{aligned} \tag{16}$$

cf. Heath, Jarrow, and Morton (1992) for details. For notational convenience define

$$\sigma_{P_T}(t) = -\int_t^T \sigma_f(t,s)ds, \tag{17}$$

the date t instantaneous volatility of the return of the zero-coupon bond with maturity date T. By writing up the SDE for $X(t,T)$ defined as

$$\begin{aligned} X(t,T) &:= \int_t^T f(u,u)du \\ &= \int_t^T f(t,s)ds + \int_t^T \int_u^T \mu_f(u,s)dsdu + \int_t^T \Big(\int_u^T \sigma_f(u,s)ds \Big) \cdot dW_u, \end{aligned}$$

the zero-coupon bond prices, $P(t,T)$, from Equation (16) can be written in the following two ways

$$\begin{aligned} P(t,T) &= \frac{P(0,T)}{P(0,t)} e^{-\int_0^t \int_t^T \mu_f(u,s)dsdu - \int_0^t \left(\int_t^T \sigma_f(u,s)ds \right) \cdot dW_u} \\ &= P(0,T) e^{\int_0^t \left(f(u,u) - \int_u^T \mu_f(u,s)ds \right) du - \int_0^t \left(\int_u^T \sigma_f(u,s)ds \right) \cdot dW_u}. \end{aligned}$$

The first way is used by Amin and Jarrow (1992), whereas we continue to work with the second.

The same arguments that was used by Heath, Jarrow, and Morton (1992) to derive the dynamics of the zero-coupon bond prices, cf. our Equation (16), can be used on $Y(t,T)$ from Equation (11) using both the SDEs of the forward interest rates from Equation (13) and the future convenience yields from Equation (14).

That is,

$$
\begin{aligned}
Y(t,T) = Y(0,T) + \int_0^t Y(u,T)\Big(&-f(u,u) + \epsilon(u,u) + \int_u^T \mu_f(u,s)ds - \int_u^T \mu_\epsilon(u,s)ds \\
&+ \frac{1}{2}\Big\|\int_u^T \sigma_f(u,s)ds\Big\|^2 + \frac{1}{2}\Big\|\int_u^T \sigma_\epsilon(u,s)ds\Big\|^2 \\
&- \Big(\int_u^T \sigma_f(u,s)ds\Big)\cdot\Big(\int_u^T \sigma_\epsilon(u,s)ds\Big)\Big)du \\
+ \int_0^t Y(u,T)\Big(\int_u^T \sigma_f(u,s)ds &- \int_u^T \sigma_\epsilon(u,s)ds\Big)\cdot dW_u.
\end{aligned} \tag{18}
$$

Now, Itô's lemma on the expression of the futures price given by Equation (12) with the SDEs of the spot commodity price from Equation (15) and $Y(t,T)$ from Equation (18) gives the dynamics (or the SDE) of the futures prices

$$
\begin{aligned}
G(t,T) = S_0Y(0,T) + \int_0^t S_uY(u,T)\Big(&-f(u,u) + \epsilon(u,u) \\
&+ \int_u^T \mu_f(u,s)ds - \int_u^T \mu_\epsilon(u,s)ds \\
&+ \frac{1}{2}\Big\|\int_u^T \sigma_f(u,s)ds\Big\|^2 + \frac{1}{2}\Big\|\int_u^T \sigma_\epsilon(u,s)ds\Big\|^2 \\
&- \Big(\int_u^T \sigma_f(u,s)ds\Big)\cdot\Big(\int_u^T \sigma_\epsilon(u,s)ds\Big)\Big)du \\
+ \int_0^t S_uY(u,T)\Big(\int_u^T \sigma_f(u,s)ds &- \int_u^T \sigma_\epsilon(u,s)ds\Big)\cdot dW_u \\
+ \int_0^t Y(u,T)S_u\mu_S(u)du &+ \int_0^t Y(u,T)S_u\sigma_S(u)\cdot dW_u \\
+ \int_0^t Y(u,T)S_u\Big(\sigma_S(u)\cdot\Big(\int_u^T &(\sigma_f(u,s) - \sigma_\epsilon(u,s))ds\Big)\Big)du \\
= G(0,T) + \int_0^t G(u,T)\Big(-\int_u^T \mu_\epsilon(u,s)ds &+ \Big\|\int_u^T \sigma_f(u,s)ds\Big\|^2 \\
&+ \frac{1}{2}\Big\|\int_u^T \sigma_\epsilon(u,s)ds\Big\|^2 - \Big(\int_u^T \sigma_f(u,s)ds\Big)\cdot\Big(\int_u^T \sigma_\epsilon(u,s)ds\Big) \\
&+ \sigma_S(u)\cdot\Big(\int_u^T (\sigma_f(u,s) - \sigma_\epsilon(u,s))ds\Big)\Big)du \\
+ \int_0^t G(u,T)\Big(\sigma_S(u) + \int_u^T &(\sigma_f(u,s) - \sigma_\epsilon(u,s))ds\Big)\cdot dW_u.
\end{aligned} \tag{19}
$$

Again, for notational convenience define

$$
\sigma_{G_T}(t) = \sigma_S(t) + \int_t^T (\sigma_f(t,s) - \sigma_\epsilon(t,s))ds, \tag{20}
$$

the date t instantaneous volatility of the percentage change in the futures price.

Under an equivalent martingale measure, the futures price process is a martingale, cf., e.g., Equation (3), hence,

$$(21)\quad -\int_t^T \mu_\epsilon(t,s)ds + \left\|\int_t^T \sigma_f(t,s)ds\right\|^2 + \frac{1}{2}\left\|\int_t^T \sigma_\epsilon(t,s)ds\right\|^2 - \left(\int_t^T \sigma_f(t,s)ds\right)\cdot\left(\int_t^T \sigma_\epsilon(t,s)ds\right) + \sigma_S(t)\cdot\left(\int_t^T \big(\sigma_f(t,s)ds - \sigma_\epsilon(t,s)\big)ds\right) = 0,$$

which imply that the drift of the future convenience yield process is given by

$$(22)\quad \begin{aligned}\mu_\epsilon(t,T) &= \sigma_f(t,T)\cdot\left(\int_t^T \sigma_f(t,s)ds\right)\\ &\quad + \big(\sigma_f(t,T) - \sigma_\epsilon(t,T)\big)\cdot\left(\sigma_S(t) + \int_t^T \big(\sigma_f(t,s) - \sigma_\epsilon(t,s)\big)ds\right).\end{aligned}$$

This can be derived from Equation (21) by differentiating with respect to T and collecting terms.

