Dynamic Asset Allocation with Stochastic Transaction Costs

Pierre Collin-Dufresne
SFI@EPFL
e-mail: pierre.collin-dufresne@epfl.ch
Kent Daniel
Columbia University
e-mail: kd2371@columbia.edu
Mehmet Sağlam
University of Cincinnati
e-mail: mehmet.saglam@uc.edu

First Draft: November 9, 2017
This Draft: November 28, 2017

Abstract

We solve for the optimal dynamic asset allocation when expected returns, volatilities, and trading costs follow a regime switching model. The optimal policy is to trade partially towards an aim portfolio, which is a weighted average of the conditional mean-variance portfolios in every state. The aim portfolio puts more weight on states that are more frequently visited, that have higher persistence, risk and trading costs than the current state. The trading speed is higher in states that are more persistent, where return volatility is higher and trading costs are lower. It can be optimal to deviate substantially from the mean-variance efficient portfolio and to underweight high Sharpe ratio assets, in anticipation of an increase in their volatility and trading costs. We implement our approach in an empirical exercise to trade the value weighted market index of US common stocks. Estimation of a regime switching model applied to this portfolio yields evidence for time variation in both means and variances; using a data-set on institutional trading costs, we find that realized trading costs are significantly higher when market volatility is high. The optimal dynamic strategy significantly outperforms a myopic trading strategy in an out-of-sample experiment. The highest gains arise from timing the changes in volatility and trading costs rather than expected returns.

*For valuable comments and suggestions we thank...
1 Introduction

Mean-variance efficient portfolio optimization, introduced by Markowitz (1952), is still widely used in practice and taught in business schools. When expected returns change over time then so does the conditional mean-variance efficient ‘Markowitz’ portfolio. In the presence of transaction costs however, it may not be optimal for investors to constantly rebalance to perfectly track the Markowitz portfolio. In a recent paper, Gårleanu and Pedersen (2013) (GP) show that in the presence of quadratic transaction costs (that is linear price impact), an investor with mean-variance preferences should adopt a trading rule that only partially rebalances from her current position towards an aim portfolio at a fixed trading speed.\(^1\) They derive closed-form expressions for both the optimal aim portfolio and the trading speed that depend on the dynamics of expected returns, the quantity of and aversion to risk, and the magnitude of price impact. Importantly, their model assumes that covariances of price changes and price-impact parameters are constant. In this paper we derive a closed-form solution for the optimal portfolio trading rule in a similar setting but where, in addition to expected returns, volatility and transaction costs may be stochastic. Indeed, there is substantial evidence that the volatility of stock returns is stochastic and that transaction costs covary with the level of stock volatility (going back at least to Rosenberg (1972) for the former and to Stoll (1978) for the latter).

We assume that stock returns, covariances, and price impact parameters follow a multi-state Markov Switching model. We assume, similar to GP, that investors maximize the expected discounted sum of mean-variance preferences over portfolio returns net of trading costs. We show that the optimal trading rule looks similar to that derived in GP, namely, to partially trade from the current position towards an aim portfolio. However, both the aim portfolio and the trading speed vary conditional on the state. We characterize the aim portfolio as a weighted average of conditional Markowitz portfolios. That is the investor trades towards a portfolio that reflects the possible optimal mean-variance efficient portfolios in all the states that it may transition too. The relative weight put on the different states depends on the persistence of these states and the likelihood of transitioning to these states, as well as on the relative risk and transaction costs faced in these state relative to the current one. Similarly, the optimal trading speed depends on the relative magnitude of the transaction costs in various states and their transition probabilities.

To illustrate, suppose that there are only two states, a state L with low volatility and zero transaction costs and a state H with higher volatility and positive transaction costs. Then it is clearly optimal to trade (at infinite speed) all the way to the aim portfolio in the L state, whereas the trading speed will be finite in the H state. Further, the aim portfolio in the H-state will equal the conditional Markowitz portfolio in that state, that is it puts zero weight on the L-state Markowitz portfolio. Intuitively, in the H-state the investor need not consider the future investment opportunity set faced in the L-state, since she will face no cost to rebalance (to the conditional Markowitz portfolio) in that state. However, the aim portfolio in the L-state will be a

\(^1\)Litterman (2005) makes a similar point in an unpublished note.
weighed average of both H- and L-conditional Markowitz portfolios, and where the weight on the H-conditional Markowitz portfolio increases with the likelihood of transitioning from L to H, the persistence of the H state, and with ratio of the volatilities in the H- and L-states.

One immediate implication of our model is that the optimal portfolio can deviate significantly from the Markowitz benchmark, purely in anticipation of a possible future shift in relative risk and transaction costs. Consider two assets, say a ‘Treasury’ and a ‘Corporate’ bond portfolio, and suppose that in state L, the Corporate portfolio has a higher Sharpe ratio than Treasuries. However, suppose further that if the economy transitions to state H, the risk and transaction costs of Corporate bonds will dramatically increase, but will remain unchanged for Treasuries. Then the optimal aim portfolio in the L state would hold a significant position in Treasury bonds and, depending on parameters, could even hold a larger fraction of Treasuries than of Corporates. It can be optimal to hold less of the high Sharpe ratio asset because a change in regime would lead to an increase in both its volatility and its trading costs. The former would require deleveraging, but the perspective of deleveraging at a higher cost, can make it optimal to preemptively reduce the position of the high-Sharpe ratio Corporate portfolio in the L-state.

We consider several other analytic and numerical examples to provide more intuition about the implications of our model.

Lastly, we propose an empirical application of our framework to optimally time the market versus cash, taking into account time varying expected returns, volatility and transaction costs. We estimate a four state Markov regime switching model and find, both in-sample and out of sample, evidence of time variation in first and second moments. To estimate the transaction cost parameters, we use a proprietary data set on trading costs, incurred by a large financial institution trading on behalf of clients, as measured by the Implementation Shortfall of their trade executions (as described in Perold (1988)). We show that the trading costs vary significantly across the different regimes, identified using the (highest) smoothed probabilities of the regimes. Trading costs tend to be higher the higher the volatility estimated in the regime. We then test our trading strategy both in-sample and out-of-sample (when the regime is estimated in real time using only past data up to the day preceding the trading date). We compare the performance of our optimal dynamic strategy to two alternatives: a constant dollar investment in the risky asset corresponding to an unconditional estimate of the sample mean and variance of returns, and a myopic one-period mean-variance problem optimized for current transaction costs (but that ignores the future dynamics of the Markov regime switching model).

The results show that the dynamic trading strategy significantly outperforms the other two strategies in the presence of transaction costs. Additionally, in an out-of-sample experiment, we examine what source of time-variation leads to the biggest gains for the dynamic strategy. Specifically, we compare the gains obtained from timing changes in expected returns, in volatility, and in transaction costs. In this out of sample experiment we find that the biggest benefits arise from accounting for time variation in volatilities and in transaction costs, while the benefits from timing variation in mean returns is mixed. We interpret this as reflecting the well-known fact that
estimates of first moments are more imprecise than second moment estimates, as pointed out by Merton (1980). Therefore, our strategy which assumes parameters are estimated without error, trades too aggressively on the predicted changes in expected returns. Further, as pointed out by Moreira and Muir (2017), there are gains to scaling down the position in response to an increase in the market’s variance, which suggests that the conditional mean of the market moves less than one-for-one with its variance. Thus, since our model captures the time-variation in volatilities and the corresponding changes in transaction costs more accurately, it is able to manage the risk-exposure and the incurred transaction costs more reliably, which directly contribute to increasing the net performance.

