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Competition in Multiechelon Assembly Supply Chains

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In this paper, we study competition in multiechelon supply chains with an assembly structure. Firms in the supply chain are grouped into homogenous sectors (nodes) that contain identical firms with identical production capabilities that all produce exactly one undifferentiated product (that may itself be a "kit" of components). Each sector may use several inputs to produce its product, and these inputs are supplied by different sectors. The production process within any sector is taken to be pure assembly in fixed proportions. The number of firms in each sector is known. The demand curve for the final product is assumed to be linear, as are production costs in all sectors. Competition is modeled via a "coordinated successive Cournot" model in which firms choose production quantities for their downstream market so as to maximize profits, given prices for all inputs and all complementary products. Production quantities for sectors supplying the same successor are coordinated through pricing mechanisms, so that complementary products are produced in the right proportions. Under these assumptions, equilibrium prices for any multiechelon assembly network are characterized by a system of linear equations. We derive closed-form expressions for equilibrium quantities and prices in any two-stage system (i.e., a system with multiple input sectors and a single assembly sector). We show that any assembly structure can be converted to an equivalent (larger) structure in which no more than two components are assembled at any node. Finally, large structures can be solved either by direct solution of the characteristic linear equations or through an iterative reduction (compression) to smaller structures.

Key words: supply chains; pricing; game theory; equilibrium; production; coordination
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1. Introduction

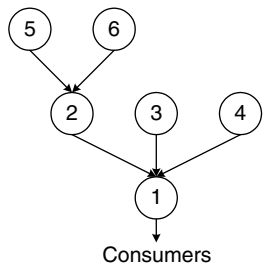
Many products, from digital equipment to apparel, are composed of modules, components, parts, and materials that travel through multiple vertically arranged industrial sectors that perform fabrication or assembly operations. These are often complex products with extensive bills of materials, so the sectors within the supply chain may have a complex arrangement in the horizontal dimension as well. Additional complexity exists in the form of competition among firms within each sector, yet the intensity of this competition can vary quite substantially across the sectors. For example, in the manufacture of personal computers, there are many final-product assemblers and many circuit board assemblers, while only a few chip manufacturers dominate that key fabrication stage of the chain. In automotive manufacturing, on the other hand, there are few final assemblers, but many parts manufacturers, at least for certain parts.

We can also observe that these supply chains exhibit a degree of coordination that is necessary for goods to be produced in the designed proportions, with appropriate materials, and at reasonable cost, but previous research has not produced models of individual firms' self-serving decisions achieving such coordination across these complex supply chains. Such a model is the first goal of this research.

Undoubtedly, the prices paid by consumers for finished goods and paid by supply chain participants for raw materials or semifinished goods are influenced by the structure of the supply chain, and the distribution of profits across the many firms in the supply chain is equally related to the chain's structure. We thus provide several techniques that facilitate analytical investigation of these effects, and we give numerous comparative statics results as well. Finally, but equally important, this model is easily solvable for supply chains of any size.

This paper specifically addresses assembly networks. Firms are grouped into sectors where all firms within a sector are identical. The number of firms in each sector is known. Each sector produces exactly one identical product from one or more inputs. The production process within a sector is pure assembly of the inputs, in fixed (given) proportions. The result can be pictured as a network that has a multiechelon or arborescent structure in which each sector is represented by a node. Any node or sector can supply

Figure 1 **Assembly Network Example**



only one downstream node, but may purchase inputs from several sectors (nodes). The structure of the network is taken as given and essentially duplicates the bill of materials for the end product at an appropriate level of granularity. An example of such an assembly network is sketched in Figure 1.

Much of the existing research into "horizontal interaction" addresses interactions or integration between firms that are within the same sector (e.g., Nault and Tyagi 2001 and Colangelo 1995). Two-sector models in which the sectors interact through the bundling of components by consumers include Matutes and Regibeau (1988), who focus on component compatibility, and Economides and Salop (1992), who focus on ownership and control issues. Neither of these streams addresses coordination across sectors.

Our analysis of "vertical interaction" between firms is closest in approach to the analysis of serial supply chains by Corbett and Karmarkar (2001), which can be seen as a special case of the problem addressed here. As in that case, we make some simplifying assumptions about the functional forms of demand and cost functions to arrive at general results for networks of any size. Also like that work, our analysis is in the tradition of the successive Cournot oligopoly literature, although for the most part that has been limited to two tiers with one or two entrants at each tier. In previous work, the upstream tier most often consists of a single monopolist. Furthermore, most of the existing literature is directed towards issues of vertical integration with the upstream monopolist integrating forward. Machlup and Taber (1960) present an early discussion of successive oligopoly and vertical integration. Greenhut and Ohta (1979) and Abiru (1988) show that vertical integration by a monopolist in the supplying sector by and large leads to higher outputs and lower prices. The seeming paradox here is that a monopolist integrating forward can drive out competitors from downstream markets, and yet social welfare can be increased. Essentially, this happens because vertical integration avoids double marginalization.

Waterson (1982), wishing to consider substitution possibilities in the downstream case, models a situation where downstream producers combine two inputs in variable proportions. One input is supplied by a monopolist, while the other is available from a competitive sector in unlimited quantities and a fixed (given) price. Examining vertical integration by the monopolist, he demonstrates that for low elasticity of substitution, social welfare rises, although prices may also rise. At higher elasticities, the results can be negative. Abiru et al. (1998) consider a more general structure with a Cournot oligopoly in the input-supplying sector as well, and with possible integration between firms in the upstream and downstream sectors. A prisoner's dilemma can arise as an equilibrium outcome in certain situations. In the dynamic case, there may be no integration. As is common (e.g., Cachon and Lariviere 1999; Colangelo 1995; Economides and Salop 1992; Salinger 1988, 1989; Stidham 1992; Van Miegham and Dada 1999), we employ a linear demand function, but this is not without loss of generality; for example, Tyagi (1999a, 1999b) focuses largely on situations in which comparative statics from linear and nonlinear demand functions differ. Other related literature includes the papers by Waterson (1980), Bresnahan and Reiss (1985), Choi (1991), Spence (1977), Vickers (1995), and Ziss (1995).

As we have noted, none of these papers consider assembly networks where there are multiple inputs supplied to assemblers by oligopolistic sectors. This situation requires imposing some type of coordination structure, especially for the fixed-proportions, pure-assembly case, so we introduce coordination in production quantities as part of the equilibrium criteria. Furthermore, unlike the large majority of the literature, we analyze supply networks with multiple stages because one of our objectives is to develop methods for large-scale networks.

We develop a model of competition in such pure-assembly supply networks that we term "coordinated, successive Cournot" competition. We obtain explicit expressions for prices and quantities for the two-stage case. We construct recursive methods based on aggregation of subnetworks that allow us to solve and analyze larger examples. We also show that prices for any structure satisfy a certain characteristic system of linear equations, which can be solved numerically.

We discuss the implications of our analyses for these questions in a later section. In the following, we first discuss coordination of multiple input-supplying sectors within the context of a successive Cournot model. Next, we analyze small examples where we can obtain complete solutions, and then extend the results to large systems. Finally, we examine the impact of changing parameters (such as demand curve or cost parameters), and changing the number of entrants, on equilibrium quantities and prices.

2. Successive Cournot Oligopoly and Quantity Coordination

The successive Cournot model has proved to be an effective tool for modeling multistage industry structures such as supply chains and networks with undifferentiated products. Consider a two-stage serial structure with one sector at each stage (e.g., one oligopoly supplying goods to another oligopoly that then serves consumers). Given the demand faced by the downstream sector (consisting of a number of identical firms), the standard Cournot framework is applied. Each firm in the sector chooses its production quantity so as to maximize profits, given the production decisions of other firms in the sector and the price of the input. In effect, the sector does not have oligopsonistic power over input price. The resulting equilibrium gives production quantities for each firm, and therefore for the sector in aggregate, as a function of the input price. This is then the derived demand curve for the input. The process can thus be repeated for the sector that provides the input material.

In the assembly case, where there are multiple input-providing sectors, the traditional successive oligopoly model requires extension. The analysis of the assembly sector is as before, except that there are multiple inputs. We assume without loss of generality that one unit of each input is required. Given the prices of these inputs, the resulting equilibrium again gives the derived inverse demand function for the downstream stage in terms of the total quantity as a function of the sum of input prices. However, it is not clear what demand curve is faced by each of the upstream supplying sectors. Furthermore, it is apparent that there must be some mechanism to effect quantity coordination across upstream suppliers because their outputs are combined in fixed proportions. We impose this coordination by defining the equilibrium solution for the supplying sectors jointly, in terms of the following criteria.

Each supplying sector chooses production quantities to maximize profits as in the standard Cournot model, with the added assumption that prices for the other (complementary)[1] inputs are given.

All sectors produce the same quantity (i.e., the market clears without any excess production in any sector).

It turns out that these conditions result in a set of equations in the prices for inputs, with a well-defined solution. We term this concept a "coordinated successive Cournot" equilibrium.

While we use this approach in the rest of this paper, it raises some obvious questions that bear discussion:

Why do suppliers know prices for complementary inputs, but not production quantities?

How are quantities actually coordinated? Is there some explicit mechanism, or is this simply something swept under the *tâtonnement* rug?

Is there an alternative model where suppliers know their complementors' quantity decisions rather than prices? This would be more consistent with the basic Cournot model.

Is this particular solution concept favorable in some way to suppliers? Does it imply some form of collusion? Or is it favorable to the buyer?

First, consider a situation where, instead of knowing prices, a firm in a supplying sector takes quantity decisions by other firms in its own sector as well as in other supplying sectors as given. In aggregate, because inputs are assembled in fixed (equal) proportions, firms in one sector cannot sell more units than their complementary sectors' output. Thus, unequal output from two complementary sectors means the high-output sector's marginal unit has no value and the low-output sector's marginal unit takes on the value of the entire parts "kit." It follows that (nondegenerate) equilibrium is never achieved—complementary sectors having equal aggregate production means that all firms have an incentive to cut output to command higher prices; unequal production means that the high-output firms should reduce quantity to eliminate unsold goods. The result is a collapse with no equilibrium solution.

