MODELING INTRA AND INTER CORRELATIONS IN CREDIT DEFAULT LOSSES

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Abstract

Comparing alternatives for a simultaneous incorporation of intra and inter correlations into the credit portfolio loss distribution within the asymptotic single risk factor (ASRF) model and showing that the resulting distribution depends on the type of a dominant correlation: whether it is of intra d-d/lgd-lgd or inter d-lgd type. Showing that the classic Vasicek distribution (derived originally for intra d-d correlations only), modified to embrace both intra and inter correlation types by properly constructing composite mean and correlation parameters, offers an analytic solution which covers an entire range of correlations and is easy to use as opposed to a more formal approach of a bivariate joint default probability density which requires numerical averaging over the underlying latent factors and applies only when intra correlations dominate.

1. Introduction

Modeling credit default losses in large portfolios is materially complicated by correlations of loss variables – defaults and recoveries. While modeling intra correlated defaults (d-d correlation) is more or less well developed– at least, in terms of asymptotic single risk factor (ASRF) approach, the situation is much less elaborated for intra correlated loss-given-defaults (lgd-lgd correlation), and even less so for inter d-lgd correlations. At the same time, a pressing need does exist – and especially for large and diversified portfolios – for practically efficient and computationally less demanding models incorporating simultaneous correlations of generic default parameters, even at the expense of using heavily stylized approaches. In this context the role of analytically tractable models rises substantially as they allow to focus on capturing leading trends and thus
significantly reduce processing time, while furnishing an accuracy quite comparable with much more sophisticated counterparts.

The authors benefited from a number of prior empirical and theoretical studies (acknowledged here without pretending for any particular order and/or completeness) which facilitated a recent evolution of ideas and which help better understand the motivation behind current work. On the empirical side there is a considerable variety of works demonstrating a positive correlation between default and loss-given-default rates (e.g. Gupton et al (2000), Frye (2003), Altman et al (2004)) and others). On the modeling side the first and foremost is the path-breaking, now classic solution for correlated default distribution by Vasicek (1987, 2002) (within ASRF) in infinitely grained uniform portfolios. Further, in the last decade there was a number of models utilizing a single systematic factor, but differing in number of idiosyncratic factors and the way they were incorporated into the resulting loss rate model (e.g. Frye (2000), Pykhtin (2003), Tasche (2004), among others, and for the review see as well Witzany(2009). Witzany (2011) also discusses a two systematic factors model to better capture the effect of correlations between defaults and recoveries. On the alternative venue, Sanchez et al (2008) modeled the impact of d-lgd correlations on the loss distribution variance by using the conditional variance with conditioning of the lgd expectation analogous to the one from the ASRF approach. We will return to this conditioning aspect in more detail later in the text. Quite recently Frye and Jacobs (2012) argued that the model where the conditional lgd rate is a deterministic function of the default rate – as opposed to traditional models without that functional dependency – quite often offers a better fit to the empirical data than more traditional results from the statistical regression lgd rates on default rates.

With that prior work in mind, we aim here at showing how correlations in the default prone portfolio can be classified and systematically accounted for within a similarly stylized default and loss-given-default models. At the same time, all aspects related to empirical data suitable as inputs to the model are intentionally left aside as this should be a subject of a separate study. In Sec. 2 we review a standard technique of ASRF models for intra d-d correlations and Vasicek distribution for default rates. Sec. 3 does the same for lgd-lgd correlations and lgd rates. In Sec. 4 we introduce and compare two different approaches to inter d-lgd correlations: individual and collective modes. Sec. 5 presents some comparison of results for loss rates stemming from both models. Brief conclusions are summarized in Sec. 6. For the ease of reading and in the interest of the broader readership/audience, most of the calculations are moved to Appendices and the main text only refers to the corresponding results as needed.
2. Brief review of ASRF correlated defaults model

An individual obligor default is driven by a Bernoulli (binary) random variable \( d \) with values only \((0,1)\) and unconditional mean equal \( pd = E(d) \). The fractional number of uncorrelated defaults in a uniform portfolio of \( N \) identical assets is given then by a Binomial random variable \( x \): a normalized sum of uncorrelated Bernoulli variables \( x = \sum d_i / N \). In the limit of \( N \to \infty \) \( x \) follows a delta–function distribution peaked at \( E(x) = pd \).

Further, in Merton picture, the default onsets as the hidden/latent idiosyncratic variable (say, normally distributed) crosses some threshold level \( K \), and therefore \( pd = E(d) = \Phi(K) \), where \( \Phi \) is a normal cdf. In ASRF, default correlations are modeled parametrically, i.e defaults are governed by a function (say, an indicator function) of a latent rv \( W^d_i \), incorporating a common for all assets market latent factor “u” which mixes with idiosyncratic obligor specific latent rv “\( e_i \)“ via a one-factor copula model as in \( W^d_i := \alpha^{1/2} u + (1 - \alpha)^{1/2} e_i \), with \( \text{Var}(W_i)=1 \) where \( \alpha^{1/2} \) is a factor loading and \( \alpha \) is the latent process covariance/correlation coefficient. Although the latent factor distribution is arbitrary, meeting only some general convergence requirements, we would assume it normal.