There is large academic literature on portfolio choice that has extended Markowitz’s one period mean-variance setting to dynamic multiperiod setting with a time-varying investment opportunity set and more general objective functions.\(^2\) This literature has largely ignored realistic frictions such as trading costs, because introducing transaction costs and price impact in the standard dynamic portfolio choice problem tends to make it intractable. Indeed, most academic papers studying transaction costs focus on a very small number of assets (typically two) and limited predictability (typically none).\(^3\)

There is a growing literature on portfolio selection that incorporates return predictability with transaction costs. Balduzzi and Lynch (1999) and Lynch and Balduzzi (2000) illustrate the impact of return predictability and transaction costs on the utility costs and the optimal rebalancing rule by discretizing the state space of the dynamic program. Their approach runs into the curse of dimensionality and only applies to very few stocks and predictors. Longstaff (2001) studies a numerical solution to the one risky asset case with stochastic volatility when agents face liquidity constraints that force them to trade absolutely continuously. Lynch and Tan (2011) extend this to two risky assets at considerable computational cost. Brown and Smith (2011) discuss this issue and instead provide heuristic trading strategies and dual bounds for a general dynamic portfolio optimization problem with transaction costs and return predictability that can be applied to larger number of stocks.

As noted earlier, our paper is most closely related to Litterman (2005) and Gârleanu and Pedersen (2013, GP). They obtain a closed-form solution for the optimal portfolio choice in a model where: (1) expected price change per share for each security is a linear, time-invariant function of a set of autoregressive predictor variables; (2) the covariance matrix of price changes is constant; (3) trading costs are a quadratic function of the number of shares traded, and (4) investors have a linear-quadratic objective function. Their approach relies heavily on linear-quadratic stochastic programming (see, e.g., Ljungqvist and Sargent (2004)). While our approach uses a similar objective function, it allows for time-variation in means, volatilities, and transaction costs by using


\(^3\)Constantinides (1986), Davis and Norman (1990), Dumas and Luciano (1991), Shreve and Soner (1994) study the two-asset (one risky-one risk-free) case with iid returns. Liu studies the multi-asset case under CARA preferences and for \textit{i.i.d.} returns. Cvitanić (2001) surveys this literature.
the regime-switching framework. Moreover, in contrast with the GP framework, our framework is equally tractable when expected-price changes are constant in each state of the regime switching model (i.e., prices follow arithmetic Brownian motion) or when expected returns, conditional on the state, are constant (i.e., prices follow geometric Brownian motion). Since the latter is a more realistic description of historical returns, it is the one we use for our empirical implementation.

2 Regime Switching Model for Price Changes

We consider the following objective function:

$$\max_{n_t} E \left[ \sum_{t=0}^{\infty} \rho^t \left\{ n_t' \mu(s_t) - \frac{1}{2} \gamma n_t' \Sigma(s_t)n_t - \frac{1}{2} \Delta n_t' \Lambda(s_t) \Delta n_t \right\} \right]$$  \hspace{1cm} (1)

where $\Delta n_t = n_t - n_{t-1}$, $\mu(s_t), \Sigma(s_t)$ are the mean vector and covariance matrix of price changes and $\Lambda(s_t)$ is the price impact matrix. All depend on a state variable $s_t$ which follows a Markov chain with transition probabilities $\pi_{s,s'}$. For simplicity we consider first only a two-state Markov chain model, but we generalize this to more states later.

This objective function is the same as that considered by GP, namely that of an investor who maximizes a discounted sum of mean-variance criterion in every period net of trading costs. In the absence of transaction costs (when $\Lambda(s) = 0$), the optimal solution would be to hold the conditionally mean-variance optimal Markowitz portfolio $m_s = \Sigma(s)^{-1} \mu(s)$ at all times. Further, if there was no variation in the investment opportunity set (that is if $\mu(s)$ and $\Sigma(s)$ were constant), then it would be optimal to hold the mean-variance efficient Markowitz portfolio and to never trade. It becomes optimal for the investor to rebalance the portfolio, and deviate from the conditionally mean-variance efficient portfolio, when there are transaction costs and the opportunity set is time-varying. Unlike GP, who only allow the conditional mean of stock returns ($\mu(s)$) to follow an AR(1) process, we consider the case where in addition all the elements of the covariance matrix and of the transaction cost matrix can vary. Using a Markov regime switching model allows us to obtain tractable solutions even though the model is not in the standard linear quadratic framework.

We will use the following notation throughout: for all $t$ where $s_t = s \in \{H, L\}$, $s_{t+1} = z \in \{H, L\}$ and $s' = \{H, L\} \setminus s$. Then, using dynamic programming principle, the value function $V(n_{t-1}, s)$ satisfies

$$V(n_{t-1}, s) = \max_{n_t} \left( n_t' \mu(s) - \frac{1}{2} \Delta n_t' \Lambda(s) \Delta n_t - \frac{\gamma}{2} n_t' \Sigma(s)n_t + \rho E[V_t(n_t, s_{t+1})] \right).$$

We guess the following quadratic form for our value functions:

$$V(n, s) = -\frac{1}{2} n' Q_s n + n' q_s + c_s,$$

where $Q_s$ is a symmetric $N, N$ matrix and $q_s, c_s$ are $N$ dimensional vectors of constants for $s \in \{H, L\}$. 

\{H,L\}. We now define the expectation conditional on state \( s \) for any matrix \( M_s \) to be \( \overline{M_s} = \pi_{s,s} M_s + \pi_{s,s'} M_{s'} \). With this notation the right hand side of the HJB equation we are optimizing can be rewritten as a quadratic objective \(-\frac{1}{2} n_t^\top J_s n_t + n_t^\top j_s + k_s\) where

\[
J_s = \gamma \Sigma(s) + \Lambda(s) + \rho \overline{Q_s} \tag{2}
\]

\[
j_s = \mu(s) + \Lambda(s)n_{t-1} + \rho \overline{\eta}_s \tag{3}
\]

\[
k_s = -\frac{1}{2} n_{t-1} \Lambda(s)n_{t-1} + \rho \overline{c}_s \tag{4}
\]

This is optimized for \( n_t = J_s^{-1} j_s \), that is:

\[
n_t = (\gamma \Sigma(s) + \Lambda(s) + \rho \overline{Q_s})^{-1} (\mu(s) + \rho \overline{\eta}_s + \Lambda(s)n_{t-1})
\]

Further, the optimized value is simply \( \frac{1}{2} j_s^\top J_s^{-1} j_s + k_s \) thus matching coefficients we find that the matrices \( Q_s, q_s \) for \( s = H, L \) must satisfy the system of equations:

\[
Q_s = -\Lambda_s (\gamma \Sigma_s + \Lambda_s + \rho \overline{Q_s})^{-1} \Lambda_s + \Lambda_s
\]

\[
q_s = \Lambda_s (\gamma \Sigma_s + \Lambda_s + \rho \overline{Q_s})^{-1} (\mu_s + \rho \overline{\eta}_s)
\]

Note that given a solution for \( Q_{H,L} \) we can obtain \( q_{H,L} \) in closed-form as a Matrix weighted average of \( \mu_H, \mu_L \). While we are not aware of closed-form solution for \( Q_H, Q_L \) in general, it is straightforward to obtain a numerical solution to the coupled Riccati matrix equation as we discuss in lemma 3 below. Further, for a variety of special cases we consider below it is possible to obtain closed-form solutions.

