Dynamic Hedging, Expected Returns and Factor Timing

Patrick C. Kiefer*
UCLA Anderson
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Abstract

A simple equilibrium restriction identifies time-varying components of expected returns. Using data from the Fama and French three- and five-factor models and momentum, we use this restriction to predict factor returns out of sample. We derive a policy for timing exposures to these components and find the modified portfolio generates returns with higher out-of-sample Sharpe ratios than the underlying mimicking portfolio. Exposure to returns to the market timing portfolio is priced and the residual variation in the market is not. Gains from market, value and momentum timing policies are compensation for risk. We provide evidence that timing size exploits mis-pricing rather than precision of conditional risk exposures.

Keywords: Risk Factors, Forecasting, Time varying risk, Markov process, Martingale, Performance Evaluation

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1 Introduction

Empirical risk factor models are ubiquitous in academic finance and money management. Conventionally, the models are implemented by first ranking traded assets into bins according to a characteristic, such as book-to-market equity. Often, in the cross-section of assets, this characteristic is thought to proxy for exposure to an undiversifiable source of risk. Then, long-short portfolios are formed from assets in the highest and lowest bins. Returns to the resulting zero-cost portfolio mimic fluctuations in the underlying source of risk. These factor replicating portfolios are then used as the basis for benchmarking asset pricing models and evaluating performance of investment strategies.

Recent evidence that factor risk prices vary through time exemplifies a broader trend to study predictability at the portfolio level. The key object for understanding the cross-section of predictability is “the factor structure of time-varying expected returns.” In this paper, we use conditioning information from the cross-section of predictable portfolio returns to construct modified factor replicating portfolios. The benchmark modification produces a timing rule for exposure to predictable components of factor risk prices. The modified portfolio returns improve the risk-return trade-off: out of sample tests show returns to our timed portfolios have reliably higher Sharpe ratios than conventional portfolios.

There are two ingredients to our study that warrant emphasis. First, by identifying the trailing sources of common variation in realized returns as the transitory fluctuations in expected returns, we can treat the factor model of expected returns as a zero-net supply inter-temporal hedging portfolio, perfectly correlated with a latent state variable relevant to wealth. This connection allows us to distinguish between unexpected returns and changes in conditional expectations by distinguishing between the return on the fixed-weight hedging portfolio and the gains or losses from rebalancing the portfolio weights when calculated on a rolling basis. Essentially, this allows us the extract a ”time effect.”

The second ingredient is the inclusion of forecasts of factor returns calculated over several

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1 See L.H. Pedersen, 2015. Quantitative funds rely heavily on value, momentum and carry.
2 In quotations, from John Cochrane’s 2011 AFA Presidential address.
horizons separately. We justify the inclusion of these data formally in the Euler equation without reference to a parametric stochastic-discount factor model. Cochrane and Piazesi (CP) (2005) find contemporaneous bond prices are not Markov states at a one-month frequency. In the equity markets we consider, scalar first-order autoregressions are insufficient to reproduce the term structure of conditional forecasts suggested by the data.\(^4\) In (CP) and in our study, forecasts are improved by enlarging the state vector to include moving averages of past prices.

2 Related Work

Hansen and Richard (1989), Cox, Ingersoll and Ross (1985), Harrison Krepps (1979), Ross (1979), Lucas (1979), and others represent equilibrium returns data in terms of a common non-negative pricing kernel. Hansen (1982) ties the objectives within these models to an econometric objective, allowing for flexible estimation of parameters implied by possibly nonlinear equilibrium restrictions. This paper exploits the mapping in Hansen (1982) but emphasizes changes in the conditional factor structure of returns rather than estimates of deep parameters implied by particular model restrictions.

Asness (1994), Fama-French (1992, 1993, 2015), and others document key deviations from baseline unconditional single factor asset pricing model implications. Jegadeesh and Titman (1993) and Moskowitz and Grinblatt (1999) study momentum by asset and industry, respectively, as a conditional signal capable of improving the conditional efficiency properties of the Fama-French three factor models.\(^5\) We build on this insight and uncover finer grained information about conditional efficiency using equilibrium restrictions implied by basic asset pricing theory.

Bandi and Tamoni (2015) implement a decomposition of returns by projecting returns on

\(^4\)AR(1) processes produce horizon-specific forecasts that are deterministic functions of each other.

\(^5\)Unconditional mean-variance efficiency requires allocations to be conditionally mean-variance efficient at each point in time. If Fama-French factor prices of risk are treated as constants while the true risk prices are time-varying, the unconditional frontier spanned by the traded Fama-French factors is not efficient. In such a case, the empirical model does not imply conditional efficiency and hence there is scope for conditioning information to improve performance.
the Haar basis. The result is a similar representation of the time-series of returns as the sum of moving averages over $J$ non-intersecting intervals of increasing scale $2^j$ $j \in \{0, 1, ..., J\}$. Severino (2014) shows existence of decompositions based on the procedure in BT and gives general conditions for related decompositions. Our decomposition is most related methodologically to BT (2015) and Severino (2014). A key feature of the equilibrium operator-based construction we exploit is that the basis is specific to the application. The basis components can be implicit functions of the parameters, allowing for treatment of nonlinear components of unknown form.

This study is not directly subject to the separate critiques of Harvey et. al (2015) and Lewellen, Shanken and Nagel (LSN) (2010). Harvey et. al (2015) criticise the use of test statistics designed for independent experiments in cases of potentially arbitrarily mined data. In this paper, we apply restrictions to known factor returns data that intentionally exclude cross-sectional data. There is no opportunity for mining. LSN show that any series that is correlated with test asset returns (or the true factor) will appear to be priced in the absence of the true factor. Rather than propose new priced factors, our research takes as given that a particular factor prices a particular cross-section. We simply extract a refinement of this given relationship.

Bryzgolva (2014) advocates for traded (price-based) proxies for risk factors over macro factors for statistical reasons. A constrained LASSO-style regression penalizes candidate risk factors with poorly measured exposures. Traded factor returns have low levels of idiosyncratic noise, making exposures easier to measure and thus the LASSO procedure penalizes traded proxies less. This helps rationalize the incremental gain in statistical significance from the hedge portfolio returns over the extracted latent process, but does not diminish the significance gain between the hedge portfolio and the conventional factor portfolio.

Bansal Yaron (2005), Hansen Heaton Li (2008) and others propose and evaluate slow-moving latent growth factors as explanations for unconditional risk premia. Hansen and Sargent (2007, 2016) outline the similarities between long-run risk stories and the implications of robust control policies in financial markets. Our findings tie in to the long-run risk discussion by identifying the slow moving component of expected market returns as the component
relevant for cross-sectional pricing.

Cochrane (2011) summarizes modern aggregate return predictability results, and Santa Clara (2015) finds evidence that the aggregate results can be misleading if applied to portfolios, where dividend yields can also predict cash flows. Brennan and Taylor (2016) evaluate aggregate and portfolio-level return predictability, and distinguish risk-based and statistical sources of predictability. Our factor timing policies exploit predictability in the factor components and have implications for return predictability in the parent factors. We also evaluate the risk content of the underlying source of predictability by testing whether returns to the replicating portfolio have pricing power in the corresponding cross-section.

Several papers argue for improvements in the prevailing constructions of Fama-French factors. Gerokos and Lihnnainmaa (2012) argue that the HML factor returns can be decomposed into price-driven and book-driven elements, and that only the price-driven component of the HML factor returns can explain cross-sectional variation in returns. Asness and Frazzini (2013) argue that HML contains about 20% momentum, and propose a construction of HML that isolates the “pure value” component. Lihnnainmaa (2015, 2016) finds accrual, investment, and profitability factor constructions that are preferred to the Fama French (2015) constructions along some dimensions. As a sensitivity check, we implement our decomposition using various modified factor constructions.

3 Factor Models of Asset Returns

We review stochastic discount factor and arbitrage based justifications for linear factor models of asset returns. We then show how our decomposed factors fit into the linear factor model.

3.1 Stochastic Discount Factor (SDF) Models

Asset pricing models based on stochastic discount factors can be written

\[ p_t^x = E_t[M_{t+\Delta}x_{t+\Delta}] = \frac{E_t[x_{t+\Delta}]}{r_f} + Cov \left( M_{t+\Delta}, x_{t+\Delta} \right) \]
where \( M_{t+\Delta} \) is the intertemporal marginal rate of substitution (IMRS) of a marginal investor, \( p_t^x \) is the price at time \( t \) of a random payout \( x_{t+\Delta} \), \( \Delta \) is the sampling rate, and \( \mathbb{E}_t[\cdot] := \mathbb{E}[\cdot|\mathcal{F}_t] \) is the conditional expectations operator.\(^6\) Let \( R^x = (p_t^x)^{-1} x_{t+\Delta} \) be the realized gross return on the security that pays \( x_{t+\Delta} \) in period \( t + \Delta \) with strictly positive price \( p_t^x > 0 \). \( \mathbb{E}[M_{t+\Delta}] \) is one over the one-period gross return to riskless savings \( r_f \). The pricing equation can be rearranged (see e.g., Cochrane 2001)

\[
\mathbb{E}_t[R^x] - r_f = \beta_{x,m} \lambda_m
\]

where \( \beta_{x,m} \) is the OLS coefficient of returns to investments in \( x \) on innovations in \( M \) and \( \lambda_m = \mathbb{V}_t[M_{t+\Delta}] \mathbb{E}_t[M_{t+\Delta}]^{-1} \) is the (conditional) price of risk.