How, then, might quantity coordination be achieved? One possibility is that the buyer sector takes an active role in the process. Imagine that the sector is represented by a single aggregate "buyer" or market specialist. Clearly, the buyer is fundamentally concerned with the total unit cost of a "kit" of all inputs in the right proportions, and is indifferent as to the way in which that cost breaks down across inputs. However, suppose that the buyer presents different inverse demand functions to each input-supplying sector (where the separate demand functions add up to the "right" function). Independent Cournot solutions can then be obtained for each supplying sector. If the resulting aggregate quantities do not match up, imagine that the buyer rearranges the inverse demand functions by moving them upwards for the sectors with lower than average quantities, and lower for the sectors with higher quantities. If we assume that the demand curve shifts are all simple vertical translations of the inverse demand function, then it is possible to characterize the solution that results in exact coordination across suppliers. It turns out that this solution will be the same as our concept proposed above. It is also the solution that leads to the maximum quantity being available to the buyer at a given

[1] Throughout this paper, terms such as "complementary products" refer to unfinished goods that are inputs to the same assembly sector. Other forms of complementarity also exist, and other definitions of these terms can be found in the existing literature.

cost, so that there is an incentive for the buyer to undertake the coordination effort.

Now, we do not wish to attribute any particular underlying mechanism to our solution approach. However, the discussion above provides some initial thoughts about the nature of the process. It suggests, first, that in the assembly case, one natural extension of the Cournot model (based on equilibria in quantity vectors across all suppliers) does not work. It also suggests that a coordination role gives the buyers some market power. Finally, coordination by the buyer also provides some motivation for assuming knowledge of prices rather than quantities across complementary suppliers. At the end, like most other solution concepts that define an equilibrium, we can only say that the results for our approach turn out to be sensible, but we cannot assert much about exact market mechanisms or the path by which equilibrium is reached.

3. The Basic Assembly Model

This section introduces the basic modelling elements for the simplest multiechelon assembly supply chains illustrated in Figure 2. Using a subscripted 1 to denote the sector of final assemblers, the price p_1 received for the finished goods is related to the quantity Q of goods sold through the linear function

$$p_1 = a - bQ. \tag{1}$$

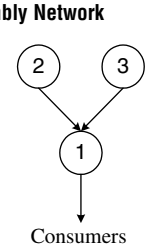
Here, a is the market reservation price, the supremum price at which demand will be positive. The other parameter b is a price sensitivity. For each node i, define

- v_i as the variable cost of production including raw materials of items in sector i; this does not include the cost of items procured from upstream sectors.
- n_i as the positive integer number of firms producing the part/assembly in sector i; infinite values are also allowed to represent perfectly competitive sectors. For convenience, we will also use a parameter N_i, defined by

$$N_i \triangleq \frac{n_i}{n_i + 1}.$$

- p_i as a common price received by all firms in sector i.
- $q_i^{[j]}$ as a quantity chosen by firm j in sector i. We use this bracketed superscript when it is necessary to refer to a specific firm in one sector. This is usually unnecessary because, under the conditions for equilibria that are given below, we are able to restrict attention to cases in which firms all select the same quantity. When this is possible, we omit the superscript and q_i denotes a production quantity within sector i that is chosen by all firms.

Figure 2 **Simple Assembly Network**



Again, we assume that competition within each node is of a Cournot fashion. That is, each firm in the node selects a quantity of goods to produce, and the total quantity produced within a sector (i.e., the sum of the individual firms' quantity decisions) then determines a price, shared by all firms within the sector, that is borne by firms at the downstream sector. Within this context, we use two criteria for determining an equilibrium. For every sector:

1. Given the prices charged by a sector's predecessors and complementors, no firm in that sector has an incentive to unilaterally deviate from its production quantity.
2. The aggregate quantity produced/assembled in every sector is the same.

We first illustrate the manner in which we apply the equilibrium criteria to solve the simple assembly network of Figure 2; here, firms in Sector 1 assemble components sourced from Sectors 2 and 3.

Applied to Sector 1 in this supply chain, the first equilibrium criterion means that each firm j in the sector will select a production quantity $q_1^{[j]}$ that maximizes its profits, given that the firms in that sector procure components at a total "kit" cost of $p_2 + p_3$ and pay variable cost v_1 for every unit produced. Firm j thus seeks to maximize revenues of $q_1^{[j]}(p_1 - p_2 - p_3 - v_1)$, when p_1 equals $a - b(q_1^{[j]} + \sum_{k \neq j} q_1^{[k]})$ from (1). Differentiation gives firm j's first-order optimality condition, which is to select quantity

$$q_1^{[j]} = \frac{a - v_1 - p_2 - p_3}{2b} - \frac{1}{2} \sum_{k \neq j} q_1^{[k]}.$$

The production decision of each of j's competitors follows this condition as well (except for the obvious reassignment of subscripts). Together this gives a system of n_i independent linear equations with a symmetric solution in which every firm produces quantity

$$q_1 = \frac{a - v_1 - p_2 - p_3}{(n_1 + 1)b}$$

and the entire sector produces an aggregate quantity

$$Q = n_1 q_1 = \frac{n_1}{n_1 + 1} \frac{(a + v_1 + p_2 + p_3)}{b}. \tag{2}$$

We now substitute this into Sector 1's demand curve (Equation (1)) to get the Sector 1 equilibrium price condition

$$p_1 = \frac{1}{n_1 + 1} a + \frac{n_1}{n_1 + 1}(v_1 + p_2 + p_3). \tag{3}$$

Next, we look at the supplier tier, Sectors 2 and 3; here the situation is more complex because the sectors are complementors. Our second equilibrium criterion requires the quantities produced at all sectors to balance, so the aggregate quantity Q must be equal to the aggregate equilibrium quantities produced in each of the supplier sectors. Our first equilibrium criterion requires that firms within Sector 2 have no incentive to deviate from their production quantities, given the price established by the complementor Sector 3. Thus, when considering production decisions in Sector 2, we take p_3 as given and solve Equation (2) for p_2 to get a demand curve for Sector 2's output:

$$p_2 = a - v_1 - p_3 - \frac{n_1}{n_1+1} bQ. \tag{4}$$

Applying the same logic as for Sector 1, which is possible because (4) is linear in Q, Sector 2 firms have no incentive to deviate from their production quantities iff they all select a quantity

$$q_2 = \frac{a - v_1 - v_2 - p_3}{n_2+1}\left(\frac{bn_1}{n_1+1}\right)^{-1},$$

which gives (because $Q = n_2 q_2$ under symmetry)

$$Q = \frac{n_2}{n_2+1}\left(\frac{bn_1}{n_1+1}\right)^{-1}(a - v_1 - v_2 - p_3).$$

Substituting this into (4) and rearranging yields the Sector 2 equilibrium price condition

$$p_2 = \frac{1}{n_2+1}(a - v_1 - p_3) + \frac{n_2}{n_2+1} v_2. \tag{5}$$

Also, Sector 3's equilibrium price condition is similarly derived as

$$p_3 = \frac{1}{n_3+1}(a - v_1 - p_2) + \frac{n_3}{n_3+1} v_3. \tag{6}$$

Equations (3), (5), and (6) are now a system of independent linear equations, the solutions of which are the equilibrium prices. Furthermore, substituting these prices into (2) gives the quantity of goods that pass through each of the sectors and to consumers. These results are formalized in the following proposition that is presented without additional proof. Note that for notational ease the "N" parameters are now used (defined previously as $N_i = n_i/(n_i+1)$).

Proposition 1. *The equilibrium values for the assembly network of Figure* 2 *are*

$$\begin{aligned} p_1 &= \left(1 - \frac{N_1N_2N_3}{N_2+N_3-N_2N_3}\right)a \\ &\quad + \frac{N_1N_2N_3}{N_2+N_3-N_2N_3}(v_1+v_2+v_3), \\ p_2 &= \frac{N_2v_2 + (1-N_2)N_3(a - v_1 - v_3)}{N_2+N_3-N_2N_3}, \\ p_3 &= \frac{N_3v_3 + N_2(1-N_3)(a - v_1 - v_2)}{N_2+N_3-N_2N_3}, \\ Q &= \frac{N_1N_2N_3}{N_2+N_3-N_2N_3}\cdot\frac{a - v_1 - v_2 - v_3}{b}. \end{aligned} \tag{7}$$

4. The Generalized Assembly Model

This solution method can also be applied to complex assembly networks (see any of the figures herein for examples). First, for each node i, define

- U_i as the set of upstream sectors; for example $U_2 = \{5, 6\}$ (here and below, examples refer to Figure 1).
- d_i as the downstream sector; e.g., $d_2 = 1$ and $d_1 = \varnothing$.
- D_i as the set of all sectors that are downstream of i—this includes d_i as well as all sectors that are downstream of d_i; e.g., $D_5 = \{1, 2\}$.
- C_i as the set of complementor nodes; these are the other sectors with the same immediate downstream sector as i. For example $C_2 = \{3, 4\}$. For cases in which i has only one complementor sector, we use c_i instead.
- DC_i as the set of "downstream complementors" to i. This is the set of all sectors that are complementors to any sector that is itself downstream of i—i.e., any sector that is a complementor of an element of D_i; e.g., $DC_5 = \{3, 4\}$.