With a solution in hand, we can define the conditional aim portfolio as the portfolio that maximizes the value function at any time \( t \) conditional on the state. A bit of algebra then allows us to characterize the optimal trading rule and the aim portfolios.
Theorem 1 The optimal trade at time $t$ in state $s$ is a matrix weighted average of the current position vector and the conditional aim portfolio:

$$n_t = (I - \tau_s)n_{t-1} + \tau_s \text{aim}_s$$

where the trading speed $\tau_s = I$ (and $Q_s = 0$) if $\Lambda_s = 0$, and else $\tau_s = \Lambda_s^{-1}Q_s \forall s = \{H, L\}$ where $(Q_H, Q_L)$ solve a system of coupled equations:

$$I - \Lambda_s^{-1}Q_s = [\Lambda_s^{-1}(\gamma \Sigma_s + \rho \pi_{ss'}Q_{s'}) + I + \rho \pi_{ss'}\Lambda_s^{-1}Q_s]^{-1}$$  \hspace{1cm} (5)

The aim portfolio, which maximizes the value function conditional on the current state, is a weighted average of the conditional Markowitz portfolios ($m_s = \Sigma_s^{-1}\mu_s$):

$$\text{aim}_s = (I - \alpha_s)m_s + \alpha_s m_{s'} \forall s = H, L$$

where

$$\alpha_s = \{(\gamma + \rho \pi_{ss'}Q_{s'}\Sigma_{s'}^{-1}Q_{s'} - 1)\Sigma_s + \rho \pi_{ss'}Q_{s'}\}^{-1}\rho \pi_{ss'}Q_{s'}$$

Proof. Optimizing the value function with respect to $n_t$ gives:

$$\text{aim}_s = (Q_s)^{-1}(q_s) = [\gamma \Sigma_s + \rho \bar{Q}_s]^{-1}(\mu_s + \rho \bar{q}_s) \forall s = H, L$$

Expanding and rearranging we obtain that the aim portfolio in state $s$ is the matrix weighted average of the current Markowitz optimal portfolio and the aim portfolio in the other state $s'$:

$$\text{aim}_s = [\gamma \Sigma_s + \rho \pi_{ss'}Q_{s'}]^{-1}(\gamma \Sigma_s m_s + \rho \pi_{ss'}Q_{s'}\text{aim}_{s'})$$ \hspace{1cm} (6)

A bit of algebra then implies

$$\{\Sigma_s + \rho \pi_{ss',s'}Q_{s'}[\gamma \Sigma_{s'} + \rho \pi_{ss'}Q_s]^{-1}\Sigma_{s'}\} \text{aim}_s = \Sigma_s m_s + \rho \pi_{ss',s'}Q_{s'}[\gamma \Sigma_{s'} + \rho \pi_{ss'}Q_s]^{-1}\Sigma_{s'} m_{s'}$$

and more algebra then gives the result.

The theorem shows that it is optimal to choose a position that is a weighted average of the current position and of the aim portfolio defined as the position that would maximize the value function at any point in time. Another interpretation of the aim portfolio is as the no-trade portfolio, i.e., the portfolio at which it is optimal to not rebalance as long as the state does not change. The speed at which we trade towards the aim portfolio is, in general, dependent on the state, that is it is typically increasing in variance and decreasing in the transaction costs, which may be state dependent in our framework. In the case (similar to GP) where only expected returns are stochastic (and covariances and transaction costs are constant) the trading speed is constant as well. The aim portfolio is state dependent. When either a state is absorbing ($\pi_{ss} = 1$) or...
transaction costs are zero ($\Lambda_s = 0$) then the aim portfolio is equal to the conditional mean-variance Markowitz portfolio ($m_s$). But in general, the aim portfolio is a weighted average of the conditional mean-variance portfolio across states, where the weight on each state is typically higher the higher the variance of returns in that state and the higher the transaction costs.

We now consider a few special cases to gain further insights into the optimal trading rule.

2.1 The case where only $\mu_s$ changes with the state (GP)

If only $\mu_s$ changes with the state, then the solution $Q_s = Q$ is independent of the state and satisfies:

$$I - \Lambda^{-1}Q = [\gamma \Lambda^{-1} \Sigma + I + \rho \Lambda^{-1}Q]^{-1}$$

This equation has an explicit solution.

**Lemma 2** Consider the diagonalization of the matrix $\Lambda^{-1} \Sigma = F \text{diag}(\ell_i) F^{-1}$ in terms of its eigenvalues $\ell_i \forall i = 1, \ldots, n$. Then note that

$$I - F^{-1} \Lambda^{-1} Q F = [\gamma \text{diag}(\ell_i) + I + \rho F^{-1} \Lambda^{-1} Q F]^{-1}$$

It follows that $Q = \Lambda F \text{diag}(\eta_i) F^{-1}$ such that the $\eta_i$ solve the quadratic equations ($\forall i = 1, \ldots, n$):

$$1 - \eta_i = [\gamma \ell_i + 1 + \rho \eta_i]^{-1}$$

that is:

$$\eta_i = \frac{\rho - 1 - \ell_i \gamma + \sqrt{(\rho - 1 - \ell_i \gamma)^2 + 4 \ell_i \gamma \rho}}{2 \rho}$$

This implies that the trading speed $\tau_s = \Lambda_s^{-1} Q_s = F \text{diag}(\eta_i) F^{-1}$ is independent of the state. That is, investors trade at a constant speed towards their aim portfolio independent of the state. The speed of trading for specific stock $i$ is increasing in the agent’s time discount rate and in the agents risk-aversion. Furthermore, for the special case where $\Lambda$ and $\Sigma$ are diagonal matrices, then speed of trading stock $i$ is increasing in $\ell_i = \Sigma_{ii}/\Lambda_{ii}$, that is the ratio of a stock’s variance to its cost of trading.

While the trading speed is constant, the aim portfolios differ across states. Indeed, the aim portfolio in a state $s$ can be computed as:

$$\text{aim}_s = (I - \alpha_s) m_s + \alpha_s m_s'$$

with

$$\alpha_s = (\gamma Q^{-1} \Sigma + \rho \pi_{s's} + \rho \pi_{ss'})^{-1} \rho \pi_{ss'}$$

$$= F \text{diag}(\frac{\rho \pi_{ss'}}{\gamma \ell_i/\eta_i + \rho \pi_{s's} + \rho \pi_{ss'}}) F^{-1}$$
We see that the aim portfolio towards which we trade in state \( s \) is a weighted average of the Markowitz portfolios where the weight we put on the conditional Markowitz portfolio in the current state is increasing in the persistence of that state \( \pi_{s,s} \) and in risk-aversion \( \gamma \), but decreasing in the time discount factor \( \rho \), and the persistence of the other state \( \pi_{s',s'} \). Furthermore, the weight is also stock-specific and increasing for stock \( i \) in \( \eta_{i,\ell,i} \), which captures the notion that the more risky a stock is relative to its trading cost the more weight we should put on the conditional Markowitz portfolio for computing the aim portfolio.

To a large extent these results are consistent with the findings of GP, albeit with a different model of the time-variation in expected returns. The more interesting case, is when we also allow covariances and transaction costs to change across states. In that case, both trading speed and aim portfolios change across states.