Options on futures can now be priced using standard methods, cf., e.g., Harrison and Kreps (1979) and Harrison and Pliska (1981). Say, e.g., that we would like to price a European call option with exercise price K and maturity date t on the date T futures price ($t \leq T$). At date zero, this European call has the price

$$(23)\quad C = E\Big[e^{-\int_0^t f(s,s)ds}\big(G(t,T) - K\big)^+\Big].$$

To further develop this expression, we need to specify the functional form of the volatilities in the underlying stochastic processes. This is what we do in the next section.

4. The Gaussian Case

In this section we will assume that all the three σ processes are deterministic functions of the time parameters. That is, we assume Gaussian continuously compounded forward interest rates and future convenience yields and log-Gaussian spot commodity prices. We show that these additional assumptions lead to a closed form Black-Scholes/Merton type pricing formula for the European call option written on either the futures price or the forward price.

4.1. Options on Futures Prices.

The derivation in this subsection is inspired by Brenner and Jarrow (1993), where they derive the closed form solutions for a European call option written on a zero-coupon bond with the same term structure of interest rate model as we have in this paper.

To evaluate the option price from Equation (23) first write

$$e^{-\int_0^t f(s,s)ds} = Ae^{-X},$$

with X defined as

$$\begin{aligned}X &= \int_0^t\int_0^s \sigma_f(u,s)\cdot dW_u ds\\ &= \int_0^t\left(\int_u^t \sigma_f(u,s)ds\right)\cdot dW_u\\ &= -\int_0^t \sigma_{P_t}(u)\cdot dW_u\end{aligned}$$

and A is residually determined. Note, moreover, that A is non-stochastic because of the way X is specified. Second, write

$$G(t,T) = Be^{Z},$$

with Z defined as

$$\begin{aligned} Z &= \int_0^t \Big(\sigma_S(u) + \int_u^T \big(\sigma_f(u,s) - \sigma_\epsilon(u,s)\big)ds\Big) \cdot dW_u \\ &= \int_0^t \sigma_{G_T}(u) \cdot dW_u \end{aligned}$$

and B is again residually determined and, per construction of Z, non-stochastic. Obviously, (X, Z) is jointly normally distributed with mean zero. The variances and covariance can be calculated as

$$\begin{aligned} \sigma_x^2 &= \int_0^t \Big\| \int_u^t \sigma_f(u,s)ds \Big\|^2 du = \int_0^t \|\sigma_{P_t}(u)\|^2 du, \\ \sigma_z^2 &= \int_0^t \Big\| \sigma_S(u) + \int_u^T \big(\sigma_f(u,s) - \sigma_\epsilon(u,s)\big)ds \Big\|^2 du = \int_0^t \|\sigma_{G_T}(u)\|^2 du, \end{aligned} \tag{24}$$

and

$$\sigma_{xz} = \int_0^t \Big(\int_u^t \sigma_f(u,s)ds\Big) \cdot \Big(\sigma_S(u) + \int_u^T \big(\sigma_f(u,s) - \sigma_\epsilon(u,s)\big)ds\Big)du = -\int_0^t \sigma_{P_t}(u) \cdot \sigma_{G_T}(u)du, \tag{25}$$

with obvious notation.

The European call price from Equation (23) can now be written as

$$\begin{aligned} C &= AE\Big[e^{-X}\big(Be^{Z} - K\big)^+\Big] \\ &= AE\Big[E\big[e^{-X} \mid Z\big]\big(Be^{Z} - K\big)^+\Big], \end{aligned} \tag{26}$$

using iterated expectations. Since in the Gaussian case the conditional distribution of X given Z is given as

$$X|Z = z \sim N\Big(z\frac{\sigma_{xz}}{\sigma_z^2}, \sigma_x^2\Big(1 - \frac{\sigma_{xz}^2}{\sigma_x^2\sigma_z^2}\Big)\Big),$$

we can calculate the conditional expectation as

$$E\big[e^{-X} \mid Z = z\big] = e^{-z\frac{\sigma_{xz}}{\sigma_z^2} + \frac{1}{2}\sigma_x^2\left(1 - \frac{\sigma_{xz}^2}{\sigma_x^2\sigma_z^2}\right)}.$$

Hence, Equation (26) can be rewritten as

$$C = Ae^{\frac{1}{2}\sigma_x^2\left(1 - \frac{\sigma_{xz}^2}{\sigma_x^2\sigma_z^2}\right)} E\Big[e^{-Z\frac{\sigma_{xz}}{\sigma_z^2}}\big(Be^{Z} - K\big)^+\Big]. \tag{27}$$

Introducing the indicator function $1_{\{Z > \log\frac{K}{B}\}}$, Equation (27) can be written

$$C = ABe^{\frac{1}{2}\sigma_x^2\left(1 - \frac{\sigma_{xz}^2}{\sigma_x^2\sigma_z^2}\right)} E\big[1_{\{Z > \log\frac{K}{B}\}} e^{Z\left(1 - \frac{\sigma_{xz}}{\sigma_z^2}\right)}\big] - AKe^{\frac{1}{2}\sigma_x^2\left(1 - \frac{\sigma_{xz}^2}{\sigma_x^2\sigma_z^2}\right)} E\big[1_{\{Z > \log\frac{K}{B}\}} e^{-Z\frac{\sigma_{xz}}{\sigma_z^2}}\big]. \tag{28}$$