Accordingly, the probability of default is no longer fixed \( pd = \Phi(K) \), but rather a conditional expectation \( E(d|u) \) as

\[
pd(u) := E(d | u) = \Phi[(\Phi^{-1}(pd) - \sqrt{\alpha}u) / \sqrt{1-\alpha}]
\] (2.1)

which is a function of the latent factor “\( u \)” and recovers initial \( pd \) for \( \alpha = 0 \). There are two important facts about \( pd(u) \) worth mentioning here. First, the underlying unperturbed value \( pd \) serves as an expected value of \( pd(u) \) over the latent factor distribution, i.e. \( E[pd(u)] = pd \). Traditionally this is viewed as the result of one-factor copula construction which holds true for any latent factor particular distribution type (with the only necessary requirements of convergence of all integrals). It is instructive though to see how this result materializes via direct but more technical derivation for, say, normal distribution, which is given in Appendix 1.

Second, the backbone advantage of introducing the correlation parametrically via one-factor copula is that for every given latent factor value \( u \) the default events of individual obligors/counterparties in the portfolio remain independent which means that the total fractional portfolio loss still follows the binomial distribution for finite \( N \) and becomes delta-function around \( E(x|u) = pd(u) \) for \( N \to \infty \). After averaging over \( u \)-distribution the portfolio fractional loss pdf becomes:

\[
\text{pdf: } f(n) = \binom{N}{n} \int [pd(u)]^n [1 - pd(u)]^{N-n} \varphi(u) du
\] (2.2)
where \( \phi(u) \) is the normal density. The important consequence of (2.2) is that for any non-vanishing correlation parameter \( \alpha \), the loss distribution for \( N \rightarrow \infty \) is no longer delta-peaked, but is spread proportionally to the correlation value \( \alpha \). This is the essence of the now classic result of Vasicek (1987, 2002):

$$\text{cdf: } F_v(pd, \alpha; z) = \Phi\left( \frac{1}{\sqrt{\alpha}} \left[ \sqrt{1-\alpha} \Phi^{-1}(z) - \Phi^{-1}(pd) \right] \right)$$

$$\text{pdf: } f_v(pd, \alpha; z) = \sqrt{\frac{1-\alpha}{\alpha}} \exp \left\{ 0.5 \left[ \Phi^{-1}(z) \right]^2 - \frac{0.5}{\alpha} \left[ \sqrt{1-\alpha} \Phi^{-1}(z) - \Phi^{-1}(pd) \right]^2 \right\} \quad (2.3)$$

which derivation is briefly reviewed in the Appendix 1. Rewriting pdf in (2.3) as

$$f_v(z) = \sqrt{\frac{1-\alpha}{\alpha}} \exp \left\{ \frac{1}{\alpha} \left[ (\alpha - \frac{1}{2}) \left[ \Phi^{-1}(z) \right]^2 - \frac{1}{\alpha} \sqrt{1-\alpha} \Phi^{-1}(z) \Phi^{-1}(pd) - \frac{1}{2\alpha} \left[ \Phi^{-1}(pd) \right]^2 \right] \right\} \quad (2.4)$$

shows the exponent containing three terms: the first component is symmetric in \( z \) and ensuring that the density vanishes at end points 0 and 1, the second adds asymmetry to the density and the last one is just a normalization constant. Generic pdf profile features are especially pronounced if \( \text{pd} = 0.5 \): last two terms disappear and the density become symmetric with \( \cup \) or \( \cap \) shape for \( \alpha < 0.5 \) and \( \alpha > 0.5 \) respectively. At \( \alpha = 0.5 \) the density undergoes a “phase transition” and is unity for any \( z \). If \( \text{pd} \neq 0.5 \) the second term only adds some asymmetry to the above features and makes the pdf for \( \alpha = 0.5 \) non-constant but still monotonous (Vasicek, 2002).

For later we would also need an explicit expression for default correlations: as is well known, they are different from latent correlations but induced by the latter. In a one-factor framework, the covariance and correlation between two default rvs are

$$\text{Cov}(d_i, d_j; \alpha) = \Phi_2(\Phi^{-1}(pd_i), \Phi^{-1}(pd_j), \alpha) - pd_i \text{pd}_j$$

$$\text{Corr}(d_i, d_j; \alpha) = \text{Cov}(d_i, d_j; \alpha) / \sqrt{\text{Var}(d_i) \text{Var}(d_j)} \quad (2.5)$$

where \( \Phi_2 \) is a bi-variate normal cdf. Therefore, for a uniform portfolio the coefficient of d-d correlation \( A = P(\alpha) \) can be written as

$$A = P(\alpha) = \frac{\text{Var}(pd(u))}{\text{Var}(d)} = \frac{1}{pd(1-pd)} \left[ \Phi_2(\Phi^{-1}(pd), \Phi^{-1}(pd), \alpha) - pd^2 \right] \quad (2.6)$$

where \( P(\alpha) \) is some universal mapping function. For further calculations we would need both direct \( P(\alpha) \) and inversed \( P^{-1}(\alpha) \). As \( P(\alpha) \) is generally non-linear some exact cases and limits are of interest. First, for \( \text{pd} = 0.5 \) being a reasonable generic estimator, we have
Further, in the limit of small $\alpha$ we have a linear asymptotic

$$\text{Cov}(d_i, d_j)_{\alpha \to 0} \to \alpha \cdot \varphi^2[\Phi^{-1}(pd)], \quad P(\alpha)_{\alpha \to 0} \to k\alpha, \quad k = \frac{\varphi^2[\Phi^{-1}(pd)]}{pd(1-pd)}$$

which is quite acceptable for small $\alpha$, say, $\alpha < 5\%$. In the opposite limit of $\alpha \to 1 \quad P(\alpha) \to 1$. These extreme cases, for the benefit of quick practical estimations of $P(\alpha)$ and inversed $P^{-1}(\alpha)$ as well, can be combined in some workable approximations. Below is a simplest second order interpolation

$$\tilde{P}(\alpha) = \alpha[k + \alpha(1-k)]$$

which would cover the whole range [0,1] and is easily invertible when obtaining $\alpha$ for a given $A$. More refined versions are suggested in Appendix 2.