2.2 The case where \( \Lambda_L = 0 \) and \( \Lambda_H > 0 \)

When transaction cost is zero in state \( L \) then the solution implies \( Q_L = 0 \) and \( Q_H \) solves a one-dimensional equation:

\[
I - \Lambda_H^{-1}Q_H = [\gamma \Lambda_H^{-1} \Sigma_H + I + \rho \pi_{HL} \Lambda_H^{-1} Q_H]^{-1}
\]

We note that this equation is identical to that obtained in the previous section with an adjusted time discount rate \((\rho \pi_{HL})\). It follows that the solution is \( Q_H = \Lambda_H F_H \text{diag}(\eta_{H,i}) F_H^{-1} \) where \((\ell_{H,i}, F_H)\) diagonalize the matrix \( \Lambda_H^{-1} \Sigma_H = F_H \text{diag}(\ell_{H,i}) F_H^{-1} \) and the \( \eta_{H,i} \) are given by:

\[
\eta_{H,i} = \frac{\rho \pi_{HL} - \ell_{H,i} \gamma + \sqrt{(\rho \pi_{HL} - 1 - \ell_{H,i} \gamma)^2 + 4 \ell_{H,i} \gamma \rho \pi_{HL}}}{2 \rho \pi_{HL}}
\]

In turn we can calculate the optimal trading speed and the aim portfolios in both states. Naturally, in the state where there is zero transaction cost it is optimal to move instantaneously to the aim portfolio, that is \( \tau_L = I \). Instead, in the high transaction cost state, it is optimal to trade slowly, with a trading speed \( \tau_H = F_H \text{diag}(\eta_{H,i}) F_H^{-1} \), towards the aim portfolio. Interestingly, the aim portfolio in the high transaction cost state is the conditional Markowitz portfolio, that is \( \text{aim}_H = m_H = \Sigma_H^{-1} \mu_H \). Intuitively, in the high transaction cost state we do not need to take into account of the investment opportunity set in the zero-transaction cost state, since in that state we will be able to rebalance at no cost to the first best position. Instead, in the zero transaction cost state, the aim portfolio weighs both Markowitz portfolios if there is a positive probability of returning to the high transaction cost state. Specifically: \( \text{aim}_L = (I - \alpha_L)m_L + \alpha_L m_H \) where the weight put on the high transaction cost Markowitz portfolio is given by: \( \alpha_L = [\gamma \Sigma_L + \rho \pi_{LH} Q_H]^{-1} \rho \pi_{LH} Q_H \). To summarize, when there are no transaction costs in the low state the optimal trading strategy is:
\[ n_{H,t} = (I - \tau_H)n_{t-1} + \tau_H m_H \]
\[ \tau_H = F_H \text{diag}(\eta_{H,i})F_H^{-1} \]
\[ n_{L,t} = \text{aim}_L = (I - \alpha_L)m_L + \alpha_L m_H \]
\[ \alpha_L = \left[ \gamma \Sigma_L + \rho \pi_{LH}Q_H \right]^{-1} \rho \pi_{LH} Q_H \]

2.3 The case with \( \Lambda_L > 0 \) and \( \Lambda_H = \infty \)

We now consider the polar case, where transaction costs are infinite in the \( H \)-state. Clearly, it is then optimal not to rebalance in the high state. Following the derivation of our model, with no rebalancing in the \( H \)-state, we see that the equation for \( Q_H \) simplifies to:

\[ Q_H = \gamma \Sigma_H + \rho Q_H \]

In turn, this implies that the equation for \( Q_L \) becomes:

\[ I - \Lambda_{L}^{-1}Q_L = [\gamma \Lambda_{L}^{-1}(\Sigma_L + \frac{\rho \pi_{LH}}{1 - \rho \pi_{HH}}) + I + \rho L \Lambda_{L}^{-1}Q_L]^{-1} \]

with \( \rho_L = \rho(\pi_{LL} + \frac{\rho \pi_{LH}}{1 - \rho \pi_{HH}}) \). This equation admits an explicit solution as before, in terms of the diagonalization of the matrix \( \Lambda_{L}^{-1}(\Sigma_L + \frac{\rho \pi_{LH}}{1 - \rho \pi_{HH}}) = F_L \text{diag}(\ell_{L,i})F_L^{-1} \).

It follows that the solution is \( Q_L = \Lambda_L F_L \text{diag}(\eta_{L,i})F_L^{-1} \) where the \( \eta_{L,i} \) are given by:

\[ \eta_{L,i} = \frac{\rho - 1 - \ell_{L,i} \gamma + \sqrt{(\rho - 1 - \ell_{L,i} \gamma)^2 + 4\ell_{L,i} \gamma \rho L}}{2\rho L} \]

Then the optimal trading strategy is:

\[ n_{H,t} = n_{t-1} \]
\[ n_{L,t} = (I - \Lambda_{L}^{-1}Q_L)n_{t-1} + \Lambda_{L}^{-1}Q_L \text{aim}_L \]
\[ \text{aim}_L = (1 - \alpha_L)m_L + \alpha_L m_H \]
\[ \alpha_L = \left\{ (1 - \rho \pi_{HH})\Sigma_H^{-1}\Sigma_L + \rho \pi_{LH} \right\}^{-1} \rho \pi_{LH} \]

To summarize, when t-costs are infinite in state H it is clearly optimal to not rebalance in that state. Instead, in state L, both the speed of trading and the aim portfolio depend on the investment opportunity set in the H state. The aim portfolio puts more weight on the H-conditional markowitz portfolio the higher the probability to transition to that state \( (\pi_{LH}) \), the more persistent the state is \( \pi_{HH} \), and the higher the variance of returns in that state relative to the L-state \( \Sigma_L \). The trading speed on the other hand increases in both \( \Sigma_H \) and \( \Sigma_L \) as well as the persistence of the low and high states.
2.4 The general case

For the general case, we need to solve the system of coupled ODEs (5) for \((Q_H, Q_L)\):

\[
I - \Lambda_s^{-1}Q_s = [\Lambda_s^{-1}(\gamma \Sigma_s + \rho \pi_{ss} Q_s') + I + \rho \pi_{ss} \Lambda_s^{-1}Q_s]^{-1}
\]

While we cannot solve the system in general, we observe that in the special case where the eigenfactors of the covariance and transaction cost matrices remain identical across states and only the eigenvalues change, the system does admit a simple explicit solution. For the general case, this then suggests a simple and efficient algorithm to compute the solution. We summarize the result in the following

**Lemma 3** If \(\Lambda_s = F \text{diag}(\lambda_{i,s})F^{-1}\) and \(\Sigma_s = F \text{diag}(\nu_{s,i})F^{-1}\) \(\forall s = H, L\) then the solution of the system of ODEs given in equation (5) is \(Q_s = \Lambda_s F \text{diag}(\eta_{i,s})F^{-1}\) where \(\forall i = 1, \ldots, n\) the constants \((\eta_{H,i}, \eta_{L,i})\) solve the system of quadratic equations:

\[
1 - \eta_{i,s} = [\gamma \nu_{i,s} + \rho \pi_{ss} \nu_{i,s} \lambda_{i,s} + \lambda_{i,s} + \rho \pi_{ss} \nu_{i,s} \lambda_{i,s}]^{-1}
\]

In general, the solution to the system of ODEs given in equation (5) can be obtained by the following recursion.