Straight-forward manipulations of normal densities yield

$$E\big[1_{\{Z > \log\frac{K}{B}\}} e^{Z\left(1 - \frac{\sigma_{xz}}{\sigma_z^2}\right)}\big] = e^{\frac{(\sigma_z^2 - \sigma_{xz})^2}{2\sigma_z^2}} N\Big(\frac{\log\frac{B}{K} + \sigma_z^2 - \sigma_{xz}}{\sigma_z}\Big)$$

and

$$E\big[1_{\{Z>\log\frac{K}{B}\}}e^{-Z\frac{\sigma_{xz}}{\sigma_z^2}}\big] = e^{\frac{\sigma_{xz}^2}{2\sigma_z^2}}N\Big(\frac{\log\frac{B}{K}-\sigma_{xz}}{\sigma_z}\Big),$$

where $N(\cdot)$ denotes the standard cumulative normal distribution function. Observe that

$$\begin{aligned}
Ae^{\frac{1}{2}\sigma_x^2\left(1-\frac{\sigma_{xz}^2}{\sigma_x^2\sigma_z^2}\right)}e^{\frac{\sigma_{xz}^2}{2\sigma_z^2}} &= Ae^{\frac{1}{2}\sigma_x^2}\\
&= AE\big[e^{-X}\big]\\
&= E\big[Ae^{-X}\big]\\
&= E\big[e^{-\int_0^t f(s,s)ds}\big]\\
&= P(0,t)
\end{aligned}$$

and that

$$\begin{aligned}
ABe^{\frac{1}{2}\sigma_x^2\left(1-\frac{\sigma_{xz}^2}{\sigma_x^2\sigma_z^2}\right)}e^{\frac{(\sigma_z^2-\sigma_{xz})^2}{2\sigma_z^2}} &= ABe^{\frac{1}{2}(\sigma_x^2+\sigma_z^2-2\sigma_{xz})}\\
&= ABE\big[e^{-X+Z}\big]\\
&= E\big[Ae^{-X}Be^{Z}\big]\\
&= E\big[e^{-\int_0^t f(s,s)ds}G(t,T)\big].
\end{aligned}$$

Moreover,

$$Be^{\frac{1}{2}\sigma_z^2-\sigma_{xz}} = \frac{E\big[e^{-\int_0^t f(s,s)ds}G(t,T)\big]}{P(0,t)},$$

implying that

$$\log\frac{B}{K}+\frac{1}{2}\sigma_z^2-\sigma_{xz} = \log\frac{E\big[e^{-\int_0^t f(s,s)ds}G(t,T)\big]}{P(0,t)K}.$$

Finally, defining

$$G(0,t,T) = E\big[e^{-\int_0^t f(s,s)ds}G(t,T)\big],$$

and substituting into Equation (28), we have the European call option price as

$$C = G(0,t,T)N\Big(\frac{\log\frac{G(0,t,T)}{P(0,t)K}+\frac{1}{2}\sigma_z^2}{\sigma_z}\Big) - P(0,t)KN\Big(\frac{\log\frac{G(0,t,T)}{P(0,t)K}-\frac{1}{2}\sigma_z^2}{\sigma_z}\Big), \tag{29}$$

where σ_z is defined in Equation (24).

With the normality assumptions stated, we can calculate $G(0,t,T)$ in the following way

$$\begin{aligned}
G(0,t,T) &= ABe^{\frac{1}{2}(\sigma_x^2+\sigma_z^2-2\sigma_{xz})}\\
&= AE\big[e^{-X}\big]BE\big[e^{Z}\big]e^{-\sigma_{xz}}\\
&= P(0,t)E\big[G(t,T)\big]e^{-\sigma_{xz}}\\
&= P(0,t)G(0,T)e^{-\sigma_{xz}},
\end{aligned} \tag{30}$$

since the futures price, $G(\cdot,T)$, is a martingale under an equivalent martingale measure. σ_{xz} is defined in Equation (25). With this expansion of $G(0,t,T)$ Equation (29) can be simplified to

$$C = P(0,t)\Big(G(0,T)e^{-\sigma_{xz}}N\Big(\frac{\log\frac{G(0,T)}{K} - \sigma_{xz} + \frac{1}{2}\sigma_z^2}{\sigma_z}\Big) - KN\Big(\frac{\log\frac{G(0,T)}{K} - \sigma_{xz} - \frac{1}{2}\sigma_z^2}{\sigma_z}\Big)\Big), \tag{31}$$

which provides a closed form expression for the price of a European call option with maturity t and exercise price K written on the commodity futures price with maturity T.

4.2. **The Difference between Forward and Futures Prices.** In the Gaussian case we can also explicitly compute the difference between forward and futures prices by the following argument:

From Equation (2) we have

$$\begin{aligned}
F(t,T) &= \frac{E\big[e^{-\int_t^T f(u,u)du}S_T \mid \mathcal{F}_t\big]}{P(t,T)} \\
&= \frac{E\big[e^{-\int_t^T f(u,u)du}S_t e^{\int_t^T (f(u,u)-\epsilon(u,u))du}e^{-\frac{1}{2}\int_t^T \|\sigma_S(u)\|^2 du}e^{\int_t^T \sigma_S(u)\cdot dW_u} \mid \mathcal{F}_t\big]}{P(t,T)} \\
&= \frac{S_t}{P(t,T)}e^{-\frac{1}{2}\int_t^T \|\sigma_S(u)\|^2 du}E\big[e^{-\int_t^T \epsilon(u,u)du}e^{\int_t^T \sigma_S(u)\cdot dW_u} \mid \mathcal{F}_t\big] \\
&= \frac{S_t}{P(t,T)}e^{-\frac{1}{2}\int_t^T \|\sigma_S(u)\|^2 du}e^{-\int_t^T \epsilon(t,s)ds}e^{-\int_t^T\int_u^T \mu_\epsilon(u,s)dsdu} \\
&\quad E\big[e^{\int_t^T (\sigma_S(u)-\int_u^T \sigma_\epsilon(u,s)ds)\cdot dW_u} \mid \mathcal{F}_t\big] \\
&= \frac{S_t}{P(t,T)}e^{-\int_t^T \epsilon(t,s)ds}e^{-\frac{1}{2}\int_t^T \|\sigma_S(u)\|^2 du}e^{-\int_t^T\int_u^T \mu_\epsilon(u,s)dsdu}e^{\frac{1}{2}\int_t^T \|\sigma_S(u)-\int_u^T \sigma_\epsilon(u,s)ds\|^2 du} \\
&= G(t,T)e^{-\frac{1}{2}\int_t^T \|\sigma_S(u)\|^2 du}e^{-\int_t^T \left(\|\int_u^T \sigma_f(u,s)ds\|^2 + \frac{1}{2}\|\int_u^T \sigma_\epsilon(u,s)ds\|^2\right)du} \\
&\quad e^{\int_t^T \left((\int_u^T \sigma_f(u,s)ds)\cdot(\int_u^T \sigma_\epsilon(u,s)ds) - (\int_u^T \sigma_f(u,s)ds - \int_u^T \sigma_\epsilon(u,s)ds)\cdot\sigma_S(u)\right)du} \\
&\quad e^{\frac{1}{2}\int_t^T \|\sigma_S(u)-\int_u^T \sigma_\epsilon(u,s)ds\|^2 du} \\
&= G(t,T)e^{-\int_t^T (\int_u^T \sigma_f(u,s)ds)\cdot(\sigma_S(u)+\int_u^T (\sigma_f(u,s)-\sigma_\epsilon(u,s))ds)du}.
\end{aligned} \tag{32}$$