3. Modeling correlated recoveries or loss given defaults.

For loss given defaults the situation is more complex in that loss-given-default $lgd$ is no longer binary, and, there exists a substantial wealth of empirical studies on particular distribution profiles (see, for instance, Schuermann [2004]). For instance, one of the pragmatically popular approaches fits the empirical data with the two-parametric Beta (a,b)-function. For our purposes the specific distribution type is not as important, and therefore we will use a heavily stylized approximation for $lgd$ rv (see, for instance, Witzany(2009)): a binary (0,1) model – in a full analogy to the defaults case - with a number of loss-given-defaults as a random variable $y = \sum lgd_i / N$ following a binomial distribution. In the limit of $N \to \infty$ the latter becomes a delta-function distribution peaked at $E(y) = \text{plgd}$ (we use plgd for an expected lgd instead of $\text{elgd}$ for more immediate analogy with defaults). Thus, analogously to the one factor copula default model from Sec. 2 we introduce a latent process $W^{lgd}_i = \beta^{/2} v + (1 - \beta)^{1/2} e_i$, which is different from $W^d_i$ and where “v” is a new latent common factor and $\beta$ is the correlation for different obligor latent factors. Hence, $\text{plgd}(v) = \Phi[(\Phi^{-1}(lgd) - \sqrt{\beta}v) / \sqrt{1-\beta}]$ gives a conditional mean $E(lgd|v) = \text{plgd}(v)$ and the unconditional mean over all possible $v$ is $E[\text{plgd}(v)] = \text{lgd}$ as in Sec. 2. Obviously, formulas (2.3 - 2.6) apply with $lgd$, replacing $d_i$, $\text{plgd}$ replacing $pd$, $\beta$ replacing $\alpha$, $B = P(\beta)$ replacing $A = P(\alpha)$ and so on.

Sanchez et al. (2008) argued in favor of keeping convenient ASRF expressions for conditional means $\text{plgd}(v)$ in conjunction with distributions for $lgd$ other than binary, and therefore less restrictive and more realistic than binary. For instance, in terms of Beta-function distribution
that would essentially mean calibrating one of the Beta lgd distribution parameters according to

\[ plgd(v) = \int_0^1 lgd \ Beta(a(v), b; lgd) \ d(lgd) \]

while leaving the other parameter free for further adjustments.


The way the portfolio loss rate formation takes place depends on correlation – or better say, coupling – between constituent loss variables: defaults and loss-given-defaults. Conceptually, we may first obtain random variables of portfolio level fractional default rate \( x \) and fractional loss-given-default rate \( y \) as defined above in Sec. 2 and 3, and then multiply them \( x \times y \) to arrive at a portfolio level loss rate variable. With that, the correlation between defaults and loss-given-defaults enters at the portfolio/collective level. Alternatively, we multiply individual asset default and loss-given-default variables into individual loss variables \( d_i \times lgd_i \) and combine them into the portfolio loss rate variable. This way d-lgd correlation enters at the individual level. The choice between these two options, which will be referred to as an individual mode (with notation \((d*lgd)- (d^*lgd)\) and a collective mode (with notation \((d-d) * (lgd-lgd)\) respectively, can be formalized as follows.

Consider two-argument coupling function \( C(1, 2) \) signifying the strength of the correlation/covariance between defaults \( d_i \) and \( d_j \) (d-d correlations), between loss given defaults \( lgd_i \) and \( lgd_j \) (lgd-lgd correlations), and between defaults \( d_i \) and loss given defaults \( lgd_j \) (d-lgd correlations). The individual mode then corresponds to the condition \( C(d_i, d_j) \sim C(lgd_i, lgdj) << C(d_i, lgdj) \), meaning the dominance of inter correlations, while the opposite case \( C(d_i, lgd_i) \sim C(lgd_i, lgd_j) >> C(d_i, lgd_j) \) points out to the collective mode and dominance if intra correlations, where “\( \sim \)” stands for the same order of magnitude and “\( << “/” >> “ \)” “ for much smaller/bigger respectively.

Generally speaking, individual and collective modes produce expectedly different portfolio loss distributions. Question arises, what these distributions are and how material is the divergence between them. We preamble here with some conceptual reasoning. First of all, in the absence of any correlation, neither intra nor inter, individual and collective modes immediately produce the same portfolio loss distribution of a binomial type - simply because any binomial variable can be represented as a sum of binary variables and the product of binary variables is again a binary variable. Obviously, this result holds in the limit of \( N \rightarrow \infty \) where a binomial distribution becomes a delta-function. The incorporation then of intra-correlations impacts both modes but in a different way: the individual mode leads to the Vasicek distribution with the composite mean, variance and correlation parameter constructed from intra and inter covariances, while in the collective mode the result differs from Vasicek because the multiplicative convolution does not
preserve the Vasicek type of distributions being convoluted. Naturally, this difference extends further in the presence of inter correlations. We will now consider both scenarios in more detail.

