Pricing and Production Planning for the Supply Chain Management

A dissertation submitted in partial satisfaction of the requirements for the degree Doctor of Philosophy in Management

by

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2007
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2007
To my family
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Acknowledgments

First and foremost, I would like to express my deepest gratitude to my advisor and my friend, Professor Christopher S. Tang. I appreciate his continuous guidance, support, and care to me during the past few years. I also owe my heartfelt gratitude to my other dissertation committee members: Professor Kumar Rajaram, who led me into the research problem of joint pricing and inventory control; Professor John W. Mamer, who provided me valuable comments on the research presented in this dissertation; and Professor Lieven Vandenberghe, who is an excellent lecturer and researcher in mathematical programming.

My special thanks go to all the faculty, staff and Ph.D. students in the Decisions, Operations and Technology Management area at the UCLA Anderson School of Management. It is their support and help that make the past few years a pleasant journey for me. I would also like to thank Ms. Lydia Heyman in the Doctoral Program at the UCLA Anderson School, for her great efforts in creating a wonderful intellectual environment for us.

I am forever grateful to my parents, Guixiang Yin and Jingxu Yin, for their unconditional love. I also thank my sister, Li Yin, for taking care of our parents in China.

Finally, I owe my dearest thanks to my husband, Yalin Wang, for his love, care, and encouragement. It is him who has always been on my side during my Ph.D. journey. I hereby dedicate this dissertation to him with all my heart.
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In recent years, many retailing and manufacturing companies have adopted many innovative pricing strategies to increase their profits. My dissertation focuses on four aspects of the joint pricing and production planning problem in the supply chain management, namely, retailer, product, supplier, and customer.

Chapter 1 contains a general theoretical framework for the joint pricing and inventory control problem in which a retailer orders and sells a single product over a finite horizon. The demand distribution in each period is determined by an exogenous Markov chain. Pricing and ordering decisions are made at the beginning of each period and all shortages are backlogged. The surplus costs as well as fixed and variable costs are state dependent. I show the existence of an optimal \((s, S, p)\)-type feedback policy for the additive demand model. I compute the optimal policy for a class of Markovian demand and illustrate the benefits of dynamic pricing over fixed pricing through numerical examples. The results indicate that it is more beneficial to implement dynamic pricing in a Markovian demand environment with a high fixed ordering cost or with high
demand variability.

Chapter 2 focuses on two different pricing strategies to manage multiple substitutable products. I consider a situation in which a retailer would either charge the same retail price for all products if he adopts a ‘fixed’ pricing strategy or charge different prices for different products if he adopts a ‘variable’ pricing strategy. I develop a base model with deterministic demand that is intended to examine how a retailer should jointly determine the order quantity and the retail price of two substitutable products under the fixed and variable pricing strategies. The analysis indicates that the optimal retail price under the variable pricing strategy is equal to the optimal retail price under the fixed pricing strategy plus or minus an adjustment term. This adjustment term depends on product substitutability and price sensitivity. I also present two different extensions of the base model. In the first extension, the analysis indicates that the underlying structure of the optimal retail price and order quantity is preserved when there is a limit on the total order quantity. The second extension deals with the issue of retail competition. Relative to the base case, I show that the underlying structure of the optimal retail price and order quantity is preserved in a duopolistic environment. Moreover, the analysis suggests that both retailers would adopt the variable pricing strategy at the equilibrium.

Chapter 3 focuses on the aspect of suppliers. I consider a situation in which a retailer orders a seasonal product from a supplier and sells the product over a selling season. While the product demand is known to be a linear function of price, the supply yield is uncertain and is distributed according to a general discrete probability distribution. In this chapter, we present a two-stage stochastic model for analyzing two pricing policies: No Responsive Pricing and Responsive Pricing. Under the No Responsive Pricing policy, the retailer would determine
the order quantity and the retail price before the supply yield is realized. Under the Responsive Pricing policy, the retailer would specify the order quantity first and then decide on the retail price after observing the realized supply yield. Therefore, the Responsive Pricing policy enables the retailer to use pricing as a response mechanism for managing uncertain supply. My analysis suggests that the retailer would always obtain a higher expected profit under the Responsive Pricing policy. I also examine the impact of yield distribution and system parameters on the optimal order quantities, retail prices, and profits under these two pricing policies. Moreover, I extend our analysis of the Responsive Pricing to examine three issues. The first issue deals with the case in which the retailer can place an emergency order with an alternative source after observing the realized yield. The second issue relates to the issue of supplier selection, and the third issue deals with a situation in which the retailer has to allocate his order among multiple suppliers.

In Chapter 4, we consider customer purchasing behavior when facing a specific price markdown strategy. Consider a retailer announces both the regular price and the post-season clearance price at the beginning of the selling season. Throughout the season, customers arrive in accord with a Poisson process. I analyze the impact of two types of customer purchasing behavior and two common in-store display formats on the retailer’s optimal expected profit and optimal order quantity. I consider the case when all customers are either myopic (purchase immediately upon arrival) or strategic (either purchase at the regular price upon arrival or attempt to purchase at the clearance price after the season ends). In addition, I consider the case when the retailer would display either all available units or one unit at a time on the sales floor. When all customers have identical valuation, we show that, in equilibrium, each strategic customer’s purchasing decision is based on a threshold policy that depends on the inventory level at
the time of arrival. I prove analytically that the retailer would obtain a higher expected profit and would order more when the customers are myopic. Also, I show analytically that the retailer would earn a higher expected profit and would order more under the display one unit format when the customers are strategic. I illustrate numerically the penalty when the retailer mistakenly assumes that the strategic customers are myopic. I extend our analysis to the case in which customers belong to multiple classes, each of which has a class-specific valuation, and also to the case in which the post-season clearance price depends on the actual end-of-season inventory level.
CHAPTER 1

Joint Pricing and Inventory Control with a Markovian Demand Model

1.1 Introduction

The joint pricing and inventory control problem has been studied extensively in the operations management literature, starting with the work of Whitin (1955). The basic idea is to integrate the pricing decision with the replenishment policy when managing product inventory. In this problem, retailers act as price setters and can adjust prices dynamically to influence demand and potentially gain higher profits. Other well-known examples in the service industry are found in revenue management, which has been adopted by all major airlines, many hotel chains and car rental companies. See Talluri and van Ryzin (2004) for a comprehensive review of revenue management.

Most of the recent papers that address the pricing and inventory control coordination problem with periodic review assume that demand in different periods are independent random variables. In practice, demand usually fluctuates and depends on many exogenous factors such as economic conditions, natural disasters, strikes, etc. In addition, when a competitor introduces a new product to the market, some customers may switch to the new product and consequently, the retailer’s average demand may drop dramatically during some periods. In
these cases, a “state-dependent” demand model seems to be more appropriate to capture such randomly changing environmental factors. Furthermore, if demand is highly price-sensitive, retailers could combine pricing decisions with replenishment planning and use price as an effective tool to hedge against demand uncertainty. Therefore there is a need to consider the joint pricing and inventory control problem in a fluctuating demand environment, and a Markovian demand modeling approach provides an effective mechanism to address this problem.

The purpose of this chapter is to characterize the structure of the optimal replenishment and pricing decisions with a Markovian demand model, and to illustrate the benefits of dynamic pricing through numerical examples. Specifically, we consider a single product, periodic review system with a finite horizon, where demand is price-dependent and its distribution at each time period is determined by an exogenous Markov chain. The ordering cost consists of a fixed cost and a variable cost, and all the cost parameters are state and time dependent. Under the assumptions of an additive demand function and full backlogging, we establish the structure of an optimal Markov (feedback) policy. We also present an algorithm to compute and analyze this policy.

There are two streams of literature that are related to the work in this chapter. The first stream is the coordination of pricing and inventory control with independent demand, as mentioned above. In this stream of research, demand is a random variable that depends on price. Under the assumption that unsatisfied demand in each period is fully backlogged, Federgruen and Heching (1999) and Chen and Simchi-Levi (2004a, 2004b) have considered periodic review models with both finite and infinite horizons. In Federgruen and Heching (1999), the ordering cost is proportional to the order quantity, and there is no setup cost. They prove a base-stock list price policy is optimal. In this policy, the optimal
replenishment policy in each period is characterized by an order-up-to level, and the optimal price depends on the initial inventory level at the beginning of the period. Furthermore, the optimal price is a nondecreasing function of the initial inventory level. Chen and Simchi-Levi (2004a, 2004b) include a fixed ordering cost in their models. They prove an \((s, S, p)\)-type policy is optimal for the finite horizon model with additive demand, and a stationary \((s, S, p)\) policy is optimal for the discounted and average profit models with general demand functions in the infinite horizon problem. In such a policy, the period inventory is managed using the classical \((s, S)\) policy, and the optimal price depends on the inventory position at the beginning of the period. Feng and Chen (2004) consider a long-run average profit model with periodic review and an infinite horizon. The optimality of an \((s, S, p)\)-type policy is also established.

When unsatisfied demands are assumed to be lost, Polatoglu and Sahin (2000) characterize the form of the optimal replenishment policy under a general price-demand relationship and provide a sufficient condition for it to be of the \((s, S)\)-type. For a finite horizon system, Chen, Ray and Song (2003) and Huh and Janakiraman (2005) have proved the optimality of an \((s, S, p)\) policy under assumptions of stationary parameters and a salvage value that is equal to the unit purchasing cost. In the area of continuous review models, Feng and Chen (2003) assume demand follows a Poisson process with price-sensitive intensities, while Chen, Wu and Yao (2004) model the demand process as a Brownian motion with a drift rate that is a function of price. Furthermore, in terms of techniques to prove the optimality of an \((s, S, p)\)-type policy, Chen and Simchi-Levi (2004a), and Chen, Ray and Song (2003) use induction and the dynamic programming formulations, which are similar to that in Scarf (1960) in the classic stochastic inventory control problem. Huh and Janakiraman (2005) propose an alternative approach for the optimality proof, which is based on the method used in Veinott.
(1966). For a review of other work in the pricing and inventory literature, the reader is referred to Petruzzi and Dada (1999), Elmaghraby and Keskinocak (2003) and Chan et al. (2004).

The second stream of related literature is the inventory control problem with a Markovian demand model. Song and Zipkin (1993) present a Markovian-modulated model to capture the fluctuating demand environment. Specifically, they assume that demand in each period follows a Poisson process whose rate depends on the demand state. Sethi and Cheng (1997) analyze a general finite horizon inventory model with a Markovian demand process. They show that under certain technical assumptions, the optimal policy for the finite horizon problem is still of \((s, S)\) type, with \(s\) and \(S\) dependent on the demand state and the time remaining. Cheng and Sethi (1999) extend their previous work to the lost sales case and establish the optimality of \((s, S)\)-type policies based on certain weak conditions on the holding, shortage and unit ordering costs. Another paper that is related to ours is Beyer, Sethi and Taksar (1998), which establishes the existence and verification theorems of an optimal feedback policy.

Recently, Gayon et al. (2004) consider a Markov Modulated Poisson Process that is similar to Song and Zipkin (1993), except that the fluctuating intensities are functions of price. The unit ordering cost is given, there is no fixed ordering cost and all shortages are lost. They generalize certain structural results in Li (1998) and prove the existence of an optimal base-stock policy for the discounted infinite horizon Markov decision process. The base-stock policy is similar to that in Federgruen and Heching (1999).

To the best of our knowledge, this is the first work in the literature to address the joint pricing and inventory control problem with a Markovian demand in a periodic-review system and a fixed ordering cost. This chapter makes the follow-
ing contributions. First, under assumptions of an additive demand function and full backlogging, we establish the optimality of a feedback policy of $(s, S, p)$-type. Second, we extend the basic model to the case when the unsatisfied demand at the end of a period is filled by an emergency order. Under certain practical assumptions on the holding cost, the regular and emergency ordering cost functions, we prove the state-dependent $(s, S, p)$ policy is still optimal for the case with additive demand. Third, we develop an algorithm to compute the optimal policy for a class of Markovian demand with an arbitrary probability transition matrix and a discrete, uniformly distributed random noise. Finally, we use this algorithm to illustrate the benefits of dynamic pricing over fixed pricing through extensive numerical examples. The results indicate that it is more beneficial to implement dynamic pricing in a Markovian demand environment with a high fixed ordering cost or with high demand variability.

This chapter is organized as follows. In Section 1.2, we introduce the notations and assumptions used in this paper and develop a general finite horizon inventory model with a Markovian demand process. In Section 1.3, we state the dynamic programming equations for the problem and establish the existence of an optimal feedback policy. In Section 1.4, the additive demand function is analyzed and the optimality of an state-dependent $(s, S, p)$ policy is proved. An extension of the basic model to the case of emergency orders is presented in Section 1.5. In Section 1.6, we discuss the computation of the optimal policy for a class of Markovian demand and present numerical examples to illustrate the benefits of dynamic pricing over fixed pricing. Section 1.7 summarizes the chapter and presents future research directions.
1.2 Model Formulation

Consider a firm that has to make production and pricing decisions simultaneously at the beginning of every period over a finite time horizon with \(N\) periods. The demand distribution at each period is determined by an exogenous Markov chain. In order to specify the pricing and inventory control problem, we introduce the following notations:

\(< 0, N > = < 0, 1, 2, \cdots, N >, \) the horizon of the problem;

\( I = \{1, 2, \cdots, L\}, \) a finite collection of possible demand states;

\( i_k = \) the demand state in period \(k\);

\( \{i_k\} = \) a Markov chain with the \((L \times L)-\)transition matrix \(P = (p_{ij})\);

\( \xi_k = \) demand at the end of period \(k, k = 0, 1, \cdots, N - 1;\)

\( p_k = \) selling price in period \(k;\)

\( p^l_k = \) the lower bound on \(p_k;\)

\( p^u_k = \) the upper bound on \(p_k;\)

\( u_k = \) the non-negative order quantity in period \(k;\)

\( x_k = \) the surplus (inventory/backlog) level at the beginning of period \(k \)

before the ordering;

\( y_k = \) the inventory position at the beginning of period \(k \)

after the ordering;

\( \delta(z) = \begin{cases} 
0 & \text{if } z \leq 0, \\
1 & \text{otherwise.} 
\end{cases} \)

Throughout this chapter, we assume that demand \(\xi_k \geq 0\) and \(\xi_k\) depends on the demand state \(i_k\). Specifically, when demand is in state \(i \in I\), and the selling price is \(p\), the demand functions have the following additive forms:

\[ \xi^i_k = D_k(i, p) + \beta^i_k, \quad (1.1) \]
where \( D_k(i, p) \) is the non-negative, strictly decreasing deterministic or riskless demand function. We assume that this is a continuous function of \( p \). \( \beta_k^i \) is the only random component and we assume it is independent of the price \( p \). Note (1.1) is a direct translation of the demand function in Chen and Simchi-Levi (2004a). For example, when demand is in state \( i \), one commonly uses the riskless linear demand function \( D_k(i, p) = a_k^i - b_k^i p \) for \( p \leq a_k^i / b_k^i, (a_k^i, b_k^i > 0) \), as in Petruzzi and Dada (1999). We also assume that when \( i_k = i \), \( \beta_k^i \) is distributed over the interval of \([t_1, t_2]\) with the density function \( \phi_{i,k}(\cdot) \). Without loss of generality, we assume that \( E(\beta_k^i) = 0 \) and the probability of negative demand is zero.

Notice that when the price is \( p \), the expected demand in period \( k \) given \( i_k = i \) is:

\[
E(D_k(i, p) + \beta_k^i) = D_k(i, p) + E(\beta_k^i) = D_k(i, p).
\]

We assume that the expected demand is finite for every \( p \in [p_k^l, \bar{p}_k] \). Since \( D_k(i, p) \) is a strictly decreasing function of \( p \), there is a one-to-one correspondence between the price and the expected demand. Also, when the firm charges price \( p \) in period \( k \), the expected revenue given \( i_k = i \) is:

\[
R_k(i, p) = E((D_k(i, p) + \beta_k^i) p) = D_k(i, p)p.
\]

We make the following assumption on the expected revenue functions, which is similar to Chen and Simchi-Levi (2004a). This assumption is used in the discussion of preliminary results in Section 3 and in the proof of Theorem 3.

**Assumption 1.** For all \( k, k = 0, 1, \cdots, N - 1 \), the expected revenue in period \( k \) given demand state \( i_k = i, i \in I, R_k(i, p) \), is a concave function of the price \( p \).

At the beginning of period \( k \), an order \( u_k \geq 0 \) is placed with the knowledge that the demand state is \( i_k \) and it will be delivered at the end of period \( k \).
but before the demand is realized. We assume that unsatisfied demand is fully backlogged. Thus the model dynamics can be expressed as:

\[
\begin{align*}
    x_{k+1} &= x_k + u_k - \xi_{ik}, \quad k = n, \ldots, N - 1, \\
    x_n &= x, \\
    i_k, \quad k = n, \ldots, N - 1, &\text{ follows a Markov chain with transition matrix } P, \\
    i_n &= i.
\end{align*}
\] (1.2)

Equation (1.2) describes the dynamics from period \(n\) onward, given the initial inventory level \(x\) and the demand state \(i\).

For each period \(k = 0, 1, \ldots, N - 1\), and demand state \(i \in I\), we define the following costs:

(a) \(c_k(i, u) = K_i^k \delta(u) + c_i^k u\), the cost of ordering \(u \geq 0\) units in period \(k\) when \(i_k = i\), where the fixed ordering cost \(K_i^k \geq 0\) and the variable cost \(c_i^k\) are also state dependent.

(b) \(f_k(i, x)\), the surplus cost when \(i_k = i\) and \(x_k = x\). We assume \(f_k\) is convex in \(x\) and there exists \(\tilde{f} > 0\), such that \(f_k(i, x) \leq \tilde{f}(1 + |x|)\).

(c) \(f_N(i, x)\), the penalty or disposal cost for the terminal surplus. We assume \(f_N\) is convex in \(x\) with \(f_N(i, x) \leq \tilde{f}(1 + |x|)\).

The objective of our model is to decide on ordering and pricing policies in order to maximize total expected profit over the entire planning horizon. Thus, given \(i_n = i\) and \(x_n = x\), the objective function to be maximized during the interval \(< n, N >\) is:

\[
J_n(i, x; U) = E\left\{\sum_{k=n}^{N-1} [p_k \xi_{ik} - c_k(i_k, u_k) - f_k(i_k, x_k)] - f_N(i_N, x_N)\right\},
\] (1.3)

where \(U = (u_n, p_n, \ldots, u_{N-1}, p_{N-1})\) is a history-dependent admissible decision for the problem.
Define the value function for the problem over the interval <n, N> with \(x_n = x\) and \(i_n = i\) to be:

\[
v_n(i, x) = \sup_{U \in \mathcal{U}} J_n(i, x; U),
\]

where \(\mathcal{U}\) denotes the class of all admissible decisions.

The objective function (1.3) is slightly different from the one used in Chen and Simchi-Levi (2004a). We assume the surplus costs \(f_k(i_k, x_k)\) are charged at the beginning of the periods as in Sethi and Cheng (1997), while Chen and Simchi-Levi (2004a) and most other literature use \(f_k(i_k, x_{k+1})\). Note these two formulations are essentially similar, since \(x_{k+1}\) is also the ending inventory of period \(k\). Furthermore, when we start with zero initial inventory at the beginning of the entire horizon, the difference between these two formulations is even smaller. It also turns out that in the emergency order case that we will discuss in Section 5, our formulation would be more convenient for the analysis. Thus to keep consistent notations with the emergency order case, we will charge the surplus cost at the beginning of every period.

### 1.3 Preliminary Results

Using the principle of optimality, we can write the following dynamic programming equations for the value function. For \(n = 0, 1, \cdots, N - 1\) and \(i \in I\),

\[
v_n(i, x) = -f_n(i, x) + \sup_{u \geq 0, p_n \geq p \geq \underline{p}_n} \left\{ R_n(i, p) - c_n(i, u) + E[v_{n+1}(i_{n+1}, x + u - \xi_n)|i_n = i] \right\}
\]

\[
= -f_n(i, x) + c_n^i x + G_n(i, x),
\]

(1.5)
where
\[
G_n(i, x) = \sup_{y \geq x, \bar{p}_n \geq p \geq \underline{p}_n} \left[ -K^i_n \delta (y - x) + g_n(i, y, p) \right], \quad \text{and} \tag{1.6}
\]
\[
g_n(i, y, p) = R_n(i, p) - c_i^i y + E\left[v_{n+1}(i_{n+1}, y - D_n(i_{n}, p) - \beta_n^i) | i_n = i \right]. \tag{1.7}
\]

Clearly, \( v_N(i, x) = -f_N(i, x) \).

Let \( B_0 \) denote the class of all continuous functions from \( I \times R \) into \( R^+ \) and the pointwise limits of sequences of these functions (Feller 1971), where \( R = (-\infty, \infty) \) and \( R^+ = [0, \infty) \). Note that this includes upper semicontinuous functions. Let \( B_1 \) be the subspace of functions in \( B_0 \) that are of linear growth, i.e., for any \( b \in B_1 \), \( 0 \leq b(i, x) \leq C_b(1 + |x|) \) for some \( C_b > 0 \). Let \( B_2 \) be the subspace of functions in \( B_1 \) that are upper semicontinuous. Then for any \( b \in B_1 \), define:
\[
F_{n+1}(b)(i, z) = E[b(i_{n+1}, z - \beta_n i)| i_n = i]
= \sum_{j=1}^{L} p_{ij} \int_{t_1}^{t_2} b(j, z - t) \phi_{i,n}(t) \, dt. \tag{1.8}
\]

By Lemma 2.1 in Beyer, Sethi and Taksar (1998), \( F_{n+1} \) is a continuous linear operator from \( B_1 \) into \( B_1 \). Thus if \( v_{n+1}(i, x) \) is continuous in \( x \), then \( E[v_{n+1}(i_{n+1}, y - D_n(i_n, p) - \beta_n)| i_n = i] = F_{n+1}(v_{n+1})(i, y - D_n(i_n, p)) \) is jointly continuous in \( (y, p) \) since \( y - D_n(i, p) \) is continuous in \( (y, p) \). From (1.7), we know that \( g_n(i, y, p) \) is jointly continuous in \( (y, p) \). Therefore \( -K^i_n \delta (y - x) + g_n(i, y, p) \) is upper semicontinuous in \( (y, p) \) and its maximum over a compact set is attained. Specifically, for any \( y \geq x \), there exists \( p_n(i, y) \in [\underline{p}_n, \bar{p}_n] \), such that
\[
G_n(i, x) = \sup_{y \geq x \atop \bar{p}_n \geq p_n \geq \underline{p}_n} \left[ -K^i_n \delta (y - x) + g_n(i, y, p) \right],
= \sup_{y \geq x \atop \bar{p}_n \geq p_n} \left[ -K^i_n \delta (y - x) + g_n(i, y, p_n) \right],
\]
where
\[
g_n(i, y, p_n(i, y)) = \max_{p_n \geq p \geq p_n} g_n(i, y, p) \nonumber
\]
\[
= R_n(i, p_n(i, y)) - c_n^i y + F_{n+1}(v_{n+1})(i, y - D_n(i, p_n(i, y))),
\]
(1.9)

and \(g_n(i, y, p_n(i, y))\) is continuous in \(y\). In view of Proposition 4.2 in Sethi and Cheng (1997) with \(A = -\infty\) and \(B = \infty\), we know that \(G_n(i, x)\) is continuous in \(x\). Therefore \(v_n(i, x)\) is continuous in \(x\) and the original dynamic programming equation (1.5) can be rewritten as
\[
v_n(i, x) = -f_n(i, x) + c_n^i x + \sup_{y \geq x} [-K_n^i \delta(y - x) + g_n(i, y, p_n(i, y))],
\]
(1.10)
\[
v_N(i, x) = -f_N(i, x).
\]
(1.11)

From (1.10) and (1.11), the original two-variable, joint optimization problem is transformed to the traditional periodic review inventory problem with a given price. Therefore we are only left to determine the replenishment policy. Next, we present two verification theorems similar to Theorems 2.1 and 2.2 in Beyer, Sethi and Taksar (1998), which establish the existence of an optimal feedback policy.

We need the following assumption on the cost functions, which is similar to that in Beyer, Sethi and Taksar (1998).

**Assumption 2.** For each \(n = 0, 1, \cdots, N - 1\) and \(i \in I\), we have
\[
c_n^i x + F_{n+1}(f_{n+1})(i, x) \to +\infty, \text{ as } x \to \infty.
\]
(1.12)

Assumption 2 is not very restrictive in practice. It rules out the unrealistic and trivial case of ordering an infinite amount, if \(c_n^i = 0\) and \(f_n(i, x) = 0\) for each \(i\) and \(n\). It is also useful in proving the first part of Theorem 1 that follows. Moreover, in the proof of the \((s, S, p)\) policy in Theorem 3 of Section 4, we do not
need to impose a condition like (1.12) for \( x \to -\infty \), as Assumption 3 in Chen and Simchi-Levi (2004). See Remark 4.4 in Sethi and Cheng (1997).

From an analytical perspective, adding the price decision significantly complicates the traditional inventory control model. For ease of analysis, like most of the joint pricing and inventory control literature, we will assume that prices are continuous and restricted in a closed interval on the real line. As in the previous discussion of this section, the compact set of the feasible prices plays a critical role to generalize the existence and verification theorems in Beyer, Sethi and Taksar (1998) to our price-inventory Markovian demand model.

Now we are ready to state the two verification theorems as below. The proofs are similar to those in Beyer, Sethi and Taksar (1998). We omit the details here.

**Theorem 1.** The dynamic programming equations (1.10) and (1.11) define a sequence of functions in \( B_1 \). Moreover, for each \( n = 0, 1, \cdots, N - 1 \) and \( i \in I \), there exists a function \( \hat{y}_n(i, x) \in B_0 \), such that the supremum in (1.10) is attained at \( y = \hat{y}_n(i, x) \) for any \( x \in \mathbb{R} \).

To solve the problem of maximizing \( J_0(i, x; U) \), we use \( \hat{y}_n(i, x) \) of Theorem 1 to define

\[
\hat{y}_k = \hat{y}_k(i_k, \hat{x}_k), \quad k = 0, 1, \cdots, N - 1 \quad \text{with} \quad i_0 = i,
\]

\[
\hat{x}_{k+1} = \hat{x}_k - \xi_k, \quad k = 0, 1, \cdots, N - 1 \quad \text{with} \quad \hat{x}_0 = x,
\]

\[
\hat{u}_k = \hat{y}_k - \hat{x}_k, \quad k = 0, 1, \cdots, N - 1, \quad \text{and}
\]

\[
\hat{p}_k = p_k(i_k, \hat{y}_k), \quad k = 0, 1, \cdots, N - 1.
\]

We have the following verification theorem.

**Theorem 2.** The policy \( \hat{U} = (\hat{u}_0, \hat{p}_0, \hat{u}_1, \hat{p}_1, \cdots, \hat{u}_{N-1}, \hat{p}_{N-1}) \) maximizes \( J_0(i, x; U) \)
over the class $\mathcal{U}$ of all admissible decisions. Moreover,

$$v_0(i, x) = \max_{U \in \mathcal{U}} J_0(i, x; U).$$

### 1.4 Optimality of $(s, S, p)$ Policies

To prove the optimality of an $(s, S, p)$ policy, we will use a similar approach as Chen and Simchi-Levi (2004a) based on the concept of $K$-convexity introduced by Scarf (1960). See Propositions 4.1 and 4.2 in Sethi and Cheng (1997) for a summary of the properties of $K$-convex functions. We make the following assumption, which is required in the proof of Theorem 3 that follows.

**Assumption 3.** For $n = 0, 1, \ldots, N - 1$ and $i \in I$, we have

$$K^i_n \geq \bar{K}^i_{n+1} = \sum_{j=1}^{L} p_{ij} K^j_{n+1} \geq 0. \quad (1.13)$$

Condition (1.13) includes the cases of the constant ordering costs ($K^i_n = K, \forall i, t$) and the non-increasing ordering costs ($K^i_n \geq K^j_{n+1}, \forall i, j, n$). See Remark 4.1 in Sethi and Cheng (1997) for a discussion.

**Theorem 3.** (a) For $i \in I, 0 \leq n \leq N - 1$, $g_n(i, y, p_n(i, y))$ is continuous in $y$, and

$$\lim_{y \to -\infty} g_n(i, y, p_n(i, y)) = -\infty.$$  

(b) For $i \in I, 0 \leq n \leq N - 1$, $g_n(i, y, p_n(i, y))$ and $v_n(i, x)$ are $K^i_n$-concave.

(c) For $i \in I$, there exists a sequence of numbers $s^i_n, S^i_n, n \in 0, 1, \cdots, N - 1$, with $s^i_n \leq S^i_n$, such that the optimal replenishment policy is:

$$\hat{u}_n(i, x) = (S^i_n - x) \delta(s^i_n - x), \quad (1.14)$$
and the optimal selling price is:

\[
\hat{p}_n^i = \begin{cases} 
p_n(i, S_n^i), & \text{if } x_n < S_n^i, \\
p_n(i, x_n), & \text{if } x_n \geq S_n^i.
\end{cases}
\] (1.15)

**Proof:** For part (a), the upper semicontinuity of \(g_n(i, y, p_n(i, y))\) was proven in Section 3 and the latter part follows from Assumption 2.

Next, we prove part (b) by induction. Notice that \(v_N(i, x)\) is \(K\)-concave for any \(K \geq 0\) since \(v_N(i, x) = -f_N(i, x)\) and \(f_N(i, x)\) is assumed to be convex in \(x\), for \(i \in I\). Now we assume that \(v_{k+1}(i, x)\) is \(K_{k+1}\)-concave in \(x\). By the definition of \(F_{k+1}\) in (1.8) and Proposition 4.1 in Sethi and Cheng (1997), it is easy to see that \(F_{k+1}(v_{k+1})(i, z) = \sum_{j=1}^L p_{ij} K_{k+1}^j\)-concave in \(z\). By Assumption 3, we know that \(F_{k+1}(v_{k+1})(i, z)\) is \(K^i\)-concave in \(z\). For any \(y < y'\), let \(z = y - D_k(i, p_k(i, y))\) and \(z' = y' - D_k(i, p_k(i, y'))\). Thus by Lemma 2 and Definition 2.2 in Chen and Simchi-Levi (2004), we have

\[
E[v_{k+1}(i, (1-\lambda)z + \lambda z') \geq (1-\lambda)F_{k+1}(v_{k+1})(i, z) + \lambda F_{k+1}(v_{k+1})(i, z') - \lambda K^i].
\] (1.16)

In addition, the concavity of \(R_k(i, p)\) and \(-c_k^i y\) implies that

\[
R_k(i, (1-\lambda)p_k(i, y) + \lambda p_k(i, y')) \geq (1-\lambda)R_k(i, p_k(i, y)) + \lambda R_k(i, p_k(i, y'))
\] (1.17)

and

\[
-c_k^i((1-\lambda)y + \lambda y') = (1-\lambda)(-c_k^i y) + \lambda(-c_k^i y')
\] (1.18)

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Adding (1.16), (1.17) and (1.18), and by (1.9), we get
\[
g_k(i, (1 - \lambda)y + \lambda y', (1 - \lambda)p_k(i, y) + \lambda p_k(i, y')) \geq (1 - \lambda)g_k(i, y, p_k(i, y)) + \\
\lambda g_k(i, y', p_k(i, y')) - \lambda K^i_k.
\]

Since \(p_k(i, (1 - \lambda)y + \lambda y')\) is the optimal price corresponding to \((1 - \lambda)y + \lambda y'\) in (1.9), we have
\[
g_k(i, (1 - \lambda)y + \lambda y', p_k(i, y)) \geq g_k(i, (1 - \lambda)y + \lambda y', (1 - \lambda)p_k(i, y) + \lambda p_k(i, y')).
\]

Therefore,
\[
g_k(i, (1 - \lambda)y + \lambda y', p_k(i, (1 - \lambda)y + \lambda y')) \geq (1 - \lambda)g_k(i, y, p_k(i, y)) + \\
\lambda g_k(i, y', p_k(i, y')) - \lambda K^i_k,
\] (1.19)

by which we have proven that \(g_k(i, y, p_k(i, y))\) is a \(K^i_k\)-concave function of \(y\).

Finally, we consider part (c). By Proposition 4.2 in Sethi and Cheng (1997) and equation (1.10), we can conclude that there exist \(s^i_k < S^i_k\), such that \(S^i_k\) maximizes \(g_k(i, y, p_k(i, y))\) and \(s^i_k\) is the smallest value of \(y\) for which \(g_k(i, y, p_k(i, y)) = g_k(i, S^i_k, p_k(i, S^i_k)) - K^i_k\), and

\[
v_k(i, x) = -f_k(i, x) + c^i_k x + \begin{cases} 
-K^i_k + g_k(i, S^i_k, p_k(i, S^i_k)), & \text{if } x < s^i_k, \\
g_k(i, x, p_k(i, x)), & \text{if } x \geq s^i_k.
\end{cases}
\] (1.20)

According to Theorem 2, the \((s, S, p)\)-type policy defined in (1.14) and (1.15) is optimal. \(\square\)

Theorem 3 extends Theorem 3.1 in Chen and Simchi-Levi (2004a) to a Markov modulated demand model. While Theorem 3 is similar to Theorem 4.1 in Sethi and Cheng (1997) in terms of optimal ordering policies, adding the price decision complicates the induction proof. This now requires a similar result as Lemma 2 in Chen and Simchi-Levi (2004a), which is stated in the above proof.
1.5 Extensions

In Section 4, we assumed that unsatisfied demand in each period is fully backlogged. In practice, sometimes an emergency order could be placed and delivered at the end of the period when a stockout occurs. This ensures that a 100% service level is achieved in each period. In this section, we will prove the optimality of the \((s, S, p)\) policies when the retailer is allowed to use emergency orders.

The difference between the model with emergency orders and the one with full backlogging is that when the on-hand inventory \(x_k\) at the beginning of period \(k\) and the amount \(u_k\) delivered in period \(k\) is less than the demand \(\xi_k\), the portion \(\xi_k - x_k - u_k\) could be satisfied immediately by an emergency order. In this case, the next period starts with zero on-hand inventory. Thus, the model dynamics over the interval \(< n, N >\) can be expressed as:

\[
\begin{align*}
    x_{k+1} &= (x_k + u_k - \xi_k)_{+}, \quad k = n, \cdots, N - 1, \\
    x_n &= x, \\
    i_k, \quad k = n, \cdots, N - 1, &\text{ follows a Markov chain with transition matrix } P, \\
    i_n &= i.
\end{align*}
\]

(1.21)

For \(k = 0, 1, \cdots, N - 1\) and \(i \in I\), we use the same function \(c_k(i, u) = K_i \delta(u) + c_i^1 u\) as the regular ordering cost. Let \(h_k(i, x)\) be the surplus (holding) cost in period \(k\) if \(x_k = x\) and \(i_k = i\). This is defined from \(I \times R\) into \(R^+\). We assume \(-h_k(i, x) \in B_2\), \(h_k(i, x)\) is convex and nondecreasing in \(x\), and \(h_k(i, x) = 0, \forall x \leq 0\). Let \(q_k(i, x)\) be the emergency ordering cost in period \(k\) if \(i_k = i\). This is also defined from \(I \times R\) into \(R^+\). We assume \(-q_k(i, x) \in B_2\), \(q_k(i, x)\) is convex and nonincreasing in \(x\), and \(q_k(i, x) = 0, \forall x \geq 0\). Furthermore, we assume \(q_k(i, x)\) is state-independent, i.e., \(q_k(i, x) = q_k(j, x), \forall i, j \in I\). A commonly used emergency ordering cost function is the linear function \(q_k(i, x) = \hat{c}_k^1 x\), where \(\hat{c}_k^1\) is the unit
emergency ordering cost in period \( k \) when \( i_k = i \), and usually \( \hat{c}_k > c_k \). See Chiang and Gutierrez (1998) for a similar state-independent emergency ordering cost function.