The equilibrium prices of any assembly network can be calculated by solving a system of equations, each of which specifies one sector's equilibrium price (common across all firms in the sector) as a linear function of other sectors' prices. It can be iteratively shown, and follows from the fact that quantity-setting equilibria preserve the linearity of the demand function as exemplified by the linearity of (5) in Q, that each of these equations takes the form

$$\begin{aligned} p_i &= (1-N_i)\left(a - \sum_{k\in D_i} v_k - \sum_{k\in C_i} p_k - \sum_{k\in DC_i} p_k\right) \\ &\quad + N_i\left(v_i + \sum_{k\in U_i} p_k\right). \end{aligned} \tag{8}$$

These equations, one per sector, generalize the above expressions (cf., Equations (3), (5), and (6)) to account for the additional sectors and tiers in complex assembly networks. In most cases, the system of equations is most conveniently expressed in matrix notation, as given in the next proposition.

Proposition 2. *An assembly network's equilibrium prices are the solution to*

$$\mathbf{S}\cdot\mathbf{p} = \mathbf{R}, \tag{9}$$

where $\mathbf{p}$ *is the vector of prices, one per sector,* $\mathbf{R}$ *is a column vector (one element per sector) with*

$$R_i = (1-N_i)\left(a - \sum_{j\in D_i} v_j\right) + N_i v_i,$$

and **S** *is square and populated with elements*

$$S_{ij} = \begin{cases} 1 & \text{if } j = i, \\ 1 - N_i & \text{if } j \in C_i, \\ 1 - N_i & \text{if } j \in DC_i, \\ -N_i & \text{if } j \in U_i, \\ 0 & \text{otherwise}. \end{cases}$$

The proof of this is omitted for brevity, as it follows from the above discussion. It can easily be seen that the **S** matrix contains sufficient information to completely specify the structure, i.e., the precedence relationships of its associated network; we thus refer to **S** as the "structure matrix." We refer to (9) as the network's "characteristic equation" because it contains all information about the network's structure within **S** and has a solution which is the network's equilibrium prices.

Example. The network of Figure 1 has the characteristic equation $\mathbf{S} \cdot \mathbf{p} = \mathbf{R}$, with

$$\mathbf{S} = \begin{bmatrix} 1 & -N_1 & -N_1 & -N_1 & 0 & 0 \\ 0 & 1 & 1-N_2 & 1-N_2 & -N_2 & -N_2 \\ 0 & 1-N_3 & 1 & 1-N_3 & 0 & 0 \\ 0 & 1-N_4 & 1-N_4 & 1 & 0 & 0 \\ 0 & 0 & 1-N_5 & 1-N_5 & 1 & 1-N_5 \\ 0 & 0 & 1-N_6 & 1-N_6 & 1-N_6 & 1 \end{bmatrix},$$

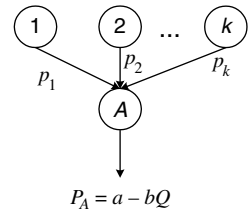
$$\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \end{bmatrix}, \text{ and } \mathbf{R} = \begin{bmatrix} (1-N_1)a + N_1 v_1 \\ (1-N_2)(a-v_1) + N_2 v_2 \\ (1-N_3)(a-v_1) + N_3 v_3 \\ (1-N_4)(a-v_1) + N_4 v_4 \\ (1-N_5)(a-v_1-v_2) + N_5 v_5 \\ (1-N_6)(a-v_1-v_2) + N_6 v_6 \end{bmatrix}.$$

After solving this for equilibrium prices, the equilibrium quantity of goods that flows through all sectors and to consumers is a simple matter of applying p_1 to the demand function (1).

5. Solving Two-Tier Assembly Networks in Closed Form

This model of assembly networks does not in general provide closed-form solutions for equilibrium prices and quantities. When there are only two tiers to the network, however, we can give such solutions, albeit in a rather complex form. The general two-tier assembly supply chain has a single assembly sector and k supplier sectors, as illustrated in Figure 3—note that to simplify the notation that follows the assembly sector is designated with an "A" and the supplier sectors denoted numerically beginning with 1. Also, in this section only we will use the following definition to simplify notation:

Figure 3 **Two-Tier Assembly Network**



$$M_i \triangleq 1 - N_i \quad \forall i.$$

We will also use a function $f_t^{\{\cdot\}}$, which is defined as follows—beginning with the set of indices $\{1, 2, \ldots, k\} \backslash \{\cdot\}$ (i.e., the set of all supplier indices except those in the set $\{\cdot\}$), find each combination of t of those indices, for each such combination calculate the product of the "M" terms with those indices, and take the sum of those products. Examples (with $k = 5$) are

$$\begin{aligned} f_1^{\{3\}} &= M_1 + M_2 + M_4 + M_5, \\ f_3^{\{5\}} &= M_1 M_2 M_3 + M_1 M_3 M_4 + M_2 M_3 M_4, \\ f_2^{\varnothing} &= M_1 M_2 + M_1 M_3 + M_1 M_4 + M_1 M_5 + M_2 M_3 \\ &\quad + M_2 M_4 + M_2 M_5 + M_3 M_4 + M_3 M_5 + M_4 M_5. \end{aligned}$$

In this case, the characteristic equation can be written in the following form:

$$\left[\begin{array}{c|ccc} 1 & -N_A & \cdots & -N_A \\ \hline 0 & & & \\ \vdots & & U & \\ 0 & & & \end{array}\right] \cdot \left[\begin{array}{c} p_A \\ \hline \\ P_U \\ \\ \end{array}\right] = \left[\begin{array}{c} M_A a + N_A v_A \\ \hline \\ B_U \\ \\ \end{array}\right]$$

where

$$U \triangleq \begin{bmatrix} 1 & M_1 & \cdots & M_1 \\ M_2 & 1 & \cdots & M_2 \\ \vdots & \vdots & \ddots & \vdots \\ M_k & M_k & \cdots & 1 \end{bmatrix}, \quad P_U \triangleq \begin{bmatrix} p_1 \\ p_2 \\ \vdots \\ p_k \end{bmatrix}, \quad \text{and}$$

$$B_U \triangleq \begin{bmatrix} M_1(a - v_a) + N_1 v_1 \\ M_2(a - v_a) + N_2 v_2 \\ \vdots \\ M_k(a - v_a) + N_k v_k \end{bmatrix}$$

and the equilibrium prices for the supplier sectors can be calculated by

$$P_U = U^{-1} \cdot B_U,$$

where the inverse of U is

$$U_{ij}^{-1} = \frac{1}{D} \cdot \begin{cases} 1 - f_2^{\{i\}} + 2f_3^{\{i\}} - 3f_4^{\{i\}} \\ \quad + \cdots + (-1)^{n-1}(n-2)f_{n-1}^{\{i\}} & \text{if } i = j, \\ -M_i(1 - f_1^{\{i,j\}} + f_2^{\{i,j\}} \\ \quad - \cdots + (-1)^{n-1} f_{n-2}^{\{i,j\}}) & \text{if } i \neq j, \end{cases}$$

$$\text{with } D = 1 - f_2 + 2f_3 - 3f_4 + \cdots + (-1)^{n-1}(n-1)f_n. \tag{10}$$

(The proof of this is provided in the appendix.) The equilibrium price for the assembled goods and the quantity of those goods is

$$p_A = M_A a + N_A\left(v_A + \sum_{i=1,\ldots,k} p_i\right) \quad \text{and} \quad Q = \frac{1}{b}(a - p_A).$$

6. Structural Properties of Assembly Networks

Thus far, we have described our model and how it can be used to represent network structure and solve for equilibrium prices through its characteristic equation. We also wish to investigate the general manner in which changes to a network's structure or to its parameters influence these equilibrium values—for example, how do prices and profits change as a sector becomes more or less concentrated? How does the structure of the network influence the profits enjoyed by the final assembly sector? In this section, we present several properties of assembly networks; in the next section, we will use these tools to address questions such as these.

Equivalence. We say that two networks are equivalent with respect to consumers when both networks provide the same quantity of goods to the market at the same price. We say the equivalence is with respect to a pair of nodes, one in each network, when these nodes receive the same equilibrium price and produce the same equilibrium quantity. When comparing networks for equivalence, we will adopt the convention that equivalent (or potentially equivalent) sectors are given the same label. This allows shorthand such as, "networks are equivalent with respect to nodes 3 and 4," which means that the networks are equivalent with respect to two pairs of nodes, those labelled "3" in each network and those labelled "4."

Expansibility. This property means that a network that is expanded by inserting "dummy sectors" is equivalent to the original network with respect to all of the nondummy sectors and also with respect to consumers. A dummy sector D is a node that is described by $N_D = 1$ and $v_D = 0$, and it is inserted between assembly and supplier nodes. Essentially, a dummy is a perfectly competitive sector in that its N_D value maps into an infinite number of competitors.