**Individual mode, (d*lgd)-(d*lgd)**

In the individual mode we obtain the loss rate rv of the individual obligor \( d \times lgd \) (obligor indices are dropped in the spirit of uniform portfolio) which is again binary rv according to Sec. 2 and 3. The rest, in essence, is a standard ASRF one-factor parameterization of a uniform correlation between individual loss rvs resulting in the distribution type \((2.2)\) with composite mean, variance and correlation parameter.

More specifically, modeling losses in an individual mode includes following simple steps.

**First**, set desirable latent *intra* correlation coefficients \( \alpha \) and \( \beta \) as in Sec. 2 and 3, as well as latent *inter* correlations for diagonal \( \gamma_0(d_i, lgd_i) \), and non-diagonal \( \gamma_1(d_i, lgd_i) \) coefficients, respectively (i.e. the d-lgd correlation for the same and different obligors, respectively).

**Second**, following \((2.6-2.9)\), map latent correlations into default correlations: i.e. convert latent correlations \( \alpha \) and \( \beta \) to default correlations, A, B, and latent \( \gamma_0 \) and \( \gamma_1 \) into default correlations \( \Gamma_0, \Gamma_1 \).

**Third**, use simple algebra of covariances in Appendix 2, to obtain the resulting composite default loss rvs correlation \( \Lambda(\lambda) \), where \( \lambda \) is a correlation for an underlying latent factor.

**Fourth**, by inverting an equation of the \((2.6)\) type, i.e. using \( \Lambda^{-1}(\lambda) \), recover \( \lambda \) and use it in the Vasicek type distribution for portfolio losses.

The details of the derivation are given in Appendix 2, with the final expression for the loss distribution is given below

\[
cdf: F_v(\nu, \lambda; z) = \Phi(\frac{1}{\sqrt{\lambda}}[\sqrt{1-\lambda}\Phi^{-1}(z) - \Phi^{-1}(\nu)])
\]

\[
pdf: f_v(\nu, \lambda; z) = \sqrt{\frac{1-\lambda}{\lambda}} \exp\left\{0.5 [\Phi^{-1}(z)]^2 - 0.5 \frac{[\sqrt{1-\lambda}\Phi^{-1}(z) - \Phi^{-1}(\nu)]^2}{\lambda}\right\}
\]

(4.1)

and the first two moments \( \bar{z} \) and \( \bar{z}^2 \) (mean loss and mean square loss) are

\[
\bar{z} = \nu + l = d \times lgd = d_i \times lgd_i + C_0
\]

\[
C_0 = Cov(d_i, lgd_i; \gamma_0) = \Gamma_0 \sqrt{Var(d_i)Var(lgd_i)} = \Phi_2(\Phi^{-1}(lgd_i), \Phi^{-1}(\nu)) - \nu, pld_i
\]

(4.2)

\[
\bar{z}^2 = \Phi_2[\Phi^{-1}(\nu), \Phi^{-1}(\nu)]
\]
Here $pl$ (probability of loss, $\overline{I}$) is a composite mean, $\lambda$ is a composite latent correlation for losses in the Vasicek type distribution as described above, $\gamma_0$ and $\Gamma_0$ are diagonal (i.e. the correlation for the same obligor) latent factor and loss correlations, respectively, and $\Phi_2$ is a bi-variate normal cdf.

Formulas (4.1-4.2, A 2.6-2.9) demonstrate the following essential trends. While the mean of the portfolio loss is affected only by the diagonal inter correlation $\Gamma_0$, the variance and composite correlation parameter are impacted by both intra ($d$-$d$/$l$gd-$l$gd) and inter $d$-$l$gd inter correlations. Specifically, intra-correlations increase the correlation $\Lambda$ (and $\lambda$) and hence make the distribution wider, increasing tail losses in the process. The role of inter correlations is different: non-diagonal inter correlations $\Gamma_1$ directly increase the loss correlation $\Lambda$, while the diagonal inter correlations $\Gamma_0$ do not contribute on its own, but only enhance or reduces (depending on the relative values of default parameters indicated in Appendix 2) the contribution of all other correlation types.

**Collective mode, (d-$d$)*($l$gd-$l$gd)**

In a collective mode the portfolio loss rate $rv$ is a product of correlated (via inter correlations) portfolio $d$ and $lgd$ rate, and, therefore, we can not merely take a multiplicative convolutions of $d$ and $lgd$ Vasicek distributions. However, we can do that for delta-distributions for each pair of latent factors ($u,v$) and then average the result over the distribution of correlated pairs of ($u,v$). Details are given in Appendix 3 and the resulting cdf and pdf according to (A 3.3) are

$$
cdf \quad F(z) = \iiint \frac{dudv}{2\pi\sqrt{1-\gamma^2}} \exp\left(-\frac{u^2 + v^2 - 2\gamma uv}{2(1-\gamma^2)}\right) \mathbb{1}[z - pd(u)plgd(v)]
$$

$$
pdf \quad f(z) = \iiint \frac{dudv}{2\pi\sqrt{1-\gamma^2}} \exp\left(-\frac{u^2 + v^2 - 2\gamma uv}{2(1-\gamma^2)}\right) \delta[z - pd(u)plgd(v)]
$$