Breeden (1979) points out that for state vector process \( X_t \in \mathcal{D} \subset \mathbb{R}^{J-1} \), and smooth functional \( M, M = M(X_t) \) can be expanded into its constituent state variables, say \( M_j, j \in J \). The linear form in each margin \( j \) recovers the familiar relationship\(^7\)

\[
\mathbb{E}_t[R^x] - r_f = \sum_j \beta_{x,m_j} \lambda_{m_j}
\]

where \( \beta_{x,m_j} \) is the exposure of asset \( x \) to movements in factor \( j \), and \( \lambda_{m_j} \) is the price of exposure to factor \( j \).

For each factor \( j \), \( f_{j,t+\Delta} := (V_W(W_t, X_t))^{-1} V_{W,X,j_0}(W_{t+\Delta}, X_{t+\Delta}) \), a factor mimicking portfolio can be constructed from traded assets. Call returns on each of the \( J \) mimicking portfolios \( R_{t+\Delta}^j \). When unexpected marginal valuations are not dependent on the level of any state

\[^6\]The IMRS is commonly modeled \( M(X_t) \Delta = d \log(u'(C(X_t))) \) for concave utility function \( u \) and adapted process \( C(X_t) \), usually consumption of a numeraire good. A process is adapted if it is measurable with respect to the filtration \( \mathcal{F}_t \) for every \( t \geq 0 \).

\[^7\]Formally, this holds up to \( o(\Delta^{-1}) \) if not log-linear. For small \( \Delta \), \( M(X_{t+\Delta}) \Delta \approx d \log(u'(C(X_t))) = d \log(V_W(W_t, X_{t,j_0})) \) for \( j_0 \in \{1, \ldots, J - 1\} \). Then,

\[
V_W(W_t, X_t) M(X_{t+\Delta}) = V_{W,W}(W_{t+\Delta}, X_{t+\Delta}) + \sum_{j_0} V_{W,X,j_0}(W_{t+\Delta}, X_{t+\Delta}) + o(\Delta^{-1})
\]

where an envelope condition was used to equate marginal utility of consumption with marginal valuation of wealth \( V_W(W, X) \), so that \( -\mathbb{E}[dV_W(W, X)] \approx r_f \Delta V_W(W, X) \) also recovers the gross return to riskless savings.
variable, the commonly used Fama-French factor models follow,

$$E[R^x] - r_f = \sum_j \beta_{x,j} E[R^j]$$  \hspace{1cm} (1)

where the risk price of the traded risk factor is given by the unconditional expected return of the factor mimicking portfolio.

In (1), the traded factor returns proxy for innovations in aggregate state variables that drive dynamics in the investment opportunity set (IOS), in the spirit of Merton (1972). In general equilibrium, dynamics in the IOS map onto dynamics in the marginal valuations of wealth. Moreover, the traded factor risk price clears the market compensating exposures to variation in the factor returns

$$\beta'_{k} \overline{R} = (\beta'_{k} \beta_{k}) E[f^k]$$ \hspace{1cm} (2)

These are the normal equations associated with the Fama-MacBeth cross-sectional regressions. $\overline{R}$ is the vector of cross-sectional averages of portfolio returns. As in Bryzgolova (2015), when the $\beta'_{k} \beta_{k}$ matrix is invertible and not too sparse, we can obtain reliable tests of factor risk prices.

We treat the constant risk price model (1) as true under the null hypothesis. The motivation for this is ultimately empirical because our performance benchmarks are the Fama-French empirical factor models. Nonetheless, a large class of parametric SDF models are consistent with (1). Time variation in the IOS does not imply the factor risk prices vary over time. In models with affine state dynamics and homogeneity of valuation in wealth, if volatility does not predict returns the IOS varies but compensation for exposure to this variation does not.\textsuperscript{8,9}

\textsuperscript{8}In contrast, affine models with stochastic volatility generate time variation in risk prices for representative agent models but also imply volatility predicts asset returns. It is noteworthy that the latter prediction is counterfactual.

\textsuperscript{9}State variables driving the IOS impact the equilibrium rate at which the numeraire good is substituted intertemporally but unexpected changes in the IOS reduce to white noise scaled by a constant function of model parameters.
3.2 Arbitrage Pricing Theory

The arbitrage pricing theory (APT) derived by Ross (1979) is implied by a purely statistical model of factor returns. In particular, if all returns have an approximate factor structure

\[ R_{t+\Delta}^x = a_{0,x} + a_x \cdot f_{t+\Delta} + u_{t+\Delta}^x \]

and, portfolios with the same factor loadings have the same returns, then

\[ E[R_{t+\Delta}^x] = a_{0,x} + a_x \cdot E[f_{t+\Delta}] \]

shows that (1) follows as a constraint on expected returns implied by no-arbitrage without reference to a parametric economic model. In particular, Ross (1979) shows that \( a_{0,x} = r_f \) for any \( x \), individual asset loadings on \( f_{t+\Delta} \) are OLS coefficients \( a_x = \beta_{x,f} \), and expected factor replicating portfolio returns are \( E[f_{t+\Delta}] = \lambda_f \).

The Fama-French factor models also follow from the APT condition, although it is not necessarily the case that each model factor is a priced risk factor. For example, linear factor models based on industry returns can be successful in explaining time series variation in realized returns and cross-sectional variation in average returns, but industry portfolio returns are not necessarily priced risk factors. In such models, (1) is still true, but the cross-sectional condition (2) is not in general true.

4 Tests

We evaluate the factor structure of conditional forecasts of asset returns. When the excess asset returns \( j \) are factor replicating portfolio returns we substitute \( f = R^{je} \) into the Euler equation to obtain

\[ E_t[M_{t+\Delta}f_{t+\Delta}] = 0 \]

The pricing kernel \( m_t M_{t+\Delta+n} = m_{t+\Delta+n} \) arises from assets paying 1 in all states \( \Delta + n \) periods in the future

\[ m_{t+\Delta} = E_{t+\Delta}[m_{t+\Delta+n} r_{\Delta,n}] \]
for any \( n \geq 0 \), and where \( m_t, m_{t+\Delta} \) are the levels of marginal valuation for any investor in periods \( t, t + \Delta \), and where the marginal valuation growth \( M_{t+\Delta} \) is enforced to be common across investors.\(^{10}\) We can write the traded risk factor return condition

\[
0 = \mathbb{E}_t \left( \mathbb{E}_{t+\Delta} \left[ \frac{m_{t+\Delta+n\Delta,n}}{m_t} f_{t+\Delta} \right] \right) =: \mathbb{E} \left[ M_{t+\Delta+n} f_{t+\Delta} \right]
\]

The unscripted \( M_{t+\Delta+n} \) is the conditional forecast of the pricing kernel made at time \( t + \Delta \) for realization at time \( t + \Delta + n \). We have separate conditions for every \( n \). To exploit these additional conditions, we stack \( N + 1 \) conditions into a column vector \( \mathbf{g} \)\(^{11}\)

\[
\mathbf{g} = \mathbb{E} \left[ \begin{bmatrix} M_{t+\Delta+n} f_{t+\Delta} \\ M_{t+\Delta+n+1} f_{t+\Delta} \\ \vdots \\ M_{t+\Delta+N} f_{t+\Delta} \end{bmatrix} \right] = 0
\]

We extract the factor structure from a symmetric positive definite \((N + 1) \times (N + 1)\) operator \( \Gamma := \mathbf{g} \mathbf{g}' \).\(^{12}\) This yields

\[
\Gamma = \mathbf{U} \Lambda \mathbf{U}^{-1}
\]

for diagonal \( \Lambda \) and \( \mathbf{U}^{-1} = \mathbf{U}' \).

From here, for any time \( t \), the scalar-valued time series of realized factor returns \( f_k(x, \tau_m, t) \), expressed as a \( T_m \times 1 \) column vector ranging along \( \tau_m := [t - T_m, t] \), can be expressed as “fitted values” \( F_k(x, \tau_m, t) \) of the projection along the \( N_0 \) component estimates of \( \mathbf{g} \) and a \( T_m \times 1 \) column of ones \( \mathbf{1} \)

\[
F_k(x, \tau_m, t) = \left[ \mathbf{1} T_m^{-1} \mathbf{1}' + \mathbf{g}' (\mathbf{g} \mathbf{g}')^{-1} \mathbf{g} \right] f_k(x, \tau_m, t)
\]

\[
= \hat{q}_{0,k}(x, t) + \left[ (\mathbf{U} \mathbf{g})' \Lambda^{-1} (\mathbf{U} \mathbf{g}) \right] f_k(x, \tau_m, t)
\]

\(^{10}\)The SDF \( M > 0 \) is common across investors within non-segmented markets where there is no arbitrage. If in an incomplete market an agent’s IMRS does not equal the market IMRS, this agent must be constrained by one-step-ahead borrowing limits as in Chien, Cole and Lustig (2007), in which case asset prices are still well-defined.