With cost functions defined, a similar value function \( v_n^e(i, x) \) over \( < n, N > \) with \( x_n = x \) and \( i_n = i \) satisfies the following dynamic programming equations:

\[
v_n^e(i, x) = -h_n(i, x) + c_n^i x + \sup_{y \geq x, p_n \geq p \geq p_k} [-K_n^i \delta(y - x) + g_n^e(i, y, p)], \quad (1.22)
\]

\[
v_N^e(i, x) = -h_N(i, x), \quad (1.23)
\]

where

\[
g_n^e(i, y, p) = R_n(i, p) - c_n^i y + E[-q_n(i, y - D_n(i, p) - \beta_n^i)] + v_{n+1}^e(i_{n+1}, (y - D_n(i_n, p) - \beta_n^i)^+)|i_n = i]. \quad (1.24)
\]

If \( v_{n+1}^e(i, x) \) is upper semicontinuous in \( x \), \( E[v_{n+1}^e(i_{n+1}, (y - D_n(i_n, p) - \beta_n^i)^+)|i_n = i] \) and \( g_n^e(i, y, p) \) are upper semicontinuous in \((y, p)\). Thus for any \( y \geq x \), there exists \( p_n(i, y) \in [p_n, \bar{p}_n] \), such that

\[
v_n^e(i, x) = -h_n(i, x) + c_n^i x + \sup_{y \geq x} [-K_n^i \delta(y - x) + g_n^e(i, y, p(i, y))], \quad (1.25)
\]

\[
g_n^e(i, y, p(i, y)) = \sup_{p_n \geq p \geq p_n} g_n^e(i, y, p) = R_n(i, p(i, y)) - c_n^i y + E[-q_n(i, y - D_n(i, p(i, y)) - \beta_n^i)] + v_{n+1}^e(i_{n+1}, (y - D_n(i_n, p(i, y)) - \beta_n^i)^+)|i_n = i]. \quad (1.26)
\]

It is easy to check that \( g_n^e(i, y, p_n(i, y)) \) and \( v_n^e(i, x) \) are upper semicontinuous. Thus we will have a similar existence theorem as Theorem 1 in the full backlog case.
Theorem 4. The dynamic programming equations (1.25) and (1.26) define a sequence of functions in $B_2$ in the emergency order case. Moreover, for each $n = 0, 1, \cdots, N - 1$ and $i \in I$, there exists a function $\hat{y}_n(i, x) \in B_0$, such that the supremum in (1.25) is attained at $y = \hat{y}_n(i, x)$ for any $x \in R^+$. Furthermore, we can prove that the Verification Theorem 2 still holds in the emergency order case and the optimality of the $(s, S, p)$ policy is established in the following theorem.

Theorem 5. Assume for each $n = 0, 1, \cdots, N - 1$ and $i \in I$,

\[
q_n^-(i, 0) \leq h_{n+1}^+(i, 0) - c_{n+1}^i, \quad \text{and} \quad c_{n+1}^i x + F_{n+1}(h_{n+1}(i, x) \to +\infty, \text{as } x \to \infty, \tag{1.27}
\]

where $q_n^-(i, 0) = \lim_{x \to 0} \frac{\partial}{\partial x} q_n(i, x)$, and $h_{n+1}^+(i, 0) = \lim_{x \to 0} \frac{\partial}{\partial x} h_{n+1}(i, x)$. Then an $(s, S, p)$ policy is optimal for the emergency order case.

Proof: To prove the optimality of an $(s, S, p)$ policy, since $g_n^e(i, y, p_n(i, y))$ is upper semicontinuous in $y$ and $g_n^e(i, y, p_n(i, y)) \to -\infty$, as $y \to \infty$ (by Assumption (1.28)), we only need to show that $g_n^e(i, y, p_n(i, y))$ is $K_n^e$-concave in $y$, by Proposition 4.2 in Sethi and Cheng (1997). This is done by induction. We assume that $v_{n+1}^e(i, x)$ is $K_{n+1}^e$-concave in $x$ and define:

\[
Q_n(i, z) = q_n(i, z) - v_{n+1}^e(i, z^+), \tag{1.29}
\]

Thus,

\[
e^{-}q_n(i, y - D_n(i, p(i, y)) - \beta_n^i) + v_{n+1}^e(i_{n+1}, (y - D_n(i_n, p(i_n, y)) - \beta_n^{i_n})^+)|i_n = i] \\
= E[-q_n(i_{n+1}, y - D_n(i_{n+1}, p(i_{n+1}, y)) - \beta_n^{i_{n+1}}) + v_{n+1}^e(i_{n+1}, (y - D_n(i_n, p(i_n, y)) - \beta_n^{i_n})^+)|i_n = i] \\
= F_{n+1}(-Q_n)(i, y - D_n(i, p(i, y))), \quad \text{and}
\]

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\[ g_n^e(i, y, p(i, y)) = R_n(i, p_n(i, y)) - c_n y + F_{n+1}(-Q_n)(i, y - D_n(i, p(i, y))), \tag{1.30} \]

where \( F_{n+1} \) is defined as (1.8). Notice (1.30) is of the exact same form as (1.9) except with \( v_{n+1} \) replaced by \(-Q_n\). It is easy to check that \( y - D_n(i, p_n(i, y)) \) is also nondecreasing in \( y \) in the emergency order case. Thus if we could prove 
\(-Q_n(i, z)\) is \( K_{n+1}^i \)-concave in \( z \), then following the same argument as in the proof of Theorem 3, we could conclude that \( g_n^e(i, y, p_n(i, y)) \) is \( K_n^i \)-concave in \( y \).

To prove \( Q_n(i, z) \) is \( K_{n+1}^i \)-convex in \( z \), since \( q_n(i, z) \) is convex and nonincreasing with \( q_n(i, x) = 0, \forall x \geq 0, \) and \( -v_{n+1}^e(i, z) \) is \( K_{n+1}^i \)-convex in \( z \), by Proposition 3.1 in Cheng and Sethi (1999), it is sufficient to verify that \( q_n'(i, 0) \leq -v_{n+1}^e(i, 0) \).

From the \( K_{n+1}^i \)-concavity of \( v_{n+1}^e(i, z) \) and (1.25), we know there exist \( s_{n+1}^i \) and \( S_{n+1}^i \), with \( 0 \leq s_{n+1}^i \leq S_{n+1}^i \), such that

\[ v_{n+1}^e(i, x) = -h_{n+1}(i, x) + c_{n+1} x + \begin{cases} 
-K_{n+1}^i \delta(y - x) + g_{n+1}^e(i, S_{n+1}^i, p_{n+1}(i, S_{n+1}^i)) & \text{if } x < s_{n+1}^i, \\
g_{n+1}^e(i, x, p_{n+1}(i, x)) & \text{if } x \geq s_{n+1}^i. 
\end{cases} \tag{1.31} \]

Thus we have 
\(-v_{n+1}^e(i, 0) = h_{n+1}(i, 0) - c_{n+1} \). By Assumption (1.27), we have proven \( q_n'(i, 0) \leq -v_{n+1}^e(i, 0) \). This completes the proof. \( \square \)

Assumption (1.27) means that the marginal emergency ordering cost in one period is larger than or equal to the regular unit ordering cost less the marginal inventory holding cost in any state of the next period. See Remark 3.2 in Cheng and Sethi (1999) for a discussion. To better understand this assumption, we consider a special case when all cost functions are state and time independent, 
\( q_n(i, x) \equiv \tilde{c} x, \ h_n(i, x) \equiv h x + c^i_n \equiv c, \) where \( \tilde{c}, h, c > 0 \) are the unit emergency ordering, holding and regular ordering costs. Therefore Assumption (1.27) implies that \( \tilde{c} \geq c - h \), which is always true in practice since \( \tilde{c} \geq c \).

Our basic model can also be extended to the case with capacity and service level constraints. See Yin and Rajaram (2005) for details.
1.6 Computation of Optimal \((s, S, p)\) Policies

In this section, we discuss the computation of the state-dependent optimal \((s, S, p)\) policy for our basic model and present illustrative numerical examples.

Suppose for each \(n, 0 \leq n \leq N - 1\), and any state \(i \in I\), \((s^i_n, S^i_n, p^i_n)\) are the parameters for the optimal policy. To guarantee global convergence of a computational algorithm, we need to find bounds for these parameters of the optimal policy. Following the similar method of Chen and Simchi-Levi (2004b), we can develop state-dependent bounds for the reorder point \(s\) and order-up-to level \(S\). See Yin and Rajaram (2005) for details.

We assume that all the input parameters (demand processes, costs and revenue functions) are state dependent, but time independent. Thus we can omit the subindex of period \(n\) from these parameters. Furthermore, we assume that the unit ordering cost is also state independent, which implies that \(c^i_n \equiv c\), for all \(n, 0 \leq n \leq N - 1\), and \(i \in I\). Also, time independence of \(K^i_n\) and Assumption 3 imply that \(K^i \geq \sum_{j=1}^{L} p_{ij} K^j, \forall i \in I\).

We assume the riskless demand in state \(i\) is a linear function of price \(p\) :
\[
D(i, p) = a^i - b^i p,
\]
where \(a^i\) represents the market size and \(b^i\) represents customer price sensitivity. When demand is in state \(i\), we assume the error term \(\beta^i\) follows a discrete uniform distribution on interval \([-\lambda^i, \lambda^i]\), where \(\lambda^i\) is a non-negative integer. Specifically, when demand is in state \(i\), \(\beta^i\) has the following probability mass function:

\[
P(\beta^i = k) = \begin{cases} 
\frac{1}{2\lambda^i + 1}, & \text{for } k \text{ integer}, \quad -\lambda^i \leq k \leq \lambda^i, \\
0, & \text{otherwise.}
\end{cases}
\] (1.32)

We also assume the inventory holding/penalty cost function takes the follow-
ing linear form in state $i$:

$$f(i, x) = h^i \max(0, x) + q^i \max(0, -x),$$

where $h^i, q^i > 0$ are the state-dependent unit inventory holding and penalty costs, respectively.

In the following examples, the entire horizon length is $N = 24$ months and there are 3 demand states, i.e., $L = 3$. We assume that all parameters take integer values and are time independent. The parameter specifications are:

$$(a^1, a^2, a^3) = (60, 30, 75), \quad (b^1, b^2, b^3) = (2, 1, 3), \quad c = 6, \quad K^i \equiv K = 100;$$

$$(h^1, h^2, h^3) = (5, 1, 4), \quad (q^1, q^2, q^3) = (12, 8, 10), \quad (\lambda^1, \lambda^2, \lambda^3) = (4, 2, 4).$$

The demand state transition matrix can take the following forms:

$$P_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix} \quad \text{and} \quad P_3 = \begin{pmatrix} 0 & 0.6 & 0.4 \\ 0.7 & 0.1 & 0.2 \\ 0.3 & 0.3 & 0.4 \end{pmatrix}.$$

Notice $P_1$ represents a special case of the Markovian demand, which is called cyclic or seasonal demand.

Assuming the initial demand state $i_0 = 1$, we develop an algorithm to calculate the optimal $(s^i_n, S^i_n, p^i_n)$ policy for the above example using data in (1.33) with the cyclic demand corresponding to transition matrix $P_1$. As an approximation for the continuous range of the prices, we assume that price can only take discrete integer values with lower bound $\underline{p} = 4$ and upper bound $\bar{p} = 23$. The details of the algorithm can be found in Yin and Rajaram (2005).

We report the optimal $(s^i_n, S^i_n)$ values in Table 2.1 for each pair of $(n, i)$, $0 \leq n \leq 23$, $1 \leq i \leq 3$. An interesting observation is that under the assumption of
time independent parameters, as time horizon $N$ becomes larger, the model tends to have stationary $(s^i, S^i)$ policy for each state $i$ and the optimal replenishment policy is cyclic. This is a similar result with Corollary 7.1 in Sethi and Cheng (1999), which states that a cyclic optimal policy exists for a cyclic demand model with an infinite horizon, when there are no pricing decisions.

To illustrate the change in optimal price across periods, we take 20 sample paths of demand realization over 24 periods and calculate the percentage change for optimal prices from period $t - 1$ to $t$, where $t = 1, 2, \ldots, 23$ for each sample path. Then we plot the average percentage change over these 20 sample paths of demand realization for period $t$, where $t = 1, 2, \ldots, 23$. Figure 1.1 shows that there are significant changes of the optimal prices across periods under the dynamic pricing, ranging from -20% to 30%. However, it is difficult to characterize the direction and magnitude of this change as it depends on the specific demand realization in the previous period and consequently, the starting inventory level at the beginning of each period.

![Figure 1.1: Average Percentage Change of Optimal Prices Across Periods.](image-url)
<table>
<thead>
<tr>
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<td>23</td>
<td>(-15, 23)</td>
<td>(-90, 11)</td>
<td>(-27, 25)</td>
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</tbody>
</table>

Table 1.1: Optimal \((s^i_n, S^i_n)\) values for \(N = 24, \ L = 3\).
Next, we consider the impact of demand variability on the benefit of dynamic pricing over fixed pricing. Notice that in the above examples, we assume that the demand distribution is determined by a cyclic transition matrix $P_1$ defined in (1.34), which is a special class of Markovian demand. As long as the starting state is known, we immediately know the demand states in all periods over the entire time horizon. Since we allow for any arbitrary transition matrix in our model, we calculate the relative profit gains of dynamic pricing over fixed pricing when the transition matrix takes the forms of $P_2$ and $P_3$ defined in (1.34). Here, $P_2$ denotes the class of Markovian demand where demand will move to the other two states in the next period with equal probabilities and $P_3$ is an arbitrary transition matrix. Notice these three transition matrices have the same stationary probabilities (1/3 in each demand state); $P_3$ represents the highest demand variability among the three, while the demand variability in class $P_2$ is higher than the class of cyclic demand. Observe from Table 1.2 that the profits due to dynamic pricing is greater than fixed pricing by an average of 10.65% ranging from 9.18% to 11.51%. Table 1.2 also shows that when the demand variability increases from $P_1$ to $P_3$, as expected, profits will decrease under dynamic pricing and under fixed pricing. However, the rate of decrease is smaller for the case with dynamic pricing, thus enhancing the gains over fixed pricing.

<table>
<thead>
<tr>
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<th>Dynamic Pricing</th>
<th>Fixed Pricing</th>
<th>Relative Profit Gain</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case 1</td>
<td>2729.78</td>
<td>2500.15</td>
<td>9.18%</td>
</tr>
<tr>
<td>Case 2</td>
<td>2551.22</td>
<td>2293.24</td>
<td>11.25%</td>
</tr>
<tr>
<td>Case 3</td>
<td>2548.77</td>
<td>2285.66</td>
<td>11.51%</td>
</tr>
</tbody>
</table>

Table 1.2: Demand variability impact on the benefit of dynamic pricing over fixed pricing.
Finally, we consider the effect of the fixed ordering cost $K$ on the benefit of dynamic pricing over fixed pricing across two different environments: Markovian demand and independent demand. The fixed price is chosen as the best price in $[p, \bar{p}]$, at which the retailer’s optimal expected profit in the entire horizon is maximized. The independent demand $i$ corresponds to the case when demand always stays in state $i$, for $i = 1, 2$ and $3$. In Figure 1.2, We plot the relative profit gains of dynamic pricing over fixed pricing when $K$ changes from 0 to 300, in a Markovian demand case and three cases of independent demand. There are two observations. First, for any value of fixed ordering cost $K$, the benefit of dynamic pricing over fixed pricing is always higher in the Markovian demand setting than the three independent demand settings. This confirms the intuition that it is more effective to use price to hedge against demand uncertainty in a fluctuating demand environment. Second, the benefit of dynamic pricing over fixed pricing increases for large values of $K$. This is true for both the Markovian demand and the independent demand cases. For instance, for $K \geq 40$, the profit gains in the Markovian demand case increase from 1.11% to 16.22% with an average of 8.05%; while profit gains in independent demand case 3 increase from 0 to 11.8% with an average of 4.63%. In particular, for a high value of the fixed ordering cost, for example, $K = 300$, the relative profit gain of dynamic pricing can be very significant as over 16%. This result is interesting. While a higher fixed ordering cost turns to have negative effects on the retailer’s expected profit, dynamically adjusting the prices provides the retailer an opportunity to manipulate the demand in a fluctuating demand environment. This overcomes the negative effects of this high ordering cost.
Figure 1.2: Impacts of fixed ordering cost $K$ on the benefit of dynamic pricing over fixed pricing in a Markovian Demand Case and 3 Cases of Independent Demand.
In summary, assuming that there is no transactional costs associated with price change, we can make the following conclusions from our numerical study. First, the profits resulting from dynamic pricing is greater than the profits resulting from fixed pricing in any demand environment. Second, we can also conclude that the profit gain of dynamic pricing over fixed pricing is higher with higher demand variability or with a higher fixed ordering cost.

1.7 Conclusions

In this chapter, we extend the results in Chen and Simchi-Levi (2004a) to a Markovian demand model where the demand distribution at every period is determined by an exogenous Markov chain. We show that for additive demand functions, under the assumptions of backlogging and state-dependent cost functions, there exists an optimal Markov policy of the \((s, S, p)\)-type which is also state dependent. We then extend the basic model to the case of emergency orders. We develop an algorithm to compute the optimal policy for a class of Markovian demand model with an arbitrary state transition matrix and a random noise term that follows a discrete uniform distribution. We consider the effects of fixed ordering cost and demand variability on the benefits of dynamic pricing over fixed pricing through extensive numerical examples. The results show that it is more beneficial to implement dynamic pricing in a Markovian demand environment with a high fixed ordering cost or with high demand variability.

Our work assumes that inventory replenishment and pricing decisions are made simultaneously at the beginning of each period, before demand is realized. In practice, firms may adjust prices after they observe some demand information, which is referred to as “responsive pricing” in Chod and Rudi (2005). In their paper, they consider a single-period, two-product, and two-stage optimization
problem, where the demand functions for both products take linear forms and the intercept terms are the only random variables following a bivariate normal distribution. Firms make the capacity decision at the first stage and the pricing decision at the second stage after the demand intercept terms materialize for both products. It might be interesting to extend our multi-period model to allow for responsive pricing. Another area for future work arises when we relax the assumption of perfect supply. This could occur when suppliers may not be able to deliver orders in full due to emergency factors, like power shutdowns and natural disasters. This introduces the concept of “Markovian supply” into our model. Parlar, Wang and Gerchak (1995) consider a periodic review inventory model with Markovian supply availability. They show that a state-dependent \((s, S)\) policy is optimal for an independent demand model under the assumption that the supplier can either deliver the orders in full amount or nothing. Extending their model to the case of Markovian demand with pricing could be another direction for future research.

1.8 References


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Song, J.-S. and P. Zipkin., “Inventory Control in a Fluctuating Demand Envi-


CHAPTER 2

Joint Ordering and Pricing Strategies for Managing Substitutable Products

2.1 Introduction

To compete for market share, many manufacturers increase product variety as a way to satisfy the need of different market segments or as a mechanism to entertain the variety seeking consumer behavior. Over the last 10 years, many manufacturers expanded their product lines by differentiating a basic product across a key attribute such as flavor or color. For instance, Ho and Tang (1998) report that most consumer package goods companies increase their stock-keeping units by more than 10% each year. While product variety is welcomed by most consumers, it can create many challenges for the retailers. First, different variants (different colors or flavors) of the same product are mutually substitutable to a certain extent. In this case, how should a retailer determine the assortment of products so that the profit is maximized? Agrawal and Smith (2003) develop a model for determining optimal assortments of substitutable products. Second, given a set of variants, how should a retailer decide on the order quantity of each variant so that the profit is maximized? Third, when different variants are substitutable, the underlying demand of different variants can be manipulated via differential pricing. Therefore, instead of charging the same price for all variants,
the retailer may charge different prices to manipulate the product demand so that the retailer’s profit is maximized. For example, when shopping for a particular dishwasher at three different retailers in Los Angeles, we found two retailers charged a higher price for the dishwasher that is painted in black instead of ivory. As the demand for each variant depends on relative prices, how should a retailer set the retail price for each variant so that the profit is maximized?

In this chapter, we present a multi-product model in which a retailer has to decide jointly on the order quantity and retail price for each variant. For ease of exposition, we shall use the term ‘product’ and the term ‘variant’ interchangeably throughout this chapter. As the products are mutually substitutable, we examine the case in which the retailer is considering two different pricing strategies: fixed and variable. Under the fixed pricing strategy, the retailer charges the same retail price for all products. On the contrary, the retailer charges different retail prices for different products under the variable pricing strategy. Our multi-product model with mutually substitutable products is intended to examine the following questions:

1. How should the retailer determine the optimal retail price and order quantity for each product under the fixed and variable pricing strategies?

2. How would the optimal retail price and order quantity change as the retailer changes from adopting the fixed pricing strategy to the variable pricing strategy?

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1Sogomonian and Tang (1993) present a single-product model for determining the order quantity and retail price jointly. For more details about single-product models that deal with joint ordering and pricing issues, the reader is referred to Eliashberg and Steinberg (1993) for a comprehensive review. In contrast, ours is a multi-product model with mutually substitutable products, and we analyze the issue of retail competition and the issue of capacity constraint.
3. What is the impact of product substitutability on the optimal retail price and order quantity under the fixed and variable pricing strategies?

4. Which pricing strategy should a retailer adopt when facing competition?

5. How should the retailer determine the optimal retail price and order quantity under different pricing strategies when there is a limit on the total order quantity?

In order to address the above questions and to gain some insights into pricing and ordering policies for substitutable products, we shall develop a model based on the following scenario. Consider a retailer that sells two substitutable products over a selling season. Prior to the selling season, the retailer utilizes market information to generate a demand function for each product. This demand function represents the estimated total number of units to be sold during the selling season. Moreover, we assume that the demand function of a product depends on the retail price of the product and the retail price of the other product. Based on the deterministic demand function of each product, the retailer needs to determine the order quantity and the (regular) retail price of each variant so that the retailer’s profit is maximized. We assume that the replenishment lead time is longer than the selling season so that no replenishment orders are allowed. However, the retailer may be able to adjust the retail price dynamically after

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2 For ease of exposition, we shall restrict our attention to the two-product case. However, the same approach presented in this chapter can be used to analyze the case when there are more than two products.

3 In retailing, (regular) retail price is referred to the actual retail price to be charged at the beginning of the selling season.
observing the actual sales volume associated with the (regular) retail price at the beginning of the selling season.\footnote{In the event of price markdown, most retailers usually discount the (regular) retail price. The reader is referred to Elmaghraby and Keskinocak (2003) for a comprehensive review of dynamic pricing literature.} In order to study the implications of the fixed and variable pricing strategies, we shall develop a deterministic model to examine two issues (i.e., order quantity and retail price) that need to be addressed prior to the selling season. However, our model does not deal with the dynamic pricing issue that may occur during the selling season.

This chapter is organized as follows. Section 2.2 provides a review of relevant literature. In Section 2.3, we first present the base model and then develop the characteristics of the optimal retail price and order quantity under the fixed and variable pricing strategies. Our analysis indicates that the optimal retail price (optimal order quantity) under the variable pricing strategy is equal to the optimal retail price (optimal order quantity) under the fixed pricing strategy plus or minus an adjustment term. This adjustment term depends on product substitutability and price sensitivity. In Section 2.4 we extend our base model to the case when there is a limit on the total order quantity. We show that the optimal retail price and order quantity possess a similar structure as the base case. Section 2.5 extends our base model to address the issue of retail competition. Specifically, our result suggests that the underlying structure of the optimal retail price and order quantity for the monopolistic case is preserved in a duopolistic environment. Moreover, we show that both retailers would adopt the variable pricing strategy at the equilibrium. Section 2.6 concludes this chapter.
2.2 Literature Review

There is a significant body of work in the operations management literature that addresses three types of product substitution: stock-out based substitution, assortment-based substitution, and price-based substitution. First, stock-out based substitution corresponds to a situation in which a customer may purchase another product as a substitute when the preferred product is out of stock. Most of the stock-out based substitution models focused on ways to determine the optimal order quantity of each product. In the context of single period model, there are two streams of literature related to our work. The first stream of work consists of single-period models. For example, Bassok, Anupindi, and Akella (1999) develop a model that deals with multiple classes of products with full downward substitution. They show that the greedy allocation policy is optimal when the demands of all classes of products are realized at the beginning of the period. However, as customers arrive randomly within the time period, stockouts and product substitution can take place at any point in time. As such, the exact analysis is extremely complex. By considering an approximate scheme, Mahajan and van Ryzin (2001) determine near-optimal order quantities by developing an iterative stochastic optimization method. In the same vein, Smith and Aggrawal (2000) develop a probabilistic demand model for capturing the effect of product substitution. They construct a methodology for determining product assortment and its associated order quantities so as to maximize the expected profit. In the second stream of work, various researchers consider the case in which the demand occurs at the end of the period so that they can avoid dealing with the issue of dynamic stockouts and product substitution within the time period. Even with this simplification, several researchers encountered technical difficulties when the underlying product demand is stochastic. McGillivray and Silver (1978) consider
the case in which the substitutable products have identical cost and each product has a fixed substitution probability. When there is a cost for substituting one product by other products, they acknowledge that there is no known tractable analysis, and they develop simulation and heuristics for evaluating the optimal order quantity. Parlar and Goyal (1984) consider the same problem and show that the total expected profit is concave for a wide range of problems. Moreover, Parlar (1988) proves the existence of Nash equilibrium for the case of two substitutable products with stochastic demands. When the products are fully substitutable (i.e., the customer will always accept a substitutable product when the target product is not available), Pasternack and Drezner (1991) show numerically that the associated optimal order quantities can be larger or smaller when compared to the case when the products are not substitutable. More recently, Netessine and Rudi (2003) determine the optimal order quantities for substitutable products under retail competition. They develop first-order conditions that can be used to estimate the optimal order quantities. Moreover, their analysis shows that the optimal order quantities are higher under retail competition. Rajaram and Tang (2001) examine the impact of product substitution on the optimal order quantity of each variant when demand is uncertain. Since the exact analysis is intractable, they develop an approximate heuristic based on the traditional newsvendor model that is intended to analyze how the level of demand uncertainty and the degree of substitutability affect the optimal order quantity of each product.

Second, assortment-based substitution occurs when products with similar attributes are substitutable. For example, different flavors of toothpaste are substitutable to a certain extent. The issue of assortment-based substitution usually arises in the context of assortment planning. By assuming that the retail prices are given exogenously, Cachon et al. (2005), Gaur and Honhon (2005), Kok and
Fisher (2004), and Van Ryzin and Mahajan (1999) develop different assortment planning models by capturing the assortment-based substitution effect.

Third, price-based substitution corresponds to a situation in which a retailer uses differential pricing to make certain products substitutable. For example, a retailer may sell those one-day old doughnuts at half price so that the old doughnuts are substitutes for the fresh doughnuts. By utilizing the Multinomial Logit consumer choice model, Chong et al. (2001) develop an empirical model and Hopp and Xu (2005) present an analytical model by considering both assortment-based and price-based substitution effects.

To obtain tractable analytical results, we take a different approach and a different perspective in this chapter. Our approach is based on the assumption that the underlying product demand is deterministic. This assumption is reasonable for the case when the retailer uses the forecasted demand function to determine the order quantity and the (regular) retail price prior to the beginning of the selling season. When the underlying demand function is deterministic, we are able to develop closed form expressions for the optimal retail price and order quantity. These closed form expressions enable us to extend our base model to address the issue of retail competition and the issue of capacity constraint. Our work differs from Hopp and Xu (2005) in the following ways. First, our model is based on a fixed set of products with deterministic demand, while Hopp and Xu’s model examines the issue of product assortments with stochastic demand. Second, our model is based on a linear demand function, while Hopp and Xu focus on the Multinomial Logit consumer choice model. Third, our model examines the issues of capacity constraints and retail competition while Hopp and Xu’s model does not address these issues. In the context of retail competition and capacity constraint, Goyal and Netessine (2004) develop a model to analyze the impact
of competition on a firm’s technology choice and capacity investment decisions. Our model differs from Goyal and Netessines model in the following manner. Specifically, our model is intended to characterize the underlying structure of the optimal retail price and order quantity for the case when a retailer can charge different retail prices so as to maximize his profit by manipulating the demand of mutually substitutable products.

2.3 The Base Model

Consider a retailer that sells two mutually substitutable products, namely, products 1 and 2. We assume that the demand function is known prior to the selling season so that the retailer can utilize this demand function to determine the order quantity $Q_w$ and the retail price $p_w$ for product $w$, where $w = \{1, 2\}$. Specifically, we assume that the retail demand functions for products 1 and 2, denoted by $D_1$ and $D_2$, possess the following structure:

$$D_1 = a_1 - b_1p_1 + \delta_1(p_2 - p_1), \quad \text{and} \quad D_2 = a_2 - b_2p_2 + \delta_2(p_1 - p_2). \quad (2.1)$$

To ensure that the total demand $D_1 + D_2$ is decreasing in $p_1$ and $p_2$, we shall assume that: $b_1 + \delta_1 - \delta_2 > 0$ and $b_2 + \delta_2 - \delta_1 > 0$. The above functional form has been considered in the marketing literature. For example, Raju et al. (1995) utilized the above demand function to evaluate the impact of store brand introduction on the sales of national brands, while Tang et al. (2004) utilized the above demand function to evaluate the cost and benefit of the advance booking discount program.

Notice from the demand functions given in (2.1) that $a_w > 0$ represents the primary demand and $b_w > 0$ represents the price sensitivity of product $w$, $w = \{1, 2\}$. For any given values of $a_w$ and $b_w$, we can always index the products
so that $\frac{a_1}{b_1} \geq \frac{a_2}{b_2}$. Without loss of generality, we shall assume that $\frac{a_1}{b_1} \geq \frac{a_2}{b_2}$ throughout this chapter. Observe that $\delta_w \geq 0$ captures the notion of product substitutability.\footnote{To ensure that $D_1, D_2 \geq 0$, $\delta_w$ is bounded from above.} For instance, if $\delta_w = 0$ for $w = 1, 2$, then these two products are not substitutable in the sense that customers will not switch to a different product regardless of the relative price $(p_1 - p_2)$. However, as $\delta_w$ increases, these two products would become more substitutable in the sense that the demand of each product would become more sensitive toward the relative price $(p_1 - p_2)$.

In the base case, we assume that there is no limit on the order quantity. As such, it is always optimal for the retailer to order according to the retail demand; i.e., $Q_w = D_w$ for $w = 1, 2$. However, in the next section, we shall consider the case when there is a limit on the total order quantity so that $Q_w \leq D_w$. When $Q_w = D_w$ for $w = 1, 2$, it suffices for the retailer to determine the retail price $p_w$ so as to maximize his net profit. In this case, the retailer’s optimal profit can be expressed as $\pi^*$, where:

$$\pi^* = \max_{p_1, p_2 \geq 0} (p_1 - c)D_1 + (p_2 - c)D_2. \quad (2.2)$$

Notice that we assume the unit cost $c$ is the same for both products. This assumption is reasonable especially because both products are two variants of the same basic product.

### 2.3.1 Fixed Retail Pricing Strategy

Let us consider the case in which the retailer adopts the fixed pricing strategy so that $p_1 = p_2 = p$. In this case, it is easy to check from (2.1) and (2.2) that the retailer’s problem can be written as:

$$\pi^*_f = \max_{p \geq 0} (p - c)(a_1 + a_2 - b_1p - b_2p). \quad (2.3)$$
By using simple calculus, we can determine the optimal price and optimal order quantity under the fixed pricing strategy. For ease of reference, we shall highlight the results in the following Proposition. (All proofs are given in the Appendix.)

**Proposition 1.** Under the fixed pricing strategy, the optimal fixed price, denoted by \( p^*_f \), is given by:

\[
p^*_f = \frac{a_1 + a_2}{2(b_1 + b_2)} + \frac{c}{2},
\]

(2.4)

The optimal order quantity for product \( w \), denoted by \( Q^*_w \), can be expressed as:

\[
Q^*_1 = \frac{a_1 - b_1 c}{2} + \frac{a_1 b_2 - a_2 b_1}{2(b_1 + b_2)}, \quad \text{and}
\]

\[
Q^*_2 = \frac{a_2 - b_2 c}{2} - \frac{a_1 b_2 - a_2 b_1}{2(b_1 + b_2)}.
\]

(2.5)

Finally, the retailer’s optimal profit, denoted by \( \pi^*_f \), can be expressed as:

\[
\pi^*_f = \frac{(a_1 + a_2 - b_1 c - b_2 c)^2}{4(b_1 + b_2)}.
\]

(2.6)

Notice from Proposition 1 that the optimal fixed price \( p^*_f \) and the optimal profit \( \pi^*_f \) are increasing in the sum of the primary demands \((a_1 + a_2)\). Next, before we interpret the optimal order quantity \( Q^*_w \), let us consider the case in which the retailer sells a single product \( w \) with demand \( D_w = a_w - b_w p \). By considering the objective function \((p - c)(a_w - b_w p)\), it is easy to check that the optimal order quantity for this single product \( w \) is equal to \( \frac{a_w - b_w c}{2} \). Suppose we view this single-product order quantity \( \frac{a_w - b_w c}{2} \) as the base level. Then the optimal order quantity \( Q^*_w \) for product \( w \) given in (2.5) is equal to this base level \( \frac{a_w - b_w c}{2} \) plus or minus an adjustment term \( \frac{a_1 b_2 - a_2 b_1}{2(b_1 + b_2)} \) that depends on the difference in the demand characteristics of products 1 and 2. Also, since \( \frac{a_1}{b_1} \geq \frac{a_2}{b_2} \), this adjustment term \( \frac{a_1 b_2 - a_2 b_1}{2(b_1 + b_2)} \geq 0 \).
2.3.2 Variable Retail Pricing Strategy

Suppose the retailer adopts the variable retail pricing strategy. Then it is easy to check from (2.1) and (2.2) that the retailer’s problem can be written as:

\[ \pi^*_{v} = \max_{p_1, p_2 \geq 0} (p_1 - c)(a_1 - b_1p_1 + \delta_1(p_2 - p_1)) + (p_2 - c)(a_2 - b_2p_2 + \delta_2(p_1 - p_2)). \] (2.7)

By differentiating the objective function with respect to \( p_1 \) and \( p_2 \) and by considering the first order conditions, we can establish the following Proposition:

**Proposition 2.** Under the variable pricing strategy, the optimal retail price for product \( w \), denoted by \( p^*_{wv} \), is given by:

\[
p^*_{1v} = \frac{2(b_2 + \delta_2)(a_1 - b_1c) + (\delta_1 + \delta_2)(a_2 - b_2c)}{4b_1b_2 + 4b_1\delta_2 + 4b_2\delta_1 - \delta_1^2 - \delta_2^2 + 2\delta_1\delta_2} + c
\]
\[= p^*_f + (2b_2 - (\delta_1 - \delta_2))A, \text{ and} \]
\[
p^*_{2v} = \frac{2(b_1 + \delta_1)(a_2 - b_2c) + (\delta_1 + \delta_2)(a_1 - b_1c)}{4b_1b_2 + 4b_1\delta_2 + 4b_2\delta_1 - \delta_1^2 - \delta_2^2 + 2\delta_1\delta_2} + c
\]
\[= p^*_f - (2b_1 + (\delta_1 - \delta_2))A. \] (2.8)

where
\[
A = \frac{2(a_1b_2 - a_2b_1) - (\delta_1 - \delta_2)(a_1 + a_2 - b_1c - b_2c)}{2(4b_1b_2 + 4b_1\delta_2 + 4b_2\delta_1 - \delta_1^2 - \delta_2^2 + 2\delta_1\delta_2)(b_1 + b_2)}. \] (2.9)

The optimal order quantity for product \( w \), denoted by \( Q^*_{wv} \), can be expressed as:

\[
Q^*_1v = Q^*_1f - (b_1\delta_2 + b_1\delta_1 + 2b_2\delta_1 + 2b_1b_2)A, \quad \text{and} \quad Q^*_2v = Q^*_2f + (b_2\delta_1 + b_2\delta_2 + 2b_1\delta_2 + 2b_1b_2)A. \] (2.10)

Finally, the retailer’s optimal profit, denoted by \( \pi^*_v \), can be expressed as:

\[
\pi^*_v = \pi^*_f + (4b_1b_2 + 4b_1\delta_2 + 4b_2\delta_1 - \delta_1^2 - \delta_2^2 + 2\delta_1\delta_2)(b_1 + b_2)A^2. \] (2.11)

By comparing the results stated in Proposition 2 and Proposition 1, we can check that the retailer will always obtain a higher profit under the variable pricing
strategy, which is guaranteed by the assumptions $b_1 + \delta_1 - \delta_2 > 0$ and $b_2 + \delta_2 - \delta_1 > 0$. This result is expected because the retailer has more price flexibility under the variable pricing strategy. However, it is interesting to observe from (2.8) and (2.10) that the optimal retail price and the optimal order quantity of each product under the variable pricing strategy are equal to the optimal retail price and the optimal order quantities under the fixed pricing strategy plus or minus an adjustment term, respectively. It is interesting to note that Petruzzi and Dada (1999) have obtained a similar structural result when analyzing joint pricing and ordering decisions for a newsvendor model with linear demand. Specifically, they show that the optimal price under stochastic demand is equal to the optimal price under deterministic demand plus a constant.