Hence,

$$\int_t^T \big(\delta(t,s) - \epsilon(t,s)\big)ds = \int_t^T \Big(\int_u^T \sigma_f(u,s)ds\Big)\cdot\Big(\sigma_S(u) + \int_u^T \big(\sigma_f(u,s) - \sigma_\epsilon(u,s)\big)ds\Big)du. \tag{33}$$

By differentiating with respect to T in Equation (33), we derive the following expression for the forward convenience yield, $\delta(t,T)$

$$\begin{aligned}
\delta(t,T) = \epsilon(t,T) + \int_t^T \Big(&2\sigma_f(u,T)\cdot\Big(\int_u^T \sigma_f(u,s)ds\Big) - \sigma_f(u,T)\cdot\Big(\int_u^T \sigma_\epsilon(u,s)ds\Big) \\
&- \sigma_\epsilon(u,T)\cdot\Big(\int_u^T \sigma_f(u,s)ds\Big) + \sigma_f(u,T)\cdot\sigma_S(u)\Big)du.
\end{aligned} \tag{34}$$

That is, Gaussian future convenience yields imply Gaussian forward convenience yields and vice versa, and the relation between the two is given by Equation (34).

4.3. **Options on Forward Prices.** Similar to the derivation of the option on futures prices from Equation (23), the price, at date zero, of the European call option with maturity date t and exercise price K

written on the date T forward price is

$$C^F = E\Big[e^{-\int_0^t f(s,s)ds}\big(F(t,T)-K\big)^+\Big]. \tag{35}$$

Let $H(t,T)$ denote

$$H(t,T) = e^{-\int_t^T\left(\int_u^T \sigma_f(u,s)ds\right)\cdot\left(\sigma_S(u)+\int_u^T(\sigma_f(u,s)-\sigma_\epsilon(u,s))ds\right)du}. \tag{36}$$

That is, from Equation (32)

$$F(t,T) = G(t,T)H(t,T).$$

Hence, $H(t,T)$ denotes the ratio of futures prices to forward prices.

Using this ratio of futures prices to forward prices, the European call option on the forward price, C^F from Equation (35), can be written

$$C^F = H(t,T)E\Big[e^{-\int_0^t f(s,s)ds}\Big(G(t,T)-\frac{K}{H(t,T)}\Big)^+\Big].$$

We can, therefore, price the option on the forward price using our formula for options on futures prices from Equation (29)

$$\begin{aligned} C^F &= H(t,T)G(0,t,T)N\Big(\frac{\log\frac{H(t,T)G(0,t,T)}{P(0,t)K}+\frac{1}{2}\sigma_z^2}{\sigma_z}\Big) \\ &\quad - H(t,T)P(0,t)\frac{K}{H(t,T)}N\Big(\frac{\log\frac{H(t,T)G(0,t,T)}{P(0,t)K}-\frac{1}{2}\sigma_z^2}{\sigma_z}\Big), \end{aligned} \tag{37}$$

where σ_z is still defined in Equation (24). Moreover, defining $F(0,t,T)$ as

$$F(0,t,T) := H(t,T)G(0,t,T) = E\big[e^{-\int_0^t f(s,s)ds}F(t,T)\big], \tag{38}$$

then $F(0,t,T)$ can also be written as

$$\begin{aligned} F(0,t,T) &= H(t,T)P(0,t)G(0,T)e^{-\sigma_{xz}} \\ &= \frac{H(t,T)}{H(0,T)}e^{-\sigma_{xz}}P(0,t)F(0,T) \\ &= e^{\int_0^t\left(\int_t^T \sigma_f(u,s)ds\right)\cdot\left(\sigma_S(u)+\int_u^T(\sigma_f(u,s)-\sigma_\epsilon(u,s))ds\right)du}P(0,t)F(0,T), \end{aligned} \tag{39}$$

by using the expressions for $G(0,t,T)$, H, and σ_{xz} from Equations (30), (36), and (25). That is, if we define α as

$$\alpha := \int_0^t\Big(\int_t^T \sigma_f(u,s)ds\Big)\cdot\Big(\sigma_S(u)+\int_u^T\big(\sigma_f(u,s)-\sigma_\epsilon(u,s)\big)ds\Big)du, \tag{40}$$

then the European call option price from Equation (37) can be simplified to

$$C^F = P(0,t)\Big(F(0,T)e^{\alpha}N\Big(\frac{\log\frac{F(0,T)}{P(0,t)K}+\alpha+\frac{1}{2}\sigma_z^2}{\sigma_z}\Big) - KN\Big(\frac{\log\frac{F(0,T)}{P(0,t)K}+\alpha-\frac{1}{2}\sigma_z^2}{\sigma_z}\Big)\Big),$$

which gives a closed form expression for the price of a European call option with maturity t and exercise price K written on the commodity forward price with maturity T.

5. Special cases

In this section we will demonstrate that our model includes as special cases many of the models known in the option pricing literature.