\(^{11}\)For out of sample testing, the estimates are obtained over a window with size denoted \( T_m \). For clarity, we fix \( T_m = 15 \) years. The stability condition is discussed in more detail here Appendix B

\(^{12}\)We extract the components from the sample covariance matrix. This is not the GMM residual covariance matrix which would in principal have common variation removed.
where \( \hat{q}_{0,k}(x,t) \) is the time \( t \) trailing \( T_m \)-moving average of the factor returns and is thus a constant within each representation window \( \tau_m \).

The projection implies the exact multi-variable representation for each scalar realization of the time-\( t \) factor return\(^{13}\)

\[
F_k(x,t) = \hat{q}_{0,k}(x,t) + \sum_{n=1}^{N_0} \hat{q}_{n,k}(x,t) \tag{3}
\]

where \( \hat{q} \) are the estimates of the factor components with in particular \( \hat{q}_{n} \) for \( 1 \leq n \leq N_0 \) corresponding to weighted columns of \( U \).\(^{14,15}\) \( N_0 \leq N \) is the dimensionality of the representation.

The representation (3) immediately implies the factor model of returns

\[
E_t[R^x] - r_f = \beta_{x,0} E_t[\hat{q}_{0,k}(x)] + \sum_{n=1}^{N_0K} \beta_{x,n} \sqrt{\lambda_n} \tag{4}
\]

by appealing to model (1). We have used \( \sqrt{\lambda_n} = E_t[\hat{q}_{n,k}(x)] \). We use (4) in our empirical analysis. When the price of risk of a component \( q_n \) varies through time, we can derive allocation policies based its predictable variation. Additional theoretical details are given in Appendix A. Details for the empirical implementation are given in section 12.

There are three key ingredients revealed in our analysis. Linear principal components of covariance matrices is not new, so we do not allocate space to it specifically. In general, the eigenvectors implied by a rotation depend on the action of the operator, so the linear PC is a specific case. The eigen-decomposition of a transition operator is different than of its sample CVC, but they are similar enough that insights can be gained by comparison.

\[^{13}\text{Empirically, the representation is exact up to trailing eigenfunctions of maximum order } 10^{-16}.\]
\[^{14}\text{The } q_n \text{ for } n \geq 1 \text{ are roughly the eigenfunctions of the equilibrium asset pricing operator. Formal conditions are outlined in appendix B1.}\]
\[^{15}\text{The estimated components } q_n \text{ are decreasing in time-series variation in the index } n \text{ and are heterogeneous in smoothness. Stable ordering of the components is managed by our fixed window size } T_m \text{ and is key for our extraction of the timing rule. Rolling windows also allows for variation in the conditional prices of risk.}\]
**Question 1**  What would happen to the fitted Euler equations if an econometrician conditioned on a strictly smaller refinement $\mathcal{G}_t \subset \mathcal{F}_t = X_{1,t}$? Suppose conditional on $X$ not $F$, problems;

**Question 2**  What would happen to the sample moment conditions if slow-moving regimes are not accounted for, e.g., what if an econometrician held the hedge portfolios fixed when the true regimes fluctuate in every finite sample?

The following filter isolates the changing portfolio weights on fifteen-year average rolling economies. The MA term changes a little in a period, the eigenvalues also change a little, so the vector contents are most sensitive. This margin proves useful.

**Discussion**

Given the representation (4), the remaining steps in large part mirror standard empirical asset pricing procedures, as in Fama MacBeth (1973). We implement standard errors via GMM or generalized empirical likelihood (GLE) where necessary for small-sample performance. We evaluate exposure of test asset returns to the components of the risk factor time-series representation in (3) by estimating time-series $\beta$’s shown in (4). We analyze the risk content of the components in (3) by testing whether cross-sectional variation in component-specific $\beta$’s explains variation in average returns.

We form new portfolios by sorting test assets based on exposures to specific components. We isolate a subset of components in (3) based on persistence and cross-sectional pricing power. We form zero-cost hedging portfolios for these components. The portfolio exploits the predictable variation in risk prices. By design, a hedging portfolio improves conditional mean-variance efficiency relative to the parent factor trading policy.

Finally, Fama and French (2016) take an agnostic view on whether the additional factors $OP$ and $RMW$ proxy for returns on hedging portfolios that track undiversifiable changes in the investment opportunity set (IOS), or whether they simply capture characteristics important for measuring firm performance. Either way, our decomposition is applicable and
the inclusion of additional factors improves portfolio Sharpe ratios. However, if the model factors are not priced, some of the timing rules derived in this paper will not be applicable.

The empirical algorithm used in the following tests implements the decomposition outlined above. Complete details are given here in the Empirical Appendix.

5 Empirical Analysis

5.1 Data

For priced risk factors, we use monthly returns data on the Fama-French three-factor, Fama French three factor plus momentum (Carhart), and Fama-French 5-factor models. The Fama-French three factors are the Market, Value (HML) and size (SMB), rebalanced annually. Details are in Fama and French (1993). Momentum is constructed from a portfolio long 2-12 month winners and short 2-12 month losers ranked in the cross-section (rather than time series) and truncated at the 30% and 70% percentiles. Each reported momentum sorted portfolio is an average of small and large-cap momentum stocks. Momentum is rebalanced monthly.

We consider several cross sections of test assets including the 25 size-BTM portfolios, the 25 size-BTM plus 10 Momentum portfolios, the 25 size-operating profit portfolios and the 32 size-BTM-OP portfolios. The size-BTM portfolios are annually rebalanced and are comprised of the intersection between five market-cap sorted stocks with five book-equity to market-equity (BTM) sorted stocks. The size-BTM plus Momentum add 10 portfolios sorted on 2-12 month returns cross-sectional rankings. The size-OP portfolios sort annually on operating profits (OP): “annual revenues minus COGS, interest expense, and SG&A”, normalized by trailing book-equity, and intersect with size. The factor returns data and the test asset returns data are obtained from Ken French’s website.

We use predictor variables from Goyal and Welch (2007). Data are available on Amit Goyal’s website. We use monthly data for the rolling average 12-month dividends, the rolling av-

\footnote{http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html}

\footnote{http://www.hec.unil.ch/agoyal/}

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Figure 1: Market decomposition and latent factors.

(a) The components are labeled $ImageCond_j$ for $j = 1, 2, 3, 4$ and are ordered according to their contributions to time series variation of market returns. $ImageConstant$ is a moving-average expectation of market returns over window size $T_m$. Factor data are the FF3 factor returns. Data are quarterly from 1927 Q1 to 2015 Q3.

Average 12-month earnings, and the index level for the S&P500. Data are from 1926-2016. Annual data on the dividends, earning and the stock index as well as per-capita consumption going back to 1871 are from Bob Shiller, available on his website \(^{18}\). Monthly Index returns ex-dividend from 1926-2016 are from CRSP.

5.2 Findings

6 Results

6.1 Market

6.1.1 Market Decomposition

We first obtain the conditional representation for the market return time series. Figure 1 plots the market returns and the components of the decomposition. The components 1-4 are decreasing in order of their contribution to time-series variation. Component 5 is a residual.

model component, picking up a moving average over window size $T_m$. The sum of all component levels equals the realized market level in every period. The sum of the conditional volatility of the first four components equals the conditional volatility of the market in every period.

The persistence of market components, measured by autoregressive coefficients, are reported below.

Table 1: Persistence of Market components.

<table>
<thead>
<tr>
<th>Term</th>
<th>Estimate</th>
<th>Statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_0$</td>
<td>0.090</td>
<td>0.215</td>
</tr>
<tr>
<td>ImageCond_1lag</td>
<td>0.072</td>
<td>1.241</td>
</tr>
<tr>
<td>$\phi_0$</td>
<td>-0.013</td>
<td>-0.091</td>
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<tr>
<td>ImageCond_2lag</td>
<td>0.171</td>
<td>3.112</td>
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<td>$\phi_0$</td>
<td>0.076</td>
<td>0.912</td>
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<tr>
<td>ImageCond_3lag</td>
<td>-0.049</td>
<td>-0.836</td>
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<tr>
<td>$\phi_0$</td>
<td>0.014</td>
<td>0.354</td>
</tr>
<tr>
<td>ImageCond_4lag</td>
<td>0.037</td>
<td>0.632</td>
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<td>$\phi_0$</td>
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<tr>
<td>ImageConstantlag</td>
<td>0.951</td>
<td>52.280</td>
</tr>
</tbody>
</table>

(a) Autoregressive coefficients for each of the market components. Data on the Fama-French three-factors plus Momentum are quarterly from 1927Q1 to 2016Q1.

### 6.1.2 Exposures to Market Components

We report the exposure of the 25 size/BTM cross-section to the market components. We also decompose the exposure of components in terms of exposure to the original series used in the GMM moment conditions. For convenience we report unconditional exposures.

Exposure to the latent market factor is monotonically increasing in size for growth stocks. There is no consistent pattern of exposure across value holding size fixed. Small growth assets are known to provide pricing difficulties. A GMM optimal (efficient) weighting matrix minimizes total pricing errors by placing less weight on small growth assets when using consumption-based stochastic discount factors (Parker and Julliard, 2005). The spread in
latent factor $\beta$ for growth stocks as size varies suggests small growth pricing errors arise from differential exposure of growth and size assets to the latent market factor.