Given an initial \((Q_H^{n-1}, Q_L^{n-1})\), perform the eigenvalue decomposition (for \(s = H, L\)) of \(\Lambda_s^{-1}(\gamma \Sigma_s + \rho \pi_{ss} Q_s^{n-1}) = F_s \text{diag}(\eta_{i,s})F_s\). Then set \(Q_s^n = F_s \text{diag}(\ell_{i,s})F_s^{-1}\) where the \(\eta_{i,s}\) solve the equation

\[
1 - \eta_{i,s} = [\gamma \ell_{i,s} + \rho \pi_{ss} \nu_{i,s}]^{-1},
\]

that is:

\[\eta_{i,s} = \frac{\rho - 1 - \ell_{i,s} \gamma + \sqrt{(\rho \pi_{ss} - 1 - \ell_{i,s} \gamma)^2 + 4 \ell_{i,s} \gamma \rho \pi_{ss}}}{2 \rho \pi_{ss}},\]

and iterate until convergence. It is natural to use as an initial guess for \(Q_s^0\) either the zero matrix, or the solution corresponding to \(\pi_{ss} = 1\).

Given a numerical solution of the \(Q_H, Q_L\) matrices we can analyze the optimal trading rule and aim portfolios.

2.4.1 Example: Corporate bond versus Treasuries

To illustrate the implications of the model, we consider a case with two assets and two states: Low-risk (L) and High-risk (H). In the low-risk state, Asset 1 has higher Sharpe ratio than Asset 2. However, in the High-risk state, Asset 2 has higher Sharpe ratio. In both states, Asset 2 is cheap to trade whereas Asset 1 is cheap to trade in only in the low-risk state.

Table 1 shows a simple calibration for this example.

Figure 1 below shows the resulting positions in the aim portfolios in both states as we vary the transaction cost of the first asset in the high risk-state. We examine the impact of increasing Asset
Table 1: Parameter Values

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
<td>$10^{-8}$</td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.9996</td>
</tr>
<tr>
<td>$\pi_{LL}$</td>
<td>0.95</td>
</tr>
<tr>
<td>$\pi_{HH}$</td>
<td>0.9</td>
</tr>
<tr>
<td>$\mu_L$</td>
<td>$\begin{bmatrix} 10 \ 8 \end{bmatrix}$</td>
</tr>
<tr>
<td>$\mu_H$</td>
<td>$\begin{bmatrix} 12 \ 16 \end{bmatrix}$</td>
</tr>
<tr>
<td>$\Sigma_L$</td>
<td>$\begin{bmatrix} 100 &amp; 50 \ 50 &amp; 100 \end{bmatrix}$</td>
</tr>
<tr>
<td>$\Sigma_H$</td>
<td>$\begin{bmatrix} 900 &amp; 450 \ 450 &amp; 900 \end{bmatrix}$</td>
</tr>
<tr>
<td>$\Lambda_L$</td>
<td>$\begin{bmatrix} 10^{-8} &amp; 0 \ 0 &amp; 10^{-8} \end{bmatrix}$</td>
</tr>
<tr>
<td>$\Lambda_H$</td>
<td>$\begin{bmatrix} \text{Variable} &amp; 0 \ 0 &amp; 10^{-8} \end{bmatrix}$</td>
</tr>
</tbody>
</table>

1’s transaction cost in the High-risk state on the aim portfolios. The interesting result is that it is optimal to hold more of Asset 2 even in the low-risk state when Asset 1’s trading cost exceeds a certain threshold.

This example captures some salient features of the Corporate versus Treasury bond returns. Like asset 1 Corporate bonds typically offer higher expected rates of returns in expansions (good states) than Treasury bonds (asset 2). However, during recessions (bad states) their risk increases dramatically and the higher probability of default leads to lower returns.\(^4\) Further, it is also a fact that corporate bonds become a lot costlier to trade in bad states than Treasuries, whose liquidity remains very high. As the stylized example demonstrates, because it is optimal to reduce the position in the corporates in the high risk state when these are very costly to trade, it can be optimal to hold a larger share of the Treasuries already in the good state even though in that state the conditional sharpe ratio of Corporates dominates that of Treasuries.

2.4.2 Equal contribution to risk

The equal contribution to risk allocation strategy has received a lot of attention among practitioners, not the least because it is applied in the very successful “All-weather” fund of Bridgewater. As a rational for such a strategy, it is sometimes argued that such a strategy reflects the difficulty in mea-

\(^4\) Of course, it is arguable whether the expected return is actually lower, since expected returns are hard to measure. For illustration we assume that in the bad states the risk of asset 1 is higher and its sharpe ratio is lower than that of asset 2.
suring expected returns of and correlations between asset classes. With all expected returns equal and constant ($\mu = 1$ say) and all correlation coefficients equal to zero, the mean-variance efficient Markowitz portfolio becomes an equal contribution to risk portfolio ($m_s = \Sigma_s^{-1}1 = \text{diag}(1/v_{i,s})$).

Here we illustrate that it is actually optimal to deviate from the optimal contribution to risk allocation under these same assumptions, if transaction costs of various asset classes move predictably with their risk. Let’s assume that $\Sigma_s = \text{diag}(v_{i,s})$ and $\Lambda_s = \text{diag}(\lambda_{i,s}$ and $\mu_s = \mu$. Then we can solve for the optimal aim portfolio in closed-form from lemma 3 with $F = \text{diag}(1)$.

We illustrate in figure 2 how the aim portfolio in $aim_s$ can deviate from the equal contribution to risk portfolio as $t$-costs in state $s'$ increase.

Following much of the literature (e.g., GP, Litterman) the model in this section assumes that conditional on a state, the expectation and covariance matrix of price changes are constant. This leads to a very tractable solution, because in a mean-variance framework the only motive to rebalance the portfolio conditional on holding the mean-variance efficient portfolio, is if there is a change in the state, that is if there is a change in the expectation of price changes or their covariance matrix. Unfortunately, it is not a very plausible model for returns empirically, as it assumes counterfactual dynamics for the return covariances. Empirically, the “conditional log-normal” model of price changes is preferable to the “conditional normal” model assumed in this section. Interestingly, in our framework the “log-normal” model, which assumes that the expectation and covariance matrix of dollar returns is constant within a state, is very tractable as well. In the next section we present
this regime switching model of returns. In the final section we apply this model to timing the market portfolio in taking into account time varying transaction costs and stochastic volatility.

3 Regime Switching Model for Returns

Suppose $x_t$ is vector of dollar holdings in risky shares and $u_t$ is dollar trade at time $t$. $R_f$ is the risk-free rate and $R_t$ is the vector of Gross returns. The net returns are given by $r_t = R_t - 1$ and $r_f = R_f - 1$.

Then we have with the convention that we trade at the end of the period:

$$x_{t+1} = x_t \ast R_{t+1} + u_{t+1}$$  \hspace{1cm} (7)

$$W_{t+1} = W_t R_f + x_t^\top (R_{t+1} - R_f) - \frac{1}{2} u_{t+1} \Lambda u_{t+1}$$  \hspace{1cm} (8)

We also suppose that the return follows a regime switching model:

$$R_{t+1} = \mu(s_t) + \epsilon_{t+1}$$  \hspace{1cm} (9)

where $\epsilon_{t+1}$ is a $(d,1)$ vector random variables with zero conditional mean $E_t[\epsilon_{t+1}] = 0$ and covariance matrix

$E_t[\epsilon_{t+1} \epsilon_{t+1}^\top] = \Sigma(s_t)$

The state process $s_t = \{H, L\}$ follows a two-state Markov chain and is independent of $\epsilon_t$. We also have $P(s_{t+1} = H|s_t = H) = p^H$ and $P(s_{t+1} = L|s_t = L) = p^L$. 