5.1. **The Merton (1973) Model.** To get the model of Merton (1973), we assume that the option and the underlying futures contract matures at the same date, i.e. $t = T$, and that the spot convenience yield, δ_s, is zero, for all s. Hence, also $\delta(t,s) = 0$, for all t and s such that $t \leq s$. This imply that

$$G(0,t,T) = E\big[e^{-\int_0^t f(s,s)ds} G(t,t)\big] = E\big[e^{-\int_0^t f(s,s)ds} S_t\big] = S_0.$$

Moreover, $\epsilon(t,s)$ is deterministic, as it can be seen from Equation (34), implying that $\sigma_\epsilon(t,s) = 0$, for all t and s such that $t \leq s$. Hence, from Equation (24)

$$\begin{aligned}
\sigma_z^2 &= \int_0^t \left\| \sigma_S(u) + \int_u^t \sigma_f(u,s)ds \right\|^2 du \\
&= \int_0^t \left\| \sigma_S(u) - \sigma_{P_t}(u) \right\|^2 du \\
&= \int_0^t \left\| \sigma_S(u) \right\|^2 du + \int_0^t \left\| \sigma_{P_t}(u) \right\|^2 du - 2\int_0^t \sigma_S(u) \cdot \sigma_{P_t}(u) du.
\end{aligned}$$

Inserting these values in the option valuation formula (31) reproduces the results of Merton (1973) and Amin and Jarrow (1992).

Of course this special case also includes the Black-Scholes model, cf. Black and Scholes (1973), by setting $\sigma_{P_t}(u) = 0$, for all u.

5.2. **Non-Stochastic Interest Rates.** If we assume non-stochastic interest rates, no-arbitrage restrictions imply that $f(t,s) = r_s$, for all t and s such that $t \leq s$, where the spot rate process, r, is now a deterministic process. Hence, $\sigma_f(t,s) = 0$, for all t and s such that $t \leq s$. In this case futures and forward prices are identical, cf., e.g., Equation (5). The drift of the future convenience yields under the equivalent martingale measure, μ_ϵ, from Equation (22) reduces, therefore, to

$$\mu_\epsilon(t,T) = -\sigma_\epsilon(t,T) \cdot \Big(\sigma_S(t) - \int_t^T \sigma_\epsilon(t,s)ds\Big),$$

which is similar to the findings of Reismann (1992), Cortazar and Schwartz (1994), and Amin, Ng, and Pirrong (1995).

In the Gaussian case σ_z and σ_{xz} from Equations (24) and (25) reduce to

$$\sigma_z^2 = \int_0^t \left\| \sigma_S(u) - \int_u^T \sigma_\epsilon(u,s)ds \right\|^2 du$$

and

$$\sigma_{xz} = 0.$$

Inserting these values in the option valuation formula (31) reproduces the result of Amin, Ng, and Pirrong (1995).

5.3. **Zero Spot Convenience Yields.** If the spot convenience yields, δ_s, is zero for all s, we have as noted earlier that $\sigma_\epsilon(t,s) = 0$, for all t and s such that $t \leq s$. In this case we reproduce the findings of Amin and

Jarrow (1992). Amin and Jarrow (1992) derive option prices on futures and forwards in a Gaussian model identical to our model except that they do not consider convenience yields.

First note that the ratio of futures prices to forward prices from Equation (32) reduces to

$$\frac{G(t,T)}{F(t,T)} = e^{\int_t^T \left(\int_u^T \sigma_f(u,s)ds\right)\cdot\left(\sigma_S(u)+\int_u^T \sigma_f(u,s)ds\right)du},$$

which corresponds to e^{λ} in Amin and Jarrow (1992, p. 225). Moreover, to compare the option pricing formulas the following calculation helps explaining the ξ in Amin and Jarrow (1992, p. 224–225):[4]

$$E\left[e^{-\int_0^t f(s,s)ds}F(t,T) \mid \mathcal{F}_t\right] = S_0\frac{P(0,t)}{P(0,T)}e^{\int_0^t\left(\int_t^T \sigma_f(u,s)ds\right)\cdot\left(\sigma_S(u)+\int_u^T \sigma_f(u,s)ds\right)du}.$$

The argument is similar to the derivation of Equation (32). The result can also be derived by combining Equations (38) and (39) and simplifying.

Moreover, Equation (34) gives an expression for the future convenience yield, $\epsilon(t,T)$, in this case of zero spot convenience yield

$$\epsilon(t,T) = -\int_t^T \sigma_f(u,T)\cdot\left(\sigma_S(u) + 2\int_u^T \sigma_f(u,s)ds\right)du.$$

Hereby confirming that the future convenience yield captures the correlation between the spot commodity price and the spot interest rate.

5.4. **The Schwartz (1997) Model.** To get the model of Schwartz (1997), we assume a three factor Gaussian model, i.e. $d = 3$, with the three deterministic diffusion terms (the σ's) defined as the following:

$$\sigma_S(t) = \sigma_S\begin{pmatrix}1\\0\\0\end{pmatrix}, \tag{41}$$

$$\sigma_\epsilon(t,s) = \sigma_\epsilon e^{-\kappa_\epsilon(s-t)}\begin{pmatrix}\rho_{S\epsilon}\\ \sqrt{1-\rho_{S\epsilon}^2}\\ 0\end{pmatrix}, \tag{42}$$

and

$$\sigma_f(t,s) = \sigma_f e^{-\kappa_f(s-t)}\begin{pmatrix}\rho_{Sf}\\ \frac{\rho_{\epsilon f}-\rho_{S\epsilon}\rho_{Sf}}{\sqrt{1-\rho_{S\epsilon}^2}}\\ \sqrt{1-\rho_{Sf}^2-\frac{(\rho_{\epsilon f}-\rho_{S\epsilon}\rho_{Sf})^2}{1-\rho_{S\epsilon}^2}}\end{pmatrix}. \tag{43}$$

[4]A simple calculation shows that Amin and Jarrow's ξ is the same as our α from Equation (40) with $\sigma_\epsilon(t,s) = 0$, for all t and s such that $t \leq s$.