### 6.1.3 Replication of Market Components

We replicate the market latent factor by ranking test assets according to their conditional exposures to this time series. The exposures are recalculated each quarter over a window of size $T_m = 15$ years. For each quarter, we resort the rankings based on the updated conditional $\beta$s, and obtain a new portfolio. We form a zero-cost hedge by shorting the least exposed 20% and using the proceeds to take a long position in the most exposed 20%. The portfolio is rebalanced at a higher frequency than the parent factor mimicking portfolios. The portfolio composition has some stable constituents, but exhibits non-trivial heterogeneity through time. We report summary statistics for the replicating portfolio below.

### 6.1.4 Performance Gains

To use the replicating portfolio for hedging in the case of the Market latent factor, we must extract signals that inform us about the how to position our hedge. The timing rule we exploit is derived from mean-variance optimality given the portfolio $a_0$ that tracks the slow-moving component of market returns.

For the other factors reported in this paper, this step in unnecessary. In the case of the market latent factor, when treated as a standalone series the price of risk changes sign three times throughout the postwar period.\(^{19}\) For the purposes of trading, we exploit the fact that the logarithm of the conditional spectral gap changes signs whenever the price of risk for the latent component changes sign.

The market hedging portfolio significantly improves mean-variance efficiency, raising the monthly Sharpe ratio from 0.182 to 0.48 with the latent factor replication. The latent factor replication that throws away remaining spectral data achieves 0.42, and the joint component replicating portfolio has a monthly Sharpe ratio of 0.23.

\(^{19}\)This presents a challenge to asset pricing theory. It also presents an opportunity, because the sign of the price of risk for this component tracks the sign of the derivative of the aggregate dividend yield.
Table 2: Market Hedge Sharpe Ratio Comparisons.

(a) Full information timing rule. Replicating of single factor \( \lambda \) from exposures to latent factor only. Replication of single factor \( \Lambda \) when exposures are calculated with the full representation. Replication of joint components (LS2) with exposures calculated under the full representation.

<table>
<thead>
<tr>
<th>Strategies</th>
<th>Market Hedging</th>
<th>LS2 Hedging</th>
<th>( \lambda ) Hedging</th>
<th>( \Lambda ) Hedging</th>
<th>Value</th>
<th>Momentum</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.1823</td>
<td>0.2315</td>
<td>0.4201</td>
<td>0.4807</td>
<td>0.1767</td>
<td>0.2781</td>
</tr>
</tbody>
</table>

(b) Market Hedge Sharpe Ratios for Log Spectral Gap Rule.

(c) Sharpe Ratios are reported for replicating portfolios constructed from exposures calculated in the presence of no other market factor components (Exposure 1) to the full component representation (Exposure 4). Components are added in order of their contribution to time-series variation. Restricted sample returns calculated beginning in Q11968.

<table>
<thead>
<tr>
<th>Market ( \log(\frac{\lambda_1}{\lambda_2}) ) Exposure 1</th>
<th>( \log(\frac{\lambda_1}{\lambda_2}) ) Exposure 2</th>
<th>( \log(\frac{\lambda_1}{\lambda_2}) ) Exposure 3</th>
<th>( \log(\frac{\lambda_1}{\lambda_2}) ) Exposure 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1767</td>
<td>0.3396</td>
<td>0.3725</td>
<td>0.3815</td>
</tr>
</tbody>
</table>

(d) Additional Statistics for Market Hedge Returns

<table>
<thead>
<tr>
<th>mean (Timed)</th>
<th>s.e. (Timed)</th>
<th>( t ) (Timed)</th>
<th>( \sigma ) (Timed)</th>
<th>mean (Un-timed)</th>
<th>( \sigma ) (Un-timed)</th>
</tr>
</thead>
</table>

(e) Means, standard errors, volatility and t-statistics for returns to the timed and untimed hedging portfolio returns and Market returns. Standard errors are HAC Robust using GMM. Data are quarterly from 1927 Q1 to 2015 Q3.
We also implement the rote timing rule based on the logarithm of the conditional spectral gap of the conditional market return representation. The results vary based on the amount of spectral data used to calculate the exposures required to construct the replication. The log spectral gap rule improves the monthly Sharpe ratio from 0.177 to 0.34, 0.37, 0.38 and 0.35 depending on whether only the latent component is used to estimate $\beta$’s, or whether incremental components are included for $\beta$ estimation, respectively. The results indicate the importance of the spectral data even when the components themselves are not the target of replication. This is because inclusion of the components affects the exposure of the test assets to the target process.
Figure 2: Logarithm of Spectral Gap Timing Rule

(a) Logarithm of latent factor and unsigned replicating portfolio returns.

(b) Logarithm of latent factor and signed replicating portfolio returns. Timing rule reverses long and short components in replicating portfolio, invested in \( t \) for returns in period \( t + \Delta \), when the logarithm of the latent factor is negative in period \( t \). Negative log latent factor realizations correspond to inversions in spectral gap constituents. Test assets are the Fama-French FF25 Size/BTM portfolios. Factor data are the FF3 plus Momentum factor returns. Data are quarterly from 1927 Q1 to 2015 Q3.
6.2 Value

6.2.1 Value Decomposition

The value decomposition shows more dependence on lower components than the market, which was largely dominated by a single component. Returns to value have a much less pronounced latent trend. We will continue to see that long-horizon slow-moving predictability is a feature specific to the market. Persistence of these components are measured by estimating $AR(1)$ coefficients.

6.2.2 Exposures

We report unconditional exposures to the penultimate eigenfunction of the HML factor returns. Plots of the t-statistics are reported.

Table 3: Persistence of returns to component replicating portfolios, value decomposition.

<table>
<thead>
<tr>
<th>term</th>
<th>estimate</th>
<th>statistic</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi_0$</td>
<td>1.126</td>
<td>14.442</td>
</tr>
<tr>
<td>RepMAlag</td>
<td>0.134</td>
<td>2.305</td>
</tr>
<tr>
<td>$\phi_0$</td>
<td>0.036</td>
<td>0.128</td>
</tr>
<tr>
<td>Rep_1lag</td>
<td>0.138</td>
<td>2.380</td>
</tr>
<tr>
<td>$\phi_0$</td>
<td>-0.113</td>
<td>-2.511</td>
</tr>
<tr>
<td>Rep_2lag</td>
<td>0.006</td>
<td>0.102</td>
</tr>
<tr>
<td>$\phi_0$</td>
<td>-0.005</td>
<td>-0.554</td>
</tr>
<tr>
<td>Rep_3lag</td>
<td>-0.006</td>
<td>-0.101</td>
</tr>
<tr>
<td>$\phi_0$</td>
<td>-0.010</td>
<td>-4.570</td>
</tr>
<tr>
<td>Rep_4lag</td>
<td>-0.029</td>
<td>-0.492</td>
</tr>
</tbody>
</table>

(a) Autoregressive coefficients for value-component replicating portfolio returns. Test assets are the Fama-French FF25 Size/BTM portfolios. Factor data are the FF3 factor returns. Data are quarterly from 1927 Q1 to 2015 Q3.

6.2.3 Replication and Performance Gains

The hedging portfolios constructed below will always improve efficiency of the base factor portfolios. Given the traded factor Sharpe ratio over a particular subsample, conditioning will raise that Sharpe ratio. Clearly, other factors can dominate over any window. We
Figure 3: HML exposure anomalies are evident around the edges of the size-BTM sorts where known cross-sectional pricing difficulties arise.

(a) Significance of exposures to slowest moving components of HML returns. FF25 size-BTM test assets. Novel exposure patterns (relative to HML) are evident in the two slowest moving components ImageCond4 and ImageConstant. The remaining components mirror exposure to the parent factor. The red line moves from S1B1 to S5B1. Data are quarterly from 1927 Q1 to 2015 Q3.
extract conditional improvements from any factor-based trading strategy. The properties of the replicating portfolio and exposure to the replicating portfolio returns are reported below.

Table 4: Sharpe ratios for value and several component replication strategies.

<table>
<thead>
<tr>
<th></th>
<th>HLR</th>
<th>HML</th>
<th>Hybrid</th>
<th>EW 1.2</th>
<th>EW 1.3</th>
<th>AVG 2.1</th>
<th>AVG 3.1</th>
<th>AVG 1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>Full Sample</td>
<td>0.177</td>
<td>0.287</td>
<td>0.195</td>
<td>0.192</td>
<td>0.228</td>
<td>0.275</td>
<td>0.202</td>
<td></td>
</tr>
<tr>
<td>Pre-Crisis</td>
<td>0.234</td>
<td>0.375</td>
<td>0.238</td>
<td>0.232</td>
<td>0.301</td>
<td>0.330</td>
<td>0.245</td>
<td></td>
</tr>
</tbody>
</table>

(a) HLR are returns on long-short portfolios sorted on exposure to the representation components (high-low returns). Hybrid is a composite portfolio of three component replications. EW $i,j$ is a two-component $i,j$ portfolio with each component weighted according to their relative contributions to total variation. AVG $i,j$ are two-component portfolios that average the two components. Test assets are the Fama-French FF25 Size/BTM portfolios. Factor data are the FF3 factor returns. Data are quarterly from 1927 Q1 to 2015 Q3.