It follows from (2.8) and (2.9) that the impact of product substitutability $\delta_1$ and $\delta_2$ on the optimal retail price depends on the values of $a_1, a_2, b_1, b_2,$ and $c$. To obtain a general idea about the impact of product substitutability on the optimal retail price, we construct a numerical example by setting $a_1 = 200$, $a_2 = 120$, $b_1 = 10$, $b_2 = 8$, $c = 4$, and $\delta_2 = 2$. Figure 2.1 reports the values of $p_{1v}^*$ and $p_{2v}^*$ when we vary the value of $\delta_1$ from 0 to 4. As shown in Figure 2.1, it is easy to check that the gap between $p_{1v}^*$ and $p_{2v}^*$ (i.e., $(p_{1v}^* - p_{2v}^*)$) is decreasing in $\delta_1$. Since $\delta_2 = 2$, we can also conclude that $(p_{1v}^* - p_{2v}^*)$ is decreasing in $(\delta_1 - \delta_2)$. We observe similar patterns for different sets of values of $a_1, a_2, b_1, b_2,$ and $c$. 

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To examine the impact of product substitutability on the optimal retail price and optimal order quantity analytically, let us consider the case when the products are equally substitutable; i.e., when $\delta_1 = \delta_2 = \delta$. In this case, the term $A$ given in (2.9) can be simplified as:

$$A = \frac{a_1 b_2 - a_2 b_1}{4(b_1 b_2 + b_1 \delta + b_2 \delta)(b_1 + b_2)}.$$ 

Since $\frac{a_1}{b_1} \geq \frac{a_2}{b_2}$, $A \geq 0$. Thus, Proposition 2 can be simplified as follows:

**Corollary 1.** Suppose $\delta_1 = \delta_2 = \delta$. Then the optimal prices, the optimal order quantities, and the optimal profit under the variable pricing strategy, can be
expressed as:

\[ p_{1v}^* = p_f^* + \frac{(a_1b_2 - a_2b_1)b_2}{2(b_1 + b_2)(b_1\delta + b_2\delta + b_1b_2)}. \]

\[ p_{2v}^* = p_f^* - \frac{(a_1b_2 - a_2b_1)b_1}{2(b_1 + b_2)(b_1\delta + b_2\delta + b_1b_2)}. \]  \hspace{1cm} (2.12)

\[ Q_{1v}^* = \frac{a_1 - b_1c}{2} = Q_{1f}^* - \frac{a_1b_2 - a_2b_1}{2(b_1 + b_2)}, \]

\[ Q_{2v}^* = \frac{a_2 - b_2c}{2} = Q_{2f}^* + \frac{a_1b_2 - a_2b_1}{2(b_1 + b_2)}, \]  \hspace{1cm} and

\[ \pi_v^* = \pi_f^* + \frac{(a_1b_2 - a_2b_1)^2}{4(b_1 + b_2)(b_1\delta + b_2\delta + b_1b_2)}. \]  \hspace{1cm} (2.13)

Corollary 1 has the following implications. Let us first examine the optimal retail price \( p_{1v}^* \) and \( p_{2v}^* \) given in (2.12). As \( \frac{a_1}{b_1} \geq \frac{a_2}{b_2} \), \( p_{1v}^* > p_{2v}^* \). Also, notice that the optimal variable price for product 1, \( p_{1v}^* \), is higher than the optimal fixed price \( p_f^* \) by an amount that is equal to \( \frac{(a_1b_2 - a_2b_1)b_2}{2(b_1 + b_2)(b_1\delta + b_2\delta + b_1b_2)} \). The opposite can be said about the optimal variable price for product 2. Next, since \( p_f^* \) is independent of \( \delta \), one can check from (2.12) that the optimal price \( p_{1v}^* \) is convex and decreasing in \( \delta \), while the optimal variable price \( p_{2v}^* \) is concave and increasing in \( \delta \). Moreover, notice that:

\[ p_{1v}^* + p_{2v}^* = 2p_f^* + \frac{(a_1b_2 - a_2b_1)(b_2 - b_1)}{2(b_1 + b_2)(b_1\delta + b_2\delta + b_1b_2)}. \]

This implies that the sum of the optimal retail prices under the variable pricing strategy is higher (or lower) than the sum of the optimal retail prices under the fixed pricing strategy when \( b_2 > b_1 \) (or when \( b_2 < b_1 \)).

Next, let us examine the optimal order quantities \( Q_{1v}^* \) and \( Q_{2v}^* \) given in (2.13). Since \( \frac{a_1}{b_1} \geq \frac{a_2}{b_2} \), we have \( Q_{1v}^* < Q_{1f}^* \), \( Q_{2v}^* > Q_{2f}^* \), and \( Q_{2v}^* - Q_{1v}^* = \frac{a_1b_2 - a_2b_1}{b_1 + b_2} \geq 0 \). Also, notice from (2.13) and (2.5) that the total optimal order quantity under the variable pricing strategy is equal to the total optimal order quantity under the fixed pricing strategy; i.e., \( Q_{1v}^* + Q_{2v}^* = Q_{1f}^* + Q_{2f}^* \). Next, by comparing
(2.12) with (2.4) and by comparing (2.13) with (2.5), we can draw the following conclusion. When the retailer switches from the fixed pricing strategy to the variable pricing strategy, it is optimal for him to increase the price for product $1$ (i.e., $p^*_v > p^*_f$), reduce the order quantity for product $1$ (i.e., $Q^*_v < Q^*_f$), reduce the price for product $2$, and increase the order quantity for product $2$. Finally, (2.14) implies that the retailer will always obtain a higher profit under the variable pricing strategy.

We now examine the impact of product substitutability $\delta$ on the optimal retail price, optimal order quantity, and optimal profit when the retailer adopts the variable pricing strategy. To do so, let us make the following observations from Corollary 1. First, observe from (2.13) that the optimal order quantity $Q^*_w$ for product $w$ is independent of the substitutability factor $\delta$. Thus, under the variable pricing strategy, it is optimal for the retailer to order the same quantity for each product regardless of the value of substitutability factor. Second, notice from (2.12) that the optimal price for product $1$ is decreasing in $\delta$ while the optimal price for product $2$ is increasing in $\delta$. These two observations imply that, as the product substitutability $\delta$ changes, it is optimal for the retailer to adjust the retail price instead of changing the order quantity for each product. Furthermore, observe from (2.12) that $p^*_1 - p^*_2 = \frac{a_1 b_2 - a_2 b_1}{2(b_1 + b_2)(b_1 \delta + b_2 \delta + b_1 b_2)} \geq 0$ and that the price difference $p^*_1 - p^*_2$ decreases as product substitutability $\delta$ increases. Finally, by comparing the optimal profit under the variable pricing $\pi^*_v$ in (2.14) and the optimal profit under the fixed pricing strategy $\pi^*_f$ in (2.6), the retailer can obtain an additional profit in the amount of $\frac{(a_1 b_2 - a_2 b_1)^2}{4(b_1 + b_2)(b_1 \delta + b_2 \delta + b_1 b_2)}$ under the variable pricing strategy. This additional profit $\frac{(a_1 b_2 - a_2 b_1)^2}{4(b_1 + b_2)(b_1 \delta + b_2 \delta + b_1 b_2)}$ is convex and decreasing in $\delta$. Thus, as the products become more substitutable (i.e., as $\delta$ increases), the benefit of variable pricing strategy diminishes. The following corollary highlights our key findings.
Corollary 2. When operating under the variable pricing strategy, it is optimal for the retailer to adjust the retail price while keeping the order quantity fixed as the product substitutability changes. Also, as the products become more substitutable, the benefit of variable pricing strategy diminishes.

2.4 Capacity Constraint

In this section, we extend our base model to the case in which the total order quantity \((Q_1 + Q_2)\) is limited by \(K\), where \(K\) can be interpreted as the manufacturer’s production capacity or the manufacturer’s export or import quota for both products.\(^6\) Given the capacity constraint, it may not be feasible for the retailer to order according to the retail demand for each product. As such, the retailer needs to take the capacity constraint \(Q_1 + Q_2 \leq K\) into consideration when deciding on the retail price and order quantity for each product.

2.4.1 Fixed Pricing Strategy with Capacity Constraint

By considering the demand functions \(D_1\) and \(D_2\) given in (2.1) and the fact that \(p_1 = p_2 = p\) under the fixed pricing strategy, we can formulate the retailer’s problem \((P1)\) with capacity constraint as follows:

\[
\pi^K_{f} = \max_{p, Q_1, Q_2 \geq 0} (p - c)(Q_1 + Q_2)
\]

subject to \(Q_1 \leq a_1 - b_1 p\), \(Q_2 \leq a_2 - b_2 p\), and \(Q_1 + Q_2 \leq K\).

\(^6\)It is common that the export or import quota is based on the product category but not on individual products. For example, there may be a quota for woolen sweaters but not woolen sweaters of a specific color.
Suppose we relax the capacity constraint \((2.15)\). Then the retailer’s relaxed problem can be written as:

\[
\pi^K_f(\lambda) = \max_{p,Q_1,Q_2 \geq 0} (p - (c + \lambda))(Q_1 + Q_2) + \lambda K \\
\text{s.t. } Q_1 \leq a_1 - b_1 p, \; Q_2 \leq a_2 - b_2 p,
\]

where \(\lambda \geq 0\) corresponds to the Lagrangian multiplier of constraint \((2.15)\). Suppose that \(p - (c + \lambda)) \geq 0\).\(^7\) Then it is optimal to have both constraints to be binding (i.e., \(Q_1 = a_1 - b_1 p, Q_2 = a_2 - b_2 p\)). As such, the retailer’s relaxed problem can be simplified as:

\[
\pi^K_f(\lambda) = \max_{p \geq 0} (p - \hat{c})(a_1 + a_2 - b_1 p - b_2 p) + \lambda K,
\]

where \(\hat{c} = c + \lambda\). In this case, we can determine the optimal order quantities, denoted by \(Q^K_{1f}\) and \(Q^K_{2f}\), the optimal retail price, denoted by \(p^K_f\), by solving the following dual problem:

\[
\pi^K_f = \min_{\lambda \geq 0} \{\lambda K + \{\max_{p \geq 0} (p - \hat{c})(a_1 + a_2 - b_1 p - b_2 p)\}\}.
\] \(2.16\)

Notice that the inner problem \(\max_{p \geq 0} (p - \hat{c})(a_1 + a_2 - b_1 p - b_2 p)\) is identical to the retailer’s problem in \((2.3)\) when we replace \(c\) with \(\hat{c}\). This observation enables us to first solve the inner problem by using the results in Proposition 1 (namely \((2.4)\), \((2.5)\) and \((2.6)\)), and then solve the outer problem to determine the optimal Lagrangian multiplier \(\lambda^*\). By considering certain boundary conditions, we can prove the following Proposition. In preparation, let:

\[
\bar{K}_f = \frac{a_1 + a_2 - b_1 c - b_2 c}{2}.
\] \(2.17\)

In this case, we have:

\(^7\)We shall prove that this assumption holds when we prove Proposition 3.
Proposition 3. Suppose $\frac{a_1 b_2 - a_2 b_1}{b_2} \leq \bar{K}_f$. Suppose there is a limit $K$ on the total order quantity. Then the optimal price $p^K_f$, the optimal order quantity $(Q^K_{1f}, Q^K_{2f})$, and the optimal profit $\pi^K_f$ under the fixed pricing strategy can be expressed as follows:

1. If $K \geq \bar{K}_f$, then $p^K_f = p^*_f$ as given in (2.4), $Q^K_{w_f} = Q^*_w$ as given in (2.5) for $w = 1, 2$, and $\pi^K_f = \pi^*_f$ as given in (2.6).

2. If $\frac{a_1 b_2 - a_2 b_1}{b_2} \leq K < \bar{K}_f$, then:

$$p^K_f = \frac{a_1 + a_2 - K}{b_1 + b_2},$$

$$Q^K_{1f} = \frac{b_1 K}{b_1 + b_2} + \frac{a_1 b_2 - a_2 b_1}{b_1 + b_2},$$

$$Q^K_{2f} = \frac{b_2 K}{b_1 + b_2} - \frac{a_1 b_2 - a_2 b_1}{b_1 + b_2}, \quad \text{and}$$

$$\pi^K_f = \pi^*_f - \frac{(a_1 + a_2 - b_1 c - b_2 c - 2K)^2}{4(b_1 + b_2)}.$$  

3. If $K < \frac{a_1 b_2 - a_2 b_1}{b_2}$, then $p^K_f = a_2/b_2$, $Q^K_{1f} = K$, $Q^K_{2f} = 0$, and $\pi^K_f = (a_2/b_2 - c)K$.

It is easy to check from the definition of $\bar{K}_f$ given in (2.17) that the supposition $\frac{a_1 b_2 - a_2 b_1}{b_2} \leq \bar{K}_f$ holds if and only if $\frac{a_1 b_2 - a_2 b_1}{b_1 + b_2} \leq a_2 - b_2 c$. For the case when $b_1 = b_2 = b$, this condition becomes $\frac{a_1 - a_2}{2} \leq a_2 - bc$, which is likely to be true. A similar result can be established for the case when $\frac{a_1 b_2 - a_2 b_1}{b_2} > \bar{K}_f$. To avoid redundancy, we omit the details.

Proposition 3 has the following implications. First, when the capacity $K$ is sufficiently large (i.e., $K \geq \bar{K}_f$), the optimal retail price $p^K_f$, the optimal order quantity $(Q^K_{1f}, Q^K_{2f})$, and the retailer’s optimal profit $\pi^K_f$ are the same as the base case presented in Proposition 1. Next, suppose we reduce the capacity $K$ so
that $K \in \left[ \frac{a_1 b_2 - a_2 b_1}{b_2}, \bar{K}_f \right)$. Then the optimal order quantity $Q^K_{w_f}$ in (2.19) is equal to a ‘base level’ $\frac{b_w K}{b_1 + b_2}$ plus or minus an adjustment term $\frac{a_1 b_2 - a_2 b_1}{b_1 + b_2}$. Moreover, Proposition 3 indicates that the retailer’s optimal profit $\pi^K_f$ will drop below the retailer’s optimal profit for the base case $\pi^*_f$, and the reduction in profit is equal to $(a_1 + a_2 - b_1 c - b_2 c - 2K)^2$. Moreover, by considering the case when $K < \bar{K}_f$, it is easy to show that $p^K_f > p^*_f$, where $p^*_f$ is given in (2.4), and that $Q^K_{w_f} < Q^*_{w_f}$ for $w = 1, 2$, where $Q^*_{w_f}$ is given in (2.5). This implies that the retailer will charge a higher retail price and will order less as the capacity $K$ is reduced. Finally, when we reduce the capacity $K$ further so that $K < \frac{a_1 b_2 - a_2 b_1}{b_2}$, the third statement in Proposition 3 suggests that it is optimal for the retailer to order only product 1 up to the capacity $K$.

2.4.2 Variable Pricing Strategy with Capacity Constraint

We now analyze the case when the retailer adopts the variable pricing strategy. By considering the demand functions $D_1$ and $D_2$ given in (2.1), we can formulate the retailer’s problem (P2) with capacity constraint under the variable pricing strategy as:

$$\pi^K_v = \max_{p_1, p_2; Q_1, Q_2 \geq 0} (p_1 - c)Q_1 + (p_2 - c)Q_2$$

s.t. $Q_1 \leq a_1 - b_1 p_1 + \delta_1 (p_2 - p_1)$, $Q_2 \leq a_2 - b_2 p_2 + \delta_2 (p_1 - p_2)$, and

$$Q_1 + Q_2 \leq K. \quad (2.21)$$

Suppose we relax the capacity constraint (2.21). Then the retailer’s relaxed problem can be written as:

$$\pi^K_v (\lambda) = \max_{p_1, p_2; Q_1, Q_2 \geq 0} (p_1 - (c + \lambda))Q_1 + (p_2 - (c + \lambda))Q_2 + \lambda K$$

s.t. $Q_1 \leq a_1 - b_1 p_1 + \delta_1 (p_2 - p_1)$, and $Q_2 \leq a_2 - b_2 p_2 + \delta_2 (p_1 - p_2)$,
where $\lambda \geq 0$ corresponds to the Lagrangian multiplier of the constraint (2.21).

Suppose $(p_w - (c + \lambda)) \geq 0$ for $w = 1, 2$.\footnote{We shall prove that this assumption holds when we prove Proposition 4.} Then it is optimal to have both constraints to be binding (i.e., $Q_1 = a_1 - b_1p_1 + \delta_1(p_2 - p_1)$, and $Q_2 = a_2 - b_2p_2 + \delta_2(p_1 - p_2)$). As such, the retailer’s relaxed problem can be simplified as:

$$
\pi^K_v(\lambda) = \max_{p_1, p_2 \geq 0} \left( (p_1 - \hat{c})(a_1 - b_1p_1 + \delta_1(p_2 - p_1)) + (p_2 - \hat{c})(a_2 - b_2p_2 + \delta_2(p_1 - p_2)) + \lambda K \right),
$$

(2.22)

where $\hat{c} = c + \lambda$. Hence, we can determine the optimal order quantities, denoted by $Q^K_{1v}$ and $Q^K_{2v}$, the optimal retail price, denoted by $(p^K_{1v}, p^K_{2v})$, by solving the following dual problem:

$$
\pi^K_v = \min_{\lambda \geq 0} \{ \lambda K + \max_{p_1, p_2 \geq 0} \left( (p_1 - \hat{c})(a_1 - b_1p_1 + \delta_1(p_2 - p_1)) + (p_2 - \hat{c})(a_2 - b_2p_2 + \delta_2(p_1 - p_2)) \right) \}.
$$

(2.23)

Notice that the inner problem $\max_{p_1, p_2 \geq 0} \left( (p_1 - \hat{c})(a_1 - b_1p_1 + \delta_1(p_2 - p_1)) + (p_2 - \hat{c})(a_2 - b_2p_2 + \delta_2(p_1 - p_2)) \right)$ is the same as the retailer’s problem in (2.7) when we replace $c$ with $\hat{c}$. This observation enables us to first solve the inner problem by using the results provided in Proposition 2 (namely, (2.8), (2.10) and (2.11)), and then solve the outer problem by determining the optimal value of $\lambda$. By considering certain boundary conditions, we can prove the following Proposition.

In preparation, let:

$$
\tilde{K}_v = \frac{(a_1 - b_1c)[b_2(\delta_1 + \delta_2) + 2b_1(\delta_1 + \delta_2)] + (a_2 - b_2c)[b_1(\delta_1 + \delta_2) + 2b_2(\delta_1 + \delta_1)]}{4b_1\delta_2 + 4b_2\delta_1 + 4b_1b_2 - \delta_1^2 - \delta_2^2 + 2\delta_1\delta_2}.
$$

(2.24)

In this case, we have:

**Proposition 4.** Suppose $\frac{a_1b_2 - a_2b_1}{2b_2 - \delta_1 + \delta_2} \leq \tilde{K}_v$. Suppose there is a limit $K$ on the total order quantity. Then the optimal price $(p^K_{1v}, p^K_{2v})$, the optimal order quantity $(Q^K_{1v}, Q^K_{2v})$, and the optimal profit $\pi^K_v$ under the variable pricing strategy can be expressed as follows:
1. If \( K \geq \tilde{K}_v \), then \( p_{wv}^K = p_{wv}^* \) as given in (2.8), \( Q_{wv}^K = Q_{wv}^* \) as given in (2.10) for \( w = 1, 2 \), and \( \pi_v^K = \pi_v^* \) as given in (2.11).

2. If \( \frac{a_1b_2 - a_2b_1}{2b_2 - \delta_1 + \delta_2} \leq K < \tilde{K}_v \), then:
   \[
   p_{1v}^K = \frac{a_1 + a_2 - K}{b_1 + b_2} + \frac{(b_2 + \delta_2 - \delta_1)(a_1b_2 - a_2b_1 - K(\delta_1 - \delta_2))}{2(b_1 + b_2)(b_1b_2 + b_1\delta_2 + b_2\delta_1)},
   \]
   \[
   p_{2v}^K = \frac{a_1 + a_2 - K}{b_1 + b_2} - \frac{(b_1 + \delta_1 - \delta_2)(a_1b_2 - a_2b_1 - K(\delta_1 - \delta_2))}{2(b_1 + b_2)(b_1b_2 + b_1\delta_2 + b_2\delta_1)} \quad (2.25)
   \]
   \[
   Q_{1v}^K = \frac{b_1K + (a_1b_2 - a_2b_1)}{b_1 + b_2} - \frac{a_1b_2 - a_2b_1 - K(\delta_1 - \delta_2)}{2(b_1 + b_2)},
   \]
   \[
   Q_{2v}^K = \frac{b_2K - (a_1b_2 - a_2b_1)}{b_1 + b_2} + \frac{a_1b_2 - a_2b_1 - K(\delta_1 - \delta_2)}{2(b_1 + b_2)}, \quad \text{and} \quad (2.26)
   \]
   \[
   \pi_v^K = \frac{(a_1 + a_2 - b_1c - b_2c - K)K}{b_1 + b_2} + \frac{(K\delta_1 - K\delta_2 + a_2b_1 - a_1b_2)^2}{4(b_1 + b_2)(b_1b_2 + b_1\delta_2 + b_2\delta_1)}. \quad (2.27)
   \]

3. If \( \frac{a_1b_2 - a_2b_1}{2b_2 - \delta_1 + \delta_2} < K < \tilde{K}_v \). Then:
   \[
   Q_{1v}^K = \min \left( K, \frac{a_1b_2 + a_1\delta_2 + a_2\delta_1 - b_1b_2c - b_1\delta_2c - b_2\delta_1c}{2(b_2 + \delta_2)} \right),
   \]
   \[
   Q_{2v}^K = 0,
   \]
   \[
   p_{1v}^K = \frac{b_2a_1 + a_1\delta_2 + a_2\delta_1 - (b_2 + \delta_2)Q_{1v}^K}{b_1b_2 + b_1\delta_2 + b_2\delta_1}, \quad \text{and} \quad (2.28)
   \]
   \[
   \pi_v^K = (p_{1v}^K - c)Q_{1v}^K.
   \]

A similar result can be established for the case when \( \frac{a_1b_2 - a_2b_1}{2b_2 - \delta_1 + \delta_2} > \tilde{K}_v \). To avoid redundancy, we omit the details.

Proposition 4 suggests that the impact of product substitutability \( \delta_1 \) and \( \delta_2 \) on the optimal retail price depends on the values of \( a_1, a_2, b_1, b_2, c \) and \( K \). To obtain a general idea about the impact of product substitutability on the optimal retail price, we construct a numerical example by using the same set of values as reported in Section 2.3.2. In addition, we set \( K = 100 \). In this case, one can
check that $\frac{a_1 b_2 - a_2 b_1}{b_2 - \delta_1 + \delta_2} \leq K < \bar{K}_v$. Hence, we only need to examine the impact of $\delta_1$ and $\delta_2$ on the optimal retail price given in (2.25). As before, we fix $\delta_2 = 2$, but we vary $\delta_2$ from 0 to 4. Figure 2.2 reports the values of $p_{1v}^K$ and $p_{2v}^K$ when we vary the value of $\delta_1$ from 0 to 4. Similar to the observations obtained from Figure 1, it is easy to check from Figure 2.2 that the gap between $p_{1v}^K$ and $p_{2v}^K$ (i.e., $(p_{1v}^K - p_{2v}^K)$) is decreasing in $(\delta_1 - \delta_2)$. We observe similar patterns for different sets of values of $a_1, a_2, b_1, b_2, c$.

![Figure 2.2: Impact of $\delta_1$ on $p_1$ and $p_2$ with capacity constraint.](image)

To examine the results stated in Proposition 4 analytically, let us consider the case when $\delta_1 = \delta_2 = \delta$. In this case, it is easy to check from (2.24) that $\bar{K}_v$ reduces to $\bar{K}_f$ given in (2.17). Therefore, Proposition 4 can be simplified as follows:

**Corollary 3.** Suppose $\frac{a_1 b_2 - a_2 b_1}{b_2} \leq \bar{K}_f$. Suppose there is a limit $K$ on the total order quantity. Then the optimal retail price $(p_{1v}^K, p_{2v}^K)$, the optimal order quantity $(Q_{1v}^K, Q_{2v}^K)$, and the optimal profit $\pi_v^K$ under the variable pricing strategy can be expressed as follows:
1. If \( K \geq \bar{K}_f \), then \( p^K_{wv} = p^*_wv \) as given in (2.12), \( Q^K_{wv} = Q^*_wv \) as given in (2.13) for \( w = 1, 2 \), and \( \pi^K_v = \pi^*_v \) as given in (2.14).

2. If \( \frac{a_1b_2-a_2b_1}{2b_2} \leq K < \bar{K}_f \), then:

\[
\begin{align*}
\pi^K_v &= \left( a_1 + a_2 - \frac{K}{b_1 + b_2} \right) \frac{a_1b_2 - a_2b_1}{b_1b_2 + b_1\delta + b_2\delta} + \frac{(a_2b_1 - a_1b_2)^2}{4(b_1 + b_2)(b_1b_2 + b_1\delta + b_2\delta)^2}, \\
n(K) &= \frac{(a_1 + a_2 - b_1c - b_2c - K)K}{b_1 + b_2} + \frac{(a_2b_1 - a_1b_2)^2}{4(b_1 + b_2)(b_1b_2 + b_1\delta + b_2\delta)^2}.
\end{align*}
\]

3. If \( K < \frac{a_1b_2-a_2b_1}{2b_2} \). Then:

\[
\begin{align*}
Q^K_{1v} &= K, \\
Q^K_{2v} &= 0, \\
p^K_{1v} &= \frac{b_2a_1 + a_1\delta + a_2\delta - (b_2 + \delta)K}{b_1b_2 + b_1\delta + b_2\delta}, \quad \text{and} \\
\pi^K_v &= (p^K_{1v} - c)K.
\end{align*}
\]

It is easy to check from the definition of \( \bar{K}_f \) given in (2.17) that the supposition \( \frac{a_1b_2-a_2b_1}{2b_2} \leq K \) holds if and only if \( 0 \leq a_2 - b_2c \). This condition is likely to be true.

Corollary 3 is analogous to Corollary 1 in the following sense. First, consider the case when the capacity \( K \) is sufficiently large (i.e., \( K \geq \bar{K}_v \)), the optimal retail price \( (p^K_{1v}, p^K_{2v}) \), the optimal order quantity \( (Q^K_{1v}, Q^K_{2v}) \), and the retailer’s optimal profit \( \pi^K_v \) are the same as the base case as presented in Corollary 1. Let us consider the case when we reduce the capacity \( K \) so that \( \frac{a_1b_2-a_2b_1}{2b_2} \leq K < \bar{K}_f \).
First, observe from (2.29) in Corollary 3 that the optimal retail price for product 1, \( p^K_{1v} \), is equal to the optimal fixed retail price \( p^K_f \) given in (2.18) of Proposition 3 plus an adjustment term \( \frac{b_2(a_1b_2-a_2b_1)}{2(b_1+b_2)(b_1b_2+b_1\delta+b_2\delta)} \). Notice that this adjustment term is non-negative because \( \frac{a_1}{b_1} \geq \frac{a_2}{b_2} \). The opposite can be said about the optimal retail price for product 2. Moreover, observe from (2.29) that the price difference \( p^K_{1v} - p^K_{2v} = \frac{a_1b_2-a_2b_1}{2(b_1\delta+b_2\delta+b_1b_2)} \), which is identical to the price difference obtained from Corollary 1 for the uncapacitated case.

Second, observe from (2.30) that the optimal order quantity for product 1, \( Q^K_{1v} \), is equal to the optimal order quantity \( Q^K_{1f} \) given in (2.19) of Proposition 3 minus an adjustment term \( \frac{a_1b_2-a_2b_1}{2(b_1+b_2)} \). The opposite can be said about the optimal order quantity for product 2. By comparing (2.29) with (2.18) and by comparing (2.30) with (2.19), we can draw the following conclusion: when the retailer switches from the fixed pricing strategy to the variable pricing strategy, it is optimal for him to increase the price for product 1 (i.e., \( p^K_{1v} > p^K_f \)) and reduce the order quantity for product 1 (i.e., \( Q^K_{1v} < Q^K_{1f} \)). The opposite can be said for product 2. This conclusion is analogous to the uncapacitated case when we compare the results stated in Corollary 1 and Proposition 1. Finally, the retailer’s optimal profit under the variable pricing strategy \( \pi^K_v \) is higher than the optimal profit under the fixed pricing strategy \( \pi^K_f \). Moreover, when operating under the variable pricing strategy, the optimal profit \( \pi^K_v \) for the capacitated case is lower than the optimal profit \( \pi^K_v \) for the uncapacitated case.

Finally, let us consider the case when we reduce the capacity \( K \) so that \( K < \frac{a_1b_2-a_2b_1}{2b_2} \). In this case, the third statement in Corollary 3 is analogous to the corresponding one in Proposition 3, which suggests that it is optimal for the retailer to order only product 1 up to the capacity \( K \).
2.5 Retail Competition

We now extend our base model to address the issue of retail competition. Consider two retailers $i$ and $j$ that sell products 1 and 2 in a duopolistic environment. We assume that the product demand function of product $w$ at retailer $r$, where $r \in \{i, j\}$, possesses the following form:

$$D_{r1} = a_{r1} - b_{r1}p_{r1} + \delta_{r,21}(p_{r2} - p_{r1}) + \theta_{sr,1}(p_{s1} - p_{r1}) + \gamma_{s2,r1}(p_{s2} - p_{r1}), \text{ and}$$

$$D_{r2} = a_{r2} - b_{r2}p_{r2} + \delta_{r,12}(p_{r1} - p_{r2}) + \theta_{sr,2}(p_{s2} - p_{r2}) + \gamma_{s1,r2}(p_{s1} - p_{r2}),$$

(2.33)

where $r, s \in \{i, j\}, s \neq r$. Essentially, the above demand function depicts a situation in which the demand for product 1 at retailer $r$ depends on the retail prices of products 1 and 2 at both stores. Just like the base model, the parameter $a_{r1}$ represents the primary demand of product 1 at retailer $r$, $b_{r1}$ represents price sensitivity of product 1 at retailer $r$, and $\delta_{r,21}$ represents the product substitutability at retailer $r$. Also, the parameter $\theta_{sr,1}$ corresponds to the store switching behavior associated with product 1. For example, suppose the retail price of product 1 at retailer $r$ is lower; i.e., $p_{s1} > p_{r1}$. Then the term $\theta_{sr,1}(p_{s1} - p_{r1})$ corresponds to the additional demand of product 1 that store $r$ can gain from those customers who switch from store $s$ to store $r$. Similarly, the parameter $\gamma_{s2,r1}$ corresponds to the product substitutability across stores. To ensure that the total demand $D_{i1} + D_{i2} + D_{j1} + D_{j2}$ is decreasing in all the prices, we shall assume that $b_{r1} + \delta_{r,21} - \delta_{r,12} + \theta_{sr,1} - \theta_{rs,1} + \gamma_{s2,r1} - \gamma_{r1,s2} > 0$ and $b_{r2} + \delta_{r,12} - \delta_{r,21} + \theta_{sr,2} - \theta_{rs,2} + \gamma_{s1,r2} - \gamma_{r2,s1} > 0$, where $r, s \in \{i, j\}, r \neq s$.

Notice that there are 20 parameters associated with the demand function displayed in (2.33). As such, the competitive analysis for the general demand function case is intractable. To obtain tractable results, we shall analyze a special case in which $\delta_{i,21} = \delta_{i,12} = \delta_{j,21} = \delta_{j,12} = \delta$; $\theta_{ji,1} = \theta_{ji,2} = \theta_{ij,1} = \theta_{ij,2} =$
θ; and \( \gamma_{j2,i1} = \gamma_{j1,i2} = \gamma_{i2,j1} = \gamma_{i1,j2} = \gamma \). For the general demand function case, we shall conduct the competitive analysis numerically. For the special case, the demand function given in (2.33) can be simplified as:

\[
\begin{align*}
D_{r1} &= a_{r1} - bp_{r1} + \theta(p_{s2} - p_{r1}) + \theta(p_{s1} - p_{r1}) + \gamma(p_{s2} - p_{r1}), \\
D_{r2} &= a_{r2} - bp_{r2} - \delta(p_{r2} - p_{r1}) + \theta(p_{s2} - p_{r2}) + \gamma(p_{s1} - p_{r2}),
\end{align*}
\] (2.34)

where \( r, s \in \{i, j\} \), \( s \neq r \).

Since there is no limit on the order quantity, it is optimal for retailer \( r \) to order according to the retail demand \( D_{rw} \) for each retailer. Thus, given the demand function \((D_{r1}, D_{r2})\) in (2.34), each retailer \( r = \{i, j\} \) would need to determine the optimal retail price for each product so as to maximize his profit. Let \( \pi^*_r \) be the optimal profit of retailer \( r \), where:

\[
\pi^*_r = \max_{p_{r1}, p_{r2} \geq 0} (p_{r1} - c)D_{r1} + (p_{r2} - c)D_{r2}.
\] (2.35)

Under retail competition, each retailer’s optimal retail price would depend on the retail price of the other retailer. For this reason, we first examine retailer \( j \)'s optimal pricing strategy when retailer \( i \) operates under the fixed pricing strategy. Then we continue our analysis for the case when retailer \( i \) operates under the variable pricing strategy.

### 2.5.1 Retailer \( i \) Adopts the Fixed Pricing Strategy

First, suppose retailer \( j \) chooses the fixed pricing strategy. Then both retailers \( i \) and \( j \) adopt the fixed pricing strategy. As such, the retail price for retailer \( r, r = \{i, j\} \), can be denoted by \( p_{r_1}^{(f,f)} = p_{r_2}^{(f,f)} = p_{r}^{(f,f)} \), where the superscript corresponds to the pricing strategy chosen by retailers \( i \) and \( j \), respectively.\(^9\) By considering

\[9\text{For example, } (f,f) \text{ corresponds to the case when both retailers } i \text{ and } j \text{ adopt the fixed pricing strategy. Similarly, the superscript } (f,v) \text{ corresponds to the case when retailer } i \text{ adopts the fixed pricing strategy while retailer } j \text{ adopts the variable pricing strategy.}\]
the fact that $p_{r1}^{(f,f)} = p_{r2}^{(f,f)} = p_r^{(f,f)}$, the demand function given in (2.34) and the profit function given in (2.35), we can differentiate the corresponding profit functions for retailers $i$ and $j$, denoted by $\pi_i^{(f,f)}$ and $\pi_j^{(f,f)}$, with respect to $p_i^{(f,f)}$ and $p_j^{(f,f)}$, respectively. The first order condition for retailer $r$ is given as:

$$p_r^* = \frac{(a_{r1} + a_{r2}) + 2(\theta + \gamma)p_r^{(f,f)} + 2c(b + \theta + \gamma)}{4(b + \theta + \gamma)}$$

where $r, s = \{i, j\}$, $r \neq s$. By solving the above equation, the optimal retail price for retailer $r$ can be expressed as:

$$p_r^{* (f,f)} = \frac{2(b + \theta + \gamma)(a_{r1} + a_{r2}) + (\theta + \gamma)(a_{s1} + a_{s2}) + c(b + \theta + \gamma)}{2(2b + \theta + \gamma)(2b + 3\theta + 3\gamma)}$$

where $r, s = \{i, j\}$, $r \neq s$. Notice that the optimal retail price $p_r^{* (f,f)}$ depends on the store switching factor $\theta$ and product substitutability across stores factor $\gamma$. Also, observe that the optimal retail price $p_r^{* (f,f)}$ reduces to $p_r^*$ in (2.4) when $\theta = 0$ and $\gamma = 0$. By substituting the optimal fixed price for retailer $r$ into (2.34) and (2.35), we can obtain the optimal profit for retailer $r$, denoted by $\pi_r^{*(f,f)}$. We omit the details.