3.1 The objective function

For simplicity we set \( r_f = 0 \) and we solve the finite horizon problem where the investor maximizes

\[
E \left[ \sum_{t=1}^{T} \rho^{t-1} \left\{ x_t^\top r_{t+1}(s_t) - \frac{1}{2} u_t^\top \Lambda(s_t) u_t - \frac{\gamma}{2} x_t^\top \Sigma(s_t) x_t \right\} \right]
\]

Using dynamic programming principle, the value function \( V_t(x_{t-1}, R_t, s_t) \) satisfies

\[
V_{t-1}(x_{t-1}, R_t, s_t) = \max_{x_t} \left\{ x_t^\top E[r_{t+1}(s_t)] - \frac{1}{2} u_t^\top \Lambda(s_t) u_t - \frac{\gamma}{2} x_t^\top \Sigma(s_t) x_t + \rho E[V_t(x_t, R_{t+1}, s_{t+1})] \right\}
\]

We guess the following quadratic form for our value functions:

\[
V_t(x_t, R_{t+1}, H) = -\frac{1}{2} x_t^\top \text{diag}(R_{t+1}) Q_t^H \text{diag}(R_{t+1}) x_t + x_t^\top \text{diag}(R_{t+1}) q_t^H + c_t^H
\]

\[
V_t(x_t, R_{t+1}, L) = -\frac{1}{2} x_t^\top \text{diag}(R_{t+1}) Q_t^L \text{diag}(R_{t+1}) x_t + x_t^\top \text{diag}(R_{t+1}) q_t^L + c_t^L
\]

For all \( t \), where \( s_t = s \in \{H, L\} \), \( s_{t+1} = z \in \{H, L\} \) and \( s' = \{H, L\} \setminus s \), our value functions equal

\[
V_{t-1}(x_{t-1}, R_t, s) = \max_{x_t} \left\{ x_t^\top (\mu^s - 1) - \frac{1}{2} (x_t - \text{diag}(R_t)x_{t-1})^\top \Lambda^s (x_t - \text{diag}(R_t)x_{t-1}) - \frac{\gamma}{2} x_t^\top \Sigma^s x_t \right. \\
+ \left. \rho E_t[V_t(x_t, R_{t+1}, z)] \right\}
\]

We can simplify \( \rho E_t[V_t(x_t, R_{t+1}, z)] \) as follows:

\[
\rho E_t[V_t(x_t, R_{t+1}, z)] = \rho p^s \left( -\frac{1}{2} x_t^\top E [\text{diag}(\mu^s + \epsilon_t^s) Q_t^s \text{diag}(\mu^s + \epsilon_t^s)] x_t + x_t^\top E [\text{diag}(\mu^s + \epsilon_t^s)] q_t^s + c_t^s \right)
\]

\[
+ \rho (1 - p^s) \left( -\frac{1}{2} x_t^\top E [\text{diag}(\mu^s + \epsilon_t^s) Q_t^{s'} \text{diag}(\mu^s + \epsilon_t^s)] x_t + x_t^\top E [\text{diag}(\mu^s + \epsilon_t^s)] q_t^{s'} + c_t^{s'} \right)
\]

\[
= -\frac{1}{2} x_t^\top A_t^s x_t + x_t^\top b_t^s + d_t^s
\]

where

\[
M^s = E[(\mu^s + \epsilon_t^s) (\mu^s + \epsilon_t^s)^\top] = \Sigma^s + \mu^s (\mu^s)^\top
\]

\[
A_t^s = \rho p^s (M^s \circ Q_t^s) + \rho (1 - p^s) (M^s \circ Q_t^{s'})
\]

\[
b_t^s = \rho p^s (\mu^s \circ q_t^s) + \rho (1 - p^s) (\mu^s \circ q_t^{s'})
\]

\[
d_t^s = \rho p^s c_t^s + \rho (1 - p^s) c_t^{s'}
\]
Using this expression for \( \rho \mathbb{E}_t [V_t(x_t, R_{t+1}, s)] \), we obtain

\[
V_{t-1}(x_{t-1}, R_t, s) = \max_{x_t} \left\{ x_t^\top (\mu^s - 1) - \frac{1}{2} (x_t - \text{diag}(R_t)x_t)^\top \Lambda^s (x_t - \text{diag}(R_t)x_t) - \frac{\gamma}{2} x_t^\top \Sigma x_t - \frac{1}{2} x_t^\top A_t^s x_t + x_t^\top b_t^s + d_t^s \right\},
\]

Thus, we maximize the quadratic objective \(-\frac{1}{2} x_t^\top J_t^s x_t + x_t^\top j_t^s + k_t^s \) where we define

\[
J_t^s = \gamma \Sigma^s + \Lambda^s + A_t^s, \quad j_t^s = \Lambda^s \text{diag}(R_t)x_{t-1} + (\mu^s - 1) + b_t^s, \quad k_t^s = d_t^s - \frac{1}{2} (\text{diag}(R_t)x_{t-1})^\top \Lambda^s (\text{diag}(R_t)x_{t-1})
\]

Then, the optimal \( x_t \) is given by \((J_t^s)^{-1}j_t^s\) and is given by

\[
x_t = (\gamma \Sigma^s + \Lambda^s + A_t^s)^{-1} (\Lambda^s \text{diag}(R_t)x_{t-1} + (\mu^s - 1) + b_t^s) \tag{11}
\]

The value achieved at the optimal solution is given by \(\frac{1}{2} (j_t^s)^\top (J_t^s)^{-1}j_t^s + k_t^s\) and we obtain the following recursions:

\[
Q_{t-1}^s = -\Lambda^s (\gamma \Sigma^s + \Lambda^s + A_t^s)^{-1} \Lambda^s + \Lambda^s \tag{12}
\]

\[
q_{t-1}^s = \Lambda^s (\gamma \Sigma^s + \Lambda^s + A_t^s)^{-1} (\mu^s - 1 + b_t^s) \tag{13}
\]

\[
c_{t-1}^s = d_t^s + \frac{1}{2} (\mu^s - 1 + b_t^s)^\top (\gamma \Sigma^s + \Lambda^s + A_t^s)^{-1} ((\mu^s - 1) + b_t^s) \tag{14}
\]

with the terminal conditions \(Q_T^s = 0\) and \(q_T^s = 0\).

Following our analysis in the previous section we define the aim\(_t\) portfolio as the portfolio at which it would be optimal not to rebalance at time \( t \) given the current state \( s \) and the dollar holdings \( x_{t-1}R_t \). We show the following results.

**Lemma 4** The aim\(_t\) portfolio, at which it is optimal not to rebalance at time \( t \), is:

\[
\text{aim}_t^s = (\gamma \Sigma^s + A_t^s)^{-1} (\mu^s - 1 + b_t^s)
\]

It maximizes the value function \( V(x_{t-1}, R_t, s) \) with respect to \( x_{t-1} \text{diag}(R_t) \).