Table 5: Sharpe Ratios for value component replicating returns.

<table>
<thead>
<tr>
<th>SharpeRatio_Value</th>
<th>SharpeRatio_Cond2</th>
<th>SharpeRatio_Cond3</th>
<th>SharpeRatio_Hybrid</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.177</td>
<td>0.230</td>
<td>0.283</td>
<td>0.301</td>
</tr>
</tbody>
</table>

(a) Portfolios directly replicate components. Test assets are the Fama-French FF25 Size/BTM portfolios. Factor data are the FF3 factor returns. Data are quarterly from 1927 Q1 to 2015 Q3.

Table 6: Sharpe Ratios for Value Component Replications, pre 2007 crisis.

<table>
<thead>
<tr>
<th>SharpeRatio_Value</th>
<th>SharpeRatio_Cond2</th>
<th>SharpeRatio_Cond3</th>
<th>SharpeRatio_Hybrid</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.190</td>
<td>0.259</td>
<td>0.223</td>
<td>0.243</td>
</tr>
<tr>
<td>tCond2</td>
<td>tCond3</td>
<td>tHybrid</td>
<td></td>
</tr>
<tr>
<td>3.403</td>
<td>2.920</td>
<td>3.191</td>
<td></td>
</tr>
</tbody>
</table>

(a) Test assets are the Fama-French FF25 Size/BTM portfolios. Factor data are the FF3 factor returns.

The value hedging portfolio improves mean-variance efficiency, raising the Sharpe ratio from 0.177 to 0.230 for the second component replicating returns, 0.283 for the third component replicating returns, and 0.301 for the hybrid returns. When several components are
predictable, a simple diversification argument shows why the hybrid returns will be more efficient than either of the individual component replicating returns.

6.3 Momentum

Momentum strategies (Jagadeesh and Titman (1993), Moskowitz and Grinblatt (1999)) are predicated on the scope for conditional improvements embedded in the Fama-French three factor models. By design, they improve conditional mean-variance efficiency relative to the FF3 baseline. Moreover, they exploit persistence in portfolio returns and in particular on the ability to identify portfolios that are predictably increasing or decreasing in total returns.

Nonetheless, the procedure outlined for the Market and value factors is applicable to momentum. Taking the momentum strategy as given, we isolate predictable components in the decomposition of momentum returns and choose to bear those risks in a manner that optimizes the mean-variance trade-off. A naive replication of a single component generates improvements of less than 1%. In contrast, a strategy that combines components based on their predictable interaction with momentum returns generates roughly a 5% increase in the Sharpe ratio. We report these performance statistics below. We also note that our conditioning information can be used to time the hybrid strategy and improve performance further.

6.3.1 Momentum Decomposition

The decomposition of returns to the Momentum factor are shown. The momentum returns have several nontrivial components from a time-series variation standpoint.

6.3.2 Exposures

We report unconditional exposure to the momentum factor components. The MOM test assets are monotonically increasing in exposure to the latent component. The FF25 test assets are monotonically decreasing in exposure to the second eigenfunction of Momentum.
### 6.3.3 Replication

The components of the momentum return decomposition exhibit varying degrees of predictability. Importantly, the $T_m$-window moving average has is significantly persistent with a one-quarter autoregressive coefficient of 0.96. Predictability for components used in the momentum hedge are reported.

Summary statistics for the replicating portfolio are reported below. This portfolio replicates the ImageCond$_3$ component of the momentum return decomposition.

<table>
<thead>
<tr>
<th>term</th>
<th>estimate</th>
<th>std.error</th>
<th>statistic</th>
<th>p.value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\phi}_0$</td>
<td>-0.013</td>
<td>0.123</td>
<td>-0.107</td>
<td>0.915</td>
</tr>
<tr>
<td>$\hat{\phi}_2$</td>
<td>-0.037</td>
<td>0.057</td>
<td>-0.648</td>
<td>0.517</td>
</tr>
<tr>
<td>$\hat{\phi}_0$</td>
<td>0.063</td>
<td>0.084</td>
<td>0.746</td>
<td>0.456</td>
</tr>
<tr>
<td>$\hat{\phi}_3$</td>
<td>-0.159</td>
<td>0.058</td>
<td>-2.760</td>
<td>0.006</td>
</tr>
<tr>
<td>$\hat{\phi}_0$</td>
<td>0.073</td>
<td>0.040</td>
<td>1.812</td>
<td>0.071</td>
</tr>
<tr>
<td>$\hat{\phi}_{Tm}$</td>
<td>0.965</td>
<td>0.016</td>
<td>58.620</td>
<td>0.000</td>
</tr>
</tbody>
</table>

(a) Single quarter autoregressions for each component and the latent factor. The components of the momentum factor returns exhibit very little predictability, while the latent momentum factor is highly predictable. Factor data are the FF3 plus Momentum factor returns. Data are quarterly from 1927 Q1 to 2015 Q3.

### 6.3.4 Performance

Momentum hedging based on a single component improves mean-variance efficiency slightly. The Sharpe ratio rises from 27.8% to 28.3%. The test statistic for the mean return to the momentum hedging strategy is 4.06.

The hybrid component hedging strategy significantly improves momentum efficiency. The Sharpe ratio from the hybrid component hedging strategy is 31.14%. The $t$–statistics for the hybrid hedge is 4.470.
Table 8: Sharpe Ratios: Momentum Component Hedging Performance

<table>
<thead>
<tr>
<th>Market Value</th>
<th>Momentum</th>
<th>Hedging One Component</th>
<th>Hedging Hybrid</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.182263</td>
<td>0.1766813</td>
<td>0.2780796</td>
<td>0.2825104</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td>0.3114585</td>
</tr>
</tbody>
</table>

Strategies | Momentum Replicator 1 Replicator 2 Hybrid Replicator Monthly Sharpe Ratios 0.2801325 0.3283816 0.3009401 0.3283816

(a) Momentum Single component hedge Sharpe Ratio and Momentum Hybrid Component Hedge Sharpe Ratio. Both exceed the momentum Sharpe Ratio. Lower panel excludes the financial crisis.

(b) Additional Momentum Hedge Performance Statistics.

<table>
<thead>
<tr>
<th>µ Timed</th>
<th>s.e. Timed</th>
<th>t Timed</th>
<th>σ Timed</th>
<th>µ Hybrid</th>
<th>σ Hybrid</th>
<th>s.e Hybrid</th>
<th>t Hybrid</th>
</tr>
</thead>
</table>

(c) Means, standard errors, volatility and t-statistics for returns to the timed and timed-hybrid momentum hedging portfolio returns. Standard errors are HAC Robust using GMM. Test assets are the Fama-French FF25 Size/BTM plus 10 momentum portfolios. Factor data are the FF3 plus Momentum factor returns. Data are quarterly from 1927 Q1 to 2015 Q3.

Figure 4: Time series of Momentum and hybrid replicating portfolio.

(a) Momentum component hybrid dominant strategy returns and momentum returns. For both panels, test assets are the Fama-French FF25 Size/BTM plus 10 momentum portfolios. Factor data are the FF3 plus Momentum factor returns. Data are quarterly from 1927 Q1 to 2015 Q3.
6.4 Size

6.4.1 SMB Representation

The decomposition of the size factor is reported. The components are close to white noise, with the exception of $ImageCond_2$ and the rolling conditional expectation $ImageConstant$. Autoregressive coefficients are reported

Table 9: Persistence of components, SMB representation.

<table>
<thead>
<tr>
<th>term</th>
<th>estimate</th>
<th>std.error</th>
<th>statistic</th>
<th>p.value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\hat{\phi}_0$</td>
<td>0.004</td>
<td>0.244</td>
<td>0.016</td>
<td>0.987</td>
</tr>
<tr>
<td>$\hat{\phi}_1$</td>
<td>0.044</td>
<td>0.058</td>
<td>0.761</td>
<td>0.447</td>
</tr>
<tr>
<td>$\hat{\phi}_0$</td>
<td>0.009</td>
<td>0.142</td>
<td>0.064</td>
<td>0.949</td>
</tr>
<tr>
<td>$\hat{\phi}_2$</td>
<td>0.067</td>
<td>0.058</td>
<td>1.142</td>
<td>0.254</td>
</tr>
<tr>
<td>$\hat{\phi}_0$</td>
<td>0.017</td>
<td>0.017</td>
<td>0.972</td>
<td>0.332</td>
</tr>
<tr>
<td>$\hat{\phi}_{Tm}$</td>
<td>0.973</td>
<td>0.013</td>
<td>72.633</td>
<td>0</td>
</tr>
<tr>
<td>$\hat{\phi}_0$</td>
<td>-0.011</td>
<td>0.075</td>
<td>-0.145</td>
<td>0.885</td>
</tr>
<tr>
<td>$\hat{\phi}_3$</td>
<td>0.037</td>
<td>0.057</td>
<td>0.657</td>
<td>0.512</td>
</tr>
<tr>
<td>$\hat{\phi}_0$</td>
<td>-0.036</td>
<td>0.055</td>
<td>-0.656</td>
<td>0.512</td>
</tr>
<tr>
<td>$\hat{\phi}_4$</td>
<td>-0.052</td>
<td>0.058</td>
<td>-0.891</td>
<td>0.374</td>
</tr>
<tr>
<td>$\hat{\phi}_0$</td>
<td>0.623</td>
<td>0.302</td>
<td>2.066</td>
<td>0.040</td>
</tr>
<tr>
<td>$\hat{\phi}_{SMB}$</td>
<td>-0.016</td>
<td>0.058</td>
<td>-0.274</td>
<td>0.785</td>
</tr>
</tbody>
</table>

(a) Factor data are the FF3 plus Momentum factor returns. Data are quarterly from 1927 Q1 to 2015 Q3.