Second, let us now consider the case when retailer $j$ chooses the variable pricing strategy so that his retail price is denoted by $(p_j^{(f,v)}, p_j^{(f,v)})$. Since retailer $i$ adopts the fixed pricing strategy, we have: $p_i^{(f,v)} = p_i^{(f,v)} = p_i^{(f,v)}$. By using the same approach as before, we can establish the following Lemma:

**Lemma 1.** Under the pricing strategy $(f, v)$; i.e., retailer $i$ adopts the fixed pricing strategy and retailer $j$ chooses the variable pricing strategy, the retailer’s optimal price and optimal profit can be expressed as:

1. $p_i^{*(f,v)} = p_i^{*(f,f)}$.

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2. \( p_{j1}^*(f,v) = p_j^*(f,f) + \frac{a_{j1} - a_{j2}}{4(b+2\delta+\theta+\gamma)} \).

3. \( p_{j2}^*(f,v) = p_j^*(f,f) - \frac{a_{j1} - a_{j2}}{4(b+2\delta+\theta+\gamma)} \).

4. \( \pi_i^*(f,v) = \pi_i^*(f,f) \).

5. \( \pi_j^*(f,v) = \pi_j^*(f,f) + \frac{(a_{j2} - a_{i1})^2}{8(b+2\delta+\theta+\gamma)} \).

Lemma 1 has the following implications. The first property suggests that it is optimal for retailer \( i \) to charge the same optimal fixed price \( p_i^*(f,f) \) as given in (2.36) even when retailer \( j \) changes from fixed to variable pricing. The second and third properties are consistent with the results displayed in Proposition 2, which suggests that retailer \( j \)'s optimal variable retail price would be equal to the retailer \( j \)'s optimal fixed retail price plus or minus a term that is related to the product substitutability factor \( \delta \), the store switching factor \( \theta \), and the product substitutability across stores factor \( \gamma \). Moreover, it can be easily seen that: the price difference \( p_{j1}^*(f,v) - p_{j2}^*(f,v) = \frac{a_{j1} - a_{j2}}{2(b+2\delta+\theta+\gamma)} \), which is analogous to the price difference obtained from Corollary 1 for the uncapacitated case. Under competition, it is interesting to note that the price difference \( p_{j1}^*(f,v) - p_{j2}^*(f,v) \) is independent of \( (a_{i1}, a_{i2}) \); i.e., the primary demand of retailer \( i \). Finally, the fourth property implies that retailer \( i \)'s optimal profit remains unaffected even when retailer \( j \) changes from fixed to variable pricing. However, the fifth property shows that retailer \( j \) will gain additional profit when he changes from fixed to variable pricing. By noting that the additional profit \( \frac{(a_{j2} - a_{j1})^2}{8(b+2\delta+\theta+\gamma)} \) is decreasing in \( \delta, \theta \) and \( \gamma \), we can conclude that the benefit of variable pricing diminishes as the products become more substitutable (i.e., as \( \delta \) increases) or as customers are
more eager to switch store (i.e., as $\theta$ or $\gamma$ increases). When retailer $i$ adopts the fixed pricing strategy, the optimal retail price and the optimal profit for retailer $j$ displayed in Lemma 1 possess a similar structure as those displayed in Proposition 2 for the monopolistic case.

### 2.5.2 Retailer $i$ adopts the variable pricing strategy

First, consider the case when retailer $j$ chooses the fixed price strategy. If we exchange retailer $i$ with retailer $j$, then it is easy to see that the retailer’s problem associated with pricing strategy $(v, f)$ is identical to the retailer’s problem under the pricing strategy $(f, v)$. For ease of reference, we highlight the results in the following Lemma:

**Lemma 2.** Under the pricing strategy $(v, f)$, the retailer’s optimal price and optimal profit can be expressed as:

1. $p_j^{(v,f)} = p_j^{(f,f)}$.

2. $p_{i1}^{(v,f)} = p_{i1}^{(f,f)} + \frac{a_{i1} - a_{i2}}{4(b+2\delta+\theta+\gamma)}$.

3. $p_{i2}^{(v,f)} = p_{i2}^{(f,f)} - \frac{a_{i1} - a_{i2}}{4(b+2\delta+\theta+\gamma)}$.

4. $\pi_j^{(v,f)} = \pi_j^{(f,f)}$.

5. $\pi_i^{(v,f)} = \pi_i^{(f,f)} + \frac{(a_{i2} - a_{i1})^2}{8(b+2\delta+\theta+\gamma)}$.

Now, let us consider the pricing strategy $(v, v)$ under which both retailers adopt the variable pricing strategy. In this case, the retail price for retailer $r$ can
be denoted by \( (p_{r_1}^{(v,v)}, p_{r_2}^{(v,v)}) \) for \( r = \{i, j\} \). By using the same approach as before, we can obtain the following results:

**Lemma 3.** Under the pricing strategy \((v,v)\), the retailer’s optimal price and optimal profit for retailer \( r, r = \{i, j\} \) can be expressed as:

1. \[ p_{r_1}^{(v,v)} = p_r^{(f,f)} + \frac{(\theta - \gamma)(a_s - a_r) + 2(b + 2\delta + \theta + \gamma)(a_r - a_s)}{2(2b + 3\delta + 4\theta + 3\gamma)(2b + 4\delta + \theta + 3\gamma)}, \]

2. \[ p_{r_2}^{(v,v)} = p_r^{(f,f)} - \frac{(\theta - \gamma)(a_s - a_r) + 2(b + 2\delta + \theta + \gamma)(a_r - a_s)}{2(2b + 3\delta + 4\theta + 3\gamma)(2b + 4\delta + \theta + 3\gamma)}, \quad \text{and} \]

3. \[ \pi^{(v,v)}_{r,s} = \pi_r^{(f,f)} + \frac{(b + 2\delta + \theta + \gamma)[(\theta - \gamma)(a_s - a_r) + 2(b + 2\delta + \theta + \gamma)(a_r - a_s)]^2}{2(2b + 3\delta + 4\theta + 3\gamma)^2(2b + 4\delta + \theta + 3\gamma)^2}, \quad \text{where} \quad r, s = \{i, j\}, \quad r \neq s. \]

It follows from the first and second properties that the price difference can be expressed as: \[ p_{r_1}^{(v,v)} - p_{r_2}^{(v,v)} = \frac{(\theta - \gamma)(a_s - a_r) + 2(b + 2\delta + \theta + \gamma)(a_r - a_s)}{(2b + 3\delta + 4\theta + 3\gamma)(2b + 4\delta + \theta + 3\gamma)}. \] This price difference differs from the price difference reported in Lemma 1 and Corollary 1 in the following sense. Specifically, when both retailers adopt the variable pricing strategy, the price difference \( p_{r_1}^{(v,v)} - p_{r_2}^{(v,v)} \) at retailer \( r \) would depend on the demand characteristics of both retailers.

By comparing the results reported in Lemmas 1, 2 and 3 with Propositions 1 and 2, we can conclude that the underlying structure of the optimal retail price and optimal profit for the monopolistic case is preserved in a duopolistic environment.

### 2.5.3 Competitive Equilibrium

Suppose both retailers \( i \) and \( j \) need to decide on their own pricing strategy (fixed or variable). Then we need to consider 4 pricing scenarios; namely, \( \{(f, f), (f, v), \)
\((v, f), (v, v)\). For each pricing scenario, we first utilize the results established in Lemmas 1, 2 and 3 to construct a payoff matrix associated with this competitive game, where each entry of this payoff matrix corresponds to the optimal profits of retailers \(i\) and \(j\) under a specific pricing scenario. By comparing the retailer’s profits under different pricing scenarios, we could determine the competitive equilibrium. The reader is referred to Winston and Albright (1997) for a general discussion of payoff matrix.

We now compare the retailer’s profits under different pricing scenarios. First, it follows from property 5 in Lemma 1 that:

\[
\pi_j^*(f,v) > \pi_j^*(f,f).
\] (2.37)

Next, it follows from property 3 in Lemma 3 and property 4 in Lemma 2 that:

\[
\pi_j^*(v,v) > \pi_j^*(v,f).
\] (2.38)

Also, one can observe from property 5 in Lemma 2 that:

\[
\pi_i^*(v,f) > \pi_i^*(f,f).
\] (2.39)

Finally, it is easy to check from property 3 in Lemma 3 and property 4 in Lemma 1 to show that:

\[
\pi_i^*(v,v) > \pi_i^*(f,v).
\] (2.40)

By considering the above inequalities along with the payoff matrix associated with the competitive game, we can establish the following result:

**Proposition 5.** Both retailers adopt the variable pricing strategy \((v, v)\) at the equilibrium.\(^{10}\)

\(^{10}\)In the event a fixed cost \(F\) is incurred when a retailer chooses to adopt the variable pricing strategy, one can show that both retailers will adopt the variable pricing strategy when \(F\) is sufficiently small, and adopt the fixed pricing strategy when \(F\) is sufficiently large at the equilibrium. However, one retailer will adopt the variable pricing strategy while the other will adopt the fixed pricing strategy at the equilibrium when \(F\) is moderate. We omit the details.
While both retailers will adopt the variable pricing at the equilibrium for the special demand function case, it is not clear if this result will hold when the parameters associated with product substitutability, store switching and product substitutability across store are no longer identical. To investigate this further, we now examine the retailers’ profits for the general demand case as defined in (2.33). Specifically, we first determine the retailers’ optimal profits numerically under 4 pricing scenarios; namely, \{(f, f), (f, v), (v, f), (v, v)\}. Then we check to see if inequalities (2.37) through (2.40) remain valid. Based on our experiment with different parameters, it appears that inequalities (2.37) through (2.40) continue to hold in every single case. Table 2.1 provides 4 different numerical examples associated with different values of the parameters. Overall, our numerical experiment seems to suggest that, even for the general demand case, both retailers will adopt the variable pricing strategy at the equilibrium.

2.6 Concluding Remarks

In this chapter, we have considered a case in which the retailer needs to make two decisions jointly; namely, the order quantity and the retail price of each variant. Although we consider the two-product case, the same approach can be used to analyze the n-product case, where \(n > 2\). By considering the case when the underlying product demand depends on the relative price, we have developed closed form expressions for the optimal order quantity and retail price under the fixed and variable pricing strategies. Our analysis indicates that the optimal retail price under the variable pricing strategy is equal to the optimal retail price under the fixed pricing strategy plus or minus an adjustment term, where this adjustment term is decreasing in the product substitutability factor \(\delta\). This implies that, as the products become more substitutable, the gap between
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<td>$\gamma_{j2,i1}$</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$\gamma_{j1,i2}$</td>
<td>4</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>$\gamma_{i2,j1}$</td>
<td>2</td>
<td>1</td>
<td>5</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>$\gamma_{i1,j2}$</td>
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<td>2</td>
<td>3</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>$\pi_j^{<em>}(f,v) - \pi_j^{</em>}(f,f)$</td>
<td>9.27</td>
<td>3.05</td>
<td>0.11</td>
<td>3.34</td>
<td></td>
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<tr>
<td>$\pi_j^{<em>}(v,v) - \pi_j^{</em>}(v,f)$</td>
<td>9.78</td>
<td>2.71</td>
<td>0.20</td>
<td>2.20</td>
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<tr>
<td>$\pi_i^{<em>}(v,f) - \pi_i^{</em>}(f,f)$</td>
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<td>59.67</td>
<td>79.06</td>
<td>86.29</td>
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<tr>
<td>$\pi_i^{<em>}(v,v) - \pi_i^{</em>}(f,v)$</td>
<td>12.64</td>
<td>59.74</td>
<td>78.89</td>
<td>84.20</td>
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</tr>
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Table 2.1: Retail Competition under the General Demand Function.
the optimal retail prices will become smaller under the variable pricing strategy. Also, we show that the retailer will always enjoy a higher profit under the variable pricing strategy; however, this profit gain diminishes as the products become more substitutable (i.e., as $\delta$ increases).

We have extended our base model in two different ways. The first extension deals with the case when there is a limit on the total order quantity. Relative to the base case, our analysis shows that the underlying structure of the optimal retail prices and the optimal order quantities is preserved when there is a capacity limit. The second extension examines the issue of retail competition. Our analysis shows that the underlying structure of the optimal retail prices and the optimal order quantities for the monopolistic case is preserved when the retailer operates in a duopolistic environment. Moreover, we show that both retailers will adopt the variable pricing strategy at the equilibrium.

While our model has certain limitations, it presents opportunities for future research. First, our model is based on the assumption that the demand function is deterministic. In our future research, we shall consider the case in which the values of the parameters for the demand function are uncertain. Second, we have considered the issues of retail competition and capacity separately. It would be of interest to develop a model for analyzing retail competition with capacity constraint. For the single product case, Lippman and McCardle (1997) have developed a competitive newsboy model to analyze how retailers can use order quantity to compete for higher market share. As a future research direction, one may consider developing a multi-retailer model in which each retailer uses order quantity and retail price to compete for higher profitability.
2.7 Appendix: Proof

Proof of Proposition 1: The proof is straightforward. Hence, it is omitted here.

Proof of Proposition 2: Consider the optimization problem as stated in (2.7). Since \( p_1, p_2 > c \), we can transform the decision variables \( p_1 \) and \( p_2 \) into two new variables \( u, v > 0 \), where:

\[
\begin{align*}
p_1 &= u + c, \quad \text{and} \\
p_2 &= v + c.
\end{align*}
\]

Then problem (2.7) can be simplified as:

\[
\pi_v^* = \max_{u, v > 0} [u(a_1 - b_1(u + c) + \delta_1(v - u)) + v(a_2 - b_2(v + c) + \delta_2(u - v))].
\]

Differentiate the objective function with respect to the new variables \( u \) and \( v \) and consider the first order conditions, we get

\[
\begin{align*}
u^* &= \frac{2(b_2 + \delta_2)(a_1 - b_1 c) + (\delta_1 + \delta_2)(a_2 - b_2 c)}{4b_1b_2 + 4b_1\delta_2 + 4b_2\delta_1 - \delta_1^2 - \delta_2^2 + 2\delta_1\delta_2}, \quad \text{and} \\
u^* &= \frac{2(b_1 + \delta_1)(a_2 - b_2 c) + (\delta_1 + \delta_2)(a_1 - b_1 c)}{4b_1b_2 + 4b_1\delta_2 + 4b_2\delta_1 - \delta_1^2 - \delta_2^2 + 2\delta_1\delta_2}.
\end{align*}
\]

Thus we have showed that the optimal retail prices \( p_1^* \) and \( p_2^* \) can be written in the form as stated in (2.8). Next, by substituting the optimal retail prices into the demand function in (2.1), we obtain the result stated in (2.10). Finally, by substituting the optimal retail price and order quantity into the objective function (2.7), we obtain the retailer’s optimal profit \( \pi_v^* \) as stated in (2.11). We omit the details.

Proof of Corollary 1: The proof follows immediately from Proposition 2 when we set \( \delta_1 = \delta_2 = \delta \). We omit the details.
Proof of Proposition 3: Suppose \( p - (c + \lambda) \geq 0 \) and consider the dual problem in (2.16). For any given \( \lambda \geq 0 \), we can apply Proposition 1 to obtain the corresponding solution for the inner problem that is given as:

\[
\begin{align*}
p^K_f(\lambda) &= \frac{a_1 + a_2}{2(b_1 + b_2)} + \frac{c + \lambda}{2}, \\
Q^K_{1f}(\lambda) &= \frac{a_1 - b_1(c + \lambda)}{2} + \frac{a_1 b_2 - a_2 b_1}{2(b_1 + b_2)}, \quad \text{and} \\
Q^K_{2f}(\lambda) &= \frac{a_2 - b_2(c + \lambda)}{2} - \frac{a_1 b_2 - a_2 b_1}{2(b_1 + b_2)}.
\end{align*}
\]

(2.41)

Thus, the outer problem can be simplified as:

\[
\pi^K_f = \min_{\lambda \geq 0} \pi^K_f(\lambda),
\]

where

\[
\pi^K_f(\lambda) = \lambda K + \frac{(a_1 + a_2 - (b_1 + b_2)(c + \lambda))^2}{4(b_1 + b_2)}. \tag{2.42}
\]

Considering the first order conditions, we get \( \lambda = \frac{(a_1 + a_2 - b_1c - b_2c - 2K)(b_1 + b_2)}{2} = \bar{K}_f \), then \( \lambda^* = 0 \). In this case, the optimal price and order quantities in (2.41) and the optimal profit in (2.42) are equal to those as stated in Proposition 1 for the uncapacitated case. This proves part 1.

If \( K < \bar{K}_f \), then \( \lambda^* = \frac{(a_1 + a_2 - b_1c - b_2c - 2K)/2}{b_1 + b_2} \). Substituting \( \lambda^* \) into (2.41) and (2.42), we get (2.18), (2.19), and (2.20). To guarantee non-negativity of \( Q^K_{1f} \) and \( Q^K_{2f} \) in (2.19), we must have \( K \geq \max \left( \frac{a_1 b_2 - a_2 b_1}{b_2}, \frac{a_2 b_1 - a_1 b_2}{b_1} \right) = \frac{a_1 b_2 - a_2 b_1}{b_2} \).

The last equality is due to the assumption that \( \frac{a_1}{b_1} \geq \frac{a_2}{b_2} \). When \( \frac{a_1 b_2 - a_2 b_1}{b_2} < \bar{K}_f \), the range of \( K \) in the second statement of Proposition 3 is non-empty. Furthermore, when \( K \geq \bar{K}_f \) or \( K \in \left[ \frac{a_1 b_2 - a_2 b_1}{b_2}, \bar{K}_f \right) \), we can verify that the assumption \( p - (c + \lambda) \geq 0 \) is always true at the optimal values of \( p \) and \( \lambda \). Therefore we have proved part 2.

It remains to prove part 3 for the case when \( K < \frac{a_1 b_2 - a_2 b_1}{b_2} \). Specifically, we first prove that exactly one of \( Q^K_{1f} \) and \( Q^K_{2f} \) equals to zero when \( K < \frac{a_1 b_2 - a_2 b_1}{b_2} \).
Then we show that it is optimal to have \( Q_{2j}^K = 0 \). We shall prove these two claims by contradiction. Suppose that the first claim does not hold and suppose \((p^*, Q_1^*, Q_2^*)\) is an optimal solution to problem (P1). Then we must have \( Q_1^* > 0 \) and \( Q_2^* > 0 \), since \( Q_1^* = Q_2^* = 0 \) can not be optimal. By complementary slackness, we know that at least one of the constraints in (P1) is binding. Thus we have the following cases:

**Case 1:** \( Q_1^* = a_1 - b_1 p^* \), \( Q_2^* < a_2 - b_2 p^* \), \( Q_1^* + Q_2^* < K \).

It is easy to check that there exists an \( \epsilon > 0 \) such that \((p^*, Q_1^* + \epsilon, Q_2^*)\) is a feasible solution with a higher profit. This contradicts the optimality of \((p^*, Q_1^*, Q_2^*)\).

**Case 2:** \( Q_1^* < a_1 - b_1 p^* \), \( Q_2^* = a_2 - b_2 p^* \), \( Q_1^* + Q_2^* < K \).

In this case, we can find a better feasible solution \((p^*, Q_1^* + \epsilon, Q_2^*)\) for some \( \epsilon > 0 \). This contradicts the optimality of \((p^*, Q_1^*, Q_2^*)\).

**Case 3:** \( Q_1^* < a_1 - b_1 p^* \), \( Q_2^* < a_2 - b_2 p^* \), \( Q_1^* + Q_2^* = K \).

It is easy to see that there exists an \( \epsilon > 0 \) such that \((p^* + \epsilon, Q_1^*, Q_2^*)\) is a feasible solution with higher profit than \((p^*, Q_1^*, Q_2^*)\). This contradicts the optimality of \((p^*, Q_1^*, Q_2^*)\).

**Case 4:** \( Q_1^* = a_1 - b_1 p^* \), \( Q_2^* = a_2 - b_2 p^* \), \( Q_1^* + Q_2^* < K \).

From the first two equalities, we have \( p^* = \frac{a_1 - Q_1^*}{b_1} = \frac{a_2 - Q_2^*}{b_2} \). Therefore, we get \( Q_1^* - \frac{b_1}{b_2} Q_2^* = \frac{a_1 b_2 - a_2 b_1}{b_2} > K \), which contradicts the constraint \( Q_1^* + Q_2^* < K \).

**Case 5:** \( Q_1^* = a_1 - b_1 p^* \), \( Q_2^* < a_2 - b_2 p^* \), \( Q_1^* + Q_2^* = K \).

From the first two constraints, we have \( Q_2^* < a_2 - b_2 \frac{a_1 - Q_1^*}{b_1} \), and \( Q_1^* - \frac{b_1}{b_2} Q_2^* > \frac{a_1 b_2 - a_2 b_1}{b_2} > K \). This contradicts the constraint \( Q_1^* + Q_2^* = K \).

**Case 6:** \( Q_1^* < a_1 - b_1 p^* \), \( Q_2^* = a_2 - b_2 p^* \), \( Q_1^* + Q_2^* = K \).

For the optimal solution \((p^*, Q_1^*, Q_2^*)\), we consider a variant \((\tilde{p}, \tilde{Q}_1, \tilde{Q}_2)\) given
by \( \tilde{p} = p^* + \epsilon, \ \tilde{Q}_1 = Q_1^* + b\epsilon, \ \tilde{Q}_2 = Q_2^* - b\epsilon, \) where \( 0 < \epsilon < \frac{a_1 - b_1 p^* - Q_1^*}{b_1 + b} \) and \( b = \max(b_1, b_2). \) We can check that \((\tilde{p}, \tilde{Q}_1, \tilde{Q}_2)\) is a feasible solution with higher profit than \((p^*, Q_1^*, Q_2^*)\). This contradicts the optimality of \((p^*, Q_1^*, Q_2^*)\).

Case 7: \( Q_1^* = a_1 - b_1 p^*, \ Q_2^* = a_2 - b_2 p^*, \ Q_1^* + Q_2^* = K. \)

We can get a contradiction by using the same argument as in case 4.

From the above 7 cases, we have proved the first claim that either \( Q_1^* > 0, Q_2^* = 0 \) or \( Q_1^* = 0, Q_2^* > 0 \). We now prove the second claim by showing that \( Q_1^* = 0, Q_2^* > 0 \) is not possible. Suppose \((p^*, Q_1^*, Q_2^*)\) is an optimal solution that has \( Q_1^* = 0 \) and \( Q_2^* > 0 \). By considering a variant \((\tilde{p}, \tilde{Q}_1, \tilde{Q}_2)\), where \( \tilde{p} = p^* + \epsilon, \ \tilde{Q}_1 = b_2 \epsilon, \ \tilde{Q}_2 = Q_2^* - b_2 \epsilon, \) and \( 0 < \epsilon < \frac{a_1 - b_1 p^*}{b_1 + b_2} \), we can prove that \((\tilde{p}, \tilde{Q}_1, \tilde{Q}_2)\) is a feasible solution with higher profit. Thus we conclude that \( Q_1^* > 0 \) and \( Q_2^* = 0. \) In this case, the original problem \( P1 \) becomes

\[
\pi^K_f = \max_{p, Q_1} (p - c)Q_1
\text{ s.t. } Q_1 \leq a_1 - b_1 p, \ 0 \leq a_2 - b_2 p, \ \text{and} \ Q_1 \leq K
\] (2.43)

We will show in the following that when \( a_1 b_2 - a_2 b_1 \geq 0, \) problem (2.43) can be further reduced to

\[
\pi^K_f = \max_{p, Q_1} (p - c)Q_1
\text{ s.t. } 0 \leq a_2 - b_2 p,
\ Q_1 \leq K
\] (2.44)

This can be accomplished by showing the following two sets are equivalent:

\[
A = \{(p, Q_1) \mid Q_1 \leq a_1 - b_1 p, 0 \leq a_2 - b_2 p, Q_1 \leq K\},
\]
\[
B = \{(p, Q_1) \mid 0 \leq a_2 - b_2 p, Q_1 \leq K\}.
\]
It is obvious that $A \subseteq B$. For any $(p, q_1) \in B$, we have $a_2 \geq b_2 p$ and $q_1 \leq K < \frac{a_1 b_2 - a_2 b_1}{b_2} \leq a_1 - b_1 p$, thus $B \subseteq A$. Therefore, we have proved $A = B$. Since $p$ and $q_1$ are separable in problem (2.44), it is easy to check that at the optimal solution, we have $p_f^K = a_2/b_2$, and $q_{1f}^K = K$. □

**Remark:** When $\frac{a_1 b_2 - a_2 b_1}{b_2} > \bar{K}_f$, the second statement in Proposition 3 disappears. The results in the first and the third statements remain unchanged, except changing the range of $K$ in the third statement from $K < \frac{a_1 b_2 - a_2 b_1}{b_2}$ to $K < \bar{K}_f$.

**Proof of Proposition 4:** Suppose $p - (c + \lambda) \geq 0$. For any given $\lambda \geq 0$, the solutions to the inner problem in (2.23) are similar to those in Proposition 2, except replacing $c$ by $c + \lambda$. Solving the outer problem of (2.23) and considering the first order conditions, we get $\lambda = \frac{(K_v - K)(4 b_2 b_1 + 4 b_1 \delta_2 + 4 b_2 \delta_1 - 4 \delta_2^2 - 2 \delta_2^2 + 2 \delta_1 \delta_2)}{2(b_1 + b_2)(b_1 b_2 + \delta_2 b_1 + \delta_1 b_2)}$, where $K_v$ is defined in (2.24). If $K \geq K_v$, then $\lambda^* = 0$. In this case, the optimal price, order quantities, and profit are equal to those as stated in Proposition 2 for the uncapacitated case. This proves part 1.

If $K < K_v$, then $\lambda^* = \frac{(K_v - K)(4 b_2 b_1 + 4 b_1 \delta_2 + 4 b_2 \delta_1 - 4 \delta_2^2 - 2 \delta_2^2 + 2 \delta_1 \delta_2)}{2(b_1 + b_2)(b_1 b_2 + \delta_2 b_1 + \delta_1 b_2)}$ and we can get (2.25), (2.26), and (2.27). To guarantee non-negativity of $Q_{1v}^K$ and $Q_{2v}^K$ in (2.26), we must have $K \geq \max(\frac{a_1 b_2 - a_2 b_1}{2b_2 - \delta_1 + \delta_2}, \frac{a_2 b_1 - a_1 b_2}{2b_1 - \delta_2 + \delta_1}) = \frac{a_1 b_2 - a_2 b_1}{2b_2 - \delta_1 + \delta_2}$. Furthermore, since we assume that $\frac{a_1 b_2 - a_2 b_1}{2b_2 - \delta_1 + \delta_2} < K_v$, the range of $K$ in the second statement of Proposition 3 is non-empty. We can also verify that when the conditions in the first or the second statement of Proposition 4 hold, the assumption $p_w - (c + \lambda) \geq 0$ is always true at the optimal values of $p_1, p_2$ and $\lambda$. Therefore we have proved part 2.

When $K < \frac{a_1 b_2 - a_2 b_1}{2b_2 - \delta_1 + \delta_2}$, we can show that $Q_{1v}^K > 0$ and $Q_{2v}^K = 0$ by the identical
proof presented in Proposition 3. Then Problem (P2) can be reduced to

\[
\pi^K_v = \max_{p_1, p_2; Q_1 \geq 0} (p_1 - c)Q_1 \\
\text{s.t. } Q_1 \leq a_1 - b_1 p_1 + \delta_1(p_2 - p_1), \quad 0 \leq a_2 - b_2 p_2 + \delta_2(p_1 - p_2), \quad \text{and} \\
Q_1 \leq K.
\] (2.45)

We fix \(Q_1\) first, and solve (2.45) sequentially as

\[
\pi^K_v = \max_{Q_1 \leq K} \max_{p_1, p_2 \geq 0} (p_1 - c)Q_1 \\
\text{s.t. } (b_1 + \delta_1)p_1 - \delta_1 p_2 \leq a_1 - Q_1, \\
(b_2 + \delta_2)p_2 - \delta_2 p_1 \leq a_2.
\] (2.46)

For fixed \(Q_1 \leq K\), the optimal solution to the above inner LP problem is:

\[
p_1(Q_1) = \frac{b_2 a_1 + a_1 \delta_2 + a_2 \delta_1 - (b_2 + \delta_2)Q_1}{b_1 b_2 + b_1 \delta_2 + b_2 \delta_1}, \\
p_2(Q_1) = \frac{a_1 \delta_2 + a_2 \delta_1 + a_2 b_1 - \delta_2 Q_1}{b_1 b_2 + b_1 \delta_2 + b_2 \delta_1}.
\] (2.47)

Thus the outer problem of (2.46) becomes

\[
\pi^K_v = \max_{Q_1 \leq K} \left( \frac{b_2 a_1 + a_1 \delta_2 + a_2 \delta_1 - (b_2 + \delta_2)Q_1}{b_1 b_2 + b_1 \delta_2 + b_2 \delta_1} - c)Q_1 \right).
\] (2.48)

This is a quadratic problem and the unconstrained maximizer is

\[
Q^0_1 = \frac{a_1 b_2 + a_1 \delta_2 + a_2 \delta_1 - b_1 b_2 c - b_1 \delta_2 c - b_2 \delta_1 c}{2(b_2 + \delta_2)}.
\]

Thus \(Q^K_v = \min \left(K, \frac{a_1 b_2 + a_1 \delta_2 + a_2 \delta_1 - b_1 b_2 c - b_1 \delta_2 c - b_2 \delta_1 c}{2(b_2 + \delta_2)} \right)\). For the case when \(\frac{a_1 b_2 - a_2 b_1}{2b_2 - \delta_1 + \delta_2} > K\), we will have the similar result as stated in the remark below the proof of Proposition 3. We omit the details. □

**Proof of Corollary 3:** The proof follows immediately from Proposition 4 when we set \(\delta_1 = \delta_2 = \delta\). We omit the details.
Proof of Lemma 1: Under pricing strategy \((f, v)\), the decision variables can be simplified as \(p_i, p_{j1}\) and \(p_{j2}\). Suppose we replace the variables \((p_{j1}, p_{j2})\) by two new variables \((x, y)\), where

\[
p_{j1} = x + y, \quad \text{and} \quad p_{j2} = x - y.
\]

Then the optimization problem for retailer \(j\) can be decomposed into the following two separate subproblems:

\[
\pi_j^{*(f,v)} = \max_x (x - c)(a_{j1} + a_{j2} - 2bx + 2(\theta + \gamma)(p_i - x)) \\
+ \max_y y(a_{j1} - a_{j2} - 2by - 4y\delta - 2y(\theta + \gamma)). \tag{2.49}
\]

However, retailer \(i\)'s problem can be written as:

\[
\pi_i^{*(f,v)} = \max_{p_i \geq 0} (p_i - c)(a_{i1} + a_{i2} - 2bp_i + (\theta + \gamma)(p_{j1} + p_{j2} - 2p_i)) \\
= \max_{p_i \geq 0} (p_i - c)(a_{i1} + a_{i2} - 2bp_i + 2(\theta + \gamma)(x - p_i)). \tag{2.50}
\]

Note that problem (2.50) and the first subproblem in (2.49) are identical to the retailer’s problem under pricing strategy \((f, f)\). Thus, we have \(p_i^{*(f,v)} = p_i^{*(f,f)}\) and \(x^* = p_{j1}^{*(f,f)}\). Since \(p_i^{*(f,v)} = p_i^{*(f,f)}\), we have \(\pi_i^{*(f,v)} = \pi_i^{*(f,f)}\). This proves parts 1 and 4. Solving the second subproblem in (2.49), we get \(y^* = \frac{a_{j1} - a_{j2}}{4(b + 2\delta + \theta + \gamma)}\).

Given \(x^*\) and \(y^*\), we can determine the optimal retail price \((p_{j1}^{*(f,v)}, p_{j2}^{*(f,v)})\) and the corresponding profit \(\pi_j^{*(f,v)}\) as displayed in parts 2, 3, and 5, respectively.\(\square\)

Proof of Lemma 2: The proof is identical to the proof of Lemma 1 when we exchange the index \(i\) with index \(j\). We omit the details.

Proof of Lemma 3: Under pricing strategy \((v, v)\), we transform the decision variables \(p_{i1}, p_{i2}, p_{j1}\) and \(p_{j2}\) as follows:

\[
p_{i1} = u + v, \quad p_{i2} = u - v, \quad \text{and} \quad p_{j1} = x + y, \quad p_{j2} = x - y.
\]
The retailer’s problem can be rewritten as:

\[
\pi^*_i(v,v) = \max_u (u - c)(a_{i1} + a_{i2} - 2bu + 2(\theta + \gamma)(x - u)) + \max_v v(a_{i1} - a_{i2} - 2bv - 4\delta v + 2(\theta - \gamma)y - 2(\theta + \gamma)v), \tag{2.51}
\]

\[
\pi^*_j(v,v) = \max_x (x - c)(a_{j1} + a_{j2} - 2bx - 2(\theta + \gamma)(x - u)) + \max_y y(a_{j1} - a_{j2} - 2by - 4\delta y + 2(\theta - \gamma)v - 2(\theta + \gamma)y). \tag{2.52}
\]

The first subproblems in (2.51) and (2.52) are identical to the retailer’s problem under pricing strategy \((f, f)\). Therefore we have \(u^* = p^*_i(f,f)\) and \(x^* = p^*_j(f,f)\).

Solving the second subproblems in (2.51) and (2.52), we get

\[
v^* = \frac{(\theta - \gamma)(a_{j1} - a_{j2}) + 2(b + 2\delta + \theta + \gamma)(a_{i1} - a_{i2})}{2(2b + 3\theta + 4\delta + \gamma)(2b + 4\delta + \theta + 3\gamma)}, \quad \text{and}
\]

\[
y^* = \frac{(\theta - \gamma)(a_{i1} - a_{i2}) + 2(b + 2\delta + \theta + \gamma)(a_{j1} - a_{j2})}{2(2b + 3\theta + 4\delta + \gamma)(2b + 4\delta + \theta + 3\gamma)}.
\]

Thus we have proved parts 1 and 2. The optimal profit in part 3 is easy to check. We omit the details.

**Proof of Proposition 5:** By examining the inequalities (2.37) through (2.40), it is easy to show that both retailers will adopt the variable pricing strategy \((v, v)\) at the equilibrium. We omit the details.

### 2.8 References


CHAPTER 3

Responsive Pricing Under Supply Uncertainty

3.1 Introduction

Due to long supply lead time and short selling season, retailers usually can place their orders only once before the start of the selling season. For example, in the fashion industry, it is common for retailers to place their orders many months before the selling season. The reader is referred to Fisher and Raman (1996) for an excellent description of the ordering process in the fashion industry. Since accurate supply or demand information is rarely available in advance, it is difficult for retailers to determine cost effective order quantities. For instance, due to uncertain supply yields, transportation delays, shrinkage during shipment, retailers usually do not know for sure if they would receive the exact quantity being ordered prior to the selling season. Besides uncertain supply, retailers often face uncertain demand as well. As such, many retailers have to struggle with the overstocking and understocking issues. Specifically, overstocking forces a retailer to dispose unsold items at clearance prices, while understocking creates lost sales.