The optimal trading rule is to “trade partially towards the aim” at the trading speed \( \tau_t^s = (\Lambda^s)^{-1}Q_t^s \):

\[
x_t^s = (I - \tau_t^s) \text{diag}(R_t)x_{t-1} + \tau_t^s \text{aim}_t^s
\]

**Proof.**
Maximizing the value function at time $V(x_{t-1}, R_t, s)$ with respect to $\text{diag}(R_t)x_{t-1}$ we obtain:

$$aim_t^s = (Q_{t-1}^s)^{-1}(q_{t-1}^s)$$

Substituting from the definitions in equations (12) and (13) we obtain:

$$aim_t^s = \left(-\Lambda^s(\gamma \Sigma^s + \Lambda^s + A_t^s)^{-1}\Lambda^s + \Lambda^s\right)^{-1}\left(\Lambda^s(\gamma \Sigma^s + \Lambda^s + A_t^s)^{-1}(\mu^s - 1 + b_t^s)\right)$$

$$= \left(- (\gamma \Sigma^s + \Lambda^s + A_t^s)^{-1}\Lambda^s + I\right)^{-1}(\gamma \Sigma^s + \Lambda^s + A_t^s)^{-1}(\mu^s - 1 + b_t^s)$$

$$= (\gamma \Sigma^s + A_t^s)^{-1}(\mu^s - 1 + b_t^s)$$

where the last equality obtains by noting that if we define the matrix

$$M = \left(- (\gamma \Sigma^s + \Lambda^s + A_t^s)^{-1}\Lambda^s + I\right)^{-1}(\gamma \Sigma^s + \Lambda^s + A_t^s)^{-1}$$

then

$$M^{-1} = (\gamma \Sigma^s + A_t^s)\left(- (\gamma \Sigma^s + \Lambda^s + A_t^s)^{-1}\Lambda^s + I\right) = (\gamma \Sigma^s + A_t^s).$$

Thus $M = (\gamma \Sigma^s + A_t^s)^{-1}$.

Further, starting from the definition of the optimal position $x_t$ given in equation (11), it is straightforward to obtain the optimal trade $x_t - \text{diag}(R_t)x_{t-1} = (\Lambda^s)^{-1}(\gamma \Sigma^s + A_t^s)(aim_t^s - x_t)$. Further algebra using the matrix $M$ above and equation (12) implies the definition of trading speed.

**3.2 Difference between Two Models**

Figure 3 illustrates the aim portfolios in models set-up in shares and dollars. We calibrate the model to a share price worth one dollar so that the Y-axis represents the dollar investment of both strategies (that is number of shares invested equal number of dollars invested). We see that the aim portfolio in the regime switching model of price changes always invests a larger position in the risky asset than the aim portfolio for the regime switching model of returns. The difference is larger the larger the expected return on the stock. The intuition is that when we rebalance at time $t$ in the return model, the dollar position will be affected by the risky return (see equation (7)), before we get to rebalance. Thus the aim portfolio in dollars reflects the expected dollar position after the risky one period return is realized.

**4 Empirical Application**

In this section, we implement our methodology using the modeling framework in dollars and illustrate that there are economically significant benefits using our approach both in-sample and out-of-sample.
Figure 3: Aim portfolios in models set-up in shares and dollars.
Table 2: Estimates of the Model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Parameter</th>
<th>Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_1$</td>
<td>0.0864%</td>
<td>$\sigma_1$</td>
<td>0.5512%</td>
</tr>
<tr>
<td>$\mu_2$</td>
<td>0.0340%</td>
<td>$\sigma_2$</td>
<td>0.9372%</td>
</tr>
<tr>
<td>$\mu_3$</td>
<td>0.0069%</td>
<td>$\sigma_3$</td>
<td>1.6032%</td>
</tr>
<tr>
<td>$\mu_4$</td>
<td>0.2939%</td>
<td>$\sigma_4$</td>
<td>3.9178%</td>
</tr>
<tr>
<td>$P_{11}$</td>
<td>0.9804</td>
<td>$P_{12}$</td>
<td>0.0196</td>
</tr>
<tr>
<td>$P_{13}$</td>
<td>0.0000</td>
<td>$P_{14}$</td>
<td>0.0000</td>
</tr>
<tr>
<td>$P_{21}$</td>
<td>0.0250</td>
<td>$P_{22}$</td>
<td>0.9670</td>
</tr>
<tr>
<td>$P_{23}$</td>
<td>0.0080</td>
<td>$P_{24}$</td>
<td>0.0000</td>
</tr>
<tr>
<td>$P_{31}$</td>
<td>0.0000</td>
<td>$P_{32}$</td>
<td>0.0233</td>
</tr>
<tr>
<td>$P_{33}$</td>
<td>0.9693</td>
<td>$P_{34}$</td>
<td>0.0074</td>
</tr>
<tr>
<td>$P_{41}$</td>
<td>0.0016</td>
<td>$P_{42}$</td>
<td>0.0000</td>
</tr>
<tr>
<td>$P_{43}$</td>
<td>0.0635</td>
<td>$P_{44}$</td>
<td>0.9350</td>
</tr>
</tbody>
</table>

4.1 Model Calibration

We use daily value weighted CRSP market returns from 1967 Q3 to 2017 Q2 (50 years) to estimate a regime switching model. The data is downloaded from French’s data library.

We estimate a Markov switching model with four states to describe the dynamics of market returns:

\[
 r_{t+1} = \mu(S_t) + \sigma(S_t)\epsilon_{t+1} \tag{15}
\]

where $S_t = \{1, 2, 3, 4\}$ and $\epsilon_{t+1}$ are serially independent and drawn from standard normal distribution. State transitions occur according to a Markov chain and we denote by $P_{ij}$ the probability of switching from state $i$ to state $j$.5

Table 2 displays the estimates of the model. All coefficients are statistically significant at 1% level.

The top panel in Figure 4 illustrates the corresponding smoothed probabilities for each regime and the bottom panel in Figure 4 illustrates the color-coded regimes by using the maximum smoothed probability for identification. The first regime (green) highlights the good states of the return data with high return and low volatility corresponding to the highest Sharpe ratio. This

5We have also tried a four-state constrained model with two mean and volatility coefficients (high and low) as opposed to four but this constrained model can be rejected with a likelihood test.
regime has also the highest expected duration with roughly 51 trading days. The transition from this state usually occurs to the second state (blue) with slightly lower expected return and higher volatility. The expected duration for this state is 30 trading days. The third state (yellow) is a distressed state with low expected return and high volatility. This state has the lowest Sharpe ratio and has an expected duration of approximately 33 trading days. The final state covers the crisis periods with very high expected return and very high volatility. This state is relatively short-lived with an expected duration of 15 trading days.

4.2 Calibration of the Transaction Costs

To calibrate the transaction cost multipliers of our model realistically, we use a novel proprietary execution data from the historical order databases of a large investment bank. The orders primarily originate from institutional money managers who would like to minimize the costs of executing large amounts of stock trading through algorithmic trading services. The data consists of two frequently used trading algorithms, volume weighted average price (VWAP) and percentage of volume (PoV). The VWAP strategy aims to achieve an average execution price that is as close as possible to the volume weighted average price over the execution horizon. The main objective of the PoV strategy is to have constant participation rate in the market along the trading period.