That is, we have the following structure of the diffusion terms of the model, here written up as quadratic variation terms.

$$\begin{aligned}
d\langle S\rangle_t &= \sigma_S^2 S_t^2 dt,\\
d\langle \epsilon(\cdot,s)\rangle_t &= \sigma_\epsilon^2 e^{-2\kappa_\epsilon(s-t)}dt,\\
d\langle f(\cdot,s)\rangle_t &= \sigma_f^2 e^{-2\kappa_f(s-t)}dt,\\
d\langle S,\epsilon(\cdot,s)\rangle_t &= \sigma_S\sigma_\epsilon\rho_{S\epsilon}e^{-\kappa_\epsilon(s-t)}S_t dt,\\
d\langle S,f(\cdot,s)\rangle_t &= \sigma_S\sigma_f\rho_{Sf}e^{-\kappa_f(s-t)}S_t dt,
\end{aligned}$$

and

$$d\langle \epsilon(\cdot,s),f(\cdot,s)\rangle_t = \sigma_\epsilon\sigma_f\rho_{\epsilon f}e^{-(\kappa_\epsilon+\kappa_f)(s-t)}dt.$$

Inserting the definitions of the σ's in Equation (20) leads to

$$\begin{aligned}
\left\|\sigma_{G_T}(u)\right\|^2 &= \sigma_S^2 + 2\sigma_S\Big(\sigma_f\rho_{Sf}\frac{1}{\kappa_f}\big(1-e^{-\kappa_f(T-u)}\big) - \sigma_\epsilon\rho_{S\epsilon}\frac{1}{\kappa_\epsilon}\big(1-e^{-\kappa_\epsilon(T-u)}\big)\Big)\\
&\quad + \sigma_\epsilon^2\frac{1}{\kappa_\epsilon^2}\big(1-e^{-\kappa_\epsilon(T-u)}\big)^2 + \sigma_f^2\frac{1}{\kappa_f^2}\big(1-e^{-\kappa_f(T-u)}\big)^2\\
&\quad - 2\sigma_\epsilon\sigma_f\rho_{\epsilon f}\frac{1}{\kappa_\epsilon}\frac{1}{\kappa_f}\big(1-e^{-\kappa_\epsilon(T-u)}\big)\big(1-e^{-\kappa_f(T-u)}\big).
\end{aligned} \tag{44}$$

Equation (44) is the same term structure of instantaneous volatilities as was derived by Schwartz (1997) in a different setting. Schwartz (1997) uses the following three-factor model for the spot commodity price, S, the spot convenience yield, δ, and the spot interest rate, r, adapted to our notation

$$\begin{aligned}
dS_t &= (r_t-\delta_t)dt + \sigma_S(t)\cdot dW_t\\
d\delta_t &= \kappa_\epsilon(\hat{\alpha}-\delta_t)dt + \sigma_\epsilon(t,t)\cdot dW_t\\
dr_t &= \kappa_f(m^*-r_t)dt + \sigma_f(t,t)\cdot dW_t,
\end{aligned}$$

where $\hat{\alpha}$ respectively m^* are risk adjusted mean reversion levels for the spot convenience yield and spot interest rate, respectively.

Thus, using Equation (24),

$$\begin{aligned}
\sigma_z^2 &= \sigma_S^2 t + 2\sigma_S\Big(\sigma_f\rho_{Sf}\frac{1}{\kappa_f}\Big(t-\frac{1}{\kappa_f}e^{-\kappa_f T}\big(e^{\kappa_f t}-1\big)\Big) - \sigma_\epsilon\rho_{S\epsilon}\frac{1}{\kappa_\epsilon}\Big(t-\frac{1}{\kappa_\epsilon}e^{-\kappa_\epsilon T}\big(e^{\kappa_\epsilon t}-1\big)\Big)\Big)\\
&\quad + \sigma_\epsilon^2\frac{1}{\kappa_\epsilon^2}\Big(t+\frac{1}{2\kappa_\epsilon}e^{-2\kappa_\epsilon T}\big(e^{2\kappa_\epsilon t}-1\big) - 2\frac{1}{\kappa_\epsilon}e^{-\kappa_\epsilon T}\big(e^{\kappa_\epsilon t}-1\big)\Big)\\
&\quad + \sigma_f^2\frac{1}{\kappa_f^2}\Big(t+\frac{1}{2\kappa_f}e^{-2\kappa_f T}\big(e^{2\kappa_f t}-1\big) - 2\frac{1}{\kappa_f}e^{-\kappa_f T}\big(e^{\kappa_f t}-1\big)\Big)\\
&\quad - 2\sigma_\epsilon\sigma_f\rho_{\epsilon f}\frac{1}{\kappa_\epsilon}\frac{1}{\kappa_f}\Big(t-\frac{1}{\kappa_\epsilon}e^{-\kappa_\epsilon T}\big(e^{\kappa_\epsilon t}-1\big) - \frac{1}{\kappa_f}e^{-\kappa_f T}\big(e^{\kappa_f t}-1\big)\\
&\qquad + \frac{1}{(\kappa_\epsilon+\kappa_f)}e^{-(\kappa_\epsilon+\kappa_f)T}\big(e^{(\kappa_\epsilon+\kappa_f)t}-1\big)\Big).
\end{aligned} \tag{45}$$

Inserting the definitions of the σ's in Equations (17) and (20) leads to

$$\sigma_{P_t}(u)\cdot\sigma_{G_T}(u) = -\sigma_f\frac{1}{\kappa_f}\left(1-e^{-\kappa_f(t-u)}\right)\Big(\sigma_S\rho_{Sf}+\sigma_f\frac{1}{\kappa_f}\left(1-e^{-\kappa_f(T-u)}\right) - \sigma_\epsilon\rho_{\epsilon f}\frac{1}{\kappa_\epsilon}\left(1-e^{-\kappa_\epsilon(T-u)}\right)\Big).$$