where $\mathbb{1}(s)$ is an indicator function equal 0 for negative $s$ and equal 1 for non-negative $s$ and $\delta(s)$ is a delta-function. It follows that (4.3) delivers a generalized bi-variate cdf off the curve $pd(u)plgd(v) - z = 0$. According to (4.3), the first two moments $\overline{z}$ and $\overline{z^2}$ are given by

$$
\overline{z} = \iiint \frac{dudv}{2\pi\sqrt{1-\gamma^2}} pd(u)plgd(v) \exp\left(-\frac{u^2 + v^2 - 2\gamma uv}{2(1-\gamma^2)}\right)
$$

$$
\overline{z^2} = \iiint \frac{dudv}{2\pi\sqrt{1-\gamma^2}} [pd(u)plgd(v)]^2 \exp\left(-\frac{u^2 + v^2 - 2\gamma uv}{2(1-\gamma^2)}\right)
$$

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which are bi- and quadra-variate normals (see Sanchez et al, 2008).

Equations (4.3) allow for some verifiable exact limits in their domain of applicability. For example, setting in (4.3) \( \gamma = 0 \) for no inter correlations immediately reduces the loss rate pdf to

\[
f(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ds f_\nu(pd, \alpha; z) f_\nu(plgd, \beta; \frac{z}{s})
\]

where \( f_\nu \) are Vasicek pdfs from (2.3). Formula (4.5) is nothing but a multiplicative convolution of uncorrelated default and loss-given-default distributions, which must be the case in the absence of inter correlations. Taking (4.5) further by setting \( pd = plgd = 0.5 \) and \( \alpha = \beta = 0.5 \) results in a particular simple density \( f(z) = \ln(1/z) \).

On the general note, the major deficiency of the collective formalism (4.3 - 4.4) is that it is valid only when \( \gamma \ll \alpha, \beta \), i.e. when intra correlations dominate, and does not apply otherwise. The reason is that the inter correlation parameter \( \gamma \) enters (4.3 - 4.4) only via the weighting exponents but is absent from conditional averages \( pd(u) \) and \( plgd(v) \). As a result, when both \( \alpha, \beta \to 0 \) then, regardless of \( \gamma \), the loss distribution spread vanishes so that the distribution tends to a \( \delta \)-function. This problem reflects the limitations of the one-factor setting and can be overcome only by adding more latent factors.

The other weakness: because of a relatively non-trivial structure of conditional expectations \( pd(u) \) and \( plgd(v) \), the collective formalism (4.3-4.4), possesses considerably less analytical tractability than its individual mode counterparty. Therefore, even though the distribution and moments come out in a closed integral form, their practical value for, say, extracting analytic trends and losses calculations is quite limited by the need for a direct numerical implementation. In that respect an individual mode offers a clear advantage. Certain conclusions from comparing of a collective mode with an individual mode will be drawn in the next section.

5. Some comparisons between individual and collective modes.

From the general standpoint it is intuitively clear and expected, except for a trivial case of no any correlation at all, that individual and collective modes produce different distributions, at least in principle. While it can be rigorously shown that densities (4.3) in general do not render Vasicek densities (4.1), the proof is quite technical and is not heuristically critical, and therefore we do not give it here, but rather provide a direct constructive illustrations. For example, consider second moments and assume for simplicity no inter correlations \( \gamma = 0 \). Then the second moment from (4.4) factorizes into the product \( \Phi_2[\Phi^{-1}(pd), \Phi^{-1}(pd), \alpha] \cdot \Phi_2[\Phi^{-1}(lgd), \Phi^{-1}(lgd), \beta] \) while
in the individual mode it is different and equal to \( \Phi_2[\Phi^{-1}(pd \times plgd), \Phi^{-1}(pd \times plgd), \lambda] \) with \( \lambda \) given by (A 2.8) from Appendix 2. Clearly, these two expressions do not coincide in general (except, may be, for a special choice of \( \lambda \)) which illustrates the point. And further when inter correlations are present, it is even more reasons to expect differences to enhance. The numerical results given below confirm these conclusions, however, they also show that there exist a certain range where both modes agree quite closely, which is very beneficial from the practical standpoint.

Below we exemplify some numerical results where for simplicity and ease of analysis restricting ourselves within \( pd = plgd = 0.5 \), leaving a more comprehensive report for later. Figures 1-7 illustrate densities and corresponding cumulative functions for three major domains: the area 1, of intra correlation dominance \( \gamma \ll \alpha, \beta \), area 2, the opposite extreme of \( \gamma \gg \alpha, \beta \) where inter correlations dominate, and intermediate range \( \gamma \sim \alpha, \beta \) – area 3. The major differences are observed in area 2 (Fig. 5-7), which is of no surprise since a collective mode is invalid here as was explained above in Sec. 4. Therefore, the correct distribution in this area is given by an individual mode from the Vasicek formula (4.1- 4.2). Intuitively, we could have expected the opposite conclusion in the area 1, where the collective mode applies. However, in the area 1(Fig.1-4) the individual mode proves very close to a collective mode which is the result of a proper construction of a composite mean and correlation parameter. Therefore, as an individual mode gives a correct result in both extreme areas 1 and 2, it is quite natural to expect that this is also the case in the intermediate range 3 (Fig.8-9) and that 1) the individual mode gives a good coverage in the entire range of relationships between \( \gamma \) and \( \alpha, \beta \) and 2) the differences in the intermediate range 3 also stem exclusively from the limitations of the collective mode. This is a main advantage of the individual mode formulas (4.1) and is of a particular significance for applications as it allows to use simple Vasicek formulas regardless of the relationship between \( \gamma \) and \( \alpha, \beta \).