Figure 4 **Network Expansibility**

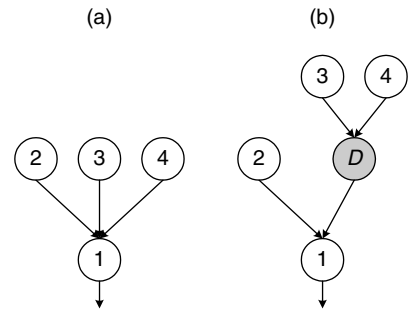


Figure 4 illustrates the equivalence of a network (a) and an expanded version of the same network (b). To show that these networks are equivalent, we show that the characteristic equation for (b) can be reduced to the characteristic equation of (a) through elementary row operations, and this will imply that the equilibrium prices of the nondummy sectors are the same. The characteristic equation of (b) is

$$\begin{bmatrix} 1 & -N_1 & 0 & 0 & -N_1 \\ 0 & 1 & 0 & 0 & 1-N_2 \\ 0 & 1-N_3 & 1 & 1-N_3 & 0 \\ 0 & 1-N_4 & 1-N_4 & 1 & 0 \\ 0 & 1-N_D & -N_D & -N_D & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_D \end{bmatrix} = \begin{bmatrix} (1-N_1)a + N_1 v_1 \\ (1-N_2)(a - v_1) + N_2 v_2 \\ (1-N_3)(a - v_1 - v_d) + N_3 v_3 \\ (1-N_4)(a - v_1 - v_d) + N_4 v_4 \\ (1-N_D)(a - v_1) + N_D v_D \end{bmatrix}.$$

Substituting that $v_D = 0$ and $N_D = 1$, this becomes

$$\begin{bmatrix} 1 & -N_1 & 0 & 0 & -N_1 \\ 0 & 1 & 0 & 0 & 1-N_2 \\ 0 & 1-N_3 & 1 & 1-N_3 & 0 \\ 0 & 1-N_4 & 1-N_4 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_D \end{bmatrix} = \begin{bmatrix} (1-N_1)a + N_1 v_1 \\ (1-N_2)(a - v_1) + N_2 v_2 \\ (1-N_3)(a - v_1) + N_3 v_3 \\ (1-N_4)(a - v_1) + N_4 v_4 \\ 0 \end{bmatrix}.$$

Now, change row 1 by adding row 5 multiplied by N_1 to it; also, change row 2 by adding row 5 multiplied by $-(1-N_2)$. This gives

$$\begin{bmatrix} 1 & -N_1 & -N_1 & -N_1 & 0 \\ 0 & 1 & 1-N_2 & 1-N_2 & 0 \\ 0 & 1-N_3 & 1 & 1-N_3 & 0 \\ 0 & 1-N_4 & 1-N_4 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_D \end{bmatrix} = \begin{bmatrix} (1-N_1)a + N_1 v_1 \\ (1-N_2)(a-v_1) + N_2 v_2 \\ (1-N_3)(a-v_1) + N_3 v_3 \\ (1-N_4)(a-v_1) + N_4 v_4 \\ 0 \end{bmatrix}. \tag{11}$$

It can be seen by inspection that (11) can be solved by first ignoring its last row and solving for p_1 through p_4; afterwards, p_D can be calculated from $p_D = p_3 + p_4$. However, the first four rows of (11) are exactly the characteristic equation of the original nonexpanded network of Figure 4(a). Thus, the two networks are equivalent with respect to the four numbered sectors and consumers.

Expansibility implies that any assembly network can be transformed into an equivalent network in which all sectors have at most two supplier sectors, and this is the basis for the solution method developed later.

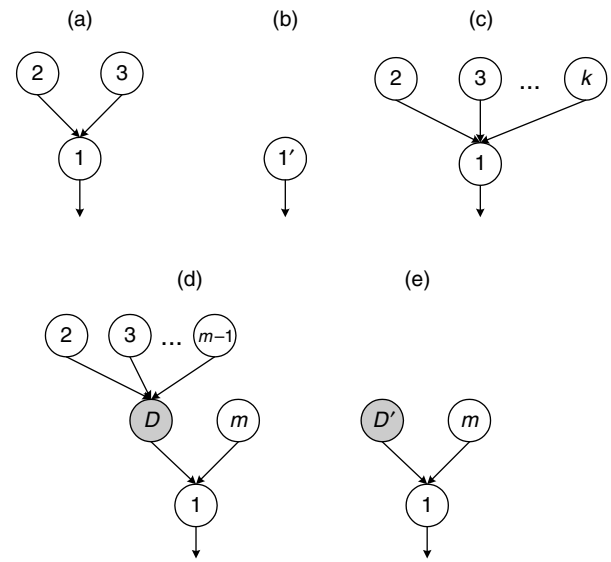
Compressibility. This property means that a network can be compressed into a smaller equivalent network. When there are only two tiers (i.e., a single assembler and one tier of suppliers), the compressed network is a single sector with redefined "N" and "v" values. The simplest example of this is to show the equivalence between (a) and (b) in Figure 5; here the prime ($'$) is used to indicate that Sector 1 in (b) is a compression of the original Sector 1 and also its supplier sectors (nodes 2 and 3). The two networks are equivalent under the definitions:

$$N_1' \triangleq \frac{N_1 N_2 N_3}{N_2 + N_3 - N_2 N_3} \quad \text{and} \quad v_1' \triangleq v_1 + v_2 + v_3,$$

as shown below.[2]

[2] As these definitions relate to the compressed node $1'$, a more precise notation would be, for example, $N_{1'}$ instead of N_1'. The less precise notation is less cumbersome, however, so we adopt it throughout. Also, as a result of the additional complexity of the assembly network relative to a single node, N_1' values typically do not map into an integer n_1, and this is also the case for the more general N_i' defined later. Earlier results such as Equation (8) still apply when "prime" values are used.

Figure 5 **Network Compressibility**



For equivalence between (a) and (b), we require that each has the same equilibrium p_1 and also that both sectors deliver the same quantity of goods to consumers. This second requirement follows directly from the first, however, and is redundant; it is only necessary to show that the p_1s are the same. It can be seen from Equation (7) and the definitions just above that (a) will deliver items at an equilibrium price

$$p_1 = (1 - N_1') \cdot a + N_1' v_1'. \tag{12}$$

It only remains to note that because (b) has only one sector, its characteristic equation from Proposition 2 is one dimensional and is exactly (12)—hence, the equivalence.

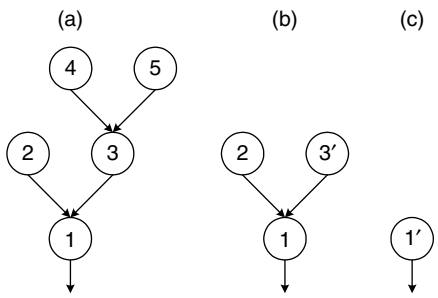
Compressibility still applies in the general two-tier case of (c), but the definitions of N_1' and v_1' take a more complex form. First, define a function $e_t^{\{\cdot\}}$ as follows: Beginning with the set of indices $\{\cdot\}$, find each combination of t of those indices, for each such combination calculate the product of the "N" terms with those indices, and take the sum of those products. The function e is quite similar to the function f defined earlier, and it is used instead of f to minimize the notational burden in the following analysis. Examples are

$$\begin{aligned} e_1^{\{1,\ldots,5\}} &= N_1 + N_2 + N_3 + N_4 + N_5, \\ e_3^{\{1,\ldots,4\}} &= N_1 N_2 N_3 + N_1 N_2 N_4 + N_2 N_3 N_4, \\ e_3^{\{1,3,4,5\}} &= N_1 N_3 N_4 + N_1 N_3 N_5 + N_3 N_4 N_5, \\ e_2^{\{2,4,5\}} &= N_2 N_4 + N_2 N_5 + N_4 N_5. \end{aligned}$$

Networks (b) and (c) are equivalent (for $k \geq 3$) when

$$N_1' = \frac{\prod_{i=1}^k N_i}{e_{k-2}^{\{2,\ldots,k\}} - (k-2)\prod_{i=2}^k N_i} \quad \text{and} \quad v_1' = \sum_{i=1}^k v_i. \tag{13}$$

Figure 6 **Iterative Compressibility**



The proof of this claim is in the appendix. Note, however, that the proof relies on the iterative compressibility of multitier networks as described next. As should be expected, the definitions in (13) reduce to those previously given for the two-supplier case.

Networks with more than two tiers exhibit *iterative compressibility*. This means that the network can be compressed by applying the previous definitions, but the compression is tier by tier, working from the top down. Figure 6 illustrates this; moving from (a) to (c), Sectors 3 through 5 first compress into Sector 3′; Sectors 1, 2, and 3′ then compress into Sector 1′. As before, the prime indicates a compressed sector representing the entire subnetwork of that sector and everything upstream.

The three networks in Figure 6 are equivalent with

$$N_3' \triangleq \frac{N_3N_4N_5}{N_4+N_5-N_4N_5}, \quad N_1' \triangleq \frac{N_1N_2N_3'}{N_2+N_3'-N_2N_3'},$$
$$v_3' \triangleq v_3+v_4+v_5, \quad \text{and} \quad v_1' \triangleq v_1+v_2+v_3'.$$

To show the equivalence between (a) and (b), we start with the characteristic equation for (a):

$$\begin{bmatrix} 1 & -N_1 & -N_1 & 0 & 0 \\ 0 & 1 & 1-N_2 & 0 & 0 \\ 0 & 1-N_3 & 1 & -N_3 & -N_3 \\ 0 & 1-N_4 & 0 & 1 & 1-N_4 \\ 0 & 1-N_5 & 0 & 1-N_5 & 1 \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{bmatrix} = \begin{bmatrix} (1-N_1)a+N_1v_1 \\ (1-N_2)(a-v_1)+N_2v_2 \\ (1-N_3)(a-v_1)+N_3v_3 \\ (1-N_4)(a-v_1-v_3)+N_4v_4 \\ (1-N_5)(a-v_1-v_3)+N_5v_5 \end{bmatrix}. \tag{14}$$

Elementary row operations are now applied to manipulate this into an equivalent characteristic equation that demonstrates compressibility. Specifically, the row vector

$$[0\ 0\ 1\ N_3N_5(N_4+N_5-N_4N_5)^{-1}\ N_3N_4(N_4+N_5-N_4N_5)^{-1}]$$

multiplied by (14) (i.e., **P** and **R**) replaces the third row. The equation then becomes (after substantial rearranging)

$$\begin{bmatrix} 1 & -N_1 & -N_1 & 0 & 0 \\ 0 & 1 & 1-N_2 & 0 & 0 \\ 0 & 1-N_3' & 1 & 0 & 0 \\ 0 & 1-N_4 & 0 & 1 & 1-N_4 \\ 0 & 1-N_5 & 0 & 1-N_5 & 1 \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{bmatrix} = \begin{bmatrix} (1-N_1)a+N_1v_1 \\ (1-N_2)(a-v_1)+N_2v_2 \\ (1-N_3')(a-v_1)+N_3'v_3' \\ (1-N_4)(a-v_1-v_3)+N_4v_4 \\ (1-N_5)(a-v_1-v_3)+N_5v_5 \end{bmatrix}. \tag{15}$$

p_1 through p_3 are now independent of p_4 and p_5, so they can be found by solving

$$\begin{bmatrix} 1 & -N_1 & -N_1 \\ 0 & 1 & 1-N_2 \\ 0 & 1-N_3' & 1 \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} (1-N_1)a+N_1v_1 \\ (1-N_2)(a-v_1)+N_2v_2 \\ (1-N_3')(a-v_1)+N_3'v_3' \end{bmatrix}. \tag{16}$$

This is exactly the characteristic equation of network (b) in Figure 6, which demonstrates the equivalence of (a) and (b). Additionally, we can see from (15) that p_4 and p_5 can be calculated immediately once p_2 is known. The equivalence of (b) and (c) then follows from the arguments given earlier for the two-tier case that give a market clearing price $p_1=(1-N_1')\cdot a+N_1'v_1'$.