6.4.2 SMB Decomposition Exposures

Key exposure patterns are reported. While overall exposure of the test assets to SMB returns are decreasing in size, the components pick up slight shifts in asset risk profiles. Value stocks decrease exposure to size as market cap grows, but much less so than growth stocks. Value stocks maintain a positive exposure to the $T_m$ SMB return moving average as market cap grows, while all other book-to-market bins become negatively exposed once they become large.
Table 10: Sharpe Ratios: Size Hybrid Component Hedging Performance

<table>
<thead>
<tr>
<th>Hybrid Size Hedge</th>
<th>Size</th>
<th>Market</th>
<th>Value</th>
<th>Momentum</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.1352322</td>
<td>0.182263</td>
<td>0.1766813</td>
<td>0.2780796</td>
</tr>
<tr>
<td></td>
<td>0.2138347</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(a) Size Hedge Sharpe ratio. Mean-variance efficiency improves by 63%. Test assets are the Fama-French FF25 Size/BTM plus 10 momentum portfolios. Factor data are the FF3 plus Momentum factor returns. Data are quarterly from 1927 Q1 to 2015 Q3.

### 6.4.3 Latent Size factor replication

Properties of the size latent factor replicating portfolio are reported.

### 6.4.4 SMB Hedge Performance

The latent factor replicating portfolio improves mean-variance efficiency by 63%. Notably, unlike other factor hedging portfolios, the size latent hedge does not have cross-sectional pricing power. Performance reports are here:

### 7 Full Sample Implications

We find some of the returns to the replicating portfolio provide additional cross-sectional pricing power. This is consistent with the view that the efficiency gains arise through a better accounting for time variation in risk prices. Recall the conditional model can be written

$$0 = \mathbb{E}_t \left[ \mathcal{M}_{t+\Delta} R^e_{t+\Delta} \right]$$

For the case of time-varying risk prices, the conditional stochastic discount factor (SDF) takes the form $\mathcal{M}_{t+\Delta} = 1 + \delta_t \cdot f_{t+\Delta}$ (up to a scale normalization). Conditioning down gives, for $\lambda_t = -\mathbb{E}[m]^{-1} \delta_t$,

$$\mathbb{E}[R^j_{t+\Delta}] - r_{f,t} = \mathbb{E}[\lambda_t] \cdot Cov(f_{t+\Delta}, R^j_{t+\Delta}) + \sum_k Cov(\lambda_{t,k}, \mathbb{E}_t[f_{t+\Delta,k} R^j_{t+\Delta}])$$  (5)

as in Jagannathan and Wang (1996). An asset’s equilibrium unconditional return includes a contribution from the covariance of the time-varying price of risk with the discounted ex-
cess return. Full sample tests are implemented following completion of the aforementioned algorithm.

4. Empirical Algorithm Augmentation

- Collect the contemporaneous portfolio returns $F^*$ from $[0, ..., T]$
- Perform a full sample Fama-MacBeth procedure replacing the factor replicating portfolio $f$ with our hybrid portfolio $F^*$
- Test $\lambda_{F^*} > 0$ and $\lambda_{F^*} \neq \lambda_f$

The latter test checks if the price of hybrid risk is significantly different than the price of factor risk in the case where the hybrid portfolio is priced.

7.1 Full Sample Pricing Results

We evaluate pricing implications for the decomposition in an initial diagnostic stage and a final stage. The diagnostic stage looks for the component specific pricing power when the components are in a horse race. We then pick the most significantly priced component(s) and construct a replicating portfolio to track the component. The final pricing test checks whether the returns to the hedging portfolios acquire the cross-sectional pricing power of the factor.

Exposure to the second component of the HML return decomposition is priced. We derive a tracking portfolio for this component. Diagnostic tests and the pricing of the hedging factor for momentum is reported.
Table 11: Prices of exposure to returns on latent hedging portfolio are high and significant. \( \alpha \)'s become indistinguishable from zero.

<table>
<thead>
<tr>
<th>term</th>
<th>lambda</th>
<th>se</th>
<th>tstat</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>0.446</td>
<td>1.592</td>
<td>0.280</td>
</tr>
<tr>
<td>HLReturn</td>
<td>9.711</td>
<td>2.514</td>
<td>3.862</td>
</tr>
<tr>
<td>HML</td>
<td>1.307</td>
<td>0.426</td>
<td>3.066</td>
</tr>
<tr>
<td>Mkt_res</td>
<td>3.618</td>
<td>1.544</td>
<td>2.344</td>
</tr>
<tr>
<td>SMB</td>
<td>0.814</td>
<td>0.506</td>
<td>1.609</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>term</th>
<th>lambda</th>
<th>se</th>
<th>tstat</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha )</td>
<td>0.978</td>
<td>1.738</td>
<td>0.563</td>
</tr>
<tr>
<td>HLReturn</td>
<td>7.989</td>
<td>2.627</td>
<td>3.042</td>
</tr>
<tr>
<td>HML</td>
<td>1.394</td>
<td>0.405</td>
<td>3.444</td>
</tr>
<tr>
<td>Mkt_res</td>
<td>2.721</td>
<td>1.790</td>
<td>1.520</td>
</tr>
<tr>
<td>Momentum</td>
<td>0.662</td>
<td>1.862</td>
<td>0.355</td>
</tr>
<tr>
<td>SMB</td>
<td>0.771</td>
<td>0.504</td>
<td>1.529</td>
</tr>
</tbody>
</table>

(a) Factors include the market residual, HML and SMB (top panel) and the market residual, HML, SMB, and Momentum (lower panel). Test assets are the Fama-French FF25 Size/BTM portfolios. Factor data are the FF3 factor returns. Data are quarterly from 1927 Q1 to 2015 Q3.

Table 12: Prices of risk for exposure to returns to the replicating portfolio of the HML latent factor, the residual HML returns, the Market and SMB. Test assets are the FF25 size/BTM sorted portfolios. Factor data are the FF3 factor returns. Data are quarterly from 1927 Q1 to 2015 Q3.

<table>
<thead>
<tr>
<th>term</th>
<th>lambda</th>
<th>se</th>
<th>tstat</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Intercept)</td>
<td>2.293</td>
<td>1.702</td>
<td>1.348</td>
</tr>
<tr>
<td>HLReturn</td>
<td>9.982</td>
<td>2.507</td>
<td>3.982</td>
</tr>
<tr>
<td>HML_res</td>
<td>-0.486</td>
<td>0.182</td>
<td>-2.671</td>
</tr>
<tr>
<td>MktRF</td>
<td>-0.007</td>
<td>1.593</td>
<td>-0.004</td>
</tr>
<tr>
<td>Momentum</td>
<td>1.490</td>
<td>1.804</td>
<td>0.826</td>
</tr>
<tr>
<td>SMB</td>
<td>0.860</td>
<td>0.508</td>
<td>1.692</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>meanR2</th>
<th>MeanAbsError</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.569</td>
<td>2.366</td>
</tr>
</tbody>
</table>
Table 13: Cross-sectional pricing power of momentum hedging portfolio.

<table>
<thead>
<tr>
<th>term</th>
<th>lambda</th>
<th>se</th>
<th>tstat</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Intercept)</td>
<td>1.431</td>
<td>1.352</td>
<td>1.058</td>
</tr>
<tr>
<td>HLReturn</td>
<td>5.407</td>
<td>4.005</td>
<td>1.350</td>
</tr>
<tr>
<td>HML</td>
<td>1.465</td>
<td>0.398</td>
<td>3.683</td>
</tr>
<tr>
<td>MktRF</td>
<td>0.842</td>
<td>1.183</td>
<td>0.712</td>
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<tr>
<td>Momentum_res</td>
<td>1.452</td>
<td>0.894</td>
<td>1.624</td>
</tr>
<tr>
<td>SMB</td>
<td>0.833</td>
<td>0.516</td>
<td>1.614</td>
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</table>

<table>
<thead>
<tr>
<th></th>
<th>meanR2</th>
<th>MeanAbsError</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.590</td>
<td>2.305</td>
</tr>
</tbody>
</table>

Table 14: Pricing power of SMB hedging portfolio is negligible.