To reduce the overstocking and understocking costs, various researchers have examined different approaches for helping retailers to meet uncertain demand. First, Fisher and Raman (1996) consider a situation in which a retailer can place two separate orders prior to the selling season. By allowing the retailer to place the second order in a later period, the retailer can generate a more accurate
demand forecast by using the market signals observed during the time between
the first order and the second order. Fisher and Raman refer the second order as
consider a situation in which a retailer offers each customer two options: pre-
commit an order at a reduced price before the selling season, or buy the product
at the regular price during the selling season. Clearly, the reduced price serves as
an incentive for the customers to pre-commit their orders before the selling season.
Tang et al. (2004) show how the retailer can use these pre-committed orders to
generate more accurate demand forecasts, which would enable the retailer to
place more cost effective orders and manage uncertain demand more efficiently.
These two ideas articulated in Fisher and Raman (1986) and Tang et al. (2004)
would certainly improve supply chain performance; however, some suppliers and
retailers may have specific concerns about the requirements associated with these
two ideas. For example, to implement the accurate response concept, the supplier
needs to have sufficient capacity to handle the second order on short notice.
Also, to implement the early commitment program, a retailer needs to show his
customers a sample of the seasonal products before the selling season, which could
make the retailer or the manufacturer more vulnerable to copycats.

In this chapter, we consider a situation in which the accurate response and the
early commitment program are impractical due to the aforementioned concerns.
Instead of having the flexibility to place two separate orders or to sell the product
at two different retail prices, we consider a situation in which the retailer can place
exactly one order and select only one retail price before the selling season. This
concept is known as ‘responsive pricing.’ Van Mieghem and Dada (1999) is the
first to analyze the responsive pricing concept as a mechanism for a retailer to
manage uncertain demand. In their paper, they present a two-stage stochastic
model in which the retailer places an order in the first period. Then the retailer
would determine the retail price after the demand uncertainty is resolved at the end of the first period but before the selling season that starts at the beginning of the second period. They show the benefits of delaying the pricing decision until the demand uncertainty is resolved. Motivated by the product postponement concept examined in Lee and Tang (1997), Chod and Rudi (2005) extend the work of Van Mieghem and Dada to the two-product case. Specifically, they consider the case in which the retailer places an order of a ‘generic’ product in the first period. Then, after the demand uncertainty is resolved, the retailer would customize this order of generic product into two individual products and then determine the retail price for each of these two products. With the additional flexibility gained from delaying the product identity and delaying the pricing decision until demand uncertainty is resolved, Chod and Rudi present a two-period model to illustrate the benefit of product postponement under responsive pricing.

While Van Mieghem and Dada (1999) and Chod and Rudi (2005) focus on the issue of demand uncertainty, we examine the benefits of responsive pricing under supply uncertainty. As an initial attempt to analyze the issue of responsive pricing under supply uncertainty, we shall consider a situation in which the demand function is known but the supply yield is uncertain. By focusing on the issue of uncertain supply, we are able to develop tractable results. It is our hope that our model and our analysis presented in this chapter can be used as a building block for analyzing responsive pricing under uncertain supply and uncertain demand in the near future. In the context of supply uncertainty, various researchers have developed models for determining the optimal order quantity under uncertain supply. The reader is referred to Yano and Lee (1993) for a comprehensive review of lot sizing models with uncertain supply yield. To our knowledge, our model is the first that examines the joint decisions of order quantity and retail pricing under supply uncertainty. In this chapter, we first present a two-stage stochastic
model for analyzing two pricing policies: No Responsive Pricing and Responsive Pricing. Under the No Responsive Pricing policy, the retailer would determine the order quantity and the retail price at the beginning of the first period before the supply yield is realized. Under the Responsive Pricing policy, the retailer would determine the order quantity at the beginning of the first period. Then, after the actual supply yield is realized at the end of the first period, the retailer would then determine the retail price at the end of the first period but before the selling season that starts at the beginning of the second period. By delaying the pricing decision after the actual supply yield is realized, the Responsive Pricing policy enables the retailer to use pricing as a response mechanism for managing uncertain supply.

This chapter is divided into two parts. In the first part, we first show analytically that, under supply uncertainty, the retailer would always obtain a higher expected profit under the Responsive Pricing policy. (This result is consistent with the result obtained by Van Mieghem and Dada (1999) when they analyze the benefits of responsive pricing under demand uncertainty.) We also examine the impact of yield distribution and system parameters on the optimal order quantities, prices, and retailer’s profits under these two pricing policies numerically. Our numerical analysis indicates that the Responsive Pricing policy is more beneficial when the supply yield is highly uncertain (low mean or high variance) or when the unit cost is high. Since the Responsive Pricing policy dominates the No Responsive Pricing policy, we extend our analysis of the Responsive Pricing to examine three separate issues in the second part of this chapter. The first issue deals with the case in which the retailer can place an emergency order with an alternative source after observing the actual supply yield. The second issue relates to the issue of supplier selection with uncertain supply yield. This issue is related to the supplier selection models developed by Tang (1988) and Targaras
and Lee (1991) that capture the interaction of supplier’s yield and the retailer’s 
process and inspection policy. Finally, the third issue deals with a situation in 
which the retailer has to allocate his order among multiple suppliers. Ramasesh 
et al. (1991) develop a model for analyzing ways to allocate an order among dif-
ferent suppliers with different lead times. The key difference between our model 
and the models examined by Tang, Targaras and Lee, and Ramasesh et al. is 
that our model incorporates the pricing decision explicitly.

This chapter is organized as follows. In Section 3.2, we present the base model 
for analyzing the retailer’s optimal expected profits under the No Responsive 
Pricing policy and the Responsive Pricing policy. We show analytically that 
the retailer would always obtain a higher expected profit under the Responsive 
Pricing policy. We also develop numerical experiments to examine the benefits of 
the Responsive Pricing policy. Section 3.3 extends our analysis of the Responsive 
Pricing policy to the case in which the retailer can place an emergency order after 
the actual supply yield is realized. In Section 3.4, we apply our analysis of the 
Responsive Pricing policy to examine the issue of supplier selection. Section 3.5 
deals with a situation in which the retailer can order from multiple suppliers with 
different supply yield distributions. This chapter is concluded in Section 3.6.

3.2 The Base Model

Consider a situation in which a retailer orders a seasonal product from a supplier 
and sells the product over a selling season. We assume that the retailer knows 
that the product demand function is given by: \( D = \alpha - \beta p \), where \( \alpha > 0 \) 
represents the potential market size, \( \beta > 0 \) represents the price sensitivity, and \( p \) 
represents the retail price. While the retailer has perfect information about the 
demand function, he has to deal with supply uncertainty in the following manner.
First, the supply lead time is equal to one period, and hence, the retailer needs to decide on the order quantity \( Q \) at the beginning of period 1 so that the retailer will receive the order at the end of period 1 prior to the selling season that starts at the beginning of period 2 and ends at the end of period 2. Second, the supply yield is uncertain in the following sense. For any order quantity \( Q \), the retailer will receive only \( yQ \) non-defective units, where \( y \) represents the supply yield. (Essentially, the yield \( y \) accounts for the defective rate of the supplier as well as the damage or shrinkage occurred during shipment.) In our model, we assume that \( y \) is a random variable that takes on \( N \) different discrete values, say, \( y_n \) for \( n = 1, 2, \cdots, N \), where \( 0 < y_1 < y_2 < \cdots < y_{N-1} < y_N \leq 1 \). Let \( \lambda_n \) be the probability that the actual yield is equal to \( y_n \); i.e., \( \text{Prob}\{y = y_n\} = \lambda_n \) for \( n = 1, 2, \cdots, N \). Hence, \( \sum_{n=1}^{N} \lambda_n = 1 \). Third, the retailer pays the supplier \( c \) per unit at the beginning of period 1, and the retailer disposes the unsold units at \( s \) per unit at the end of period 2. Without loss of generality, it is easy to show that one can transform the cost parameters so that one can set \( s = 0 \). To simply our exposition, we shall consider the case when \( s = 0 \).

In this chapter, we consider two pricing policies: No Responsive Pricing (NRP) policy and Responsive Pricing (RP) policy. Under the NRP policy, the retailer specifies the order quantity \( Q \) and the retail price \( p \) jointly at the beginning of period 1. Under the RP policy, the retailer first decides on the order quantity \( Q \) at the beginning of period 1. Then he specifies the retail price \( p \) at the beginning of period 2 after he observes the actual supply yield realized at the

---

\(^{1}\)In our model, we assume that the uncertain supply yield has a general discrete distribution, which allows us to approximate any continuous distribution. Notice that our way of capturing uncertainty is different from the models examined by Van Mieghem and Dada (1999) and Chod and Rudi (2005). First, we examine supply uncertainty, while they consider demand uncertainty. Second, we assume that the supply yield follows a general discrete probability distribution; however, they assume that the demand function is additive (a linear function of price plus an uncertain error term) and that the distribution of the error term is Uniform or Exponential in the former and Bivariate Normal in the latter.
end of period 1. As such, the RP policy allows the retailer to delay the pricing decision until the actual supply yield is realized. In other words, the RP policy enables the retailer to use pricing as a response mechanism for managing uncertain supply. Given these two pricing strategies, we are interested in examining the following questions:

1. Would the retailer obtain a higher expected profit under the RP policy? If yes, by how much?

2. How would the yield distribution affect the retailer’s optimal expected profits, order quantities, and prices under these two pricing policies?

To answer these questions, we now formulate the retailer’s problem under these two policies.

3.2.1 Problem Formulation

First, under the NRP policy, the retailer has to determine the optimal order quantity $Q'$ and optimal retail price $p'$ at the beginning of period 1 so as to maximize the retailer’s expected profit. In this case, the retailer’s problem $P(NRP)$ can be formulated as follows:

$$\Pi' = \max_{Q,p} E_y \{-cQ + p \cdot \min\{yQ, D\}\}. \quad (3.1)$$

Next, under the RP policy, the retailer would first determine the order quantity $Q$ at the beginning of period 1. Then the retailer would determine the retail price $p$ at the beginning of period 2 after observing the actual yield realized at the end of period 1. In order to determine the optimal order quantity $Q^*$ and the optimal retail price $p^*$ so that the retailer’s expected profit is maximized, we can
formulate the retailer’s problem $P(RP)$ as follows:

$$\Pi^* = \max_Q -cQ + E_y\{\max_p \{p \cdot \min\{yQ, D\}\}\}.$$  \hfill (3.2)

Let us compare the retailer’s expected profits under these two policies. Suppose we implement the optimal order quantity $Q'$ and the optimal retail price $p'$ under the NRP policy. Then it is easy to check from (3.2) and (3.1) that

$$\Pi^* \geq -cQ' + E_y\{p' \min\{yQ', (\alpha - \beta p')\}\} = \Pi'.$$

This implies that the optimal expected profit under the RP policy is always higher than that under the NRP policy. This result is intuitive because, under the Responsive Pricing policy, the retailer has the flexibility to select a ‘more profitable’ retail price after observing the actual yield. While it is clear that the RP policy enables the retailer to obtain a higher profit, we would like to compare the retailer’s optimal expected profits, order quantities, and retail prices under both pricing policies and to examine the impact of the yield distribution on these quantities. However, such comparison is analytically intractable, but it can be done numerically. In the remainder of this section, we first determine the retailer’s optimal expected profits, order quantities, and retail prices under these two policies analytically. We then compare these quantities numerically.

### 3.2.2 Analysis of the No Responsive Pricing policy

Since $D = \alpha - \beta p$, the No Responsive Pricing problem $P(NRP)$ given in (3.1) can be rewritten as:

$$\Pi' = \max_{Q,p} E_y\{\Pi'(Q,p|y)\}, \text{ where }$$

$$\Pi'(Q,p|y) = \begin{cases} -cQ + pyQ & \text{if } y \leq \frac{\alpha - \beta p}{Q}, \\ -cQ + p(\alpha - \beta p) & \text{if } y > \frac{\alpha - \beta p}{Q}. \end{cases}$$
Essentially, $\Pi'(Q, p|y)$ represents the retailer’s profit associated with order quantity $Q$ and retail price $p$ for any given realization of yield $y$. The underlying structure of problem $P(NRP)$ resembles the joint pricing and ordering decision for the Newsvendor problem examined by Petruzzi and Dada (1999). While there is no simple closed form solution for this problem, we now present a simple approach for solving problem $P(NRP)$.

Suppose we set $y_0 = 0$ and $y_{N+1} = Y$, where $Y$ is a sufficiently large number. For any given $Q$ and $p$, there must exist a $k$, $k = 0, 1, \ldots, N$, so that the ratio $\frac{\alpha - \beta p}{Q}$ satisfies $y_k \leq \frac{\alpha - \beta p}{Q} < y_{k+1}$. For this particular $k$, we can take the expectation of $\Pi'(Q, p|y)$ to show that the retailer’s expected profit can be rewritten as:

$$E_y\{\Pi'(Q, p|y)\} = \sum_{m=1}^{k} \lambda_m(-cQ + py_mQ) + \sum_{m=k+1}^{N} \lambda_m(-cQ + p(\alpha - \beta p))$$

if $y_k \leq \frac{\alpha - \beta p}{Q} < y_{k+1}$.

By denoting the terms $\sum_{1}^{0} = 0$ and $\sum_{N+1}^{N} = 0$, we can decompose problem $P(NRP)$ into $N + 1$ subproblems $P_k(NRP)$, where $k = 0, 1, \ldots, N$ and each subproblem $P_k(NRP)$ can be expressed as:

$$\Pi'_k = \max_{Q, p} \left\{ \sum_{m=1}^{k} \lambda_m(-cQ + py_mQ) + \sum_{m=k+1}^{N} \lambda_m(-cQ + p(\alpha - \beta p)) \right\} \quad (3.3)$$

subject to $y_k \leq \frac{\alpha - \beta p}{Q} < y_{k+1}$.

Let $Q'_k$ and $p'_k$ be the optimal order quantity and retail price for subproblem $P_k(NRP)$, respectively. Also, let $k' \in \arg\max\{\Pi'_k : k = 0, 1, 2, \ldots, N\}$. In this case, one can utilize the solutions of these $N + 1$ subproblems $P_k(NRP)$ to determine the optimal expected profit and the optimal solutions to the original problem $P(NRP)$ as follows:

$$\Pi' = \Pi'_{k'}, \quad Q' = Q'_{k'}, \quad p' = p'_{k'}.$$  

(3.4)
However, the uniqueness of the above optimal solution could not be established for the general discrete distribution, which is consistent with the result obtained by Van Mieghem and Dada (1999).

It remains to determine the optimal solutions to subproblem $P_k(NRP)$. In preparation, let us define two partial sums that will become useful. Let:

$$u_k = \sum_{m=1}^{k} \lambda_m y_m \quad \text{and} \quad v_k = \sum_{m=k+1}^{N} \lambda_m.$$  \hfill (3.5)

For notational convenience, we let $u_0 = 0$ and $v_N = 0$. In this case, it is easy to check from the definitions that $u_k$ is increasing in $k$ and $u_N = \sum_{n=1}^{N} \lambda_n y_n = E(y) = \mu$. Also, note that $v_k$ is decreasing in $k$ and $v_0 = 1$. By considering the terms $u_k$ and $v_k$ along with subproblem $P_k(NRP)$ given in (3.3), we can establish the following Proposition:

**Proposition 1.** The solutions for subproblem $P_k(NRP)$ can be expressed as follows:

1. Suppose $\frac{\beta c}{\alpha} < \frac{u_k(u_k+v_ky_k+1)}{u_k+2v_ky_k+1}$. Then $p'_k = \frac{c}{2(u_k+y_kv_k)} + \frac{\alpha}{2\beta}$, $Q'_k = \frac{\alpha(u_k+v_ky_k) - \beta c}{4\beta y_k(u_k+v_ky_k)}$.

2. Suppose $\frac{u_k(u_k+v_ky_k+1)}{u_k+2v_ky_k+1} \leq \frac{\beta c}{\alpha} < \frac{u_k(u_k+v_ky_k)}{u_k+2v_ky_k}$. Then

$$p'_k = \begin{cases} \frac{\alpha}{2y_k} - \frac{\beta c}{2(u_k+v_ky_k)} & \text{if } \frac{\alpha(u_k+v_ky_k) - \beta c}{y_k(u_k+v_ky_k)} > \frac{\alpha(u_k+v_ky_k+1) - \beta c}{y_k(u_k+v_ky_k+1)} ; \\ \frac{c}{2(u_k+y_kv_k)} + \frac{\alpha}{2\beta} & \text{if } \frac{\alpha(u_k+v_ky_k) - \beta c}{y_k(u_k+v_ky_k)} \leq \frac{\alpha(u_k+v_ky_k+1) - \beta c}{y_k(u_k+v_ky_k+1)} ; \end{cases}$$

$$Q'_k = \begin{cases} \frac{\beta c}{2y_k} + \frac{\alpha}{2y_k+1} - \frac{\beta c}{2(u_k+v_ky_k+1)y_k+1} & \text{if } \frac{\alpha(u_k+v_ky_k) - \beta c}{y_k(u_k+v_ky_k)} > \frac{\alpha(u_k+v_ky_k+1) - \beta c}{y_k(u_k+v_ky_k+1)} ; \\ \frac{\alpha}{2y_k} - \frac{\beta c}{2(u_k+v_ky_k)} & \text{if } \frac{\alpha(u_k+v_ky_k) - \beta c}{y_k(u_k+v_ky_k)} \leq \frac{\alpha(u_k+v_ky_k+1) - \beta c}{y_k(u_k+v_ky_k+1)} ; \end{cases}$$

$$\Pi'_k = \begin{cases} \frac{\alpha(u_k+v_ky_k) - \beta c}{4\beta y_k(u_k+v_ky_k+1)} & \text{if } \frac{\alpha(u_k+v_ky_k) - \beta c}{y_k(u_k+v_ky_k)} > \frac{\alpha(u_k+v_ky_k+1) - \beta c}{y_k(u_k+v_ky_k+1)} ; \\ \frac{\alpha(u_k+v_ky_k) - \beta c}{4\beta y_k+1(u_k+v_ky_k+1)} & \text{if } \frac{\alpha(u_k+v_ky_k) - \beta c}{y_k(u_k+v_ky_k)} \leq \frac{\alpha(u_k+v_ky_k+1) - \beta c}{y_k(u_k+v_ky_k+1)} . \end{cases}$$
3. Suppose \( \frac{u_k (u_k + v_k y_k)}{u_k + 2v_k y_k} \leq \frac{\beta c}{\alpha} < u_k + v_k y_k + 1 \). Then \( p'_k = \frac{c}{2(u_k + v_k y_k + 1)} + \frac{\alpha}{2 \beta} \), \( Q'_k = \frac{\alpha}{2y_k + 1} - \frac{\beta c}{2(u_k + v_k y_k + 1) y_k + 1} \) and \( \Pi'_k = \frac{\alpha (u_k + v_k y_k + 1) - \beta c}{4y_k + 1(u_k + v_k y_k + 1)} \).

4. Suppose \( \frac{\beta c}{\alpha} \geq u_k + v_k y_k + 1 \). Then \( Q'_k = 0 \), \( p'_k \) is arbitrary and \( \Pi'_k = 0 \).

**Proof:** All proofs are given in the Appendix.

We can interpret Proposition 1 by considering two cases. First, consider the case when the expected yield is sufficiently low, say, when \( E(y) = \mu \leq \frac{\beta c}{\alpha} \). In this case, we have: 
\[
\frac{\beta c}{\alpha} \geq \mu = \sum_{m=1}^{k} \lambda_m y_m + \sum_{m=k+1}^{N} \lambda_m y_m \geq \sum_{m=1}^{k} \lambda_m y_m + [\sum_{m=k+1}^{N} \lambda_m] y_k + 1 = u_k + v_k y_k + 1 \text{ for } k = 0, 1, 2, \ldots, N.
\]
We can apply the fourth statement in Proposition 1 to show that \( Q'_k = 0 \) for \( k = 0, 1, 2, \ldots, N \). This implies that under the No Responsive Pricing policy, when the expected yield \( \mu \) is sufficiently low; i.e., when \( \mu \leq \frac{\beta c}{\alpha} \), it is optimal for the retailer to order nothing; i.e., \( Q' = 0 \). Second, consider the case when the expected yield is sufficiently high, say, when \( \mu > \frac{\beta c}{\alpha} \). In this case, it is easy to check from (3.5) that 
\[
\frac{u_k (u_k + v_k y_k + 1)}{u_k + 2v_k y_k + 1} = \mu > \frac{\beta c}{\alpha} \text{ when } k = N.
\]
Hence, we can apply the first statement in Proposition 1 to show that it is optimal for the retailer to order something when \( k = N \); i.e., \( Q'_N > 0 \). Combine this observation with the fact that \( \Pi' \geq \Pi'_N \), it is easy to show that the retailer’s optimal order quantity \( Q' > 0 \) when the expected yield \( \mu \) is sufficiently high; i.e., when \( \mu > \frac{\beta c}{\alpha} \). These two cases enable us to establish the threshold for \( \mu \) above which the retailer would order something in the optimal solution.

**3.2.3 Analysis of the Responsive Pricing Policy**

We now analyze the retailer’s optimal expected profit under the Responsive Pricing policy. Observe from (3.2) that problem \( P(RP) \) can be rewritten as:
\[
\Pi^* = \max_Q \{ E_y \max_p \{ \Pi(Q, p|y) \} \}, \tag{3.6}
\]
where $\Pi(Q,p|y)$ is the retailer’s profit associated with order quantity $Q$ and retail price $p$ for any given realization of $y$. The term $\Pi(Q,p|y)$ can be expressed as:

$$
\Pi(Q,p|y) = \begin{cases} 
-cQ + pyQ & \text{if } p \leq \frac{\alpha - yQ}{\beta}, \\
-cQ + p(\alpha - \beta p) & \text{if } p > \frac{\alpha - yQ}{\beta}.
\end{cases}
$$

By differentiating $\Pi(Q,p|y)$ with respect to $p$ and by considering the break points, we can determine the optimal retail price $p^*(Q|y)$ associated with order quantity $Q$ placed at the beginning of period 1 and the actual yield $y$ realized at the end of period 1 as follows:

$$
p^*(Q|y) = \begin{cases} 
\frac{\alpha - yQ}{\beta} & \text{if } Q \leq \frac{\alpha}{2y}, \\
\frac{\alpha}{2y} & \text{if } Q > \frac{\alpha}{2y}.
\end{cases}
$$

(3.7)

Substitute the optimal retail price $p^*(Q|y)$ into $\Pi(Q,p^*|y)$, getting:

$$
\Pi(Q,p^*|y) = \begin{cases} 
-cQ + (\frac{\alpha - yQ}{\beta})yQ & \text{if } Q \leq \frac{\alpha}{2y}, \\
-cQ + \frac{\alpha^2}{4y} & \text{if } Q > \frac{\alpha}{2y}.
\end{cases}
$$

(3.8)

Before we take the expectation of $\Pi(Q,p^*|y)$ with respect to $y$, let us define an additional partial sum in addition to the terms $u_k$ and $v_k$ defined in (3.5). Let:

$$
w_k = \sum_{m=1}^{k} \lambda_m y_m^2,
$$

(3.9)

where $w_0 = 0$. Notice that $w_k$ is increasing in $k$. Let us also define the following break points: $x_{N+1} = 0, x_N = \frac{\alpha}{2y_N}, \ldots, x_k = \frac{\alpha}{2y_k}$, for $k = N, N-1, \ldots, 1$, and $x_0 = X$, where $X$ is a large number. Since $y_1 < y_2 < \cdots < y_{N-1} < y_N$, $x_k$ is decreasing in $k$; i.e., $x_{N+1} < x_N < x_{N-1} < \cdots < x_1 < x_0$. In this case, there must exist a $k$ so that $x_{k+1} < Q \leq x_k$. By using the terms $u_k, v_k$ and $w_k$ and by using (3.8), we can show that the retailer’s expected profit $E_y(\Pi(Q,p^*|y))$ can be expressed as follows:

$$
E_y(\Pi(Q,p^*|y)) = \left(\frac{\alpha}{\beta}u_k - c\right)Q - \frac{w_k}{\beta}Q^2 + \frac{\alpha^2v_k}{4\beta} & \text{if } x_{k+1} < Q \leq x_k.
$$

(3.10)
where \( k = N, N - 1, \cdots, 0 \). Since the function \( (\frac{\alpha}{\beta} u_k - c)Q - \frac{w_k}{\beta}Q^2 + \frac{\alpha^2 v_k}{4\beta} \) is concave in \( Q \), it is easy to see that the objective function of problem \( P(RP) \) given in (3.6) is a piece-wise concave function. By evaluating the derivative of this piece-wise concave function at various break points \( x_k = \frac{\alpha}{2y_k} \), we can determine the unique optimal solution to the problem \( P(RP) \) as follows:

**Proposition 2.** Suppose \( \frac{\beta c}{\alpha} < \mu \). Then

\[
k^* = \arg\max \{ u_k - \frac{w_k}{y_k} < \frac{\beta c}{\alpha} : k = N, N - 1, \cdots, 1 \}.
\]

Also, the optimal order quantity \( Q^* \) and the retailer’s optimal expected profit \( \Pi^* \) under the Responsive Pricing policy can be expressed as:

\[
Q^* = \frac{\alpha u_{k^*} - \beta c}{2w_{k^*}}, \quad \text{and} \quad \Pi^* = \frac{(\alpha u_{k^*} - \beta c)^2}{4\beta w_{k^*}} + \frac{\alpha^2 v_{k^*}}{4\beta}.
\]

Moreover, suppose \( \frac{\beta c}{\alpha} \geq \mu \). Then \( Q^* = 0 \) and \( \Pi^* = 0 \).

It follows from Propositions 1 and 2 that it is optimal for the retailer to order nothing under both pricing policies when the expected yield \( \mu \) is sufficiently small, say, when \( \mu \leq \frac{\beta c}{\alpha} \). Therefore, it suffices to focus on the case when \( \mu > \frac{\beta c}{\alpha} \) throughout the chapter.

### 3.2.4 Numerical Analysis

We now utilize Propositions 1 and 2 to construct numerical experiments for comparing the retailer’s optimal expected profits, order quantities, and prices under the NRP and the RP policies. In addition, we shall examine the impact of the yield distribution and the unit cost on these quantities. In our numerical experiments, we consider the case when the supply yield follows a ‘discrete’ Uniform
distribution over $[a, b]$, where $0 < a < b \leq 1$. Specifically, we consider the case when the supply yield takes on $N$ possible values $y_1 < y_2 < \cdots < y_N$, where $y_k = a + \frac{b-a}{N-1} (k - 1)$ and $\text{Prob}(y = y_k) = 1/N$ for $k = 1, 2, \ldots, N$.\footnote{Since our analysis is based on the assumption that the supply yield has a general discrete distribution, we can use our general discrete distribution to approximate any continuous probability distribution. For example, our discrete Uniform distribution converges to the (continuous) Uniform distribution over $[a, b]$ when $N \to \infty$.} Hence, the expected supply yield $\mu = \mathbb{E}(y) = \frac{a+b}{2}$ and the variance of the supply yield $\sigma^2 = \text{Var}(y) = \frac{(a-b)^2(N+1)}{12(N-1)}$.

In our experiments, we set $N = 1000$, $\alpha = 100$, $\beta = 5$ and we vary the values of $a$, $b$ and $c$ so as to examine the impact of the yield distribution and the unit cost. In each set of experiments, we compute the retailer’s optimal expected profits and the optimal order quantities under the NRP and the RP policies. Since the optimal retail price under the Responsive Pricing $p^*(Q^*|y)$ given in (3.7) is based on the actual realization of the supply yield, we shall compare the optimal retail price under the No Responsive Pricing policy with the expected optimal retail price $E_y(p^*(Q^*|y))$ under the Responsive Pricing policy.

3.2.4.1 The Impact of Supply Yield Uncertainty

In our first set of experiments, we examine the impact of the standard deviation of the supply yield $\sigma$ on the retailer’s optimal expected profits, order quantities, and prices under the NRP and the RP policies. To do so, we set the unit cost $c = 3$ and we vary the standard deviation $\sigma$. To isolate the effect of $\sigma$, we vary the standard deviation of the supply yield $\sigma$ while keeping $\mu$ constant at $\mu = \frac{a+b}{2} = 0.5$. Specifically, we increase the value of $a$ from 0.1 to 0.48 and decrease the value of $b$ from 0.9 to 0.52 according to an increment of 0.02. As we vary the values of $a$ and $b$, we reduce the standard deviation of the supply
yield $\sigma$ from 0.23 to 0.012. The impact of $\sigma$ on various quantities are reported in Figures 4.1, 4.2 and 4.3. Figure 4.1 confirms that the retailer would always obtain a higher expected profit under the RP policy; i.e., $\Pi^* > \Pi'$. Observe from Figure 1 that the profit gap $\Pi^* - \Pi'$ increases as the supply yield becomes more variable; i.e., when $\sigma$ increases. This observation implies that the RP policy is more beneficial when the supply yield is more variable. Next, notice from Figure 4.2 that the retailer would order more under the RP policy. This result is intuitive because the retailer can afford to order more at the beginning of period 1 because he has the flexibility to set the price after the actual yield is realized. Observe from Figure 4.2 that the optimal order quantities are decreasing in $\sigma$ under both pricing strategies. This result is consistent with the well-known property of the optimal newsvendor order quantity (i.e., the optimal newsvendor order quantity decreases as the demand uncertainty increases). Finally, Figure 4.3 suggests that the optimal retail prices are increasing in $\sigma$ under both pricing strategies. However, as the supply yield becomes more uncertain, the retailer would charge a much higher price under the RP policy. Combining the results displayed in Figures 4.2 and 4.3, we can conclude that, relatively speaking, the retailer would place a larger order and charge a higher price (in expectation) under the RP policy as the standard deviation of the supply yield $\sigma$ increases. This observation explains why the profit gap $\Pi^* - \Pi'$ increases as the supply yield becomes more variable in Figure 4.1.
Figure 3.1: Impact of $\sigma$ on optimal profit

Figure 3.2: Impact of $\sigma$ on optimal order quantity

Figure 3.3: Impact of $\sigma$ on optimal price
3.2.4.2 The Impact of the Expected Supply Yield

In our second set of experiments, we examine the impact of the expected supply yield $\mu$ on the retailer’s optimal expected profits, order quantities, and prices. To do so, we set the unit cost $c = 3$ and we vary $\mu$. To isolate the effect of $\mu$, we vary $a$ from 0.1 to 0.6 according to an increment of 0.02 and set $b = a + 0.4$. This would allow us to vary the expected supply yield $\mu$ from 0.3 to 0.8 while keeping $\sigma^2$ constant, where $\sigma^2 = \frac{(a-b)^2(N+1)}{12(N-1)} = 0.0133$. The computational results are summarized in Figures 4.4, 4.5, and 3.6. Figure 4.4 suggests that, under both pricing policies, the retailer’s optimal expected profits increase as the expected supply yield $\mu$ increases. This result is intuitive because the retailer should be able to obtain a higher expected profit when the expected supply yield is higher. Notice from Figure 4.5 that, under both pricing policies, the optimal order quantities are not monotonic in $\mu$. In addition, unlike the result reported in Figure 4.2, Figure 4.5 suggests that the retailer would order less under the Responsive Pricing policy when the expected supply yield $\mu$ is sufficiently high, say, when $\mu > 0.56$. Observe from Figure 3.6 that, under both pricing policies, the optimal retail prices are decreasing in $\mu$. This result is intuitive. However, it is interesting to observe from Figure 3.6 that, under the RP policy, the retailer would charge a lower price (in expectation) when $\mu$ is sufficiently large.
Figure 3.4: Impact of $\mu$ on optimal profit

Figure 3.5: Impact of $\mu$ on optimal order quantity

Figure 3.6: Impact of $\mu$ on optimal price
3.2.4.3 The Impact of Unit Cost

In our third set of experiments, we investigate the impact of the unit cost $c$ on the retailer’s optimal expected profits, order quantities, and prices under these two policies. To do so, we set $a = 0.1$, $b = 1$, and we vary $c$ from 1 to 10 according to an increment of 0.2. The computational results are summarized in Figures 3.7, 3.8, and 3.9. Figure 3.7 confirms that the retailer would obtain a higher expected profit under the Responsive Pricing policy; however, the profit gap $\Pi^* - \Pi'$ decreases as the unit cost $c$ increases. Moreover, if one measures the relative benefit of RP policy over the NRP policy according to the relative profit gain $\Delta = \frac{\Pi^* - \Pi'}{\Pi'}$, then one can check that the relative profit gain $\Delta$ is increasing in the unit cost $c$. The corresponding figure is omitted here. Therefore, we can conclude that the RP policy becomes more beneficial when the unit cost $c$ is high. Next, Figure 3.8 indicates that the retailer would order less under the RP policy when the unit cost $c$ is sufficiently low, say, when $c < 2.6$. Figure 3.9 suggests that the optimal prices are increasing in the unit cost $c$, which is intuitive. Finally, Figure 3.9 implies that the retailer should always set a higher retail price (in expectation) under the RP policy.

![Figure 3.7: Impact of unit cost on optimal profit](image)

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In this section, we have presented a model for analyzing the retailer’s optimal expected profits, order quantities and prices under the two pricing policies. We have shown analytically and numerically that the Responsive Pricing dominates the No Responsive Pricing policy. In addition, we have demonstrated numerically that the Responsive Pricing policy is even more beneficial to the retailer when the supply yield is more uncertain (low mean or high variance) or when the unit ordering cost is high. Since the Responsive Pricing policy dominates the No Responsive Pricing policy, we shall focus our attention on the Responsive Pricing policy in the remainder of this chapter. Specifically, in the next 3 sections, we
shall extend our analysis of the Responsive Pricing policy to examine three issues: emergency order, supplier selection, and supplier order allocation.

3.3 Extension 1: Emergency Order under Responsive Pricing

Let us consider a situation in which the retailer can place an emergency order from an alternative source as follows: after the retailer receives $yQ$ non-defective units at the end of period 1, he can order $(1 - y)Q$ non-defective units from this perfect source and receive this emergency order immediately so as to bring the total non-defective units up to $Q$ at the beginning of period 2. We now examine the conditions under which the retailer would place an emergency order. Suppose that the unit cost for this emergency order is $c^e$, where $c^e > c$. By observing the fact that the retailer can always use the emergency order to ensure $Q$ units are available at the beginning of period 2, we can formulate the retailer’s problem with emergency order under the RP policy as problem $P(EO)$, where:

$$\Pi^e = \max_Q -cQ + E_y\{-c^e(1 - y)Q + \max_p \{ p \cdot \min\{Q, D\} \}\}.$$ 

Rearranging the terms, problem $P(EO)$ can be simplified as:

$$\Pi^e = \max_Q \max_p \{\Pi^e(Q, p)\}, \text{ where}$$

$$\Pi^e(Q, p) = \begin{cases} 
-\hat{c}Q + pQ & \text{if } p \leq \frac{\alpha - Q}{\beta}, \\
-\hat{c}Q + p(\alpha - \beta p) & \text{if } p > \frac{\alpha - Q}{\beta}, \text{ and}
\end{cases}$$

$$\hat{c} = c + c^e(1 - \mu).$$
By considering the first order condition, the optimal retail price $p^e(Q)$ is given as:

$$p^e(Q) = \begin{cases} \frac{\alpha - Q}{\beta} & \text{if } Q \leq \frac{\alpha}{2}, \\ \frac{\alpha}{2} & \text{if } Q > \frac{\alpha}{2}, \end{cases}$$

(3.14)

Notice that the optimal retail price is independent of the realized yield, we can conclude that the optimal retail price remains the same regardless of the pricing policy is responsive or not. Substitute the optimal retail price $p^e(Q)$ into $\Pi^e(Q, p^e(Q))$, getting:

$$\Pi^e(Q, p^e(Q)) = \begin{cases} -\hat{c}Q + \frac{(\alpha - Q)Q}{\beta} & \text{if } Q \leq \frac{\alpha}{2}, \\ -\hat{c}Q + \frac{\alpha^2}{4\beta} & \text{if } Q > \frac{\alpha}{2}. \end{cases}$$

It remains to determine the optimal order quantity $Q^e$ that maximizes the expected profit $\Pi^e(Q, p^e(Q))$. By considering the first order condition at the boundary points and assuming $\alpha > \beta \hat{c}$, it is easy to show that the optimal order quantity and the retailer’s optimal expected profit can be expressed as:

$$Q^e = \frac{\alpha - \beta \hat{c}}{2}, \text{ and}$$

$$\Pi^e = \frac{(\alpha - \beta \hat{c})^2}{4\beta}.$$  

(3.15) \hspace{1cm} (3.16)

By comparing the optimal profits given in (3.13) and (3.16), we can establish the following Proposition:

**Proposition 3.** It is optimal for the retailer to place an emergency order if

$$e^e < \frac{(\alpha - \beta c) - \sqrt{((\alpha u_k^* - \beta c)^2 + \alpha^2 u_k^*)}}{(1 - \mu)\beta},$$

where $k^*$ is given by (3.11).