In short, compressibility facilitates analysis and solution computation by allowing expression (8) to be written in a form that is independent of upstream prices:

$$p_i=(1-N_i')\left(a-\sum_{k\in D_i} v_k-\sum_{k\in C_i} p_k-\sum_{k\in DC_i} p_k\right)+N_i'v_i'. \tag{17}$$

The compressibility property provokes one additional observation: The final assembly node in any network structure, meaning the number and arrangement of sectors, has an associated set of possible N' values. The different values in this set come from different combinations of n_i values—one per sector (for nondummy sectors), each taken from the set $\{1,2,\ldots,\infty\}$. For example, a single-node network has possible N_1' values of $\{1/2, 2/3, 3/4, \ldots, 1\}$ that are generated as n_1 increases from 1 to ∞. Next, as sectors are added to a network, this set grows—that is, the original network's set is a subset of the augmented network's. In this manner,

assembly networks are partially ordered—a network is "more complex" than another if its set of this type is a superset of the other's.

7. Semi-Closed-Form Solutions

We now show how the properties described in the previous section are exploited to provide simple expressions for equilibrium prices. We refer to these expressions as "semi-closed form" because several values are calculated in an iterative albeit straightforward fashion. First note that we limit attention here to cases in which each assembly sector has at most two supplier sectors. This eases the notational burden and is without loss of generality because expansibility allows any network to be converted into an equivalent network for which this holds.

These iterative calculations, which generalize equations seen previously, are as follows[3]:

• For nonassembly sectors (i.e., nodes that do not have predecessors) $N_i' \triangleq N_i$ and $v_i' \triangleq v_i$. For assembly sectors (here, j and k are sector i's two predecessor/upstream nodes)

$$N_i' \triangleq \frac{N_i N_j' N_k'}{N_j' + N_k' - N_j' N_k'} \quad \text{and} \quad v_i' \triangleq v_i + v_j' + v_k'.$$

These values are calculated "top-down," working from the uppermost supplier nodes towards the final assembly node. This recursive definition implies that N_i' is increasing in the number of firms at sector i and at all of its upstream sectors; it is not sensitive to the number of firms at downstream sectors, however.

• If i is the final assembly sector, $a_i \triangleq a$ and $b_i \triangleq b$. Otherwise[4],

$$a_i \triangleq \frac{N_{c_i}'(a_{d_i} - v_{d_i} - v_{c_i}') + (1 - N_{c_i}')N_i' v_i'}{N_i' + N_{c_i}' - N_i' N_{c_i}'} \quad \text{and} \quad b_i \triangleq \frac{b_{d_i}}{N_{d_i}'}.$$

The recursion here is "bottom-up," working upstream from the final assembly sector.

Proposition 3. *At equilibrium,*

$$p_i = (1 - N_i')a_i + N_i' v_i'. \tag{18}$$

Additionally, if sector i has a complementor sector (c_i), *then*

$$p_{c_i} = \frac{(1 - N_{c_i}')N_i'(a_{d_i} - v_{d_i} - v_i') + N_{c_i}' v_{c_i}'}{N_i' + N_{c_i}' - N_i' N_{c_i}'} \tag{19}$$

and

$$p_i = \frac{(1 - N_i')N_{c_i}'(a_{d_i} - v_{d_i} - v_{c_i}') + N_i' v_i'}{N_i' + N_{c_i}' - N_i' N_{c_i}'}. \tag{20}$$

Thus far, the only method we have of calculating the equilibrium quantity Q that flows through the network is to first calculate the equilibrium price of the final assembly sector using (18) and to substitute that into the demand Equation (1). The next proposition allows us to directly calculate Q from the equilibrium price at any sector.

Proposition 4. *At equilibrium, the quantity of goods flowing through the network is* $(a_i - p_i)/b_i$ *for every* i.

8. Structural Results for Assembly Networks

It is intuitive that the internal characteristics of a sector, for example, the number of competitors, should influence the equilibrium selling prices and profits enjoyed by the firms in that sector. It is a bit less obvious that the internal characteristics of other sectors in the network will also influence prices and profits, as will the actual structure of the network. The two propositions that follow show the actual influence of these different factors on equilibrium prices.

Proposition 5. *Suppose that the number of firms in sector j increases* (*i.e.*, n_j, *and thus* N_j, *increase*).

(i) p_j *decreases.*

(ii) *If i is downstream of j, then* p_i *decreases.*

(iii) *If i is upstream of j and if j has no complementor or downstream complementor sectors (i.e.,* $C_j = DC_j = \varnothing$*), then* p_i *does not change.*

(iv) *If i is upstream of j and if j does have a complementor or downstream complementor sector, then* p_i *decreases.*

(v) *Otherwise,* p_i *increases.*

(vi) Q *increases.*

(i) is quite intuitive, as it is natural that the entry of new firms into a sector should, through the basic mechanisms of competition, lead to lower prices. Absent the introduction of a disruptive technology, a scenario that we do not model, it would be surprising to find a violation of this in any real situation. (ii) tells us that these price reductions are then passed along to the downstream sectors. The ultimate downstream sector is the final assembly sector, and the price reduction there leads to larger quantities traversing the entire network; hence (vi). Case (v) says that reduced industry concentration in one sector (i.e., a larger number of firms in the sector) has the effect of increasing prices in complementary sectors.

(iii) and (iv) point out a significant feature of assembly networks that is not found in the simpler structure of serial networks. In serial networks, which we model as "assembly" networks in which sectors have only a single supplier sector, an increase in industry concentration does not change upstream selling prices. This simpler structure means that all sectors fall into one of the first three cases in the proposition, with the consequence that the price reductions of (i) and (ii) increase the quantity of goods sold by the

[3] These definitions anticipate a complementor sector and two supplier sectors. If i has no complementor, set $N_{c_i}' = 1$ and $v_{c_i}' = 0$. If i has just one supplier, set $N_k' = 1$ and $v_k' = 0$.

[4] d_i and c_i are, respectively, i's downstream and complementor nodes.

upstream firms of (iii), but not their selling prices. In the presence of assembly operations, however, there are also sectors that fall under case (iv), and the higher prices enjoyed by those sectors have the effect of reducing the upstream prices. Note, however, that the proposition is a bit more general in this regard in that it captures these effects when portions of the network have sectors with only one supplier sector.

The next proposition shows how the sectors' variable cost parameters influence equilibrium prices throughout the network. Proposition 7, presented without proof for brevity, then shows how the demand parameters affect pricing.

Proposition 6. *Suppose that v_j, the variable production cost in sector j, increases.*

(i) *p_j increases.*
(ii) *If i is downstream of j, then p_i increases.*
(iii) *Otherwise, p_i decreases.*
(iv) *Q decreases.*

(i) here is, again, intuitive as it should be expected that increased production cost by firms in a sector should increase those firms' selling prices. (ii) then says that these price increases are passed along through the downstream supply chain, eventually resulting in higher prices for consumers and lower overall volume (as per (iv)). Upstream and complementary sectors fall into case (iii).

Next, our final proposition says that prices increase with the reservation price a, but are constant in the sensitivity parameter b. This result is omitted without formal proof. The change with a can easily be seen to hold by inspection, because a_i increases with a for every i and because the equilibrium prices of (18) increase with each a_i. The fact that changes in the sensitivity parameter do not affect equilibrium prices is a bit surprising; it holds because the b parameter does not appear in either a_i or (18).

Proposition 7. *Equilibrium prices of all sectors*: (i) *increase with reservation price a, and* (ii) *are constant in the demand elasticity b.*

Of course, profits are also a primary interest of any firm, so we are also interested in how profits are impacted by the various problem parameters. For the simplest case of Figure 2, the profits per firm in each sector are available in a fairly simple closed form:

$$\text{Sector 1:}\quad \text{profits} = (p_1 - p_2 - p_3 - v_1)\frac{Q}{n_1} = \frac{N_1(1-N_1)N_2^2N_3^2(a-v_1-v_2-v_3)^2}{bn_1(N_2+N_3-N_2N_3)}.$$

$$\text{Sector 2:}\quad \text{profits} = (p_2 - v_2)\frac{Q}{n_2} = \frac{N_1(1-N_2)N_2N_3^2(a-v_1-v_2-v_3)^2}{bn_2(N_2+N_3-N_2N_3)}.$$

Also, the expression for profits in Sector 3 is the same as for Sector 2 except for the obvious reassignment of subscripts. The following statements follow from differentiation of these expressions. The statements have also been proven to hold in general for the two-tier case (with an arbitrary number of supplier sectors) and have been numerically verified in general.

1. An increase in variable production costs in a sector leads to decreased profits in every sector.
2. Reduced industry concentration (i.e., more competitors within a sector) leads to smaller profits in that sector, but higher profits at all other sectors. Interestingly, total supply chain profits aggregated across all participants may rise or fall upon entry of an additional competitor.
3. Increased market size, parameterized by an increase in the market reservation price a, leads to increased profits at all sectors.
4. Increased price sensitivity, parameterized by an increase in b, leads to decreased profits at all sectors.