<table>
<thead>
<tr>
<th>term</th>
<th>lambda</th>
<th>se</th>
<th>tstat</th>
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</thead>
<tbody>
<tr>
<td>(Intercept)</td>
<td>3.276</td>
<td>1.657</td>
<td>1.977</td>
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<tr>
<td>HLReturn</td>
<td>-4.502</td>
<td>3.261</td>
<td>-1.381</td>
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<tr>
<td>HML</td>
<td>1.467</td>
<td>0.413</td>
<td>3.550</td>
</tr>
<tr>
<td>MktRF</td>
<td>-1.027</td>
<td>1.508</td>
<td>-0.681</td>
</tr>
</tbody>
</table>

(a) Test assets are the Fama-French FF25 Size/BTM portfolios. Factor data are the FF3 plus Momentum factor returns. Data are quarterly from 1927 Q1 to 2015 Q3.
8 Discussion

Our approach exploits restrictions on priced risk factors. The results can thus be interpreted as evidence that the Fama-French factors tested in this paper are proxies for priced risk factors rather than measures of characteristics that vary cross-sectionally with returns. In particular, the market, value and momentum have embedded information the priced factor restriction identifies, while SMB appears not to. This finding is consistent with the intuition put forward in Fama and French (2016).
9 Conclusion

We extract predictable components from priced risk factors and show these components can be used to improve performance. Consideration of latent components of factor risk prices can be used in real time to improve efficiency of conditional resource allocation and thus carries normative implications for traders and policy makers. The findings suggest investors are able to allocate capital with more precision using past returns data alone than previously indicated.

Characterization of the latent components also carries implications for economic modeling. Rational agents making allocations optimally in an asset pricing model must be endowed with the spectral data. In models where data are initialized at a deterministic invariant distribution this distinction is immaterial. In more general models, incorporation of spectral data into the decision makers’ information set can impact model implications.
Our findings have additional positive implications for economic theory. The predictable component in market returns must be driven by the dividend yield. This places restrictions on specific parametrizations of stochastic discount factors. Two natural theoretical benchmarks are, one, persistent growth and Harrison-Krepps preferences, and two, ambiguity aversion. Hansen (2011) points out that ambiguity averse investors ex-post look like rational expectations investors if the equilibrium data generating process is the worst-case model. For plausibly indistinguishable models, the worst case model is the long-run risk model. Thus, the findings presented in this paper provide an opportunity to discriminate between these competing theories.

References


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[8] Bandi and Andrea Tamoni, *Consumption and Wold tests*,


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[26] Campbell R. Harvey, Yan Liu, Heqing Zhu, . . . and the Cross-Section of Expected Returns, Review of Financial Studies, October 2015


10 Appendix A.1

11 Markov Asset Pricing Models

We develop stylized equilibrium (SDF) and statistical (arbitrage) factor models of asset returns to analyse common variation in expected returns. A general environment and additional technical details are provided in Appendix A.1. During the course of this development we will recover the relationships in the previous sections.

\[ 0 = -m_t p_t^i + \mathbb{E} \left[ m_{t+k} (d_{t+k} + p_{t+k}^i) \right] \]  
\[ R_{i,t} - r_f = k_{0,i} + \sum_{k=1}^{K} \lambda_{k,i} \lambda_k F_{t,k} \]

The first line is the discounted cash-flow model, and the second is a linear factor model of asset returns. Empirical tests are built on the structure suggested by the second line.

Uncertainty is driven by a Markov chain \( X_t \in \mathcal{D} \subseteq \mathbb{R}^d \) for \( 1 \leq d < \infty \), with transition matrix \( \mathbb{M} \). Define \( X_{t,1} = 1_t \) to be the indicator column vector in \( \mathbb{R}^d \) with zeros away from
the indicated coordinate \( j \): \( X_t = x_j \in X \).\(^{20}\) The indicator state can be interpreted as a list of Arrow-Debreu securities payoffs, with cash flow functional \( R(X_t) \equiv 1 \). Dynamics track the entire column as a state variable. State prices are \( Q(X_t) \).

Securities \( i \in \{1, \ldots, I\} \) with equilibrium prices written \( p_{i,t} = p_{i,t}(X_t) \) are traded competitively, and \( I \neq d \) in general. The marginal value of wealth for investor \( j \) is \( m_j(X_t) \) in state \( X_t \).\(^{21}\) Each investor adjusts asset holdings until

\[
-m_j(X_t)p_{i,t} + \mathbb{E}_t \left[ m_j(X_{t+1})p_{i,t+1}^+ \right] = 0
\]

in each \((t, X_t)\) for every asset \( i \), and where \( p_{i,t+1}(X_{t+1})^+ = p_{i,t+1}(X_{t+1}) + d_{i,t+1}(X_{t+1}) \) reflects the payout policy of asset \( i \), with \( d_{i,t+1} \geq 0 \).

Clearing markets gives the asset pricing Euler equation

\[
\mathbb{E}_t \left[ M(X_{t+1})R_{t+1}^e(X_{t+1}) \right] = 0 \tag{3.0}
\]

where \( M(X_{t+1})m_j(X_t) = m_j(X_{t+1}) \) for every investor. Marginal values are positive \( m_j(X_t) > 0 \) \( \mathbb{P} \) -a.s. so that \( M(X_{t+1})\int J m_j(X_t)v_{t+1}(dj) = \int J m_j(X_{t+1})v_{t+1}(dj) \). \( R_{t+1}^e(X_{t+1}) \) is the column vector of the \( I \) asset excess returns \( R_{t+1}(X_{t+1}) - r(X_t) \). The conditional expectation is applied to each element of \( R_{t+1}^e(X_{t+1}) \).

### 11.1 Empirical Processes

The residuals of the Markov chain are defined implicitly \( X_{t+1} = \mathbb{M} X_t + u_{t+1} \) through the sequence of conditional forecast errors \( \mathbb{M} X_t = \mathbb{E}[X_{t+1} | X_t] \in \mathbb{R}^d \) The vector \( u_{t+1} \in \mathbb{R}^d \) has serially uncorrelated mutually independent white noise constituents \( u_{d,t+1} \).

\(^{20}\) Reduction to the indicator follows Anderson (1989). Ergodic \( d \) - state Markov chains are unitarily equivalent to the irreducible \( d \) - state chain on the indicator vector.

\(^{21}\) We assume preferences \( \succ \) satisfy local nonsatiation and an Inada condition: If the investor \( j \) has \( u_j(h_j(X_t)) \) for \( du_j(h_j(X_t)) = m_j dX_t \) then the Inada condition is \( \lim_{h_j(X_t) \to 0} m_j = \infty \) for monotonic increasing \( h_j \). Convex marginal utility gives local nonsatiation.
The returns process $R(X_t)$ has the Markov property.\textsuperscript{22} It can be sufficient that $R(X_t)$ is a local martingale in order to exhibit the Markov property.\textsuperscript{23} We write $Q_k R_n(X_t) := \mathbb{E}[M(X_{t+k})R_{n,t+k}(X_{t+k})|X_t]$ for discounted expectations of lump-sum payments $R_{n,p_{n,t}}$ with delivery $k$-periods from today, conditioned on today’s realization $X_t$. For $R_n$ unadjusted for risk, we write $M^k R_n(X_t) := \mathbb{E}[R_{n,t+k}(X_{t+k})|X_t]$. Unexpected equilibrium asset returns define the residuals

$$\mathbf{R}_{t+1}(X_{t+1}) - \mathbb{M} \mathbf{R}(X_t) = \zeta_{t+1}(u_{t+1}, M)$$

where $\mathbf{R}_{t+1}, \mathbf{R}$ give the $N \times 1$ vectors of realized and forecasted asset returns, and where the $N \times 1$ vector $\zeta_{t+1}(u_{t+1}, M)$ of functions of white noise $u_{t+1}$ quantify the equilibrium conditional risk adjustments for each asset $n$ and time $t$.

We use the lag operator $\mathcal{L} R(X_{t+1}) = R(X_t)$ to obtain a simple expression

$$\mathcal{L} R(X_{t+1}) = \zeta_{t+1}(u_{t+1}, M)$$

(1.m)

The Markov transition can be written as a sequence of rank one projections

$$\mathbb{M} = H \Lambda_0 H' = \sum_{k=1}^{K(d)} \lambda_k h_k h_k'$$

The leading eigenvalue is $\lambda_1 = 1$ and the leading eigenspace is the span of the invariant distribution $\mu$ of $(\mathbb{M}, X_t)$. Because of the indicator normalization, the invariant distribution is the vector of unconditional Arrow state prices. Equivalently, it is the column vector of the marginal risk-neutral probabilities on $X_{1,t}$.

Now, take asset $i$ and project $R_i$ onto $\mathbb{M}$, e.g., run OLS. Pre-multiplying the resulting expression by $H'$ and isolating the unit eigenvector gives

$$R_{i,t} = \bar{r}_i + H (I - \Lambda_{2,i} \mathcal{L})^{-1} H' \zeta_i(u_t)$$

showing that the leading order contributions to level dynamics are not relevant for time-

\textsuperscript{22}This is true whenever $\mathbb{P}$-measurable $R$ is supported on $X$ for finite, countable or uncountable $X$.

\textsuperscript{23}The implication that the data $M(X_t)R(X_t)$ are martingales can justify its treatment as Markovian. See Appendix A.1 for a thorough discussion.
varying risk premiums. The trailing order contributions are predictable (transitory) fluctuations. The trailing rank-one projections in \( M \) drive local predictability and autocorrelation.

It is misleading to think that the trailing order drivers of the process do not matter for risk because they are transitory. In fact, they are the only thing that matters for conditional variation in risk prices, exposures and expected returns. However, the largest contribution to unconditional volatility comes from the permanent shocks in \( \mu \).