Proposition 3 suggests that it is optimal to use the emergency order when the unit cost associated with the emergency order is sufficiently small.
Proposition 3 has the following implications. Consider the case when \( c^e \) is low enough so that the retailer decides to use the emergency order. In this case, the retailer would place an order at the beginning of period 1 according to the optimal order quantity \( Q^e \) given in (3.15), i.e., \( Q^e = \frac{\alpha - \beta c^e}{2} \). Since \( Q^e < \frac{\alpha}{2} \), it is easy to check from (3.14) that \( p^e = \frac{\alpha - Q^e}{\beta} \). This implies that \( Q^e = \alpha - \beta p^e \). Combine this result with the fact that the retailer would always place an emergency order to bring the total number of non-defective units up to \( Q^e \) at the beginning of period 2, we can conclude that the retailer would set the optimal retail price \( p^e \) so as to sell all \( Q^e \) units in period 2.

### 3.4 Extension 2: Supplier Selection

In many instances, suppliers with different yield distributions would normally charge different unit costs. Consider the case in which the retailer has to select exactly one supplier among a set of potential suppliers. (We shall consider the case when the retailer may order from multiple suppliers in the next section.) In this case, the retailer can use (3.13) to evaluate the expected profits associated with different suppliers and then choose the supplier that yields the highest expected profit. We omit the details.

To examine the impact of yield distribution on supplier selection, let us consider a special case that can be described as follows. There are two potential suppliers \( i \) and \( j \), each of which has only two possible yields \( y_{s1} \) and \( y_{s2} \) with equal probability of 0.5 for \( s = i, j \). For each supplier \( s \), it is easy to check from (3.5) and (3.9) that \( u_{s0} = 0, u_{s1} = 0.5y_{s1}, u_{s2} = 0.5y_{s1} + 0.5y_{s2} = \mu_s, v_{s0} = 1, v_{s1} = 0.5, v_{s2} = 0, w_{s0} = 0, w_{s1} = 0.5y_{s1}^2, \) and \( w_{s2} = 0.5(y_{s1}^2 + y_{s2}^2) \). By considering the definition of \( k^* \) given in (3.11), it is easy to check that \( k^*_s = 2 \) if
Corollary 4. Under the RP policy, the optimal order quantity $Q^*_s$ and the optimal expected profit $\Pi^*_s$ for supplier $s$, $s = i, j$, can be expressed as follows:

$$Q^*_s = \begin{cases} 
\frac{0.5(\mu_1-\beta c_s)}{y_{s1}} & \text{if } \mu_s - \frac{0.5(y_{s1}^2+y_{s2}^2)}{y_{s2}} \geq \frac{\beta c_s}{\alpha}, \\
\frac{\alpha \mu_s-\beta c_s}{y_{s1}^2+y_{s2}^2} & \text{if } \mu_s - \frac{0.5(y_{s1}^2+y_{s2}^2)}{y_{s2}} < \frac{\beta c_s}{\alpha}, \quad \text{and} \\
\frac{(\alpha y_{s1}^2-\beta c_s)^2}{2\beta y_{s1}^2} + \frac{\alpha^2}{8\beta} & \text{if } \mu_s - \frac{0.5(y_{s1}^2+y_{s2}^2)}{y_{s2}} \geq \frac{\beta c_s}{\alpha}, \\
\frac{(\alpha \mu_s-\beta c_s)^2}{2\beta (y_{s1}^2+y_{s2}^2)} & \text{if } \mu_s - \frac{0.5(y_{s1}^2+y_{s2}^2)}{y_{s2}} < \frac{\beta c_s}{\alpha}.
\end{cases}$$

Suppose the expected yield of both suppliers are the same, say, $\mu_i = \mu_j = \mu$. Then we can compare the optimal profit for each supplier $s$ to establish the following Proposition:

Proposition 4. Suppose $\mu - \frac{0.5(y_{s1}^2+y_{s2}^2)}{y_{s2}} \geq \frac{\beta c_s}{\alpha}$ for $s = i, j$. Then it is optimal for the retailer to choose supplier $i$ if and only if: $\frac{y_{s1}}{c_i} \geq \frac{y_{s2}}{c_j}$. Moreover, suppose $\mu - \frac{0.5(y_{s1}^2+y_{s2}^2)}{y_{s2}} < \frac{\beta c_s}{\alpha}$ for $s = i, j$. Then it is optimal for the retailer to choose supplier $i$ if and only if $\frac{(\alpha \mu_s-\beta c_s)^2}{(y_{s1}^2+y_{s2}^2)} \geq \frac{(\alpha \mu_s-\beta c_s)^2}{(y_{s1}^2+y_{s2}^2)}$.

Proposition 4 can be interpreted as follows. Since $\mu_i = \mu_j = \mu$ and since the yields have equal probability, the yield distribution for each supplier $s$ can be expressed as: $y_{s1} = \mu - \sigma_s$ and $y_{s2} = \mu + \sigma_s$, where $\sigma_s$ corresponds to the standard deviation of the supplier’s yield. In this case, the first statement suggests that, when the expected supply yield $\mu$ is sufficiently high, supplier $i$ is preferred when $\frac{\mu - \sigma_i}{c_i} \geq \frac{\mu - \sigma_j}{c_j}$. Notice that the inequality $\frac{\mu - \sigma_i}{c_i} \geq \frac{\mu - \sigma_j}{c_j}$ is more likely to hold when $\sigma_i < \sigma_j$ or when $c_i < c_j$. Hence, the first statement implies that supplier $i$ is preferred when either the unit cost of supplier $i$ is lower or when supplier $i$’s yield is more stable. Next, observe that the term $(y_{s1}^2 + y_{s2}^2) = 2(\sigma_s^2 + \mu^2)$ for $s = i, j$. Hence, the second statement suggests that, when the expected supply
yield $\mu$ is sufficiently low, supplier $i$ is preferred when 
\[(\alpha \mu - \beta c_i)^2 \geq \frac{(\alpha \mu - \beta c_j)^2}{2(\sigma_i^2 + \mu^2)} \]
By observing the fact that the inequality 
\[\frac{(\alpha \mu - \beta c_i)^2}{2(\sigma_i^2 + \mu^2)} \geq \frac{(\alpha \mu - \beta c_j)^2}{2(\sigma_j^2 + \mu^2)}\]
is more likely to hold when $\sigma_i < \sigma_j$ or when $c_i < c_j$. Hence, the second statement implies that supplier $i$ is preferred when either the unit cost of supplier $i$ is lower or when supplier $i$'s yield is more stable.

### 3.5 Extension 3: Supplier Order Allocation

We now consider the case when the retailer can order from multiple suppliers with different yield distributions. To simplify our exposition, we shall consider the 2-supplier case. Consider a situation in which the retailer orders $Q_i$ units from supplier $i$ and $Q_j$ from supplier $j$. As such, the retailer will receive $y_iQ_i + y_jQ_j$ non-defective units of the same product at the end of period 1. We assume that for $s = i, j$, the supply yield $y_s$ is a discrete random variable that takes on $N$ different values of $y_{sn}$, where $n = 1, \cdots, N$. We define the probability 
\[\text{Prob}\{y_s = y_{sn}\} = \lambda_{sn}\]
so that $\sum_{n=1}^{N} \lambda_{sn} = 1$ for $s = i, j$. Under the Responsive Pricing policy, the retailer would first determine the optimal order quantities $Q_i$ and $Q_j$ at the beginning of period 1. Then he would specify the optimal retail price $p$ after the actual yield of each supplier is realized at the end of period 1. As such, there are 3 decision variables.

To simplify our exposition, we shall transform our decision variables $Q_i$ and $Q_j$ as follows. Let $Q$ be the total number of units ordered from the suppliers, where $Q = Q_i + Q_j$. Let $f \geq 0$ be the ratio between the order quantities; i.e.,
\[f = \frac{Q_i}{Q_j}\]
Notice that $f = 0$ when the retailer orders only from supplier $j$ and $f = \infty$ when the retailer orders only from supplier $i$. By noting that $Q_i = \frac{f}{1+f}Q$ and $Q_j = \frac{1}{1+f}Q$, we can formulate the retailer’s problem based on the decision
variables $Q, f$ and $p$. In preparation, let $c_s$ be the unit cost of supplier $s$, $s = i, j$. For any order quantities $Q_i$ and $Q_j$, we can express the total ordering cost in terms of $c^a Q$, where $c^a$ corresponds to the weighted average unit cost that can be expressed as $c^a = \frac{c_i f + c_j}{1 + f}$. By observing that the total number of non-defective units received by the retailer at the end of period 1 is equal to $y_{im} Q_i + y_{jn} Q_j$ for some $m, n = 1, 2, \cdots , N$, we can express the total number of non-defective units in terms of $rQ$, where $r$ corresponds to the effective yield with $N^2$ realizations $r_{mn} = \frac{y_{im} f + y_{jn}}{1 + f}$, and $m, n = 1, 2, \cdots , N$. Notice that the expected effective yield $\mu^a = E(r) = E_{y_i, y_j}(\frac{y_{im} + y_{jn}}{1 + f}) = \mu_i + \mu_j$. In this case, it is easy to check that, under the RP policy, the retailer’s supplier order allocation problem can be formulated as problem $P(SOA)$, where:

$$\Pi^a = \max_{Q, f} -c^a Q + E_r \max_p \{ \min \{ rQ, D \} \} = \max_f \{ \max_Q E_r \max_p \{ \Pi^a(Q, f, p|\cdot) \} \}. \quad (3.18)$$

Notice that $P^a(Q, f, p|\cdot)$ is the retailer’s profit associated with order quantity $Q$, ratio $f$ and retail price $p$ for any given realization of $r$.

To solve problem $P(SOA)$ given in (3.18), we first solve the subproblem for any given ratio $f$ and then we determine the optimal ratio $f^a$ using a simple search algorithm. For any given value of $f$, we can check from (3.18) that the subproblem is given by $\max_Q E_r \max_p \{ \Pi^a(Q, f, p|\cdot) \}$, where:

$$\Pi^a(Q, f, p|\cdot) = \begin{cases} -c^a Q + prQ & \text{if } p \leq \frac{\alpha - rQ}{\beta}, \\ -c^a Q + p(\alpha - \beta p) & \text{if } p > \frac{\alpha - rQ}{\beta}. \end{cases} \quad (3.19)$$

Suppose we treat the effective yield $r$ as $y$. Then it is clear that the subproblem has the same structure as problem $P(RP)$ given in (3.2). Therefore, we can use exactly the same approach presented in Section 3.2.3 to solve this subproblem first and then determine the optimal ratio $f^a$ by using a simple search algorithm.
Recall from Section 3.2.3 that the solution approach for solving problem $P(RP)$ hinges upon the boundary points $\frac{\alpha}{2y_k}$, $k = N, N - 1, \cdots, 1$, as well as the terms $u_k, v_k$ and $w_k$. In order to apply the solution approach for solving problem $P(RP)$ to solve our subproblem, we need to develop a new index $l$, $l = 1, \cdots, L = N^2$ for the effective yield $r$. To do so, we first sort the realized effective yield $r_{mn} = \frac{y_m f + y_n}{1 + f}$ in ascending order and then assign the new index $l$ so that $r_1 < r_2 < \cdots < r_l < \cdots < r_L$. After creating the new index $l$ for the effective yield $r$, we can define the terms $u_k, v_k$ and $w_k$ as follows:

$$u_k = \sum_{l=1}^{k} \lambda_l r_{l1}, \quad v_k = \sum_{l=k+1}^{L} \lambda_l, \quad \text{and} \quad w_k = \sum_{l=1}^{k} \lambda_l r_{l1}^2,$$

(3.20)

where $u_0 = 0, v_L = 0$, and $w_0 = 0$. Thus we can apply Proposition 2 in Section 3.2.3 to determine the optimal solution to the problem $P(SOA)$ for any given ratio $f$ as follows:

**Corollary 5.** Suppose $\mu_s > \frac{\beta c_s}{\alpha}$ for $s = i, j$. Then, for any given ratio $f$, the optimal solutions to the retailer’s problem (3.18) can be expressed as follows:

$$Q^a = \frac{\alpha u_{k^a} - \beta c^a}{2 w_{k^a}}, \quad \text{where}$$

$$k^a = \arg \max \{ u_k - \frac{w_k}{r_k} < \frac{\beta c^a}{\alpha} : k = L, L - 1, \cdots, 1 \},$$

(3.21) (3.22)

$$p^a = \begin{cases} \frac{\alpha - Q^a}{\beta} & \text{if } Q^a \leq \frac{\alpha}{2f}, \\ \frac{\alpha}{2\beta} & \text{if } Q^a > \frac{\alpha}{2f}, \end{cases}$$

(3.23)

$$\Pi^a = \frac{(\alpha u_{k^a} - \beta c^a)^2}{4\beta w_{k^a}} + \frac{\alpha^2 v_{k^a}}{4\beta}.$$  

(3.24)

Given the retailer’s optimal expected profit for any given ratio $f$, we can determine the optimal ratio $f^a$ by using a simple search algorithm.

While it is difficult to determine the optimal ratio $f^a$ analytically for the general case, we can determine $f^a$ for a special case. In this special case, there
are two potential suppliers $i$ and $j$. Supplier $i$’s yield is constant and it is equal to $\mu$. Supplier $j$’s yield has two possible values $y_{j1}$ and $y_{j2}$ with equal probability of 0.5 and supplier $j$’s expected yield is also equal to $\mu$. Hence, we can express $y_{j1} = \mu - \sigma$ and $y_{j2} = \mu + \sigma$. Given supply yield of both suppliers, it is easy to check that the effective yield $r = \frac{y_{i}f + y_{j}}{1+f}$ takes on 2 possible values, namely, $r_{1} = \mu - \frac{\sigma}{f+1}$ and $r_{2} = \mu + \frac{\sigma}{f+1}$ with equal probability of 0.5. Since supplier $i$ has a constant yield $\mu$ and since supply $j$ has uncertain yield with mean $\mu$, it is reasonable to assume that $c_{i} > c_{j}$.

We now investigate the conditions under which the retailer would order from exactly one supplier or both suppliers. First, since $r_{1} = \mu - \frac{\sigma}{f+1}$ and $r_{2} = \mu + \frac{\sigma}{f+1}$, it is easy to check from (3.20) that:

$$u_{0} = 0, u_{1} = 0.5(\mu - \frac{\sigma}{f+1}), u_{2} = \mu, v_{0} = 1, v_{1} = 0.5, v_{2} = 0,$$

$$w_{0} = 0, w_{1} = 0.5(\mu - \frac{\sigma}{f+1})^{2}, \text{and } w_{2} = \mu^{2} + \frac{\sigma^{2}}{(f+1)^{2}}.$$  

It follows from (3.22) in Corollary 2 that $k_{a} = 2$ if $u_{2} - \frac{w_{2}}{r_{2}^{2}} - \frac{\beta c_{a}}{\alpha} < 0$, and $k_{a} = 1$, otherwise. By considering the terms $u_{2}, w_{2}, r_{2}$ and $c_{a}$ as functions of $f$, it is easy to show that the function $h(f) = u_{2} - \frac{w_{2}}{r_{2}^{2}} - \frac{\beta c_{a}}{\alpha}$ is a quadratic function of $f$. Hence, there exists two roots $\tau_{1} \leq \tau_{2}$ that satisfy $h(f) = 0$. To simplify our exposition, we shall focus on the case when $\tau_{1} < 0$ and $\tau_{2} > 0$. By comparing the profit function $\Pi_{a}$ associated with different values of $f$, we can establish the following Proposition:

**Proposition 5.** Suppose $\beta c_{i}(\mu + \sigma) + \beta c_{j}\mu - \alpha \sigma \mu > 0$ and $\beta c_{j}(\mu + \sigma) + \alpha \sigma^{2} - \alpha \sigma \mu < 0$. Then the optimal $k_{a}$ can be expressed as follows:

$$k_{a} = \begin{cases} 
1 & \text{if } 0 \leq f \leq \tau_{2}, \\
2 & \text{if } \tau_{2} < f \leq \infty \end{cases} \quad (3.25)$$

\(^{3}\text{One can use the exact same approach to analyze other cases, say, when } \tau_{1} > 0 \text{ and } \tau_{2} > 0 \text{ or when } \tau_{1} < 0 \text{ and } \tau_{2} < 0 . \text{ We omit the details.} \)
Moreover, for any given value of $f$, the retailer’s optimal profit can be expressed as:

$$
\Pi_a(f) = \begin{cases} 
\frac{|\alpha \mu (f+1) - \alpha \sigma - 2 \beta (\sigma_i f + \sigma_j)|^2}{8 \beta (f+1)^2 |\mu - \sigma|^2} + \frac{\sigma^2}{8 \beta} & \text{if } 0 \leq f \leq \tau_2; \\
\frac{|\alpha \mu (f+1) - \beta (\sigma_i f + \sigma_j)|^2}{4 \beta (f+1)^2 |\mu^2 + \sigma|^2} & \text{if } \tau_2 < f \leq \infty.
\end{cases}
\tag{3.26}
$$

By comparing the profit function $\Pi_a(f)$ for different values of $f$, we can determine the optimal ratio $f^a$ as follows:

**Proposition 6.** Suppose $\beta c_i (\mu + \sigma) + \beta c_j \mu - \alpha \sigma \mu > 0$ and $\beta c_j (\mu + \sigma) + \alpha \sigma^2 - \alpha \sigma \mu < 0$ and suppose $\alpha > \frac{2 \beta c_i}{\mu - \sigma}$. Then:

$$
f^a = \begin{cases} 
0 & \text{if } \sigma \leq \frac{\mu (c_i - c_j)}{c_i}, \\
\in [\tau_2, \infty) & \text{if } \sigma > \frac{\mu (c_i - c_j)}{c_i}.
\end{cases}
$$

Proposition 6 can be interpreted as follows. When $\sigma$ is sufficiently small, say, $\sigma \leq \frac{\mu (c_i - c_j)}{c_i}$, it is optimal for the retailer to order only from supplier $j$ (i.e., $f^a = 0$). This result is intuitive because supplier $j$ will dominate supplier $i$ when supply $j$’s unit cost $c_j$ is sufficiently lower or when the standard deviation of supplier $j$’s yield $\sigma$ is sufficiently lower. On the other hand, when $\sigma$ is reasonably large so that $\sigma > \frac{\mu (c_i - c_j)}{c_i}$ and the suppositions hold, it is optimal for the retailer to order from both suppliers. This implication is consistent with the portfolio theory in which an investor should invest in a stock portfolio instead of a single stock. To elaborate, notice that supplier $i$ has a constant yield with a higher unit cost while supplier $j$ has an uncertain yield with a lower cost. As supplier $j$’s yield becomes more variable (i.e., when $\sigma_j$ is large), Proposition 6 suggests that it is optimal for the retailer to order from both suppliers so as to maintain an optimal tradeoff between lower unit cost and higher yield uncertainty.
3.6 Conclusion

We have developed a two-stage stochastic model for determining the optimal order quantity and optimal retail price under supply uncertainty. Specifically, we show the Responsive Pricing policy dominates the No Responsive Pricing policy in terms of the retailer’s optimal expected profit. By examining the underlying structure of the problems associated with these two pricing policy, we have developed simple approaches for determining the optimal order quantity and retail price for any discrete supply yield distribution. By using these solution approaches, we examine the impact of yield distribution on the optimal order quantity, retail price, and retailer’s expected profit. We have also shown how to extend our analysis of the Responsive Pricing policy to examine three issues including emergency order, supplier selection, and order allocation among multiple suppliers. Our model has certain limitations including deterministic demand and linear demand function. We plan to extend our model to examine the issue of responsive pricing under uncertain demand and uncertain supply in our future research.

3.7 Proofs

Proof of Proposition 1: For given $k = 0, 1, \cdots, N$, the problem $P_k(NRP)$ in (2.3) can be rewritten as:

$$
\Pi_k' = \max_{0 \leq p \leq \frac{\alpha}{\beta}} \max_{Q \geq 0} \left\{ (u_k p - c)Q + p(\alpha - \beta p)v_k \right\} \quad (3.27)
$$

subject to \( \frac{\alpha - \beta p}{y_{k+1}} < Q \leq \frac{\alpha - \beta p}{y_k} \).

Our solution approach is as follows. We first determine the optimal $Q$ for any given $p$ and then we determine the optimal $p$. For any given $p \in [0, \frac{\alpha}{\beta}]$, the
Let us consider the following cases:

Case 1: Suppose \( \frac{\alpha}{\beta} > \frac{c}{u_k} \) (or equivalently, \( \frac{\beta c}{\alpha} < u_k \)). Depending on the sign of \( (u_k p - c) \), it is easy check from (3.27) that:

\[
Q'_{k} = \begin{cases} 
\frac{\alpha - \beta p}{\beta c} & \text{if } \frac{\alpha}{\beta} \geq p > \frac{c}{u_k}, \\
\frac{\alpha - \beta p}{\beta c} & \text{if } p \leq \frac{c}{u_k}.
\end{cases}
\]

Case 2: Suppose \( \frac{\alpha}{\beta} \leq \frac{c}{u_k} \) (or equivalently, \( \frac{\beta c}{\alpha} \geq u_k \)). Since \( p \leq \frac{\alpha}{\beta} \) and \( \frac{\alpha}{\beta} \leq \frac{c}{u_k} \), we have \( u_k p - c \leq 0 \). Therefore, it is easy to check from (3.27) that \( Q'_{k} = \frac{\alpha - \beta p}{\beta c} \).

Let us examine further about two subproblems (a) and (b) associated with Case 1:

(a) Suppose \( \frac{\alpha}{\beta} \geq p > \frac{c}{u_k} \). Then we can substitute \( Q'_{k} = \frac{\alpha - \beta p}{\beta c} \) into the objective function (3.27), getting a concave function of \( p \). By considering the first order condition, it is easy to show that \( p'_{1k} = \frac{c}{2(u_k + v_k y_k)} + \frac{\alpha}{\beta} \). Notice \( \frac{\alpha}{\beta} \geq p'_{1k} \) always holds when \( \frac{\alpha}{\beta} > \frac{c}{u_k} \). Therefore, the inequality \( \frac{\alpha}{\beta} \geq p'_{1k} > \frac{c}{u_k} \) holds if and only if \( \frac{\beta c}{\alpha} < \frac{u_k(u_k + v_k y_k)}{u_k + 2v_k y_k} \). This implies that when \( \frac{\beta c}{\alpha} < \frac{u_k(u_k + v_k y_k)}{u_k + 2v_k y_k} \), the optimal price is given by \( p'_{1k} \), the optimal order quantity is \( Q'_{1k} = \frac{\alpha}{2y_k} - \frac{\beta c}{4v_k y_k} \) and the optimal expected profit is \( \Pi'_{1k} = \frac{[\alpha(u_k + v_k y_k) - \beta c]^2}{4v_k y_k(u_k + v_k y_k)} \). However, when \( \frac{\beta c}{\alpha} \geq \frac{u_k(u_k + v_k y_k)}{u_k + 2v_k y_k} \), the optimal price is given by \( p'_{3k} = \frac{c}{u_k} \) with expected profit \( \Pi'_{3k} \).

(b) Suppose \( p \leq \frac{c}{u_k} \). Then we can substitute \( Q'_{k} = \frac{\alpha - \beta p}{\beta c} \) into the objective function (3.27) and get a concave function of \( p \). By considering the first order condition, we can show that \( p'_{2k} = \frac{c}{2(u_k + v_k y_k)} + \frac{\alpha}{\beta} \). In this case, it is easy to show that the inequality \( p'_{2k} \leq \frac{c}{u_k} \) holds if and only if \( \frac{\beta c}{\alpha} \geq \frac{u_k(u_k + v_k y_k + 1)}{u_k + 2v_k y_k + 1} \). Hence, when \( \frac{\beta c}{\alpha} \geq \frac{u_k(u_k + v_k y_k + 1)}{u_k + 2v_k y_k + 1} \), the optimal price is given by \( p'_{2k} \), the optimal order quantity is given by \( Q'_{2k} = \frac{\alpha}{2y_k + 1} - \frac{\beta c}{4v_k y_k + 1} \) and the optimal expected profit is given by \( \Pi'_{2k} = \frac{[\alpha(u_k + v_k y_k + 1) - \beta c]^2}{4v_k y_k(u_k + v_k y_k + 1)} \). However, when \( \frac{\beta c}{\alpha} < \frac{u_k(u_k + v_k y_k + 1)}{u_k + 2v_k y_k + 1} \), the optimal price is given by \( p'_{3k} = \frac{c}{u_k} \) with expected profit \( \Pi'_{3k} \).
Notice that the retailer’s optimal expected profit associated with Case 1 is given as: \( \Pi'_k = \max\{\Pi'_{1k}, \Pi'_{2k}, \Pi'_{3k}\} \). Since \( y_k < y_{k+1} \), we have \( \frac{u_k(u_k + vy_{k+1})}{u_k + 2vy_{k+1}} < \frac{u_k(u_k + vy_k)}{u_k + 2vy_k} \).

We now use this observation to determine the optimal solution to problem \( P_k(NRP) \) associated with Case 1 as follows.

1. Suppose \( \frac{\beta c}{\alpha} < \frac{u_k(u_k + vy_{k+1})}{u_k + 2vy_{k+1}} \). Our observation implies that \( \frac{\beta c}{\alpha} < \frac{u_k(u_k + vy_k)}{u_k + 2vy_k} \). Hence, we can apply the results obtained in (a) to show that \( \Pi'_k = \Pi'_{1k} \), \( p'_k = p'_{1k} \), and \( Q'_k = Q'_{1k} \). This proves the first statement in the Proposition.

2. Suppose \( \frac{\beta c}{\alpha} \geq \frac{u_k(u_k + vy_{k+1})}{u_k + 2vy_{k+1}} \). Since \( \frac{u_k(u_k + vy_{k+1})}{u_k + 2vy_{k+1}} < \frac{u_k(u_k + vy_k)}{u_k + 2vy_k} \), we can apply the results obtained in (b) to show that \( \Pi'_k = \Pi'_{2k} \), \( p'_k = p'_{2k} \), and \( Q'_k = Q'_{2k} \). This proves half of the third statement in the Proposition.

3. Suppose \( \frac{u_k(u_k + vy_{k+1})}{u_k + 2vy_{k+1}} \leq \frac{\beta c}{\alpha} < \frac{u_k(u_k + vy_k)}{u_k + 2vy_k} \). Then \( \Pi'_k = \max\{\Pi'_{1k}, \Pi'_{2k}\} \) and we can determine the optimal price and order quantity accordingly. This proves the second statement in the Proposition.

We now turn our attention to Case 2; i.e., when \( \frac{\beta c}{\alpha} \leq \frac{u_k}{\beta} \). In this case, we have \( \frac{\beta c}{\alpha} \geq u_k \). Let us first substitute \( Q'_k = \frac{\alpha - \beta p}{y_{k+1}} \) into the objective function (3.27), getting a concave function of \( p \). Then, by examining the first order condition, we can show that \( p'_{2k} = \frac{\alpha}{2(u_k + y_{k+1} + 1v_k)} + \frac{\alpha}{2\beta} \). We now determine the optimal solution to problem \( P_k(NRP) \) associated with Case 2 as follows.

1. Suppose \( p'_{2k} < \frac{\alpha}{\beta} \) (or equivalently, \( \frac{\beta c}{\alpha} < u_k + vy_{k+1} \)). Then the optimal price is \( p'_k = p'_{2k} \). This proves half of the third statement in the Proposition.

2. Suppose \( \frac{\beta c}{\alpha} \geq u_k + vy_{k+1} \). Then \( Q'_k = 0, p'_k \) is arbitrary, and \( \Pi'_k = 0 \). This proves the fourth statement in the Proposition.

This completes the proof. \( \square \).
Proof of Proposition 2: Let us differentiate the objective function given in (3.10) with respect to $Q$, getting:

$$
\frac{dE_y(Π(Q, p^*|y))}{dQ} = \left( \frac{α}{β} u_k - c \right) - \frac{2w_k}{β} Q
$$

(3.28)

First, let us consider the case when $μ > \frac{βc}{α}$. Notice that the derivative is positive when $Q$ is sufficiently small, say, when $Q = x_{N+1} = 0$ and is negative when $Q$ is sufficiently large, say, $Q = x_1 = \frac{1}{2y_1}$. Also, by evaluating the derivatives at the boundary points $x_{k+1}$ and $x_k$, one can use the definitions of $u_k$ and $w_k$ given in (3.5) and (3.9) to show that:

$$
\left| \frac{dE_y(Π(Q, p^*|y))}{dQ} \right|_{Q=x_{k+1}} - \left| \frac{dE_y(Π(Q, p^*|y))}{dQ} \right|_{Q=x_k} = \frac{α}{β} \left[ \sum_{m=1}^{k} λ_my_m^{2}\left( \frac{1}{y_{k+1}} - \frac{1}{y_k} \right) \right] < 0.
$$

Since $x_{k+1} < x_k$, we can conclude that the first derivative of the objective function at the boundary points $x_{N+1}, x_N, \cdots, x_0$ are decreasing. Combine this observation with the fact that the objective function is piece-wise concave, we can conclude that there exists an $k^*$ such that $\frac{dE_y(Π(Q, p^*|y))}{dQ}|_{Q=x_{k^*+1}} \geq 0$ and $\frac{dE_y(Π(Q, p^*|y))}{dQ}|_{Q=x_{k^*}} < 0$. This implies that the optimal $k^*$ is given by (3.11). By considering the first order condition (3.28), we can show that the optimal order quantity $Q^*$ is given by (3.12). Given the optimal order quantity $Q^*$, we can compute the optimal retail price $p^*(Q^*|y)$ once the actual yield $y$ is observed. In addition, we can substitute $Q^*$ into (3.10) to show that the retailer's optimal expected profit under the responsive pricing policy is given by (3.13).

Next, let us consider the case when $μ \leq \frac{βc}{α}$. In this case, we have $\frac{dE_y(Π(Q, p^*|y))}{dQ} \leq 0$ for $Q \geq 0$. Therefore $Q^* = 0$ is the optimal solution. This completes the proof.

Proof of Proposition 3: The proof is immediate. We omit the details.

Proof of Corollary 1: The proof follows directly from Proposition 2. We omit the details.
Proof of Proposition 4: First, let us consider the case when \( \mu_s - \frac{0.5(y_i^2 + y_j^2)}{y_{i2}} \geq \frac{\beta_{c_s}}{\alpha} \) for \( s = i, j \). It follows from (3.17) that supplier \( i \) dominates supplier \( j \) if \( \frac{(\alpha(0.5y_i) - \beta c_i)^2}{2\beta y_i^3} + \frac{\sigma^2}{8\beta} \geq \frac{(\alpha(0.5y_j) - \beta c_j)^2}{2\beta y_j^3} + \frac{\sigma^2}{8\beta} \). In this case, we can prove the first statement by rearranging the terms. Second, let us consider the case when \( \mu_s - \frac{0.5(y_i^2 + y_j^2)}{y_{i2}} < \frac{\beta_{c_s}}{\alpha} \) for \( s = i, j \). It follows from (3.17) that supplier \( i \) dominates supplier \( j \) if \( \frac{(\alpha\mu_i - \beta c_i)^2}{2\beta(y_i^2 + y_j^2)} \geq \frac{(\alpha\mu_j - \beta c_j)^2}{2\beta(y_i^2 + y_j^2)} \). Since \( \mu_i = \mu_j = \mu \), we can obtain the second statement by rearranging the terms. We omit the details. \( \square \)

Proof of Corollary 2: The proof follows directly from Proposition 2. We omit the details. \( \square \)

Proof of Proposition 5: The results follow from Corollary 2. Let \( \eta_1 = \beta c_i (\mu + \sigma) + \beta c_j \mu - \alpha \sigma \mu \) and \( \eta_2 = \beta c_j (\mu + \sigma) + \alpha \sigma^2 - \alpha \sigma \mu \). Then it is easy to check that the two roots associated with the equation \( h(f) = 0 \) are: \( \tau_1 = \frac{\eta_1 - \sqrt{\eta_1^2 - 4\beta c_i \mu \eta_2}}{2 \beta c_i \mu} \) and \( \tau_2 = \frac{\eta_1 + \sqrt{\eta_1^2 - 4\beta c_i \mu \eta_2}}{2 \beta c_i \mu} \). Suppose \( \eta_1 > 0 \) and \( \eta_2 < 0 \). Then we have \( \tau_1 < 0 \) and \( \tau_2 > 0 \). First, let us consider the case when \( f > \tau_2 \). In this case, \( k^a = 2 \) and \( \Pi^a(f) = A(f) \), where:

\[
A(f) = \frac{[\alpha \mu (f+1) - \beta (c_i f + c_j)]^2}{4\beta [(f+1)^2 + \sigma^2]}.
\] (3.29)

Second, consider the case when \( 0 \leq f \leq \tau_2 \). In this case, \( k^a = 1 \) and \( \Pi^a(f) = B(f) \), where:

\[
B(f) = \frac{[\alpha \mu (f+1) - \alpha \sigma - 2\beta (c_i f + c_j)]^2}{8\beta [(f+1) \mu - \sigma]^2} + \frac{\alpha^2}{8\beta}.
\] (3.30)

Third, consider the case when \( f = \infty \). In this case, \( w_2 = \mu^2, r_2 = \mu \) and \( w_2 - \frac{w_2}{r_2} = 0 < \frac{\beta_{c_a}}{\alpha} \) holds. Thus, we can conclude that \( k^a = 2 \) and \( \Pi^a(f = \infty) = A(f = \infty) \). This completes the proof. \( \square \)

Proof of Proposition 6: It follows from (3.26) that the retailer’s optimal expected profit \( \Pi^a \) can be expressed as: \( \Pi^a = \max \{ \max_{0 \leq f \leq \tau_2} B(f), \max_{\tau_2 < f \leq \infty} A(f) \} \),
where \( A(f) \) and \( B(f) \) are given in (3.29) and (3.30), respectively. We now examine the characteristics of \( A(f) \) and \( B(f) \). First, we can check from (3.29) and (3.30) that \( B(f) \geq A(f) \) for any \( f \in [-\infty, \infty] \). Second, it is easy to check from (3.30) that:

\[
\frac{dB(f)}{df} = \frac{(\alpha(\mu - \sigma) - 2\beta c_i + (\alpha \mu - 2\beta c_i) c_i(\sigma - \mu) + c_j \mu)}{2(\mu(f + 1) - \sigma)^3}. 
\]

(3.31)

Suppose \( \alpha \) is sufficiently large so that \( \alpha(\mu - \sigma) > 2\beta c_i \). Then it is easy to use the fact that \( c_i > c_j \) and \( \mu > \sigma \) to show that \( \alpha(\mu - \sigma) - 2\beta c_i + (\alpha \mu - 2\beta c_i) f > 0 \) and that \( \mu(f + 1) - \sigma > 0 \). Applying this result to (3.31), we can conclude that

\[
\text{sign}\left(\frac{dB(f)}{df}\right) = \text{sign}(c_i(\sigma - \mu) + c_j \mu). 
\]

Let us consider the following 2 cases:

1. When \( c_i(\sigma - \mu) + c_j \mu \leq 0 \). In this case, we have \( \frac{dB(f)}{df} \leq 0 \). Therefore \( B(f) \) is decreasing in \( f \in [0, \infty] \), \( B(0) = \max_{0 \leq f \leq \infty} B(f) \geq \max \{ \max_{0 \leq f \leq \tau_2} B(f), \max_{\tau_2 < f < \infty} A(f) \} \).

Hence, \( f^a = 0 \).