Another important aspect of the comparison is that in the one-factor setting the collective mode treats d-lgd correlations in an “averaged” way and does not discriminate between diagonal and non-diagonal inter correlations i.e. \( d_i-lgd_i \) vs \( d_i-lgd_j \). This is quite counterintuitive (and unrealistic even within a limited context of a stylized modeling framework) and clearly is a defect of the collective mode compared to an individual mode which is free from that problem.
6. Conclusions

The main conclusions are as follows.

The profile of the loss rate distribution is determined by the relative strength of intra vs inter correlations and as such this strength impacts the choice of a suitable formalism to describe the distribution.

At the same time, the one-factor based Vasicek type distribution with properly constructed composite mean and correlation parameters demonstrate a stable performance regardless of the relationship between intra and inter correlations. Taken together with an analytical character of the model, this makes the individual mode an effective and flexible tool for loss estimations from the practical standpoint in particular.

Also, an additional flexibility of the individual mode is in that it allows to discriminate between diagonal and non-diagonal inter correlations, and, more broadly, permits start modeling by choosing correlations optionally either between underlying latent risk factors or directly between default related rvs.

On the other hand, it is of a considerable interest to explore further the potential of the collective mode formalism as a general framework in the two-factor settings.

Finally, as indicated in the introduction, the next step should be a study of how to relate inputs into both individual and collective modes to existing empirical data.

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Appendix 1. Review of the Vasicek distribution

In a uniform infinite $N \to \infty$ portfolio for any latent variable $u$ the conditional default rate probability density $z$ is a delta-function $\delta(z - pd(u))$ peaked around the conditional default probability $pd(u) = \Phi[(K - \sqrt{\alpha}u) / \sqrt{1 - \alpha}]$, where $K = \Phi^{-1}(pd)$ and $\Phi$ stands for a normal distribution. The unconditional default rate cdf $F(z)$ (i.e. probability of all default fractions smaller than $z$) obtains from weighting with normal density as

$$F(z) = \int \mathbb{I}[z - pd(u)]\varphi(u)du = \int_{-\infty}^{u_0} \varphi(u)du = \Phi(u_0), \quad u_0 = [\Phi^{-1}(z)\sqrt{1 - \alpha} - K] / \sqrt{\alpha} \quad (A\ 1.1)$$

and where $\mathbb{I}[z - pd(u)]$ is an indicator function and therefore

$$F_{\nu}(z) = \Phi\left(\frac{1}{\sqrt{\alpha}}[\sqrt{1 - \alpha}\Phi^{-1}(z) - \Phi^{-1}(pd)]\right) \quad (A\ 1.2)$$

Respectively the unconditional default rate pdf is

$$f(z) = \int \delta[z - pd(u)]\varphi(u)du = \int_{-\infty}^{\infty} du\ \varphi(u)\delta[u - u_0(z)] / pd'(u) = \varphi(u_0) / pd'(u_0) \quad (A\ 1.3)$$

with the final expression as

$$f_{\nu}(z) = \frac{1 - \alpha}{\alpha} \exp\left\{0.5 [\Phi^{-1}(z)]^2 - \frac{0.5}{\alpha} [\sqrt{1 - \alpha}\Phi^{-1}(z) - \Phi^{-1}(pd)]^2\right\} \quad (A\ 1.4)$$

The calculations above can be viewed as obtaining the pdf of $g = pd(u)$ given the pdf of $u$ according to $\varphi(u)du = \varphi_{\alpha}(g)dg$ and then having $\varphi_{\alpha}(g) = \varphi(u) / du$ transformed back with $u = pd^{-1}(g)$.

In conclusion of this Appendix we illustrate how to prove that $E[pd(u)] = pd$. A technically simple way would be by taking the derivative $d E[pd(u)] / dK$ over the Merton threshold level $K$. Because $\varphi(u)$ is normal, the differentiation w.r.t. $K$ and the integration over $u$ leads to $\varphi(K)$. Taking then an integration over $K$ results in $\Phi(K) = \Phi[\Phi^{-1}(pd)] = pd$. 

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Appendix 2. Correlations of individual loss rates in individual [(d*lgd)-(d*lgd)] mode

In this Appendix we calculate the covariance and correlation between loss rate rvs \( d \times lgd \). For further potential applications and ease of appreciating the formula symmetry, we will not incorporate the fact that for any binary random variables we have \( s^2 = s \) and \( \text{Var}(s) = ps(1-ps) \). This can be easily substituted in the final results.