### 11.2 Markov Processes and Martingale Excess Returns

We review conditions on the local dynamics of martingales that ensure a Markov representation is valid. Our interest is in the added structure made available for Markov processes that are not available for generic martingales. An appealing tool is the semi-parametric factor structure implied by the generator and the functionals \( R \) on \( X_t \).

In many cases, non-Markovian martingales can be made Markovian by enlarging the state-vector, and non-martingale Markov processes can be made into martingales by tilting the measure. It turns out the equivalence is very general: whenever the martingale dynamics \( X_t \) can be described by a linear operator \( \mathcal{A} \) defined on a subspace of bounded continuous functions \( C_B(X) \) over \( X \), then the law of the process obeying the martingale dynamics is the law of a Markov process with transition functions determined from the empirical distribution of \( X_t \).\(^{24}\) For simplicity we model the continuous time case.\(^{25}\) A technical definition of the economic equilibrium that encompasses this example is given in A.3.

For state space \( X \) Polish, define \( C(X) \) the space of continuous functions on \( X \). The space of positive payoffs is written \( C(X)^+ \), and the quotient space \( R := \{ p \in C(X) : \lambda(p) = 1 \} \) contains returns.

\(^{24}\)A technical description of the environment is given in A.3. The transition functions determine the Markov semigroup and the generator, the latter of which is shown to be an extension of \( \mathcal{A} \).

\(^{25}\)The equivalence holds in discrete and continuous time but is more cumbersome and involves further qualifications in discrete time (Liggett (1998)).
Price data \( p_t^i(X_t) \) are functionals of a stochastic process \( X_t \) constrained in equilibrium by

\[
p_t^i(X_t)m_t(X_t) = \mathbb{E}_t \left[ \int_t^\infty C^i(x_{t+s})m(x_{t+s}) \, ds \right]
\]

\[
= \int_t^\infty (Q_s C^i) (X_{t+s}) \, ds
\]

where the operator \( Q_s := \mathbb{E}_t m_{t+s} \) is the risk adjusted conditional expectations operator for
the process \( X_t \) given \( m(X_t) \). When \( X_t \) is explicitly Markov, \( Q_s \) is the strongly continuous single parameter \( s \in [0, \infty) \) semigroup for a tilted generator \( A^Q \) of \( X_t \).

The model implies in addition that

\[
M_p(X_t) = p_t(X_t) - \int_0^t (Q_u C) (X_{t+u}) \, du
\]

is a martingale. For simplicity, we have cleared the securities markets \( i \). \( p_t(X_t) \) is the price of a claim to aggregate consumption, i.e., \( p_t(X_t) = W_t(X_t) \).

Without loss of generality, we normalize \( M_p(X_0) \equiv 1 \). Consider the preference order \( \succ \) captured in \( m_j(x_t) \), and write the constraint sets \( B_0 := \{ a \in \mathbb{R}^N : p_t(X_t) \cdot a - W_t \leq 0 \} \) and \( B_1(a) := \{ \tilde{a} \in \mathbb{R}^N : (\tilde{a}, X_{t+1}) \succ (a, X_{t+1}) \} \). Write \( a_0 \) for \( a \in \mathbb{R}^N \) comprising the binding level set \( B_0^\dagger \) of \( B_0 \). Put \( B(a_0) := B_0 \cap B_1(a_0) \). Equilibrium enforces \( \mathbb{P}(B(a_0)) = 0 \).

From the Kolmogorov extension theorem, the law of a stochastic process is completely determined by its finite -dimensional distributions \( F(X_s), 0 \leq s < \infty \). Varadhan (1979) shows the finite dimensional distributions of the process \( X_t \) that solves the martingale problem M.1 are given by the Markov transition functions \( P_t(X_{t+s} | X_t) \) corresponding to the generator \( \mathcal{A} \). Then, by Kolmogorov extension, the law of \( X_t \) written \( \mu_0 \), is the law of the Markov process with the implied transitions, and the generator \( \mathcal{A}^+ \) associated with the Markov transitions extends the domain of \( \mathcal{A} \).

When the operator \( \mathcal{A} \) measures excess returns and the netting rate is known conditionally, the domain \( \mathcal{D} \) is closed under addition of constants, e.g., \( \mathcal{A} 1 = 0 \). In this case the conditions
for the martingale problem can be written in Novikov form

\[ z_t(X_t)p_0(X_0) = p_t(X_t)e^{-\int_0^t \log(A_{ps}(X_s)) - \log(p_s(X_s)) \, ds} \]

The \( X_t \) that makes \( z_t(X_t) \) a martingale is a Markov process when the martingale dynamics are generated locally by a linear operator \( A \).

**12 Appendix A.2: Empirical Algorithm**

We present the detailed step-by-step procedure used to implement the testing described at the beginning of section 4 and motivated throughout the paper. Readers sufficiently comfortable with the approach can skip this section.

For every \( t_s, s \in \{1, 2, ..., T\} \) in a sample of returns data of length \( T \),

1. Using realized factor returns on the subsample \( \tau_m := \{t_s - T_m, t_s\} \)
   - Construct conditional forecasts of each risk factor return over horizons \( k \in \{2, 4, ..., 2^J\} \) for chosen \( J \leq \log(T_m)/\log(2) \), for choice of predictor variable \( \overline{R}_t \)
   - \( R_{t+k} = a_0 + a_1 \overline{R}_t + u_{t+k} \)

   - Construct forecasts within each \( T_m \) by truncating the data to length \( T^* < T_m \) according to the longest horizon forecast \( h_{max} = T_m - T^* > 0 \)
   - Extrapolate to full window \( T_m \) using fitted values from truncated window and observations from full window \( T_m \)

   - Construct a column of per-period conditional forecasts using the fitted values \( \widehat{R}_t \) from the forecasting regressions

\[
g(R, t_s, T_m; \theta) = \begin{bmatrix} \widehat{R}_{t+2} \\ \vdots \\ \widehat{R}_{t+2J} \end{bmatrix}
\]
• Perform a singular value decomposition on the sample $R_{t-T_m,t}$, or perform a spectral decomposition on the symmetric matrix $\Gamma = RR'$ yielding

$$\Gamma = U \Lambda^2 U^{-1}$$

for diagonal $\Lambda$ and $U^{-1} = U'$

• Regress each realized factor returns series on the decomposed sample moment matrix to obtain

$$F_k(\tau_m, s) = q_{0,k}(\tau_m, s)1 + q_{N_0,k}(\tau_m, s)$$

for constant scalar $q_{0,k}$ and $T_m \times 1$ column of ones $1$. All units are monthly returns

2. Construct a tradeable version of the return forecast for period $t_s + 1$

• Take the contemporaneous representation of the factor return in period $t_s$ and collect the trailing contemporaneous representations for periods $t_i \in \tau_m(s)$ to define representation dynamics $F_k^\Delta$

$$F_k^\Delta(\tau_m, s) = q_{0,k}^\Delta(\tau_m, s) + q_{N_0,k}^\Delta(\tau_m, s)$$

with constituents

$$F_k(t_s) = q_{0,k}(t_s) + \sum_{n=1}^{N_0} q_{n,k}(t_s)$$

$$\vdots$$

$$F_k(t_{s-T_m}) = q_{0,k}(t_{s-T_m}) + \sum_{n=1}^{N_0} q_{n,k}(t_{s-T_m})$$

• Construct replicating portfolios $q_{n,k}^\Delta$ for components $q_{n,k} \in q_k^\Delta$ by projecting each component $T_m$ time-series onto the test-asset space

• Evaluate the autoregressive properties of the replicating portfolio returns by running

$$q_{n,k}^\Delta(t_i) = a_0 + a_1 q_{n,k}^\Delta(t_{i-1}) + u_i$$
\[ \forall t_i \in \{ t_s - T_m + 1, t_s \}, n \in \{0, 1, \ldots, N_0\} \text{ and for each factor } k \]

- Extract the forecast of factor risk prices for period \( t_s + 1 \) by forecasting the individual components

\[
\hat{R}_{t,s+1}^k = \mathbb{E}_{t_s} \left[ q_{0,k}(t_s + 1) + \sum_{n=1}^{N_0} q_{n,k}(t_s + 1) \right] := b_0 + b \cdot q
\]

for a subset of persistent components \( q \in \mathbb{R}^{N_p} \) where \( N_p \leq N_0 \) and coefficients \( \{b_0(a_0, a_1), b(a_1)\} \)

- Form a new portfolio \( F_{t_s,t_{s+1}}^* \) that minimizes variance subject to the constraint that it forecasts the one-step-ahead factor risk price \( \mathbb{E}_{t_s}[F_{t_s,t_{s+1}}^*] = b_0 + b \cdot q \)

3. **Hold this portfolio** \( F_{t_s,t_{s+1}}^* \) **between periods** \( t_s, t_{s+1} \)

4. **Iterate.**

We can only use data up to \( t \) for our forecasts of \( t + 1 \) portfolios. We choose an initialization window of \( T_0 = 15 \) years. We keep the window a fixed size to prevent the confounding effects associated with a growing sample. In particular, under mild assumptions the fixed window \( T_m(t) = 15 \) years prevents the invariant dynamics from mechanically driving out estimated contributions from transitory movements as the sample window grows.