Thus, using Equation (25),

$$\begin{aligned}\sigma_{xz} = \sigma_f\frac{1}{\kappa_f}\Big(&\sigma_S\rho_{Sf}\Big(t-\frac{1}{\kappa_f}\left(1-e^{-\kappa_f t}\right)\Big)\\ &+\sigma_f\frac{1}{\kappa_f}\Big(t-\frac{1}{\kappa_f}e^{-\kappa_f T}\left(e^{\kappa_f t}-1\right)-\frac{1}{\kappa_f}\left(1-e^{-\kappa_f t}\right)+\frac{1}{2\kappa_f}e^{-\kappa_f T}\left(e^{\kappa_f t}-e^{-\kappa_f t}\right)\Big)\\ &-\sigma_\epsilon\rho_{\epsilon f}\frac{1}{\kappa_\epsilon}\Big(t-\frac{1}{\kappa_\epsilon}e^{-\kappa_\epsilon T}\left(e^{\kappa_\epsilon t}-1\right)-\frac{1}{\kappa_f}\left(1-e^{-\kappa_f t}\right)\\ &\qquad+\frac{1}{(\kappa_\epsilon+\kappa_f)}e^{-\kappa_\epsilon T}\left(e^{\kappa_\epsilon t}-e^{-\kappa_f t}\right)\Big)\Big).\end{aligned} \tag{46}$$

With the derived expressions of σ_z and σ_{xz} from Equations (45) and (46) the European call option price can now be valued using Equation (31).

Similarly, we can use Equation (36) to calculate the ratio of futures prices to forward prices in this model

$$\begin{aligned}H(t,T) = \exp\Big(-\sigma_f\frac{1}{\kappa_f}\Big(&\sigma_S\rho_{Sf}\Big((T-t)-\frac{1}{\kappa_f}\left(1-e^{-\kappa_f(T-t)}\right)\Big)\\ &+\sigma_f\frac{1}{\kappa_f}\Big((T-t)-\frac{2}{\kappa_f}\left(1-e^{-\kappa_f(T-t)}\right)+\frac{1}{2\kappa_f}\left(1-e^{-2\kappa_f(T-t)}\right)\Big)\\ &-\sigma_\epsilon\rho_{\epsilon f}\frac{1}{\kappa_\epsilon}\Big((T-t)-\frac{1}{\kappa_\epsilon}\left(1-e^{-\kappa_\epsilon(T-t)}\right)-\frac{1}{\kappa_f}\left(1-e^{-\kappa_f(T-t)}\right)\\ &\qquad+\frac{1}{(\kappa_\epsilon+\kappa_f)}\left(1-e^{(\kappa_\epsilon+\kappa_f)(T-t)}\right)\Big)\Big)\Big).\end{aligned}$$

In the paper by Schwartz (1997) the emphasis is on the stochastic behavior of futures prices. In this paper we have shown how to value options on these futures prices.

6. Numerical Example

In this section, we demonstrate with a numerical example that even a small time lag between the maturity of the option and the underlying futures actually can play an important role in the pricing of the options. Take, as an example, a European options on COMEX High Grade Copper Futures and assume that the time lag between the maturity of the option and the underlying futures is six weeks.[5]

Assuming the stochastic processes as defined by the three diffusion terms in Equations (41)–(43), we compare three different futures option pricing models, denoted model M1, M2, and M3, with a time lag of six weeks between the maturity of the option and the maturity of the copper futures contract in mind.

[5] A time lag of six weeks is by no means unrealistic. For the traded American options on COMEX High Grade Copper Futures the prospectus describing the option defines the last trading day of the option contract as the "second Friday of the month prior to the delivery month of the underlying futures contract." Cf., e.g., Chicago Board of Trade (1989, p. 324). On the other hand, the last trading day of the underlying futures contract is defined as the "third last business day of the maturing delivery month." Cf., e.g., Chicago Board of Trade (1989, p. 323).

			K	M1	M2	Diff.	%Diff.
t	=	3m	80	15.34	15.08	0.26	1.71
T	=	3m+6w	95	4.97	4.21	0.76	15.30
P	=	0.988	110	0.92	0.52	0.40	43.73
t	=	6m	80	16.19	15.42	0.77	4.76
T	=	6m+6w	95	6.94	5.53	1.41	20.34
P	=	0.975	110	2.33	1.29	1.04	44.56
t	=	9m	80	16.99	15.70	1.29	7.60
T	=	9m+6w	95	8.39	6.37	2.02	24.11
P	=	0.963	110	3.59	1.92	1.67	46.43
t	=	12m	80	17.70	15.92	1.78	10.06
T	=	12m+6w	95	9.56	6.99	2.58	26.94
P	=	0.951	110	4.70	2.45	2.25	47.92

TABLE 1. Comparison of European copper futures option prices using model M1 and M2 for different exercise prices and maturity dates. The maturities of the option are 3, 6, 9, and 12 months and the maturities of the futures are six weeks later.

			K	M3	M2	Diff.	%Diff.
t	=	3m	80	15.00	15.08	-0.08	-0.52
T	=	3m+6w	95	3.91	4.21	-0.30	-7.67
P	=	0.988	110	0.39	0.52	-0.13	-33.69
t	=	6m	80	15.14	15.42	-0.28	-1.84
T	=	6m+6w	95	4.89	5.53	-0.64	-13.12
P	=	0.975	110	0.89	1.29	-0.40	-45.12
t	=	9m	80	15.24	15.70	-0.46	-3.01
T	=	9m+6w	95	5.49	6.37	-0.88	-16.06
P	=	0.963	110	1.29	1.92	-0.63	-48.79
t	=	12m	80	15.34	15.92	-0.58	-3.80
T	=	12m+6w	95	5.98	6.99	-1.01	-16.90
P	=	0.951	110	1.67	2.45	-0.78	-46.80

TABLE 2. Comparison of European copper futures option prices using model M2 and M3 for different exercise prices and maturity dates. The maturities of the option are 3, 6, 9, and 12 months and the maturities of the futures are six weeks later.

			K	M1	M2	M3
t	=	3m	80	15.34	15.00	14.95
T	=	3m+3m	95	4.97	3.93	3.68
			110	0.92	0.39	0.30
t	=	6m	80	16.19	15.08	14.97
T	=	6m+6m	95	6.94	4.72	4.42
			110	2.33	0.80	0.64
t	=	12m	80	17.70	15.25	15.20
T	=	12m+12m	95	9.56	5.82	5.71
			110	4.70	1.55	1.48

TABLE 3. Comparison of European copper futures option prices using model M1, M2, and M3 for different exercise prices, maturity dates, and time lags.