Parts of this list—a sector's profits increasing in market size and decreasing in degree of competition, variable cost, and price sensitivity—essentially restate conventional wisdom. This is also true for some of the earlier findings—e.g., consumer prices increase with the reservation price a and with production costs. This conventional wisdom derives from models with very simple structures, such as the classic Cournot model, and so does not translate into our arbitrarily complex networks in any rigorous fashion. We are thus reassured that many existing and appealing insights into the behavior of production and distribution systems carry over into these assembly networks that have not been modeled previously.

Of course, sans-assembly models are silent about the "horizontal interplay" across complementor sectors. For example (and referencing Figure 1) consider the following questions.

1. Given that the materials/goods flowing through Sectors 4 and 6 may be entirely dissimilar and that the sectors are connected very indirectly (through Sectors 1 and 2), why do prices and profits at 4 both suffer because of higher production costs at 6?
2. Why does additional competition within Sector 4 increase profits of firms who purchase nothing from that sector?

Such questions require a solvable assembly model, and the model here provides substantial intuition. For the first question, suppose that, beginning with an equilibrium situation, production costs at 6 increase. It is intuitive that the cost increase will be passed along through higher prices at 6, 2, and finally 1. Consumers will thus see higher prices and will reduce purchases (at 1). Sector 4 will see this as a weakened demand curve, hence lower prices and profits there. Similarly for Question 2: Increased competition

at 4 will lower 1's total procurement cost; consumer purchases will increase as some of this is passed to the market. Sector 1 will thus require additional quantities from all upstream sectors who will see a stronger demand curve, and thus higher prices and quantities.

9. Summary and Future Research

We have analyzed competition in pure-assembly multiechelon supply chains, using an approach we term "coordinated successive Cournot oligopoly." The approach is a fairly natural extension of the literature in successive Cournot oligopoly models, although it raises some basic questions about alternative equilibrium concepts for this setting. We develop closed-form expressions for prices and quantity in the two-stage case and characterize the general case as the solution to a set of linear equations. We demonstrate certain equivalence principles that allow networks to be iteratively compressed, or to be expanded to a specialized structure in which at most two components are assembled at any stage. These techniques also permit an efficient approach to the general case, in which the equilibrium solution can be obtained in near-closed form once certain parameters are computed iteratively.

Finally, we present some comparative statics results. Changes in variable costs of production have expected effects, as do changes in the parameters of the demand function. The effects of changes in sector concentration are not all as obvious. If the number of firms in a sector increases, the quantity produced increases, and the prices at the sector and at any downstream sectors, decrease. The upstream consequences are a little more complicated in that whether upstream prices increase depends on whether the now less concentrated sector has a complementor or downstream complementor. Finally, prices in sectors that are complementors, downstream complementors, or upstream of either of these sectors all show price increases. Thus, this paper rigorously extends existing intuition, largely derived from models with a single competitive sector, to complex assembly networks of arbitrary size. Additionally, this paper models interactions between complementor sectors, about which the existing literature is necessarily silent.

The present analysis not only provides equilibrium solutions and qualitative results for large-scale assembly supply chains, but also provides the basis for further analysis of network structure and entry. These issues are substantially more complex in assembly networks than in the serial networks that have been analyzed heretofore. We are pursuing research into the effect of network structure, the competitive impact of vertical and horizontal integration, and entry competition in the presence of fixed costs. We are also developing techniques similar to this paper and that of Corbett and Karmarkar (2001) to analyze multiechelon distributive networks. The eventual target of this stream of research is to assemble techniques to analyze competition in large-scale supply chains and networks, with very general structures.

Appendix. Proofs

Proof of the Form of U^{-1}. The following properties can be shown for this function for all $i, j, x \in \{1, 2, \ldots, n\}$:[5]

(f1): $\sum_{j \notin \{\cdot\}} M_j f_x^{\{\cdot\} \cup j} = (x+1) f_{x+1}^{\{\cdot\}}$ when $x < n$,

(f2): $f_x^{\{\cdot\}} = M_j f_{x-1}^{\{\cdot\} \cup j} + f_x^{\{\cdot\} \cup j}$ if $x < n - |\cdot|$, and $f_x^{\{\cdot\}} = M_j f_{x-1}^{\{\cdot\} \cup j}$ if $x = n - |\cdot|$,

(f3): $\prod_{j \notin \{\cdot\}}^{n} (1 - M_j) = 1 - f_1^{\{\cdot\}} + f_2^{\{\cdot\}} - \cdots + (-1)^{n-|\cdot|} f_{n-|\cdot|}^{\{\cdot\}}$, where $|\cdot|$ is the cardinality of $\{\cdot\}$.

The proof proceeds by showing that $U \cdot U^{-1}$ equals the identity matrix when U^{-1} is given by (10); we first prove that $(U \cdot U^{-1})_{ii}$ equals 1 for every i:

$$\begin{aligned}(U \cdot U^{-1})_{ii} &= U_{ii}^{-1} + M_i \sum_{j \neq i} U_{ji}^{-1} \\ &= \frac{1}{D}\Big\{1 - f_2^{\{i\}} + 2 f_3^{\{i\}} - 3 f_4^{\{i\}} + \cdots + (-1)^{n-1}(n-2) f_{n-1}^{\{i\}} \\ &\qquad + M_i \sum_{j \neq i} (-M_j)\Big(1 - f_1^{\{i,j\}} + f_2^{\{i,j\}} \\ &\qquad - \cdots + (-1)^{n-2} f_{n-2}^{\{i,j\}}\Big)\Big\} \text{ from (10).}\end{aligned}$$

Rearranging,

$$\begin{aligned}(U \cdot U^{-1})_{ii} &= \frac{1}{D}\Big\{1 - f_2^{\{i\}} + 2 f_3^{\{i\}} - 3 f_4^{\{i\}} + \cdots + (-1)^{n-1}(n-2) f_{n-1}^{\{i\}} \\ &\qquad - M_i \Big[\sum_{j \neq i} M_j - \sum_{j \neq i} M_j f_1^{\{i,j\}} + \sum_{j \neq i} M_j f_2^{\{i,j\}} \\ &\qquad - \cdots + (-1)^{n-2} \sum_{j \neq i} M_j f_{n-2}^{\{i,j\}}\Big]\Big\}.\end{aligned}$$

It follows from (f_1)[6] that $\sum_{j \neq i} M_j = f_1^{\{i\}}$, $\sum_{j \neq i} M_j f_1^{\{i,j\}} = 2 f_2^{\{i\}}$, $\sum_{j \neq i} M_j f_2^{\{i,j\}} = 3 f_3^{\{i\}}, \ldots$, and $\sum_{j \neq i} M_j f_{n-2}^{\{i,j\}} = (n-1) f_{n-1}^{\{i\}}$. Thus,

$$\begin{aligned}(U \cdot U^{-1})_{ii} &= \frac{1}{D}\Big\{1 - f_2^{\{i\}} + 2 f_3^{\{i\}} - 3 f_4^{\{i\}} + \cdots + (-1)^{n-1}(n-2) f_{n-1}^{\{i\}} \\ &\qquad - M_i f_1^{\{i\}} + 2 M_i f_2^{\{i\}} - 3 M_i f_3^{\{i\}} \\ &\qquad + \cdots + (-1)^{n-1}(n-1) M_i f_{n-1}^{\{i\}}\Big\}.\end{aligned}$$

It follows from (f2)[7] that $M_i f_1^{\{i\}} = f_2 - f_2^{\{i\}}$, $M_i f_2^{\{i\}} = f_3 - f_3^{\{i\}}$, $M_i f_3^{\{i\}} = f_4 - f_4^{\{i\}}, \ldots$, and $M_i f_{n-1}^{\{i\}} = f_n$, so

$$\begin{aligned}(U \cdot U^{-1})_{ii} &= \frac{1}{D}\Big\{1 - f_2^{\{i\}} + 2 f_3^{\{i\}} - 3 f_4^{\{i\}} + \cdots + (-1)^{n-1}(n-2) f_{n-1}^{\{i\}} \\ &\qquad - (f_2 - f_2^{\{i\}}) + 2(f_3 - f_3^{\{i\}}) - 3(f_3 - f_3^{\{i\}}) \\ &\qquad + \cdots + (-1)^{n-1}(n-1) f_{4n}\Big\}.\end{aligned}$$

[5] Here and below i, j, and k will be used as indices. x denotes a numerical subscript to the function f.

[6] Here, the applicable form of (f1) is $\sum_{j \neq i} M_j f_x^{\{i,j\}} = (x+1) f_{x+1}^{\{i\}}$.

[7] The applicable form of (f2) is $M_i f_x^{\{i\}} = f_x - f_x^{\{i\}}$ for $x < n-1$, and $M_i f_{n-1}^{\{i\}} = f_n$.

Cancelling, the bracketed terms reduce to D as given in (10), so $(U\cdot U^{-1})_{ii}=1$ as required.