2. When \( c_i(\sigma - \mu) + c_j \mu > 0 \). In this case, we have \( \frac{dB(f)}{df} > 0 \) and \( \tau_2 = \arg \max \{ B(f) : 0 \leq f \leq \tau_2 \} \). It follows from (3.29) that:

\[
\frac{dA(f)}{df} = \frac{(\alpha - \beta c_j) + f(\alpha \mu - \beta c_i)((\alpha \mu - \beta c_i) \sigma^2 - \beta \mu^2(1 + f)(c_i - c_j))}{2\beta(\mu^2 + \sigma^2 + \mu^2 f + \mu^2 f^2)^2} \]

(3.32)

Observe that \( (\alpha - \beta c_j) \sigma^2 - \beta \mu(1 + f)(c_i - c_j) < 0 \) when \( f \) is sufficiently large. Combine this observation with the fact that \( (\alpha \mu - \beta c_j) + f(\alpha \mu - \beta c_i) > 0 \), we can check from (3.32) that \( \frac{dA(f)}{df} < 0 \) when \( f \) is sufficiently large. Therefore, \( \arg \max \{ A(f) : \tau_2 < f \leq \infty \} < \infty \), which implies that \( f^a \in [\tau_2, \infty) \).

We can prove the Proposition by considering the results associated with these two cases. \( \Box \)
3.8 References


CHAPTER 4

The Implications of Customer Purchasing Behavior and In-store Display Formats

4.1 Introduction

Consider a retailer who sells a fashion product with uncertain customer arrivals over a single selling season. For any given selling price $p_h$ during the season and salvage value $s$, one can formulate the problem as the newsvendor problem and obtain the optimal order quantity and the corresponding optimal expected profit. The elegant newsvendor solution is based on an assumption that the customers are ‘myopic’ in the sense that they will purchase the product immediately upon arrival. However, the myopic assumption becomes questionable when the retailer deploys different pricing strategies.

To obtain a clearance price that is higher than the salvage value $s$, retailers have developed different dynamic pricing mechanisms since the late 1980s. When selling seasonal goods, a common form of dynamic pricing strategy is the markdown pricing strategy. Fisher et al. (1994) reported that 26% of fashion goods are sold at markdown prices. As retailers offer different markdown pricing mechanisms, customers would take the future price into consideration when making their purchasing decisions. As such, customers are becoming more ‘strategic’ in the sense that they might wait for a sale instead of purchasing the product
immediately upon arrival. This type of strategic purchasing behavior has been reported in Kadet (2004) and McWilliams (2004).

As customers become strategic, the newsvendor solution no longer holds. This new shift in customer purchasing behavior has motivated us to develop a model for examining the impact of strategic purchasing behavior on the retailer’s optimal order quantity and optimal expected profit. As an initial attempt to study this issue, we shall focus our analysis on the case in which the retailer adopts a simple form of markdown pricing mechanism that can be described as follows. At the beginning of the season, the retailer orders $Q \geq 1$ units and announces that the product will be sold at the reduced price $p_l < p_h$ if it is not sold at the regular price $p_h$ by the end of the season.\(^1\) A customer can either purchase the product (if available) during the season at price $p_h$ or attempt to purchase the product at the reduced price $p_l$ after the season ends. Suppose there are $k$ units available at the end of the season and suppose there are $n$ customers who decided to wait for the end-of-season sale. Then each customer will get 1 unit if $n \leq k$ and each of the $n$ customers has an equal probability of $\frac{k}{n}$ for getting 1 unit when $n > k$. This rationing policy mimics the situation when all markdown items are sold on a first-come-first-serve basis. A more general version of this markdown pricing mechanism has been adopted by the Filene’s Basement store in Boston since 1908. At the Filene’s Basement store, the “automatic markdown plan” is pre-announced: most unsold items after 2, 4 and 6 weeks will be sold at 25%, 50% and 75% off the regular price, respectively. Filene’s Basement will donate all the unsold items after 2 months to the charity. The reader is referred to Bell and Starr (1998) for more details.

In addition to markdown pricing mechanisms, in-store display format has

\(^1\)This is equivalent to the case in which the price is dropped from $p_h$ to $p_l$ within the season.
a direct impact on the strategic purchasing behavior as well. This is because different in-store display formats can create different impressions about the actual inventory level at the store. As an initial attempt to analyze the impact of in-store display formats on the retailer’s optimal order quantity and optimal expected profit when the customers are strategic, we consider two basic in-store display formats under which the retailer would either display all units or display one unit at a time on the sales floor. The ‘Display All’ format has been adopted by many fashion retailers such as Filene’s Basement and Benetton, while the ‘Display One’ format has been adopted by various high-end stores such as the Bally handbag stores in Taiwan and the Hour Glass watch stores in Singapore and Hong-Kong. Both in-store display formats have different implications. The Display All format allows a retailer to utilize the available space more effectively by maximizing the sales floor space. Also, it provides each arriving customer perfect information about the actual inventory level available for sale at the time of arrival, which has direct impact on the customers’ strategic purchasing decisions. The Display One format allows a retailer to use the limited sales space to display an assortment of different designs instead of multiple units of the same design. By displaying one unit at a time, it creates an impression of scarcity, which would urge interested customers to purchase the product immediately upon arrival.

When customers are either myopic or strategic and when the retailer adopts either the Display All or the Display One format, we are interested in examining the following questions for any given values of $p_h$ and $p_l$:

1. When the customers are strategic, how would different display formats affect their optimal purchasing behavior?

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2 There are other in-store display formats including the case in which the retailer, say, Zara, displays only a few items on the sales floor, and the case in which the retailer display the products in different strategic locations such as front of the aisle, entrance to the store, etc. The reader is referred to Ghemawat and Nueno (2003) for details.
2. When the customers are strategic, how would different display formats affect the retailer’s optimal expected profit and optimal ordering decision?

3. Suppose the retailer incorrectly assumes that the strategic customers are myopic. How would this incorrect assumption affect the retailer’s optimal expected profit and optimal ordering decision for each display format?

To answer these questions, we develop a model that incorporates stochastic customer arrivals and rational purchasing behavior. In our base model, we analyze the case in which customers with identical valuation arrive to the store in accord with a Poisson process. We first show that each customer’s optimal purchasing decision is based on a threshold that depends on the inventory level at the time of arrival. This result implies that, when the customers are strategic, the total demand for the product during the season would depend on the number of customers arrived during the season and their actual arrival times. In contrast, when the customers are myopic, the total demand depends only on the number of customers arrived during the season. Therefore, the total demand associated with the strategic customer case is more uncertain than that of the myopic customer case. Hence, one would speculate that the retailer’s optimal order quantity would be higher when the customers are strategic. However, when customers are strategic, some customers pay $p_h$ and others pay $p_l < p_h$. Hence, the effective price would be lower than $p_h$ and one would conjecture that the retailer’s optimal order quantity would be lower when customers are strategic. In light of these two opposite views, the net impact of strategic purchasing behavior on the retailer’s optimal order quantity is not obvious to us.

To investigate how customer’s purchasing behavior affects retailer’s optimal expected profit and optimal order quantity, we determine the retailer’s expected profit associated with both display formats when the customers are myopic and
strategic. We show that: (a) The retailer will order more and enjoy a higher expected profit when the customers are myopic instead of strategic; and (b) The retailer will order more and enjoy a higher profit under the Display One format than that of the Display All format when the customers are strategic. In addition, we show analytically that the retailer will over-order and will obtain a lower expected profit when the retailer incorrectly assumes that the strategic customers are myopic.\(^3\) These results imply that the customer’s purchasing behavior (myopic or strategic) and the in-store display formats (Display All or Display One) have significant impacts on the retailer’s optimal order quantity and optimal expected profit. We also extend our analysis to the case in which customers belong to different classes, each of which has a class-specific valuation. We obtain similar analytical results when the retailer adopts the Display One format. Furthermore, we extend our analysis to the case in which the post-season clearance price depends on the actual end-of-season inventory level.

### 4.2 Literature Review

Our work is related to two groups of recent papers that analyze dynamic pricing issues for the case when customers are strategic.\(^4\) In the first group, all customers are assumed to be present at the beginning of the selling season. Besides the earlier economic models developed by Stokey (1979), Besanko and Winston (1990), and Harris and Raviv (1981), we review three recent papers that are based on the assumption that all customers are present at the beginning of the selling season. Elmaghraby, Gulcu and Keshkinocak (2004) examine a situation in which

\(^3\)This analytical result is consistent with the numerical results obtained by Aviv and Pazgal (2005) and Levin et al. (2005).

\(^4\)The reader is referred to an article by Weatherford and Bodily (1992), a comprehensive review by Elmaghraby and Keskinocak (2003), and a book by Talluri and van Ryzin (2004) for more in-depth discussion about dynamic pricing.
the retailer pre-announces the price markdown schedule. The customers may demand multiple units of the product and can choose the number of units to purchase at each price drop. By determining the rational purchase behavior of each customer, they compare the retailer’s expected profit associated with different pre-announced markdown mechanisms. Levin et al. (2005) present a stochastic dynamic game formulation for the dynamic pricing problem. They prove the existence of a unique subgame perfect equilibrium dynamic pricing policy and they obtain monotonicity results for two special cases: (a) when customers are myopic; and (b) when customers are strategic but do not need to compete for the items since the inventory level is sufficient to satisfy all customers. Liu and van Ryzin (2005) study a situation when the retailer commits to a pre-announced markdown price schedule. By assuming that all customers are present simultaneously at the beginning of the selling season, they determine the optimal ordering decision and they develop conditions under which it is optimal for the retailer to create shortages by understocking products.

In the second group, customers arrive at different times throughout the selling season. Su (2005) examines a situation in which customers may belong to either the high-valuation segment or the low-valuation segment, and may either be strategic or myopic. When customers arrive continuously in a deterministic manner throughout the season, he determines the optimal dynamic pricing policy over time as well as the optimal ordering decision for the retailer. Elmaghraby et al. (2005) analyze a situation in which the retailer sells 1 unit under two operating regimes. Under the reservation regime, a buyer can either purchase the product at the regular price or reserve the product at the post-season clearance price. If the buyer reserves the product and if it remains unsold at the end of the season, he is obligated to purchase the product at the clearance price. Under the no reservation regime, a buyer can either purchase the product at the regular
price or he enters a lottery to purchase the product at the clearance price if the
product remains unsold. In the presence of Poisson customer arrivals, they show
that the retailer can always obtain a higher expected profit under the reservation
regime when there is a single class of customers with identical valuation. How-
ever, when there are multiple classes of customers with class-specific valuations,
they establish conditions under which the reservation regime dominates the no
reservation regime. Aviv and Pazgal (2005) study two pricing strategies: inven-
tory contingent discounting strategy and announced fixed-discount strategy. In
the first strategy, the retailer would only announce the clearance price after the
actual end-of-season inventory level is realized. In the second strategy, the retailer
would announce the clearance price at the beginning of the season. They assume
that each arriving customer only knows the initial order quantity $Q$, but does not
know the actual inventory level at the time of arrival. This assumption enables
them to show that it is optimal for customers to purchase according to individ-
ual thresholds that depend on the individual valuations and arrival times. This
assumption also enables them to develop a subgame perfect Nash equilibrium for
the game between the retailer and the customers.

While our model is based on a pre-announced markdown pricing scheme, the
focus is different from the aforementioned papers in the following ways. First,
most of the aforementioned papers focused on the retailer’s optimal pricing pol-
icy, while our focus is on the retailer’s optimal expected profit and optimal or-
der quantity. Second, most of the papers are based on the assumption that all
customers are present at the beginning of the season or customer arrivals are de-
terministic, while our work considers stochastic customer arrivals as in Aviv and
Pazgal (2005) and Elmagraby et al. (2005). Third, Elmagraby et al. (2005)
examines the retailer’s optimal profit associated with two operating regimes for
the case when there is only 1 unit to sell, while we consider the case when the

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retailer has $Q$ units available for sale at the beginning of the season. Also, we determine the retailer’s optimal profit and optimal order quantity for the case when all customers are either myopic or strategic and when the retailer adopts either the Display All or Display One format. Fourth, unlike the assumption considered in Aviv and Pazgal (2005), we assume that each arriving customer knows the actual inventory level at the time of arrival, which is probably more reasonable in a traditional retailing environment. Also, we consider different in-store display formats: Display All and Display One. Under both display formats, we show that it is optimal for each customer to purchase according to his threshold that depends on his valuation, his arrival time, and the actual inventory level upon the time of arrival. By taking the customer’s strategic purchasing behavior into consideration, we are able to express the retailer’s optimal expected profit and optimal order quantity in implicit functional forms for both display formats. These implicit functional forms enable us to compare the retailer’s optimal expected profits and optimal order quantities associated with different scenarios analytically.

This chapter is organized as follows. By assuming all customers belong to a single class with identical valuation, Section 4.3 examines the base models in which all customers can be either myopic or strategic and the retailer adopts either the Display All or the Display One format. When the customers are strategic, we establish the optimal purchasing rule and determine the retailer’s expected profit under both display formats. In Section 4.4, we compare the retailer’s optimal expected profits and optimal order quantities for each of the base models, and we show analytically that the optimal order quantity is higher when the customers are myopic and the optimal order quantity is higher under the Display One format when the customers are strategic. We consider two extensions in Section 4.5. In the first extension, we extend our analysis to the case in which the customers
belong to multiple classes, each of which has a class-specific valuation. Under the Display One format, we determine the optimal purchasing rule and the retailer’s expected profit when the customers are strategic. We compare the retailer’s optimal profits and optimal order quantities for the cases when customers are either myopic or strategic. In the second extension, we consider a situation in which the post-season clearance price depends on the end-of-season inventory level. Section 4.6 ends our work with some concluding remarks.

4.3 The Base Model

Consider a retailer who orders and sells $Q \geq 1$ units of a single product with unit cost $c$ over a selling season that spans over $[0, T]$. At the beginning of the selling season, the retailer announces both the price $p_h$ at which the product will be sold during the selling season and the post-season clearance price $p_l$ for the unsold items, where $c < p_l < p_h$. The retailer will obtain a salvage value $s < c$ for each unit that remains unsold after the post-season clearance. In this chapter, we consider two display formats: Display all and Display one. Under the Display All format, all available units are displayed on the sales floor at all times, and hence, each arriving customer has perfect information regarding the inventory level at the time of arrival. Under the Display One format, the retailer displays only 1 item on the sales floor, and keeps other available units in the storeroom. Once the display item is sold, the retailer will display a new item retrieved from the storeroom. We assume that each arriving customer thinks that the retailer has only 1 item (if available) for sale at the time of arrival.\footnote{This assumption is reasonable when each customer is only interested in purchasing one unit, when the customer does not ask the retailer about the actual inventory level, when the retailer does not know the actual inventory level, or when the retailer does not reveal the actual inventory level to the customers.}
During the season, customers arrive in accord with a Poisson process with rate $\lambda$, where $\lambda$ remains constant throughout the entire season. Upon arrival, each customer can either purchase one unit (if available) during the season at $p_h$ or wait and then attempt to purchase at the reduced price $p_l$ after the season ends. When the season ends, each customer who waited will get one unit at the reduced price $p_l$ if the leftover inventory exceeds the number of interested customers. Otherwise, the retailer will ration out the leftover inventory to these interested customers with equal probability.\(^6\)

In our base model, we assume that the market is comprised of a single class of customers with identical valuation $v$.\(^7\) To ensure each customer might purchase the product during the season, we assume that $v > p_h > p_l$. We also assume that the parameter values $v, p_h, p_l$ and $\lambda$ are common knowledge. In addition, each customer knows his arrival time $t$. The base model with a single class customers enables us to understand the underlying structure of the model and to generate specific insights. In a later section, we shall extend our analysis of the Display One format to the case in which the customers belong to multiple classes, each of which has a class-specific valuation. In preparation, let $B(t)$ and $A(t)$ be the number of customers who arrive ‘before’ and ‘after’ $t$, respectively. Notice that $B(t)$ and $A(t)$ are independent Poisson random variables with parameters $\lambda t$ and $\lambda (T - t)$, respectively.

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\(^6\)This rationing policy mimics the case when the clearance items are sold on a first-come-first-serve basis.

\(^7\)Our model can be easily extended to the case when there are two classes of customers. Class $i$ customers have identical valuation $v_i$, for $i = 0, 1$, where $v_1 > p_h > v_0 > p_l$, so that all arriving customers of Class 0 will always attempt to purchase the item at the reduced price $p_l$ after the season ends.
4.3.1 Myopic Customers

When the customers are myopic and when $v > p_h$, each arriving customer during the season will attempt to purchase the item at $p_h$ regardless of the display format adopted by the retailer. Hence, the effective demand for the product is equal to $B(T)$. In this case, regardless of the display format adopted by the retailer, the retailer’s expected profit is identical to the expected profit function associated with the newsvendor problem. Thus, when the customers are myopic, the retailer’s expected profit for any order quantity $Q$ can be written as:

$$
\Pi_r^M(Q) = E\{p_h \min\{Q, B(T)\} + s(Q - B(T))^+\} - cQ = (p_h - c)Q - (p_h - s)E[Q - B(T)]^+.
$$

(4.1)

4.3.2 Strategic Customers Under the Display All Format

We now determine the retailer’s expected profit function associated with the Display All format for the case when the customers are strategic. To begin, let us examine the customer’s strategic purchasing behavior.

4.3.2.1 Optimal Strategic Purchasing Rule Under the Display All Format

When the retailer adopts the Display All (DA) format, each arriving customer knows the actual number of units available for sale upon arrival. To examine how this knowledge affects a strategic customer’s purchasing decision, let us consider a customer who arrives at time $t$ and observes $k$ units available for sale, where $1 \leq k \leq Q$. He will enjoy a surplus $v - p_h$ if he purchases the item at $p_h$. Alternatively, he can wait and attempt to purchase the item at the reduced price.
If he attempts to purchase the item at the reduced price $p_l$, his expected surplus is equal to $(v - p_l)H(k, t)$, where the term $H(k, t)$ represents the expected probability of getting the item at the reduced price after the season ends. By comparing the expected surpluses associated with these two purchase options, we can establish the following DA threshold purchasing rule: For any customer who arrives at time $t$ and observes $k$ units available for sale, he should: (a) purchase one unit at $p_h$ if $t \leq t^*(k)$; and (b) attempt to purchase one unit at $p_l$ after the season ends if $t > t^*(k)$, where the threshold $t^*(k) = \max\{0, t(k)\}$ and $t(k)$ satisfies:

$$H(k, t(k)) = \frac{v - p_h}{v - p_l}, \quad (4.2)$$

and $t(Q) < t(Q - 1) < \cdots < t(1)$.

In general, the expected probability $H(k, t)$ associated with the DA threshold purchasing rule is a complex function because it depends on the customer arrival pattern throughout the entire season. However, the expected probability $H(k, t)$ can be established easily for the case when $t = t(k)$. When $t = t(k)$, the DA threshold purchasing rule implies that, in order for a customer to observe $k$ items available at time $t(k)$, no customers arrived before $t(k)$ would wait and all customers who arrive after $t(k)$ would wait. Therefore, all $k$ units available at time $t(k)$ will still be available for sale at the reduced price $p_l$ after the season ends. Under our rationing policy, the customer arriving at time $t(k)$ who decided to wait will get the item at the reduced price $p_l$ with probability 1 when $A(t(k)) \leq k - 1$ and $\frac{k}{A(t(k)) + 1}$ when $A(t(k)) \geq k$. Combine this observation with the fact that

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8 We assume that each customer who decides to wait will return to the store at time $T$. It is easy to check that our model can be extended to the case when a fixed proportion $0 < q \leq 1$ of customers who decide to wait will eventually return to the store at time $T$. For simplicity, we assume that $q = 1$ in our model.

9 This construct generalizes the analysis presented in Elmaghraby et al. (2005) for the case when $Q = 1$. 

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A(t(k)) is a Poisson random variable with parameter $\lambda(T - t(k))$, we can express the term $H(k, t(k))$ as:

$$H(k, t(k)) = \sum_{n=0}^{k-1} \text{Prob}(A(t(k)) = n) + \sum_{n=k}^{\infty} \text{Prob}(A(t(k)) = n) \frac{k}{n+1},$$

$$= \sum_{n=0}^{k-1} \frac{[\lambda(T - t(k))]^n}{n!} e^{-\lambda(T - t(k))}$$

$$+ \sum_{n=k}^{\infty} \frac{[\lambda(T - t(k))]^n}{n!} e^{-\lambda(T - t(k))} \frac{k}{n+1}. \quad (4.3)$$

By examining (4.3) and (4.2), we can prove the following Lemma:

**Lemma 4.** The threshold $t(k)$ that satisfies (4.2) is unique. Also, the threshold $t(k)$ has the following properties:

1. $t(Q) < t(Q - 1) < \cdots < t(k + 1) < t(k) < \cdots < t(1) < T$.

2. The threshold $t(k)$ is increasing in $\lambda, v$ and $p_l$, and decreasing in $p_h$.

**Proof:** All proofs are given in the Appendix.

It follows from Lemma 1 that the threshold $t(k)$ is strictly decreasing in $k$ and the fact that $t^*(k) = \max\{0, t(k)\}$, it is easy to show that:

**Proposition 7.** There exists a positive integer $\theta$ that satisfies: $\theta = \arg\min \{t(j) \leq 0 : j = 1, 2, \cdots \}$. Moreover, the threshold $t^*(k) = \max\{0, t(k)\}$ has the following properties:

1. If $Q < \theta$, then $0 < t^*(Q) < t^*(Q - 1) < \cdots < t^*(1) < T$.

2. If $Q \geq \theta$, then $t^*(Q) = t^*(Q - 1) = \cdots = t^*(\theta) = 0 < t^*(\theta - 1) < \cdots < t^*(1) < T$.

3. The threshold $t^*(k)$ is increasing in $\lambda, v$ and $p_l$, and decreasing in $p_h$. 


Proposition 1 implies that the thresholds $t^*(k)$ are decreasing in $k$ and that $t^*(Q) = 0$ when the initial order quantity $Q \geq \theta$. When $Q$ is sufficiently large so that $t^*(Q) = 0$, each customer arriving at time $t > t^*(Q)$ will observe $Q$ units available and will attempt to purchase the product at the reduced price $p_l$ under the DA threshold purchasing rule. This implication is intuitive because, when the initial order quantity is sufficiently large, the expected probability of getting the product at the reduced price after the seasons ends is high; i.e., $H(Q, t)$ is high. Hence, there is no incentive for any arriving customer to purchase the product at the regular price $p_h$. This result is consistent with that obtained by Liu and van Ryzin (2005) under the assumption that all customers are present at the beginning of the season.

By applying Proposition 1, we can compare the expected surplus of an arriving customer for the case when he follows the DA threshold purchasing rule and the case when he deviates from the DA threshold purchasing rule. This comparison enables us to prove that:

**Proposition 8.** There is a Nash equilibrium in which all arriving customers follow the DA threshold purchasing rule.

### 4.3.2.2 Expected Payoffs Under the Display All Format

We now determine the retailer’s expected profit when all arriving customers follow the DA threshold purchasing rule. Notice that the retailer’s profit depends on the purchasing decisions made by the customers who arrive during different time intervals $(t^*(j), t^*(i)]$ for $1 \leq i < j \leq Q + 1$, where $t^*(Q + 1) \equiv 0$; hence, the computation of the retailer’s expected payoff is not straightforward. However, it can be computed in a recursive manner. In preparation, for $1 \leq i < j \leq Q + 1$, ...
let:

\[ f(j, i) = \text{the retailer’s expected revenue to be obtained from } t^*(j) \text{ to } T \]
when \( i \) units are available for sale at time \( t^*(j) \), and

\[ g(i) = \text{the retailer’s expected revenue to be obtained from } t^*(i) \text{ to } T \]
when \( i \) units are available for sale at time \( t^*(i) \).

Since the retailer has \( Q \) units available for sale at time \( t^*(Q+1) \equiv 0 \), the function \( f(Q+1, Q) \) corresponds to the retailer’s expected revenue over the entire season. Hence, for any order quantity \( Q \), the retailer’s expected profit can be expressed as:

\[ \Pi_{DA}^r(Q) = f(Q+1, Q) - cQ. \tag{4.4} \]

To determine the retailer’s expected profit \( \Pi_{DA}^r(Q) \) for any given \( Q \), it suffices to focus on the function \( f(j, i) \) for \( 1 \leq i < j \leq Q + 1 \). To begin, let \( N(j, i) \) be the number of customers who arrive within the time window \((t^*(j), t^*(i)]\). Considering three mutually exclusive and exhaustive events associated with \( N(j, i) \) yields:

**Proposition 9.** For \( 1 \leq i < j \leq Q + 1 \), the recursive function \( f(j, i) \) and the function \( g(i) \) satisfy:

1. \( f(j, i) = g(i)\text{Prob}(N(j, i) = 0) + ip_h \sum_{k=1}^{\infty} \text{Prob}(N(j, i) = k) + \sum_{k=1}^{i-1} (kp_h + f(i, i - k))\text{Prob}(N(j, i) = k). \)

2. \( g(i) = ip_l - (p_l - s) \sum_{k=0}^{i-1} \binom{i-1}{k} \text{Prob}(N(i, 0) = k). \)

Since \( N(j, i) \) is a Poisson random variable with parameter \( \lambda(t^*(i) - t^*(j)) \), we can determine the functions \( f(j, i) \) and \( g(i) \), and hence, the retailer’s expected profit \( \Pi_{DA}^r(Q) \) given in (4.4).\(^{11}\)

\(^{10}\) We define \( \sum_{k=1}^{0} = 0. \)

\(^{11}\) To compute customers’ expected surplus \( \Pi_{c DA}^r \) under the Display All Policy, we can use the same approach by defining similar recursive functions \( f(j, i) \) and \( g(i) \). We omit the details.
4.3.3 Strategic Customers Under the Display One Format

We now examine the case when the retailer orders $Q$ units at the beginning of the selling season, displays only 1 item on the sales floor, and keeps the rest in the storeroom. The Display One format operates as follows: once the display item is sold, the retailer will display a new item retrieved from the storeroom. We now determine the strategic customer’s purchasing rule and the retailer’s expected profit under the Display One format.

4.3.3.1 Optimal Strategic Purchasing Rule Under the Display One Format

Under the Display One format, a customer who arrives at time $t$ will observe $k = 1$ unit available for sale. He will enjoy a surplus $v - p_h$ if he purchases the item at $p_h$. Alternatively, he can wait and attempt to purchase the item at $p_l$ after the season ends. Since he believes that there is only $k = 1$ unit available, each arriving customer would behave in accord with the case when $Q = 1$ under the Display All format. As such, the Display One format is a special case of the Display All format when $Q = 1$. Hence, we can use the same approach to prove that all customers will follow the DO threshold purchasing rule in equilibrium, where the DO threshold purchasing rule is defined as follows: (a) purchase the item at $p_h$ if $t \leq t'$; and (b) attempt to purchase at $p_l$ after the season ends if $t > t'$, where $t' = t^*(1) = \max\{0, t(1)\}$ and $t(1)$ satisfies (4.2).

4.3.3.2 Expected Payoffs Under the Display One Format

Under the DO purchasing rule, all customers arriving before $t'$ (denoted by $B(t')$) would attempt to purchase the product at $p_h$ and all customers arriving after $t'$
(denoted by $A(t')$) would attempt to purchase the product at the reduced price $p_l$ after the season ends. Therefore, for any order quantity $Q$, the retailer can generate three revenue streams from selling $\min\{Q, B(t')\}$ items at $p_h$, selling $\min\{[Q - B(t')]^+, A(t')\}$ items at $p_l$, and disposing of $[[Q - B(t')]^+ - A(t')]^+$ items at salvage value $s$. Therefore, for any order quantity $Q$, the retailer’s expected profit associated with the Display One format can be expressed as:

$$
\Pi_{DO}^Q (Q) \quad \text{= E} \left\{ p_h \min\{Q, B(t')\} + p_l \min\{[Q - B(t')]^+, A(t')\} \right\} \\
\quad \quad + s[Q - B(t') - A(t')]^+ - cQ \\
\quad \quad = (p_h - c)Q - (p_h - p_l)E[Q - B(t')]^+ - (p_l - s) E[Q - B(T)]^+ \quad (4.5)
$$

where $B(T) = A(t') + B(t')$ corresponds to the total number of customers arriving within the selling season.

4.4 Comparisons

We now compare the retailer’s expected profits and the retailer’s optimal order quantities when the customers are either myopic or strategic and when the retailer adopts either the Display All or the Display One format.

4.4.1 Comparison of the Retailer’s Expected Profits

First, let us compare the retailer’s expected profit when customers are myopic and the retailer’s expected profit associated with the case of strategic customers under the Display One format as presented in Section 4.3.3. When customers are myopic, all arriving customers (i.e., $B(T)$) will attempt to purchase the item at

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12 We can compute the customers’ expected surplus under the Display One format by observing that each of the $\min\{(Q, B(t'))\}$ customers will obtain a surplus $(v - p_h)$ and each of the $\min\{([Q - B(t')]^+, A(t'))\}$ customers will obtain a surplus $(v - p_l)$. We omit the details.
Regardless of the display format. However, under the Display One format, only those customers arriving before \( t' \) (i.e., \( B(t') \)) will attempt to purchase the item at \( p_h \) when they are strategic. Since \( B(T) \) and \( B(t') \) are Poisson random variables with parameters \( \lambda T \) and \( \lambda t' \), respectively; and since \( t' = t^*(1) < T \), \( B(T) \) is stochastically larger than \( B(t') \), and hence, \( E[Q - B(T)]^+ \leq E[Q - B(t')]^+ \).

Combine this observation with the retailer’s expected profits given in equations (4.1) and (4.5), we have shown that \( \Pi_r^M(Q) \geq \Pi_r^{DO}(Q) \) for any given order quantity \( Q \). Next, comparing the retailer’s expected profits associated with the Display One and the Display All formats yields:

**Proposition 10.** \( \Pi_r^M(Q) \geq \Pi_r^{DO}(Q) \geq \Pi_r^{DA}(Q) \).

For any order quantity \( Q \), Proposition 10 indicates that the retailer will gain more when the customers are myopic. Moreover, when customers are strategic, displaying the items one at a time instead of displaying all items will enable the retailer to obtain a higher expected profit. It follows from Proposition 10, it is easy to see that: \( \max_Q \Pi_r^M(Q) \geq \max_Q \Pi_r^{DO}(Q) \geq \max_Q \Pi_r^{DA}(Q) \). Hence, the retailer can obtain a higher optimal expected profit under the Display One format when the customers are strategic.

### 4.4.2 Comparison of Retailer’s Optimal Order Quantities

We now compare the retailer’s optimal order quantities when customers are myopic and when customers behave strategically under the two display formats. In preparation, we determine the optimal order quantity for each case. First, when customers are myopic, the retailer’s expected profit function \( \Pi_r^M(Q) \) given in (4.1) has an identical structure as in the newsvendor problem, and hence, the optimal
order quantity $Q^M$ is the smallest integer that satisfies:

$$F(Q) \geq \frac{p_h - c}{p_h - s}, \quad (4.6)$$

where $F(\cdot)$ is the cumulative distribution function of $B(T)$, a Poisson random variable with parameter $\lambda T$.

Second, when the customers are strategic and when the retailer adopts the Display One format, it is easy to check from (4.5) that the retailer’s profit function $\Pi_r^{DO}(Q)$ is a concave function of $Q$. By considering the first order condition, the retailer’s optimal order quantity $Q^{DO}$ is the smallest integer that satisfies:

$$(p_h - p_l)G(Q) + (p_l - s)F(Q) \geq p_h - c, \quad (4.7)$$

where $G(\cdot)$ is the cumulative distribution function of $B(t')$, a Poisson random variable with parameter $\lambda t'$.

Third, when customers are strategic and when the retailer adopts the Display All format, we need to consider two separate cases: $Q < \theta$ and $Q \geq \theta$. Let $Q_1 = \arg \max \{\Pi_r^{DA}(Q) : Q \text{ integer and } Q \geq \theta\}$, and $Q_2 = \arg \max \{\Pi_r^{DA}(Q) : Q \text{ integer and } Q < \theta\}$. Therefore, under the Display All format, the retailer’s optimal order quantity is:

$$Q^{DA} = \begin{cases} Q_1, & \text{if } \Pi_r^{DA}(Q_1) > \Pi_r^{DA}(Q_2); \\ Q_2, & \text{otherwise}. \end{cases} \quad (4.8)$$

Comparing the optimal order quantities given in (4.6), (4.7) and (4.8) yields:

**Proposition 11.** $Q^M \geq Q^{DO} \geq Q^{DA}$.

Proposition 11 implies that the retailer will tend to overstock if he thinks the customers are myopic while they are indeed strategic. In addition, since $\Pi_r^{DO}(Q^{DO}) \geq \Pi_r^{DO}(Q^M)$ and $\Pi_r^{DA}(Q^{DA}) \geq \Pi_r^{DA}(Q^M)$, the retailer’s expected
profit will suffer if he incorrectly assumes that the strategic customers are myopic. To illustrate numerically about the penalty associated with this incorrect assumption, we set: \( p_h = 100, p_l = 35, T = 8, \lambda = 1, c = 25, s = 10 \) and we vary \( v \) from 115 to 190. As shown Figure 4.1, the retailer’s relative profit loss under the Display All format \( \left( \frac{\Pi_{DA}(Q_{DA}) - \Pi_{DA}(Q_M)}{\Pi_{DA}(Q_{DA})} \right) \) is increasing in the customer’s valuation, ranging from 60% to 87%. However, the relative profit loss under the Display One format \( \left( \frac{\Pi_{DO}(Q_{DO}) - \Pi_{DO}(Q_M)}{\Pi_{DO}(Q_{DO})} \right) \) is decreasing in the customer’s valuation, ranging from 1% to 10%. Therefore, it is important for the retailer to gain a clearer understanding about customer’s purchasing behavior when making ordering decision for products with short selling seasons. In addition, when customers are strategic, Proposition 11 suggests that the retailer can obtain a higher expected profit by ordering more when he adopts the Display One format instead of the Display All format. This result is verified numerically in Figure 4.2. In summary, customers’ strategic purchasing behavior and retailer’s display format can have significant impacts on the retailer’s order quantity and expected profit.

![Figure 4.1: Impact of valuation on the retailer’s relative profit loss by mistakenly assuming strategic customers as myopic](image-url)
4.5 Extensions

4.5.1 Extension 1: Multiple Classes of Customers

In the base case, all arriving customers have identical valuation $v$. We now extend our analysis to the case in which there are $n$ classes of customers, each of which has a class-specific valuation $v_i$ with probability $\alpha_i$, where $i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} \alpha_i = 1$. Without loss of generality, we assume that $p_l < p_h < v_1 < v_2 < \cdots < v_n$. Since the customer arrival process is Poisson, the arrival processes for these $n$ classes of customers are independent Poisson processes with rates $\alpha_i \lambda$, where $i = 1, 2, \ldots, n$. As before, we let $B_i(t)$ and $A_i(t)$ be the number of customers of class $i$ who arrive ‘before’ and ‘after’ $t$; respectively.

First, when the customers are myopic, each arriving customer of class $i$ will purchase at $p_h$ immediately upon arrival because $v_i > p_h$. As such, regardless of the display format, the retailer’s expected profit remains the same as stated in (4.1). Next, when the customers are strategic and when the retailer adopts
the Display All format, the exact analysis for the customers’ purchasing behavior becomes intractable. For this reason, we shall restrict our attention to the case when the retailer adopts the Display One format.