M1 This model simply ignores the time lag and prices the option using a Black-Scholes model with σ equal to the volatility of the return of the underlying spot commodity price, σ_S.

M2 This model uses Equation (31) of this paper with σ_z and σ_{xz} as in Equations (45) and (46).

M3 This model prices the option using a Black-Scholes model with σ equal to the volatility of the relative price change of the underlying futures price, $||\sigma_{G_T}||$.

To value the options for the three models discussed above, we use the parameter estimates for the COMEX High Grade Copper Futures data presented in Schwartz (1997, Table 10). That is,

$$\begin{aligned} \sigma_S &= 0.266, & \rho_{S\epsilon} &= 0.805, & & \\ \sigma_\epsilon &= 0.249, & \rho_{Sf} &= 0.0964, & \kappa_\epsilon &= 1.045, \\ \sigma_f &= 0.0096, & \rho_{\epsilon f} &= 0.1243, & \kappa_f &= 0.2. \end{aligned}$$

Tables 1 and 2 show such calculations, with $G(0,T) = 95$ and $P(0,t) = e^{-0.05t}$. "m" respectively "w" are used as time units and are abbreviations for month and week, respectively. Take, for example, an at the money option with six months to maturity. Model M1 gives a price of \$4.55, whereas model M2 gives a price of \$3.90. So the price difference is \$0.65 or 14.31% of the price given by model M2. Similarly, model M3 prices this option at \$3.66. So model M3 prices this option \$0.24 below model M2 or 6.53% lower. As it can be seen from the numbers even this small time lag plays a very important role in the pricing of options on commodity futures.

In Table 3, we have allowed for even larger time lags. This table shows that the prices using model M1 diverge from the prices using model M2 and M3 as the time lag increases. On the other hand, the prices using model M2 and M3 converge as the time lag increases. This is not surprising given that the instantaneous volatility of the relative change of the futures price as function of time to maturity converge to a fixed value, as shown in Figure 1. Model M1 uses the volatility at date $T = 0$, model M2 uses the average volatility between date t and T corrected for correlation with the interest rate, which is negligible in this example, and model M3 uses the volatility at date T. As mentioned in the introduction, the reason for this drop in the instantaneous volatility of the futures price comes from the mean reversion effect due to the large correlation of .805 between the spot commodity price and the spot convenience yield.

7. Conclusion

In this paper, we developed a general model for valuing options on commodity futures. The inputs to the model are the term structure of commodity futures prices and discount bond prices. In the development of the model, we distinguished between future and forward convenience yields. This model generalized previous work by Merton (1973), Gibson and Schwartz (1990), Amin and Jarrow (1992), Reismann (1992), Cortazar and Schwartz (1994), Amin, Ng, and Pirrong (1995), and Schwartz (1997).

In the Gaussian case, we were able to obtain closed form solutions for options on commodity futures and forwards. In addition, we obtained closed form expressions for the relation between forward and futures prices and forward and future convenience yields.

Using the parameters estimated by Schwartz (1997) for copper futures, we showed that even a small time lag between the maturity of the futures contract and the option contract can have a significant effect on option prices. The reason for this large effect can be explained by the very high correlation between the spot commodity price and the spot convenience yield which induces mean reversion in the spot commodity price.

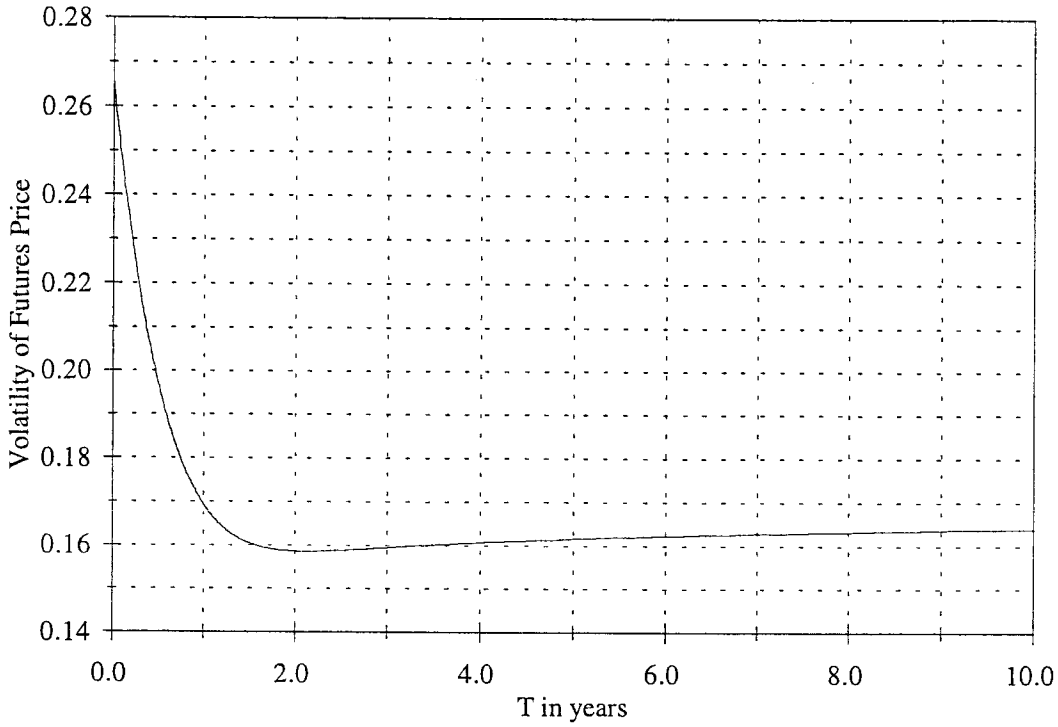

FIGURE 1. The instantaneous volatility of the futures price as function of time to maturity, T.

The methods developed in the paper can also be applied to more complicated derivatives such as American options and exotic options, but in some cases numerical methods will be required.

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