To complete the proof, it remains to show that $(U\cdot U^{-1})_{ij}=0$ for $i\neq j$. First, from the definition of U above,

$$(U\cdot U^{-1})_{ij}=\sum_{k\neq i,j} M_k U_{ik}^{-1}+M_i U_{ii}^{-1}+U_{ij}^{-1}. \tag{A.1}$$

We begin by isolating the first term:

$$\sum_{k\neq i,j} M_k U_{ik}^{-1} = \sum_{k\neq i,j} M_k\left(-\frac{M_i}{D}\right)\left(1-f_1^{\{i,k\}}+f_2^{\{i,k\}}-\cdots+(-1)^{n-2}f_{n-2}^{\{i,k\}}\right).$$

Adding additional terms to the summation, this becomes

$$\begin{aligned}\sum_{k\neq i,j} M_k U_{ik}^{-1} &= \sum_{k\neq i} M_k\left(-\frac{M_i}{D}\right)\left(1-f_1^{\{i,k\}}+f_2^{\{i,k\}}-\cdots+(-1)^{n-2}f_{n-2}^{\{i,k\}}\right)\\ &\quad+M_j\left(-\frac{M_i}{D}\right)\left(-1+f_1^{\{i,j\}}-f_2^{\{i,j\}}+\cdots+(-1)^{n-1}f_{n-2}^{\{i,j\}}\right)\\ &=-\frac{M_i}{D}\Bigg\{\sum_{k\neq i}M_k-\sum_{k\neq i}M_k f_1^{\{i,k\}}+\sum_{k\neq i}M_k f_2^{\{i,k\}}\\ &\qquad-\cdots+(-1)^{n-2}\sum_{k\neq i}M_k f_{n-2}^{\{i,k\}}-M_j+M_j f_1^{\{i,j\}}\\ &\qquad-M_j f_2^{\{i,j\}}+\cdots+(-1)^{n-1}M_j f_{n-2}^{\{i,j\}}\Bigg\}.\end{aligned}$$

It follows from (f1)[8] that $\sum_{k\neq i}M_k=f_1^{\{i\}}$, $\sum_{k\neq i}M_k f_1^{\{i,k\}}=2f_2^{\{i\}}$, $\sum_{k\neq i}M_k f_2^{\{i,k\}}=3f_3^{\{i\}},\ldots,\sum_{k\neq i}M_k f_{n-2}^{\{i,k\}}=(n-1)f_{n-1}^{\{i\}}$.

It follows from (f2):[9] that $M_j=f_1^{\{i\}}-f_1^{\{i,j\}}$, $M_j f_1^{\{i,j\}}=f_2^{\{i\}}-f_2^{\{i,j\}}$, $M_j f_2^{\{i,j\}}=f_3^{\{i\}}-f_3^{\{i,j\}},\ldots,M_j f_{n-2}^{\{i,j\}}=f_{n-1}^{\{i\}}$.

Substituting these facts into the expression above gives

$$\begin{aligned}\sum_{k\neq i,j} M_k U_{ik}^{-1} &= -\frac{M_i}{D}\Big\{f_1^{\{i\}}-2f_2^{\{i\}}+3f_3^{\{i\}}-\cdots+(-1)^{n-2}(n-1)\\ &\qquad\cdot f_{n-1}^{\{i\}}-(f_1^{\{i\}}-f_1^{\{i,j\}})+(f_2^{\{i\}}-f_2^{\{i,j\}})\\ &\qquad-(f_3^{\{i\}}-f_3^{\{i,j\}})+\cdots+(-1)^{n-1}f_{n-1}^{\{i\}}\Big\},\end{aligned}$$

rearranging and cancelling

$$\begin{aligned}\sum_{k\neq i,j} M_k U_{ik}^{-1} &= -\frac{M_i}{D}\Big\{-f_2^{\{i\}}+2f_3^{\{i\}}-\cdots+(-1)^{n-2}(n-2)f_{n-1}^{\{i\}}\\ &\qquad+f_1^{\{i,j\}}-f_2^{\{i,j\}}+f_3^{\{i,j\}}-\cdots+(-1)^{n-2}f_{n-2}^{\{i,j\}}\Big\}.\end{aligned} \tag{A.2}$$

Turning to the other terms of (A.1), we have

$$M_i U_{ii}^{-1}=\frac{M_i}{D}\left\{1-f_2^{\{i\}}+2f_3^{\{i\}}-\cdots+(-1)^{n-2}(n-2)f_{n-1}^{\{i\}}\right\} \tag{A.3}$$

and

$$U_{ij}^{-1}=-\frac{M_i}{D}\left\{1-f_1^{\{i,j\}}+f_2^{\{i,j\}}-f_3^{\{i,j\}}+\cdots+(-1)^{n-1}f_{n-2}^{\{i,j\}}\right\}. \tag{A.4}$$

[8] The applicable form of (f1) is $\sum_{k\neq i}M_k f_x^{\{i,k\}}=(x+1)f_{x+1}^{\{i\}}$.

[9] The applicable form of (f2) is $M_j f_{x-1}^{\{i,j\}}=f_x^{\{i\}}-f_x^{\{i,j\}}$ for $x<n-1$, and if $M_{if_{n-2}}^{\{i,j\}}=f_{n-1}^{\{i\}}$.

Finally, note that $(U\cdot U^{-1})_{ij}$ (see (A.1)) is the sum of (A.2), (A.3), and (A.4). This sum equals 0 as required, which can be easily seen by cancelling terms. □

Proof of Compressibility for the Two-Tier Assembly with Multiple Supplier Sectors. Here we show that the networks (b) and (c) from Figure 5 are equivalent when N_1' and v_1' are given by (13).

This claim is proved by induction. For starters, the claim for $k=3$ is shown in the text (this is the network of Figure 5(a)). Next, assume that it is true for $k=m-1\geq 3$ and show that it holds for $k=m$. By expansibility, the network is now equivalent to the network 5(d). By the induction assumption, network 5(d) is equivalent to network 5(e) with

$$N_D'=\frac{N_D\prod_{i=2}^{m-1}N_i}{e_{m-3}^{\{2,\ldots,m-1\}}-(m-3)\prod_{i=2}^{m-1}N_i}=\frac{\prod_{i=2}^{m-1}N_i}{e_{m-3}^{\{2,\ldots,m-1\}}-(m-3)\prod_{i=2}^{m-1}N_i}.$$

(The second equality holds because $N_D=1$ as D is a dummy node.) Network (e) is a case of $k=3$ for which compressibility has already been shown. Thus,

$$\begin{aligned}N_1' &= \frac{N_m N_1 N_D'}{N_m+N_D'-N_m N_D'}\\ &= \left(N_m N_1\prod_{i=2}^{m-1}N_i\right)\cdot\Bigg\{\left[e_{m-3}^{\{2,\ldots,m-1\}}-(m-3)\prod_{i=2}^{m-1}N_i\right]\\ &\qquad\cdot\Bigg[N_m+\frac{\prod_{i=2}^{m-1}N_i}{e_{m-3}^{\{2,\ldots,m-1\}}-(m-3)\prod_{i=2}^{m-1}N_i}\\ &\qquad-N_m\left(e_{m-3}^{\{2,\ldots,m-1\}}-(m-3)\prod_{i=2}^{m-1}N_i\right)\Bigg]\Bigg\}^{-1}\\ &= \frac{N_m N_1\prod_{i=2}^{m-1}N_i}{N_m[e_{m-3}^{\{2,\ldots,m-1\}}-(m-3)\prod_{i=2}^{m-1}N_i]+\prod_{i=2}^{m-1}N_i-N_m\prod_{i=2}^{m-1}N_i}\\ &= \frac{N_m N_1\prod_{i=2}^{m-1}N_i}{N_m e_{m-3}^{\{2,\ldots,m-1\}}+\prod_{i=2}^{m-1}N_i-(m-2)N_m\prod_{i=2}^{m-1}N_i}\\ &= \frac{\prod_{i=1}^{m}N_i}{N_m e_{m-3}^{\{2,\ldots,m-1\}}+\prod_{i=2}^{m-1}N_i-(m-2)\prod_{i=2}^{m}N_i},\end{aligned}$$

and it can be shown that $N_m e_{m-3}^{\{2,\ldots,m-1\}}+\prod_{i=2}^{m-1}N_i=e_{m-3}^{\{2,\ldots,m\}}$, giving

$$N_1'=\frac{\prod_{i=1}^{m}N_i}{e_{m-2}^{\{2,\ldots,m\}}-(m-2)\prod_{i=2}^{m}N_i},$$

which is sufficient to show that the statement holds for $k=m$.

Finally, $v_D'=\sum_{i=2}^{m-1}v_i=\sum_{i=1}^{m-1}v_i$ by the induction assumption and the fact that D is a dummy sector (i.e., $v_D=0$), so (13) gives $v_1'=v_1+v_D'+v_m=\sum_{i=1}^{m}v_i$, as required for the induction step. □

Proof of Proposition 3. (By induction beginning at the first tier—i.e., the final assembly sector.) This final assembly sector has no complementor, so only the first expression in the proposition is relevant, and it holds by iterative compressibility. For the induction step, we assume for an arbitrary sector i (which is not the final assembly sector) the proposition holds for i's downstream assembly sector d_i. By compressibility (cf. expression (17)) we have

$$p_{d_i}=(1-N_{d_i}')\left(a-\sum_{k\in D_{d_i}}v_k-p_{c_{d_i}}-\sum_{k\in DC_{d_i}}p_k\right)+N_{d_i}'v_{d_i}' \tag{A.5}$$

and

$$p_i = (1-N_i')\left(a - \sum_{k\in D_i} v_k - p_{c_i} - \sum_{k\in DC_i} p_k\right) + N_i' v_i'. \tag{A.6}$$

By the induction assumption we also have

$$p_{d_i} = (1-N_{d_i}')a_{d_i} + N_{d_i}' v_{d_i}'. \tag{A.7}$$

(A.5) and (A.7) together show that

$$a_{d_i} = a - \sum_{k\in D_{d_i}} v_k - p_{c_{d_i}} - \sum_{k\in DC_{d_i}} p_k. \tag{A.8}$$

Next, $D_i = D_{d_i} \cup d_i$ and $DC_i = DC_{d_i} \cup c_{d_i}$, so (A.6) can be written as

$$\begin{aligned} p_i &= (1-N_i')\left(a - \sum_{k\in D_{d_i}} v_k - v_{d_i} - p_{c_i} - \sum_{k\in DC_{d_i}} p_k - p_{c_{d_i}}\right) + N_i' v_i' \\ &= (1-N_i')(a_{d_i} - v_{d_i} - p_{c_i}) + N_i' v_i' \quad \text{from (A.8).} \end{aligned} \tag{A.9}$$