We now extend the DO threshold purchasing rule to the case of multiple classes of customers. A class \( i \) customer is said to follow the DOM threshold purchasing rule if, given his arrival time \( t \), he (a) purchases the item at \( p_h \) if \( t \leq t'_i \); and (b) wait and attempt to purchase the item at \( p_l \) after the season ends if \( t > t'_i \), where \( t'_1 < t'_2 < \cdots < t'_n \). First, let us consider a customer of class \( n \) with valuation \( v_n \) arriving at time \( t'_n \). If this customer purchases the product at time \( t'_n \), he will receive a surplus \( v_n - p_h \). Alternatively, he can wait and attempt to purchase along with \( \sum_{i=1}^{n} A_i(t'_i) \) customers (i.e., all arriving customers of class \( i \geq 1 \) who arrive after \( t'_i \)). In this case, this class \( n \) customer will get an expected surplus \( E(\sum_{i=1}^{n} \frac{1}{A_i(t'_i)}) (v_n - p_l) \). At the break-even point, \( t'_n \) must satisfy the following equation:

\[
v_n - p_h = E(\sum_{i=1}^{n} \frac{1}{A_i(t'_i)}) (v_n - p_l). \tag{4.9}
\]

Notice that \( \sum_{i=1}^{n} A_i(t'_i) \) is a Poisson random variable with parameter \( \lambda(T - \sum_{i=1}^{n} \alpha_i t'_i) \). Therefore (4.9) can be simplified as:

\[
\frac{1 - e^{-\lambda(T - \sum_{i=1}^{n} \alpha_i t'_i)}}{\lambda(T - \sum_{i=1}^{n} \alpha_i t'_i)} = \frac{v_n - p_h}{v_n - p_l}. \tag{4.10}
\]

Next, let us consider a customer of class \( j = 1, 2, \cdots, n - 1 \) with valuation \( v_j \) arriving at time \( t'_j \). He will enjoy a surplus \( v_j - p_h \) if he purchases the product at time \( t'_j \). Alternatively, he can wait and attempt to purchase after the season ends. However, since \( t'_j < t'_{j+1} < \cdots < t'_n \), any customer of class \( k \), \( k \geq j + 1 \), arriving between \( t'_j \) and \( t'_k \) will purchase the product at \( p_h \) under the DOM rule. Because the retailer displays one item at a time, this customer of class \( j \) thinks that he will get nothing by postponing his purchase if at least 1 customer of class
$k$, $k \geq j + 1$, arrives between $t'_j$ and $t'_k$. If no such arrivals occur, this customer of class $j$ will attempt to purchase the item along with $\sum_{i=1}^{n} A_i(t'_i)$ customers after the season ends. By considering the probability of having no customers of class $k$ arriving between $t'_j$ and $t'_k$ for $k = j + 1, j + 2, \cdots, n$, the threshold $t'_j$, $j = 1, 2, \cdots, n - 1$, must satisfy:

$$v_j - p_h = \prod_{k=j+1}^{n} e^{-\alpha_k \lambda (t'_k - t'_j)} \cdot E\left( \frac{1}{\sum_{i=1}^{n} A_i(t'_i) + 1} (v_j - p_i) \right). \quad (4.11)$$

By considering (4.11), it is easy to check that for $j = 1, 2, \cdots, n - 1$:

$$t'_{j+1} - t'_j = \frac{1}{\lambda \sum_{k=j+1}^{n} \alpha_k} \cdot ln\left\{ \frac{(v_{j+1} - p_h)(v_j - p_h)}{(v_j - p_h)(v_{j+1} - p_h)} \right\} > 0. \quad (4.12)$$

The last inequality results from the fact that $v_j < v_{j+1}$. By applying (4.12) repeatedly, we can express $t'_j$ in terms of $t'_n$ for $j = 1, 2, \cdots, n - 1$. By substituting these values for $t'_j$ into (4.10), we can determine $t'_n$ and then we can compute the values for $t'_j$, $j = 1, 2, \cdots, n - 1$. Notice that $t'_j$ is a complex function that depends on $\alpha_k$ and $v_k$ for $k = 1, 2, \cdots, n$. In any event, we have established the following property of $t'_k$ for $1 \leq k \leq n$:

**Proposition 12.** $0 \leq t'_1 < t'_2 < \cdots < t'_n \leq T$. For $j = 1, 2, \cdots, n - 1$, $t'_{j+1} - t'_j$ is decreasing in $\lambda, v_j$ and $p_i$, and increasing in $v_{j+1}$ and $p_h$.

By examining the expected payoff when a customer deviates from the DOM threshold purchasing rule, we can show the following Proposition.

**Proposition 13.** There is a Nash equilibrium in which all arriving customers follow the DOM threshold purchasing rule.

When each arriving customer follows the DOM threshold purchasing rule, all of the $B_i(t'_i)$ customers of class $i$ will attempt to purchase the item at $p_h$ (if

\[\text{In the event when } t'_j \text{ lies outside the range } [0, T], \text{ one can always set } t'_j = 0 \text{ when } t'_j < 0 \text{ and set } t'_j = T \text{ when } t'_j > T \text{ for } j = 1, \cdots, n.\]
available) upon arrival, and all of the $A_i(t'_i)$ customers will attempt to purchase at $p_t$ after the season ends, where $i = 1, 2, \ldots, n$. Following the similar argument as presented in Section 4.3.3.2, the retailer’s expected profit for any order quantity $Q \geq 1$ is:

$$
\Pi^{DOM}_r(Q) = (p_h - c)Q - (p_h - p_l)E[Q - \sum_{i=1}^{n} B_i(t'_i)]^+ - (p_l - s) E[Q - B(T)]^+.
$$

(4.13)

Since the retailer’s expected profit $\Pi^{DOM}_r$ given in (4.13) is a concave function of $Q$, the optimal order quantity $Q^{DOM}$ is the smallest integer that satisfies:

$$
(p_h - p_l) \hat{G}(Q) + (p_l - s) F(Q) \geq p_h - c,
$$

(4.14)

where $\hat{G}(\cdot)$ is the cumulative distribution function of $\sum_{i=1}^{n} B_i(t'_i)$, a Poisson random variable with parameter $\lambda \sum_{i=1}^{n} \alpha_i t'_i$.

We now compare the retailer’s optimal profits and optimal order quantities associated with the case of multiple customer classes when customers are myopic or strategic. By using the same argument as presented in Section 4.4.1 that the random variable $B(T)$ is stochastically larger than the random variable $\sum_{i=1}^{n} B_i(t'_i)$, we can prove the following Proposition:

**Proposition 14.** Under the Display One format, for any given order quantity $Q$, the retailer would gain a higher expected profit when the customers are myopic; i.e., $\Pi^M_r(Q) \geq \Pi^{DOM}_r(Q)$. The optimal order quantity is higher when the customers are myopic; i.e., $Q^M \geq Q^{DOM}$.

Next, we compare the retailer’s expected profit associated with the case when the retailer operates in a homogeneous market with a single class of strategic customers with identical valuation $v$ to the case when the retailer operates in a heterogeneous market with $n$ classes of strategic customers, each of which has a
class-specific valuation $v_i$, for $i = 1, 2, \cdots, n$. To establish a meaningful comparison, we shall consider the case when $v = \sum_{i=1}^{n} \alpha_i v_i$ and when all other parameters $(p_h, p_l, \lambda)$ remain the same for both markets. Because the effective demand for the product during the selling season depends on $n$ different thresholds $t'_j$, the effective demand appears to be more uncertain when operating in a heterogeneous market instead of a homogeneous market. This observation would lead one to conjecture that the retailer would obtain a lower expected profit when operating in a heterogeneous market. To examine this issue, we compare the retailer’s expected profits given in (4.5) and (4.13) by considering the case when $v = \sum_{i=1}^{n} \alpha_i v_i$, getting:

**Proposition 15.** When $v = \sum_{i=1}^{n} \alpha_i v_i$, the weighted average of the thresholds associated with the case of multiple classes of customers is higher than the threshold associated with the single class case; i.e., $\sum_{i=1}^{n} \alpha_i t'_i > t'$. In addition, for any order quantity $Q$, the retailer would obtain a higher expected profit when operating in a heterogeneous market; i.e., $\Pi^{DOM}_r(Q) \geq \Pi^{DO}_r(Q)$. Furthermore, it is optimal for the retailer to order more when operating in a heterogeneous market; i.e., $Q^{DOM} \geq Q^{DO}$.

Proposition 15 presents a counter-intuitive result in which the retailer would obtain a higher expected profit when operating in a heterogeneous market. To understand the underlying reason, observe that the weighted average of the thresholds associated with the case of multiple classes of customers is higher than the threshold associated with the single class case. This implies that, when operating in a heterogeneous market, the retailer has a longer aggregate time window (i.e., $\sum_{i=1}^{n} \alpha_i t'_i$) within which each arriving customer will purchase the product at $p_h$. This explains why the retailer would obtain a higher expected profit in a heterogeneous market.
Finally, let us consider the case in which the retailer incorrectly assumes that a heterogeneous market is homogeneous. Specifically, the retailer assumes that the market is comprised of a single class of customers with identical valuation $v$, while the actual market consists of multiple classes of customers with class specific valuation $v_i$, for $i = 1, 2, \cdots, n$. This situation could occur when the retailer aggregates the customer’s valuation into a single class so that $\sum_{i=1}^{n} \alpha_i v_i = v$.

We now evaluate the relative profit loss $\frac{\Pi_{DOM}^{DOM}(Q_{DOM}) - \Pi_{DOM}^{DOM}(Q_{DO})}{\Pi_{DOM}^{DOM}(Q_{DOM})}$ associated with this incorrect assumption. To do so, we use the same parameter values as given in Section 4.4.2. For the homogeneous market, we set $v = 117$; however, for the heterogeneous market, we set $n = 2$ and $\alpha_1 = \alpha_2 = 0.5$. To ensure that $\sum_{i=1}^{2} \alpha_i v_i = v = 117$, we set $v_2 = 117 + \Delta > v_1 = 117 - \Delta$ and we vary $\Delta$ from 1 to 16, where $\Delta$ captures the heterogeneity of customer valuations.

As shown in Figure 4.3, the relative profit loss increases as $\Delta$ increases, and hence, it is important for the retailer to obtain a clearer understanding about the heterogeneity of customer valuations.

![Figure 4.3: Impact of customer’s valuation heterogeneity on the retailer’s relative profit loss by mistakenly assuming heterogeneous market as homogeneous](image)
4.5.2 Extension 2: Inventory Dependent Clearance Price

In the base model, the post-season clearance price $p_l$ is pre-committed at the beginning of the season and is independent of the end-of-season inventory level $I$, where $0 \leq I \leq Q$. Using this markdown pricing mechanism, the retailer is unable to set the clearance price as a response to the actual end-of-season inventory level $I$. In this section, we extend our analysis to the case in which the retailer would announce that the post-season clearance price will follow a specific function $p(I)$, and that $p(I)$ is a decreasing function of $I$. In this case, the customer knows the functional form of $p(\cdot)$ but would not know the actual post-season clearance price until the end-of-season inventory level $I$ is realized.\(^\text{14}\)

Besides adopting the Display All format, let us consider the situation in which the retailer announces the regular price $p_h$ and the post-season clearance price plan $p(\cdot)$ at the beginning of the season, where $p_h > p(I)$ for $0 \leq I \leq Q$.\(^\text{15}\) Suppose a customer arrives at time $t$ and observes $k$ units available for sale, where $1 \leq k \leq Q$. Then he will enjoy a surplus $v - p_h$ if he purchases the item at $p_h$. Alternatively, he can wait and attempt to purchase the item at the reduced price $p(I)$ after the season ends. If he decides to wait and if all customers who arrive after $t$ would also wait, then $I = k$. In this case, he will get an expected surplus $(v - p(k))H(k, t)$, where the term $H(k, t)$ represents the expected probability of getting the item at the reduced price after the season.

\(^\text{14}\)Based on our discussion with a subsidiary of a high-end handbag retailer in Taiwan, we learned that it is common for their Europe headquarter to pre-announce to the subsidiaries regarding the specific functional form of $p(\cdot)$ at the beginning of the season so that the subsidiaries in different countries can coordinate their markdown prices efficiently. While we are not aware of a situation in which the retailer would pre-announce the markdown price function $p(\cdot)$ to their customers, we think this analysis can provide additional insights regarding a different post-season clearance pricing mechanism.

\(^\text{15}\)Since this price markdown plan depends on the end-of-season inventory level $I$, the customers would need to know the actual inventory level at all times. As such, the Display One format would not be appropriate for this case.
ends. By comparing the expected surpluses associated with these two purchase options, we can establish the following Inventory Dependent Clearance (IDC) threshold purchasing rule: For any customer who arrives at time \( t \) and observes \( k \) units available for sale, he should: (a) purchase one unit at \( p_h \) if \( t \leq \tilde{t}(k) \); and (b) attempt to purchase one unit at \( p(I) \) after the season ends if \( t > \tilde{t}(k) \), where the threshold \( \tilde{t}(k) = \max\{0, t(k)\} \) and \( t(k) \) satisfies the following equation:

\[
H(k, t(k)) = \frac{v - p_h}{v - p(k)},
\]

(4.15)

where \( H(k, t(k)) \) is given in (4.3).

By replacing \( p_l \) with \( p(k) \), it is easy to show that Lemma 1, Proposition 1 and Proposition 2 continue to hold. Specifically, Propositions 1 and 2 become:

**Proposition 16.** There exists a positive integer \( \tilde{\theta} \) that satisfies:

\[
\tilde{\theta} = \arg\min \{t(j) \leq 0 : j = 1, 2, \cdots \}.
\]

Moreover, the threshold \( \tilde{t}(k) = \max\{0, t(k)\} \) has the following properties:

1. If \( Q < \tilde{\theta} \), then \( 0 < \tilde{t}(Q) < \tilde{t}(Q - 1) < \cdots < \tilde{t}(1) < T \).

2. If \( Q \geq \tilde{\theta} \), then \( \tilde{t}(Q) = \tilde{t}(Q-1) = \cdots = \tilde{t}(\tilde{\theta}) = 0 < \tilde{t}(\tilde{\theta}-1) < \cdots < \tilde{t}(1) < T \).

3. The threshold \( \tilde{t}(k) \) is increasing in \( \lambda \) and \( v \) and \( p(k) \), and is decreasing in \( p_h \).

Furthermore, there exists an equilibrium in which all arriving customers will follow the IDC threshold purchasing rule.

Next, let us compare the threshold \( t^*(k) \) associated with the base case as presented in Section 4.3.2.1 and the threshold \( \tilde{t}(k) \) associated with the Inventory Dependent Clearance price case. For the purpose of comparison, let us consider the case in which \( p_l \) is bounded between \( p(Q) \) and \( p(1) \) so that there exists a \( \delta \)
so that \( p(1) \geq p(2) \geq \cdots \geq p(\delta) > p_l \geq p(\delta + 1) \geq \cdots \geq p(Q) \). In this case, we can prove the following result:

**Proposition 17.** \( \tilde{t}(k) > t^*(k) \) for \( 1 \leq k \leq \delta \) and \( \tilde{t}(k) \leq t^*(k) \) for \( \delta + 1 \leq k \leq Q \).

The result stated in Proposition 17 is intuitive. Consider the case when there are fewer items left; i.e., when \( k \) is small so that \( 1 \leq k \leq \delta \). In this case, \( p(k) > p_l \), and hence, each customer who observes \( k \) units available for sale is more eager to purchase the item at \( p_h \) under the IDC threshold purchasing rule. This explains why \( \tilde{t}(k) > t^*(k) \) when \( k \leq \delta \). We can use a similar approach to explain the result \( \tilde{t}(k) \leq t^*(k) \) when \( k > \delta \).

Finally, let us compute the retailer’s expected profit when all arriving customers follow the IDC threshold purchasing rule. In this case, we can use the same approach as presented in Section 4.3.2.2 to determine the retailer’s expected profit. To do so, we first re-define \( f(j, i) \) and \( g(i) \) in Section 4.3.2.2 as \( \tilde{f}(j, i) \) and \( \tilde{g}(i) \) by replacing the threshold \( t^*(k) \) with \( \tilde{t}(k) \) and by replacing \( p_l \) with \( p(i) \) for certain value of \( i \). Second, we can redefine \( N(j, i) \) as a Poisson random variable with parameter \( \lambda(\tilde{t}(i) - \tilde{t}(j)) \). Then we can show that the retailer’s expected profit can be expressed as:

\[
\tilde{\Pi}_r(Q) = \tilde{f}(Q + 1, Q) - cQ. \tag{4.16}
\]

To compare the retailer’s optimal order quantity and optimal expected profit under the pre-committed clearance price in the base model with the inventory dependent clearance price in this section, we consider the same numerical example as in Section 4.4.2, except we set \( p_l = 50 \) in the pre-committed clearance price case. For any order quantity \( Q \), we set the inventory dependent price function \( p(I) \) as \( 40 = p(Q) = \cdots = p(Q/2) < p(Q/2 + 1) = \cdots = p(1) = 60 \). As we can see from Figure 4.4, under both clearance pricing mechanisms, when the customer’s
valuation $v$ is below a certain threshold, it is optimal for the retailer to order a large quantity so that all arriving customers will wait for post-season clearance sale. When the customer’s valuation $v$ is sufficiently high, the retailer’s optimal order quantity increases with valuation under both clearance pricing mechanisms. Furthermore, as shown in Figure 4.4, it is optimal for the retailer to order more when the clearance price $p_l$ is pre-committed at the beginning of the season. By ordering more, the retailer may be able to obtain a higher optimal profit when the clearance price $p_l$ is pre-committed. This result is illustrated numerically in Figure 4.5. Therefore, having the flexibility to set the clearance price at the end of the season will actually reduce the retailer’s expected profit when the customers are strategic. This result is consistent with the numerical result presented in Aviv and Pazgal (2005).

Figure 4.4: Impact valuation on the retailer’s optimal order quantities under pre-committed and inventory dependent clearance prices
4.6 Conclusions

In this chapter, we have examined the retailer’s optimal order quantity and the retailer’s optimal expected profit under four scenarios: the customers are either myopic or strategic, and the retailer adopts either the Display All or the Display One format. To obtain tractable results, we have developed a model based on a situation in which the retailer announces both the regular price and the post-season clearance price at the beginning of the selling season and the customers arrive in accord with a Poisson process. When the customers have identical valuation, we have shown that, in equilibrium, each strategic customer should behave according to a threshold policy that depends on inventory level at the time of arrival. We have proved that the retailer would obtain a higher profit and would order more when the customers are myopic and that the retailer would earn a higher profit and would order more under the Display One format when the customers are strategic. We have illustrated numerically the penalty associated with the case when the retailer mistakenly assumes that the strategic customers are myopic. We have extended our analysis to the case in which customers belong
to multiple classes and the retailer adopts the Display One format. Furthermore, we have extended our analysis to the case in which the post-season clearance price depends on the actual end-of-season inventory level.

Some of our assumptions in the chapter can be relaxed. First, we have assumed a customer’s valuation is fixed throughout the entire season. However, as discussed in Aviv and Pazgal (2005), the customer’s valuation can be time-dependent: high in the beginning and decline over time. It is easy to check that our results continue to hold when all the customer have the same time-declining valuation function: $v(t) = Ve^{-\alpha t}$, where $V$ is the same base valuation and $\alpha$ is the declining rate. Second, we have assumed that customers are risk-neutral in our work. Liu and van Ryzin (2005) consider the retailer’s optimal stocking decisions when the customers are risk neutral or risk-averse and they show that the retailer will behave differently in these two cases. Assuming the same power utility function $u(x) = x^r$ as in Liu and van Ryzin (2005), our analysis still holds.

Our work has certain limitation in terms of the same valuation for all the customers under the Display All format. Since the exact analysis for customers’ strategic purchasing behavior is intractable when customers have different valuations, a different approach is needed for our future research.

4.7 Appendix: Proof

Proof of Lemma 4: In order to show that $t(k)$ that satisfies (4.2) is unique, we need to show that the equation $\hat{H}(k,t) = \frac{v-p_h}{v-p_l}$ has a unique solution, where $\hat{H}(k,t)$ is an auxiliary function associated with $H(k,t(k))$. Specifically,

$$\hat{H}(k,t) = \sum_{n=0}^{k-1} \frac{\lambda(T-t)^n}{n!} e^{-\lambda(T-t)} + \sum_{n=k}^{\infty} \frac{\lambda(T-t)^n}{n!} e^{-\lambda(T-t)} \frac{k}{n+1}$$

(4.17)
By observing the fact that $\hat{H}(k, t) = 1$ when $t = T$ and $\hat{H}(k, t)$ is strictly increasing in $t$ for any $k \geq 1$, we can conclude that there exists an unique $t(k)$ that satisfies (4.2).

Next, we prove $t(k + 1) < t(k)$ by contradiction. Suppose $t(k + 1) \geq t(k)$. By considering (4.2), (4.3), and the fact that $A(t(k + 1))$ and $A(t(k))$ are Poisson random variables with parameters $\lambda(T - t(k + 1))$ and $\lambda(T - t(k))$, respectively, we can show that:

$$\frac{v - p_h}{v - p_l} = \frac{v - p_h}{v - p_l}$$

This leads to a contradiction. Hence, we must have $t(k + 1) < t(k)$.

Finally, we prove $t(1) < T$ by contradiction. Suppose $t(1) \geq T$. By considering (4.2) and (4.3), we have $1 > \frac{v - p_h}{v - p_l} = \hat{H}(1, t(1)) = \hat{H}(1, t(1)) \geq \hat{H}(1, T) = 1$. This leads to a contradiction. Finally, to prove the third statement, we apply the implicit function theorem by taking the derivative of the equation (4.2) with respect to $\lambda$, $v$, $p_h$ and $p_l$. We omit the details. $\square$

Proof of Proposition 7: To start, it follows from Lemma 1 that the threshold $t(k)$ is strictly decreasing in $k$ and $t(1) < T$, there must exist a $\theta$ so that $\theta$ is the smallest integer that has $t(\theta) \leq 0$. By considering the definition of $\theta$ and the definition of $t^*(k)$, we can apply Lemma 1 to prove the remainder of the Proposition. $\square$.

Proof of Proposition 8: We prove our result by contradiction. Suppose not.
Then there must exist a customer who arrives at time $t$, observes $k$ units available for sale, and obtains a higher surplus by deviating from the DA threshold purchasing rule, while all other arriving customers follow the rule. To aim for a contradiction, we now show this customer cannot get a higher surplus by deviating from the DA threshold purchasing rule. Let us consider the following cases:

1. When $Q < \theta$. Since $Q < \theta$, we have $0 < t(k) = t^*(k) < T$ by Proposition 1. We now consider two scenarios:

   (a) When $t < t^*(k)$. Under the DA threshold purchasing rule, this customer would receive a surplus $(v - p_h)$ by purchasing the item at $p_h$. However, he deviates from the rule by attempting to purchase the product at $p_l$ after the season ends. By doing so, he receives a surplus $(v - p_l)H(k, t)$, where $H(k, t)$ is the expected probability of getting one item at $p_l$ after the season ends. To aim for a contradiction, it suffices to show that $H(k, t) \leq \frac{v - p_h}{v - p_l}$. The exact expression for $H(k, t)$ is quite complex because it depends on the specific customer arrival pattern after $t$. In preparation, let $N_0$ be the number of customers arriving between $t$ and $t(k)$ and let $N_i$ be the number of customers arriving between $t(k - i + 1)$ and $t(k - i)$ for $i = 1, 2, \ldots, k - 1$, where the $N_i$’s are independent Poisson random variables. Let us make the following observations. Consider the case when $N_0 = 0$ (i.e., no customers arrive between $t$ and $t(k)$), then it is easy to see that $H(k, t) = H(k, t(k))$. Next, consider the case when $N_0 = 1, N_1 = 0$. In this case, the customer who arrives between $t$ and $t(k)$ will purchase one item under the DA threshold purchasing rule, and hence, there are $k - 1$ items left at time $t(k)$. However, since $N_1 = 0$ (i.e., no customers arrive between
it is easy to see that $H(k, t) = H(k - 1, t(k - 1))$. By using the same argument, we can enumerate certain events associated with the random variables $N_i$'s so that in each event, we have $H(k, t) = H(k - j, t(k - j))$ for some $j \in \{0, 1, \cdots, k - 1\}$. By considering the probability of the occurrence of these events, it can be shown that:

$$H(k, t) = P_0H(k, t(k)) + P_1H(k - 1, t(k - 1)) + \cdots + P_{k-1}H(1, t(1)),$$

and that $P_0 + P_1 + \cdots + P_{k-1} < 1$. It follows from (4.2) that $H(i, t(i)) = \frac{v-p_h}{v-p_l}$ for $i = k, k-1, \cdots, 1$, we can conclude that $H(k, t) < \frac{v-p_h}{v-p_l}$. This leads to a contradiction.

(b) When $t > t^*(k)$. Under the DA threshold purchasing rule, this customer would receive a surplus $(v-p_l)H(k, t)$ by attempting to purchase at $p_l$ after the season ends. However, he obtains a surplus $(v-p_h)$ instead by purchasing the item at $p_h$. To aim for a contradiction, it suffices to show that $(v-p_l)H(k, t) \geq (v-p_h)$. Under the DA threshold purchasing rule, all customers arriving after $t(k)$ would wait. Hence, had the customer who arrives at time $t$ attempted to purchase the item at $p_l$ after the season ends, he would have received a surplus:

$$(v-p_l)H(k, t) = (v-p_l)\left\{ \sum_{n=0}^{k-1} \text{Prob}(N + A(t) = n) \right\} + \sum_{n=k}^{\infty} \text{Prob}(N + A(t) = n) \frac{k}{n+1},$$

where $N$ represents the number of customers arriving within the interval $(t(k), t)$, and $A(t)$ represents the number of customers arriving within the interval $(t, T]$. Since $N$ and $A(t)$ are independent Poisson random variables, the distribution of $N + A(t)$ is identical to the
distribution of $A(t(k))$, where $A(t(k))$ corresponds to the number of customers arriving within the interval $(t(k), T]$. It follows from (4.2), we have: $(v - p_t)H(k, t) = (v - p_t)H(k, t(k)) = v - p_h$. This leads to a contradiction.

2. When $Q \geq \theta$. Since $Q \geq \theta$, Proposition 1 implies that $t(Q) < 0$ and $t^*(Q) = 0$. In this case, since all the other customers follow the DA threshold purchasing rule by waiting, this deviating customer who arrives at time $t > 0$ would observe $Q$ items available. Had he waited and attempted to purchase the item after the season ends, he would have received an expected surplus $H(Q, 0)(v - p_t) = \hat{H}(Q, 0)(v - p_t) > \hat{H}(Q, t(Q))(v - p_t) = v - p_h$, where the inequality follows by the fact that $\hat{H}(k, t)$ is a strictly increasing function of $t$. Therefore, this customer cannot gain a higher surplus by deviating from the DA threshold purchasing rule.

By combining the above cases, we can conclude that any customer who deviates from the DA threshold purchasing rule cannot obtain a higher surplus. This completes the proof. □

**Proof of Proposition 9:** To show the recursive formula for $f(j, i)$ in the first statement, we consider the following three events associated with $N(j, i)$:

1. When $N(j, i) = 0$; i.e., when no customers arrive within the time window $(t^*(j), t^*(i)]$. Since there are $i$ items available for sale at time $t^*(j)$, the retailer will still have $i$ items available at time $t^*(i)$. Hence, $f(j, i) = g(i)$.

2. When $N(j, i) \geq i$. Since $i < j$ and since $t^*(j) \leq t^*(i) \leq t^*(k)$, for $1 \leq k \leq i$ (Proposition 1), each customer arriving at time $t \in (t^*(j), t^*(i)]$ has $t \leq t^*(k)$ for $1 \leq k \leq i$. Hence, each of these arriving customers will
attempt to purchase at \( p_h \) under the DA purchasing rule. Since \( N(j, i) \geq i \),
\[ f(j, i) = ip_h. \]

3. When \( N(j, i) = k \), where \( k = 1, \cdots, i - 1 \). Using the same argument as before, it is easy to show that the retailer will receive \( kp_h \) from these \( k \) arriving customers and will have \( i - k \) remaining units available at time \( t^*(i) \). In this case, \( f(j, i) = kp_h + f(i, i - k) \).

Combining the payoffs associated with the above three cases, we have proved the recursive formula for \( f(j, i) \).

To prove the second statement, it remains to determine the function \( g(i) \) when there are \( i \) units available at time \( t^*(i) \). Under the DA threshold purchasing rule, when \( i \) units are available at time \( t^*(i) \), all customers arriving after time \( t^*(i) \) will attempt to purchase the item at the reduced price \( p_l \) after the season ends. Based on the number of arrivals after time \( t^*(i) \); i.e, \( A(t^*(i)) \), the retailer will sell \( \min\{i, A(t^*(i))\} \) units at \( p_l \) and dispose of each of the \([i - A(t^*(i))]^+\) leftover units at salvage value \( s \). Noting that \( A(t^*(i)) = N(i, 0) \) as \( t^*(0) = T \), the function \( g(i) \) can be expressed as:

\[
g(i) = p_l \mathbb{E}\{\min\{i, A(t^*(i))\}\} + s \mathbb{E}\{[i - A(t^*(i))]^+\}
\]
\[
= ip_l - (p_l - s) \mathbb{E}\{[i - A(t^*(i))]^+\}
\]
\[
= ip_l - (p_l - s) \sum_{k=0}^{i-1} (i - k) \operatorname{Prob}(N(i, 0) = k).
\]

This completes the proof. \( \square \)

**Proof of Proposition 10:** It remains to show \( \Pi_r^{DO}(Q) \geq \Pi_r^{DA}(Q) \). We need to consider two cases: \( Q \geq \theta \) and \( Q < \theta \). When \( Q \geq \theta \), Proposition 1 implies that \( t^*(Q) = 0 \). Since \( t^*(Q + 1) = t^*(Q) = 0, N(Q + 1, Q) = 0 \). Hence, it is easy to check from Proposition 9 that \( f(Q + 1, Q) = g(Q) \) and \( g(Q) = \)
Qp_l − (p_l − s)E[Q − B(T)]^+, because A(t^*(Q)) = A(0) = B(T). Therefore, the retailer’s expected profit given in (4.4) can be expressed as:

\[ \Pi_r^{DA}(Q) = (p_l - c)Q - (p_l - s)E[Q - B(T)]^+. \] (4.19)

Compare the above equation with the retailer’s expected profit under the Display One format given in (4.5), it is easy to show that \( \Pi_r^{DO}(Q) \geq \Pi_r^{DA}(Q) \) when \( Q \geq \theta \).

Let us consider the case when \( Q < \theta \). In this case, Proposition 1 implies that \( 0 < t^*(Q) < \cdots < t^*(1) = t' < T \). To compare the retailer’s expected profits associated with these two display formats, let us consider the following three mutually exclusive and exhaustive events:

1. When \( B(t') \geq Q \). Under the DO threshold purchasing rule, all \( B(t') \) customers will attempt to purchase the item at \( p_h \), and hence, the retailer’s profit is equal to \( (p_h - c)Q \). Under the DA threshold purchasing rule, not all \( B(t') \) customers will attempt to purchase the item at \( p_h \) because each arriving customer’s purchasing decision depends on the actual inventory level upon arrival. As such, the retailer will sell \( k \) items at price \( p_h \) and sell the rest of the \( Q - k \) items at price \( p_l \), where \( 0 \leq k \leq Q \) and \( k \) depends on the arrival times of those \( B(t') \) customers. In this case, the retailer’s profit is equal to \( kp_h + (Q - k)p_l - cQ \). Hence, the retailer would obtain a lower profit under the Display All format.

2. When \( B(t') = m < Q \) and \( B(T) \geq Q \). Under the DO purchasing rule, \( B(t') = m \) customers will attempt to purchase the item at \( p_h \), and \( B(T) - B(t') \) customers will attempt to purchase the item at the reduced price \( p_l \) after the season ends. Since \( B(t') = m < Q \) and \( B(T) \geq Q \), the retailer’s profit is equal to \( mp_h + (Q - m)p_l - cQ \). Under the DA threshold purchasing rule, we can use the same argument as before to show that the retailer’s
profit is equal to \( (kp_h + (Q - k)p_l) - cQ \), where \( 0 \leq k \leq m \). Hence, the retailer would obtain a lower profit under the Display All format.

3. When \( B(t') = m < Q \) and \( B(T) = n < Q \). Under the DO threshold purchasing rule, \( B(t') = m \) customers will attempt to purchase the item at \( p_h \), and \( B(T) - B(t') = n - m \) customers will attempt to purchase the item at the reduced price \( p_l \) after the season ends. Since \( n < Q \), there will be \( Q - n \) units left after the post-season clearance. As such, the retailer’s profit is equal to \( (mp_h + (n - m)p_l + (Q - n)s) - cQ \). Under the Display All format, we can use the same argument as before to show that the retailer’s profit is equal to \( (kp_h + (Q - k)p_l + (Q - n)s) - cQ \), where \( 0 \leq k \leq m \). Hence, the retailer would obtain a lower profit under the Display All format.

Since the retailer obtains a lower profit under the Display All format in all three cases as stated above, we can conclude immediately that \( \Pi_{DO}^r(Q) \geq \Pi_{DA}^r(Q) \) when \( Q < \theta \). This completes the proof. \( \square \)

**Proof of Proposition 11:** We prove \( Q^M \geq Q^{DO} \) first. Since \( B(T) \) and \( B(t') \) are Poisson random variables with parameters \( \lambda T \) and \( \lambda t' \) respectively, \( B(T) \) is stochastically larger than \( B(t') \). Thus, \( G(x) \geq F(x) \) for \( x \geq 0 \), where \( F(\cdot) \) and \( G(\cdot) \) are the cumulative distribution functions associated with the random variables \( B(T) \) and \( B(t') \), respectively. Since \( G(x) \geq F(x) \), \( (p_h - p_l)G(Q^M) + (p_l - s)F(Q^M) \geq (p_h - s)F(Q^M) \geq p_h - c \), where the last inequality follows from (4.6). In this case, we have shown that \( Q^M \) satisfies (4.7). Since \( Q^{DO} \) is the smallest integer that satisfies (4.7), we can conclude that \( Q^M \geq Q^{DO} \).

We now prove \( Q^{DO} \geq Q^{DA} \). When \( Q \geq \theta \), the retailer’s expected profit \( \Pi_{DA}^r(Q) \) is given in (4.19). By considering the first order condition, the retailer’s optimal order quantity \( Q_1 \) is the smallest integer value that satisfies \( F(Q) \geq \frac{p_h - c}{p_l - s} \). Next, when \( Q < \theta \), the retailer’s expected profit \( \Pi_{DA}^r(Q) \) is given in (4.4). In this
case, we can determine the optimal order quantity $Q_2$ by evaluating the function $\Pi^{DA}(Q)$ for $Q \in [1, \theta - 1]$ and obtain the optimal expected profit $\Pi^{DA}(Q_2)$. By the definition of $Q^{DA}$ in (4.8), it suffices to show that $Q^{DO} \geq Q_1$ since $Q_1 \geq \theta > Q_2$.

Since $Q_1$ is the smallest integer that satisfies $F(Q) \geq \frac{\theta - c}{p - s}$, it suffices to show that $F(Q^{DO}) \geq \frac{\theta - c}{p - s}$. We prove this by contradiction. Suppose $F(Q^{DO}) < \frac{\theta - c}{p - s}$.

Then $(p_h - p_l) G(Q^{DO}) + (p_l - s) F(Q^{DO}) < (p_h - p_l) G(Q^{DO}) + (p_l - c) < p_h - c$, which violates the definition of $Q^{DO}$ given in (4.7). Therefore, we can conclude that $F(Q^{DO}) \geq \frac{\theta - c}{p - s}$ and that $Q^{DO} \geq Q_1$. This completes the proof. □

**Proof of Proposition 13:** It follows by the similar arguments as in the proof of Proposition 8. We omit the details. □

**Proof of Proposition 15:** Notice $t'$ satisfies \( \frac{1 - e^{-\lambda(T - t')}}{\lambda(T - t')} = \frac{v - p_h}{v - p_l} \). By considering equation (4.10) and the fact that $v_n > v$, we can conclude immediately that $\sum_{i=1}^{n} \alpha_i t'_i > t'$ and $\sum_{i=1}^{n} B_i(t'_i)$ is stochastically larger than $B(t')$. Therefore, $\Pi^{DOM}(Q) \geq \Pi^{DO}(Q)$ and $Q^{DOM} \geq Q^{DO}$ follow by considering equations (4.5), (4.13), (4.7) and (4.14). We omit the details. □

**Proof of Proposition 16:** We can use the same approach as the proof of Propositions 7 and 8 to prove our result. We omit the details. □

**Proof of Proposition 17:** Since $p(k) > p_l$ for $1 \leq k \leq \delta$, we have $\frac{v - p_h}{v - p_l} > \frac{v - p_k}{v - p_l}$. We can apply Lemma 1 and Proposition 16 to show that $\hat{t}(k) > t^*(k)$ for $1 \leq k \leq \delta$. We can complete the proof by using a similar argument. We omit the details. □

### 4.8 References