Now, assume a uniform intra correlation (coefficient A) among default rvs \( d_i \), and a uniform intra correlation (coefficient B) among loss-given-default rvs \( lgd_i \), but for now no inter-correlation between defaults and loss-given defaults. Saying differently, we look for the correlation between individual loss rates induced by intra d-d and lgd-lgd correlations only. Set \( Z_i = d_i \times lgd_i \) and consider covariance \( \text{Cov}(Z_i, Z_j) \). (below we use \( \bar{x} \) for the mean value of x). The correlation between loss rvs \( Z_i \) and \( Z_j \) is 
\[
\Lambda_{ij} = \frac{\text{Cov}(Z_i, Z_j)}{\sqrt{\text{Var}(Z_i)\text{Var}(Z_j)}}
\]

Because of \( \text{Cov}(a,b) = \overline{ab} - \overline{a}\overline{b} \) we obtain
\[
\Lambda_{ij} = \overline{lgd_i \cdot lgd_j \cdot \text{Cov}(d_i, d_j)} + \overline{d_i \cdot d_j \cdot \text{Cov}(lgd_i, lgd_j)} + \text{Cov}(d_i, d_j)\text{Cov}(lgd_i, lgd_j) \sqrt{\text{Var}(Z_i)\text{Var}(Z_j)} \tag{A 2.1}
\]

For a uniform portfolio (identical assets and correlations among all of them) \( \text{Cov}(d_i, d_j) = A \cdot \text{Var}(d) \), \( \text{Cov}(lgd_i, lgd_j) = B \cdot \text{Var}(lgd) \) and so
\[
\text{Cov}(Z_i, Z_j) = A \cdot \overline{lgd^2 \cdot \text{Var}(d)} + B \cdot \overline{d^2 \cdot \text{Var}(lgd)} + A \cdot B \cdot \text{Var}(d)\text{Var}(lgd) \tag{A 2.2}
\]

where for a notational convenience \( \overline{d} \) and \( \overline{lgd} \) stand for \( pd \) and \( plgd \) respectively.

Analogously,
\[
\sqrt{\text{Var}(Z_i)\text{Var}(Z_j)} = \overline{lgd^2 \cdot \text{Var}(d)} + \overline{d^2 \cdot \text{Var}(lgd)} + \text{Var}(d)\text{Var}(lgd) \tag{A 2.3}
\]

and, therefore, the loss rate correlation coefficient is
\[
\Lambda_{\text{intra}} = \Lambda_{ij} = \frac{A \cdot \overline{lgd^2 \cdot \text{Var}(d)} + B \cdot \overline{d^2 \cdot \text{Var}(lgd)} + AB \cdot \text{Var}(d)\text{Var}(lgd)}{\overline{lgd^2 \cdot \text{Var}(d)} + \overline{d^2 \cdot \text{Var}(lgd)} + \text{Var}(d)\text{Var}(lgd)} \tag{A 2.4}
\]

or, introducing relative variables \( R_d = \overline{d^2} / \text{Var}(d), \quad R_{lgd} = \overline{lgd^2} / \text{Var}(lgd) \)
\[ \Lambda_{\text{intra}} = \frac{B \cdot \frac{d^2}{\text{Var}(d)} + A \cdot \frac{\text{lgd}^2}{\text{Var}(\text{lgd})} + AB}{\frac{d^2}{\text{Var}(d)} + \frac{\text{lgd}^2}{\text{Var}(\text{lgd})} + 1} \]

\[ = AB \frac{\frac{d^2}{\text{Cov}(d_i,d_j)} + \frac{\text{lgd}^2}{\text{Cov}(\text{lgd}_i,\text{lgd}_j)} + 1}{\frac{d^2}{\text{Var}(d)} + \frac{\text{lgd}^2}{\text{Var}(\text{lgd})} + 1} \] (A 2.5)

Therefore, the correlation and covariance are

\[ \Lambda_{\text{intra}} = AB \frac{R_d / A + R_{\text{lgd}_d} / B + 1}{R_d + R_{\text{lgd}_d} + 1} \]

\[ \text{Cov}_{\text{intra}} = \text{Cov}(Z_i,Z_j) = AB \cdot \text{Var}(d) \cdot \text{Var}(\text{lgd}) \cdot \left[ \frac{R_d / A + R_{\text{lgd}_d} / B + 1}{R_d + R_{\text{lgd}_d} + 1} \right] \] (A 2.6)

where index “intra” points out to intra correlations. In other words, the covariance/correlation between individual loss rates is a function of intra d-d and lgd-lgd covariances/correlations, respectively. The final step now is to add here an inter correlation between defaults and loss-given defaults d-lgd. To that end we introduce matrices \( C_{ij} = \text{Cov}(d_i,\text{lgd}_j) \) of a linear and bi-linear \( \text{CC}_{ij} = \text{Cov}(d_i,\text{lgd}_j) \) covariances, respectively. For a uniform portfolio denote \( C_0 \) the diagonal covariance \( i = j \) for the same obligor and \( C_1 \) the non-diagonal covariance \( i \neq j \) between different obligors. Also, set \( \text{CC}_0 = \text{Cov}(d_i^2,\text{lgd}_j^2) \) and \( \text{CC}_1 = \text{Cov}(d_i,\text{lgd}_j,\text{lgd}_j,\text{lgd}_j) \) for diagonal and non-diagonal bi-linear covariances, respectively. With that, formula (A 2.6) modifies as follows

\[ \Lambda = \Delta_1 + \Delta_2, \quad \Delta_1 = AB \frac{R_d / A + R_{\text{lgd}_d} / B + 1}{R_d + R_{\text{lgd}_d} + 1 + S_0}, \quad \Delta_2 = \frac{S_1}{R_d + R_{\text{lgd}_d} + 1 + S_0} \]

\[ S_0 = (\text{CC}_0 - 2C_0 \bar{d} \text{lgd} - C_0^2) / [\text{Var}(d) \text{Var}(\text{lgd})] \]