The same logic can be also be applied to i's complementor sector to give

$$p_{c_i} = (1-N_{c_i}')(a_{d_i} - v_{d_i} - p_i) + N_{c_i}' v_{c_i}'. \tag{A.10}$$

After applying the definition of a_{d_i}, Equations (A.9) and (A.10) then solve to (18) and (19). Finally, (20) follows by reversing the roles of i and its complementor in (19). □

Proof of Proposition 4. (By induction beginning at final assembly sector.) At the final assembly sector, the proposition follows immediately by solving the demand equation (1) for Q. For the induction step, we assume that the proposition holds at sector d_i, the sector that is immediately downstream from an arbitrary sector i, and show this implies that it holds for i. So,

$$\begin{aligned} Q &= \frac{a_{d_i} - p_{d_i}}{b_{d_i}} \\ &= \frac{1}{b_{d_i}}(a_{d_i} - (1-N_{d_i}')a_{d_i} - N_{d_i}' v_{d_i}') = \frac{N_{d_i}'}{b_{d_i}}(a_{d_i} - v_{d_i}') \\ &= \frac{N_{d_i} N_i' N_{c_i}'}{b_{d_i}(N_i' + N_{c_i}' - N_i' N_{c_i}')}(a_{d_i} - v_{d_i} - v_i' - v_{c_i}'). \end{aligned} \tag{A.11}$$

The first equality is the induction assumption, and the next line is substituting (18) and simplifying. The last relation then follows from the definitions of N_{d_i}' and v_{d_i}'. To complete the proof, we require that $(a_i - p_i)/b_i$ also equals (A.11):

$$\begin{aligned} \frac{a_i - p_i}{b_i} &= \frac{N_{d_i}}{b_{d_i}}(a_i - p_i) \\ &= \frac{N_{d_i}}{b_{d_i}}(a_i - (1-N_i')a_i - N_i' v_i') = \frac{N_{d_i} N_i'}{b_{d_i}}(a_i - v_i') \\ &= \frac{N_{d_i} N_i'}{b_{d_i}}\left(\frac{N_{c_i}'(a_{d_i} - v_{d_i} - v_i) + (1-N_{c_i}')N_i' v_i}{N_i' + N_{c_i}' - N_i' N_{c_i}'} - v_i'\right). \end{aligned}$$

The first relation is from the definition of b_i, the next line is again from (18) and simplifying, and the last line is from the definition of a_i. This then simplifies to (A.11) as required. □

Proof of Proposition 5. Differentiation of (18) gives

$$\begin{aligned} \frac{dp_i}{dN_j} &= \frac{dN_i'}{dN_j}\frac{\partial p_i}{\partial N_i'} + \frac{da_i}{dN_j}\frac{\partial p_i}{\partial a_i} \\ &= -\frac{dN_i'}{dN_j}(a_i - v_i') + \frac{da_i}{dN_j}(1-N_i'), \end{aligned} \tag{A.12}$$

and it can be shown that $(a_i - v_i')$ and $(1-N_i')$ are both positive.

Recall that the "N'" values are calculated recursively, working down from the top of the network, with every sector's N' increasing with the N's of its supplier sectors. This implies by induction that $dN_j'/dN_j > 0$ ((i) in the Proposition 5), and $dN_i'/dN_j > 0$ for (ii) in the Proposition 5. The top-down recursion implies that N_i' is sensitive to the number of firms in sector i or its supplier (direct or indirect) sectors only; thus, $dN_i'/dN_j = 0$ for (iii), (iv), and (v).

The "a" values are calculated working from the downstream root up, as per the definition in §7. Disregarding the special case below, it follows from differentiation of its definition that the a for each sector is decreasing in the N' of that sector, increasing in the N' of the complementor sector, and increasing in the a of the downstream sector. Thus, $da_j/dN_j < 0$ (for (i) in Proposition 5); and it can be shown that $da_i/dN_j < 0$ for (ii) and (iv) in Proposition 5, and $da_i/dN_j > 0$ for (v).

As an example of why these facts hold, let k be the sector that supplies the final assembly sector and that is also downstream of j. As N_j increases, this sector will see a downstream a that is constant (because its downstream sector is the final assembly sector with a constant a value by definition), an increasing N_k' (because it is downstream of j), and a constant N_{c_k}' (k is downstream of j, so c_k is not; N_{c_k}' is thus insensitive to N_j). Thus, a_k will decrease. Working inductively and moving upwards towards sector j, it is easy to see that each sector will see its own a value decreasing because its downstream sector's a decreases (as per the induction), its N' value increases as it is still downstream of j, and its complementor sector's N' value stays constant (same logic as just above).

The special case mentioned above is the case in which a sector i has no complementor sector; the definition of a_i then reduces to $a_{d_i} - v_{d_i}$. In (iii) of Proposition 5, this special case holds for j as well as for all sectors that are downstream of j, so a_j (which is ordinarily computed inductively) actually becomes $a - \sum v_k$, where the summation is across every sector k that is downstream of j, and this is obviously constant in N_j. It follows that if i is one of the supplier sectors to j, then a_i will be constant in N_j. Indeed, all of the inputs into sector i's price computation will be constant in N_j. Because the computation of the a values is inductive, the constancy of a_i will be inherited by all firms further upstream and their prices will be constant in N_j as well.

Finally, (vi) follows immediately from the demand function because either (i) or (ii) will imply that consumer prices are decreasing. □

Proof of Proposition 6. We will require the following partials:

$$\frac{\partial a_i}{\partial v_i'} = \frac{N_i' - N_i' N_{c_i}'}{N_i' + N_{c_i}' - N_i' N_{c_i}'} \quad \text{and} \quad \frac{\partial a_i}{\partial a_{d_i}} = \frac{N_{c_i}'}{N_i' + N_{c_i}' - N_i' N_{c_i}'}. \tag{A.13}$$

Each of these is greater than 0 and less than 1; also, their sum equals 1. Additionally, it is easy to show by induction that dv'_i/dv_j equals 1 if $i=j$ or if i is downstream of j; it otherwise equals 0.

We first consider cases (i) and (ii) together using induction, and we prove two facts for every sector i that is included in these cases: (1) $0<da_i/dv_j<1$, and (2) $dp_i/dv_j>0$—the second of these is the desired result; the first is used in the induction step of the proof. For convenience, we will label the sectors that are downstream of j by the tier in which they reside. Thus, the final assembly sector is "Sector 1," the next upstream sector (that is still downstream of j) is "Sector 2," etc. The remaining sectors in the network are arbitrarily numbered.

At Sector 1, the only variable in a_1 (see (7)) that changes with v_j is an increase in v'_1; specifically, $dv'_1/dv_j=1$ as above. Thus,

$$\frac{da_1}{dv_j}=\frac{\partial a_1}{\partial v'_1}\frac{dv'_1}{dv_j}=\frac{\partial a_1}{\partial v'_1},$$

which is greater than 0 and less than 1 as per (A.13). This, together with the fact that p_1 is increasing in a_1 and v'_1 (by inspection of (18)), gives $dp_1/dv_j>0$.

Now assuming that (1) and (2) above hold when $i=(k-1)$, consider the case of $i=k$ (with $k\geq 2$ and presuming that k is still downstream of j or at least the same as j). In this case, all variables in (7) except v'_{d_k} and a_{d_k} (which here are v'_k and a_{k-1}, respectively) are constant, so

$$\begin{aligned}\frac{da_k}{dv_j} &= \frac{\partial a_k}{\partial v'_k}\frac{dv'_k}{dv_j}+\frac{\partial a_k}{\partial a_{k-1}}\frac{da_{k-1}}{dv_j}\\ &= \frac{\partial a_k}{\partial v'_k}+\frac{\partial a_k}{\partial a_{k-1}}\frac{da_{k-1}}{dv_j}<\frac{\partial a_k}{\partial v'_k}+\frac{\partial a_k}{\partial a_{k-1}}=1.\end{aligned}$$

Here, the first equality is the chain rule, and the rest follows from the induction assumption ($da_{k-1}/dv_j<1$) and from (A.13). Also, da_k/dv_j is positive because $\partial a_k/\partial v'_k$ and $\partial a_k/\partial a_{k-1}$ are greater than 0 (by direct differentiation of a_k) and $da_{k-1}/dv_j>0$ by the induction hypothesis. It remains to show that $dp_k/dv_j>0$; this is again true because p_k is increasing in both a_k and v'_k (as per (18)), which are themselves increasing in v_j.

Case (iii) has several subcases that share a distinction from the above analysis—each sector i will now have v'_i constant in v_j (rather than increasing as before). Thus, to show that p_i decreases with v_j, it is sufficient to show that a_i (as defined by (7)) decreases with v_j. The first subcase is when sector i is immediately upstream of j; i.e., j is the sector d_i, so $v_{d_i}=v_j$. Thus,

$$\frac{da_i}{dv_j}=\frac{\partial a_i}{\partial a_j}\frac{da_j}{dv_j}+\frac{\partial a_i}{\partial v_j}=\frac{da_j}{dv_j}\left(\frac{\partial a_i}{\partial a_j}-1\right).$$

Here, the first equality is the chain rule using $v_{d_i}=v_j$; the second equality holds after rearranging because a_i (again see (18)) is symmetrical in $a_{d_i}=a_j$ and $v_{d_i}=v_j$. Next, $\partial a_i/\partial a_j=\partial a_i/\partial a_{d_i}<1$ from (A.13) after noting that $da_j/dv_j=da_j/dv'_j$, and $da_j/dv_j>0$ follows from the final step of the induction used in cases (i) and (iii). The other subcases are: i further upstream of j, sectors i and j as complementors, i as a downstream complementor, or i upstream of a downstream complementor of j. These proofs are similar to that of case (iii) and are omitted for brevity.

Case (iv) follows from the fact that Q is decreasing in p_1 and is itself increasing in v_j as per case (i). □

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