\[ S_1 = (\text{CC}_1 - 2C_0 \bar{d} \text{lgd} - C_0^2) / [\text{Var}(d) \text{Var}(\text{lgd})] \approx (C_1^2 + 2C_1 \bar{d} \text{lgd}) / [\text{Var}(d) \text{Var}(\text{lgd})] \] (A 2.7)

where combinations \( S_0 \) and \( S_1 \) account for diagonal and non-diagonal inter correlations. Note that \( S_0 \) can be both positive or negative depending on relative values of \( \text{CC}_0, C_0, \bar{d} \) and \( \bar{\text{lgd}} \). Also, interestingly enough, because \( S_0 \) is located in the denominator it does not contribute on its own to the composite correlation \( \Lambda \): depending on its sign, \( S_0 \) only enhances or reduces the impact of d-d, lgd-lgd and non-diagonal d-lgd correlations. Also note, that in the last line of (A 2.7) \( S_1 \) accounts only for \( d_i \text{lgd}_j \) inter non-diagonal correlations ( intra d-d, lgd-lgd and diagonal d-lgd inter correlations are accounted for by other components of (A 2.7)) and therefore \( S_1 \) can be transformed to incorporate this fact explicitly. Finally, the latent correlation \( \lambda \) for the mixing copula underlying the loss variables \( l_i = d_i \times \text{lgd}_i \) recovers analogously to (2.8-2.9) of the main text via
$\Lambda_{\lambda \to 0} = P(\lambda)_{\lambda \to 0} \approx \lambda \cdot \frac{\varphi^2[\Phi^{-1}(pl)]}{pl(1 - pl)} = k \lambda$, \quad k = \frac{\varphi^2[\Phi^{-1}(pl)]}{pl(1 - pl)} \quad (A 2.8)$

$\tilde{P}(\lambda) = \lambda[k + \lambda(1 - k)]$

with help of a mean value

\[
pl = \bar{l} = \bar{d} \times lgd = \bar{d}_i \times lgd_i + C_0 = pd \times plgd + C_0
\]

\[
C_0 = Cov(d_i, lgd_i, \gamma) = \Phi_2(\Phi^{-1}(pd_i), \Phi^{-1}(plgd_i), \gamma) - pd_i, plgd_i
\]

(A 2.9)

We conclude this Appendix with a possible refinement of the interpolation (2.9), Sec. 2 and (A 2.8). By adding a nonlinear term of $\lambda(1 - \lambda)$ into (2.9) we have

\[
\tilde{P}(\lambda) = \lambda[k + \lambda(1 - k) + m\lambda(1 - \lambda)]
\]

(1.1)

where $m$ is an adjustable parameter. The choice of $m \approx -0.6$ for $pd = plgd = 0.5$ keeps the relative error within $\leq 2\%$. 


**Appendix 3.** *Correlation between loss rates in a collective [(d-d)*(lgd-lgd)] mode.*

In a collective mode the d-d and lgd-lgd correlated distributions are run by corresponding latent processes /copulas and to obtain the final loss distribution we, generally speaking, need to perform a multiplicative convolution of two Vasicek distributions. However, in the presence of the inter d-lgd correlations this is not as straightforward. Therefore we render each distribution with an integral representation over latent variables and then, after making a convolution for each pair of random latent factors \((u,v)\), perform an averaging over both latent factor distribution. The details are as follows.

The portfolio loss rate follows as the composition of d and lgd rate. Specifically, for any given pair of latent variables \(u,v\) corresponding to conditional probabilities \(pd(u)\) and \(plgd(v)\) the loss rate conditional pdf is a multiplicative convolution of \(\delta[z – pd(u)]\) and \(\delta[z – plgd(u)]\)

\[
\int \frac{dz'}{z'} \delta[z' – pd(u)] \delta[\frac{z}{z'} – plgd(v)] = \delta[z – pd(u)plgd(v)]
\]

And then the unconditional default rate cdf and pdf follow after weighting over the bi-variate normal density \(\psi(u,v)\) as in

\[
F(z) = \int [z – pd(u)plgd(v)]\psi(u,v)dudv, \quad f(z) = \int \delta[z – pd(u)plgd(v)]\psi(u,v)dudv \quad (A 3.2)
\]

\(F(z)\) in (A3.2) gives a generalized bi-variate normal cdf off the curve \(pd(u) \times plgd(v) = z\) and \(f(z)\) is a corresponding pdf. We note in passing, that as for one-dimensional case, obtaining pdf can be viewed as obtaining the pdf of \(z(u,v) = pd(u)plgd(v)\) given the bi-variate normal density \(\psi(u,v)\) according to \(\psi(u,v)dudv = q_1[pd(u)plgd(v)]d[pd(u)plgd(v)]\).

In particular, for a correlated bi-variate normal distribution we have

\[
cdf: \quad F(z) = \int \int \frac{dudv}{2\pi \sqrt{1-\gamma^2}} \exp\left(-\frac{u^2 + v^2 - 2\gamma uv}{2(1-\gamma^2)}\right) \mathbb{I}[z – pd(u)plgd(v)]
\]

\[
pdf: \quad f(z) = \int \int \frac{dudv}{2\pi \sqrt{1-\gamma^2}} \exp\left(-\frac{u^2 + v^2 - 2\gamma uv}{2(1-\gamma^2)}\right) \delta[z – pd(u)plgd(v)] \quad (A 3.3)
\]

where \(\gamma\) is the correlation between underlying latent factors \(u\) and \(v\).
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