The execution data covers S&P 500 stocks between January 2011 and December 2012. Execution duration is greater than 5 minutes but no longer than a full trading day. Total number of orders is 81,744 with an average size of approximately 1 million.

According to our quadratic transaction cost model, trading \( q \) dollars in state \( j \) would cost the investor \( \lambda_j q^2 \). We label each trading day with one of the four states by finding the state with maximal smoothed probability on that day from the estimated regime switching model. With this methodology, 22,946 executions occur in regime 1, 41,898 executions in regime 2, 14,502 executions in regime 3 and 2,398 in regime 4. Compared to other states, regime 4 has relatively small number of executions due to its short-lived nature.

Our execution data has information on both the order size and total trading cost by comparing the average price of the execution to the prevailing price in the market before the execution starts. This is usually referred to as implementation shortfall (IS). Formally, IS of the \( i \)th execution is given by

\[
IS_i = \text{sgn} (Q_i) \frac{P_{i,\text{avg}} - P_{i,0}}{P_{i,0}},
\]

where \( Q_i \) is the dollar size of the order (negative if a sell order), \( P_{i,\text{avg}} \) is the volume-weighted execution price of the parent-order and \( P_{i,0} \) is the mid-quote arrival price. Thus, total trading cost in dollars is \( Q_i IS_i \). According to our model, this is given by \( \lambda_{m(i)} Q_i^2 \) where \( m(i) \) maps the \( i \)th execution to the state of the trading day. Thus, we can estimate \( \lambda_j \) for each state by fitting the
Table 3: Estimated values are multiplied by $10^{10}$. (Double-clustered at the stock and calendar day level.)

<table>
<thead>
<tr>
<th>Dependent variable: TC</th>
<th>All Stocks</th>
<th>Top Decile in Market Cap</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_1$</td>
<td>1.688***</td>
<td>0.501**</td>
</tr>
<tr>
<td>(0.459)</td>
<td>(0.217)</td>
<td></td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>1.725***</td>
<td>0.793***</td>
</tr>
<tr>
<td>(0.195)</td>
<td>(0.189)</td>
<td></td>
</tr>
<tr>
<td>$\lambda_3$</td>
<td>3.037***</td>
<td>1.506***</td>
</tr>
<tr>
<td>(0.418)</td>
<td>(0.352)</td>
<td></td>
</tr>
<tr>
<td>$\lambda_4$</td>
<td>2.274</td>
<td>0.812</td>
</tr>
<tr>
<td>(1.927)</td>
<td>(1.329)</td>
<td></td>
</tr>
</tbody>
</table>

Note: *p < 0.1; **p < 0.05; ***p < 0.01

The following model:

$$IS_i = \lambda_1 Q_i \mathbf{1}_{(m(i)=1)} + \lambda_2 Q_i \mathbf{1}_{(m(i)=2)} + \lambda_3 Q_i \mathbf{1}_{(m(i)=3)} + \lambda_4 Q_i \mathbf{1}_{(m(i)=4)} + \varepsilon_i$$

Table 3 illustrates the estimated coefficients. The reported standard error are clustered at the calendar day level. We observe that $\lambda$ estimates are all highly significant (except in state 4 where we observe fewer executions in our data-set) and vary a lot across regimes and tend to increase with volatility. We find that $\lambda_3$ is the largest across all states. Using Wald tests pairwise, we find that the estimate of transaction costs in this distress state, $\lambda_3$, is statistically significanitly higher than all other coefficients at a 10% significance level.

To better understand the variation in transaction costs across our states, we present in Table 4 the average values of various liquidity proxies across states. We find that bid-ask spreads, mid-quote volatility and turnover are increasing across states. The Amihud illiquidity proxy is similar but the point estimates indicate that state 3 is more liquid than state 4 (though not statistically significantly different) even though state 4 has the highest volatility. Since volume is much larger in that state, it may act as a mitigating factor on trading costs (Admati-Pfleiderer (1988) and Foster and Viswanathan (1993)).

Since we would like to estimate the price impact of trading the market portfolio, our estimates may be overestimating the cost as it is based on the complete S&P 500 stocks. In order to address this issue, we rerun our regressions only using data corresponding to the top 10% of stocks with respect to market capitalization. We believe that this universe of stocks reflect a more natural comparison to the market portfolio.

The second column of Table 3 illustrates the estimated coefficients for this universe. We observe that the coefficients are down by a factor between two and three but preserve the same ranking across states. In this case, $\lambda_3$ is statistically different than the coefficients of the first and second state at 10% significance level. The second column of Table 4 illustrate the state average values of
Table 4: Regression of interval statistics on Regime dummies. Double-clustered at the stock and calendar day level.

<table>
<thead>
<tr>
<th></th>
<th>All Stocks</th>
<th>Top Decile in Market Cap</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Spread (bps)</td>
<td>Volatility (%)</td>
</tr>
<tr>
<td>1</td>
<td>3.80***</td>
<td>1.11***</td>
</tr>
<tr>
<td></td>
<td>(0.04)</td>
<td>(0.02)</td>
</tr>
<tr>
<td>2</td>
<td>3.95***</td>
<td>1.23***</td>
</tr>
<tr>
<td></td>
<td>(0.04)</td>
<td>(0.02)</td>
</tr>
<tr>
<td>3</td>
<td>4.95***</td>
<td>1.92***</td>
</tr>
<tr>
<td></td>
<td>(0.09)</td>
<td>(0.06)</td>
</tr>
<tr>
<td>4</td>
<td>5.62***</td>
<td>2.84***</td>
</tr>
<tr>
<td></td>
<td>(0.39)</td>
<td>(0.28)</td>
</tr>
</tbody>
</table>

Note: *p<0.1; **p<0.05; ***p<0.01

liquidity proxies for this universe of stocks.

### 4.3 Objective function

We use our model of a regime switching model for returns framework presented in section 3 rather than the model presented in section 2, since these dynamics fit our 50 years of data better. Formally, the investor’s objective function is:

\[
E \left[ \sum_{t=0}^{\infty} \rho^t \left\{ x_t \mu(S_t) - \frac{1}{2} \lambda(S_t) u_t^2 - \frac{\gamma}{2} \sigma^2(S_t) x_t^2 \right\} \right]
\]  

(17)

where \( x_t = x_{t-1}(1 + r_t) + u_t \) and \( S_t \in \{1, 2, 3, 4\} \). We calibrate \( \rho \) so that the annualized discount rate is 1%. We set \( \gamma = 1 \times 10^{-10} \) which we can think of as corresponding to a relative risk aversion of 1 for an agent with $10 billion dollars under management. And we assume daily rebalancing.

The optimal portfolio policy of the investor is given by

\[
x_t^{opt}(S_t) = (1 - Q(S_t) \lambda(S_t)^{-1}) (1 + r_t) x_{t-1}^{opt} + \frac{Q(S_t)}{\lambda(S_t)} aim(S_t) \quad \forall S_t \in \{1, 2, 3, 4\}
\]  

(18)

where \( q \) and \( Q \) solve the following system of equations \( \forall s \in \{1, 2, 3, 4\} \):

\[
Q(S_t) = -\lambda(S_t)^2 \left( \gamma \sigma^2(S_t) + \lambda(S_t) + \rho (\sigma^2(S_t) + (1 + \mu(S_t))^2) \overline{Q}(S_t) \right)^{-1} + \lambda(S_t)
\]  

(19)

\[
q(S_t) = (\mu(S_t) + \rho \mu(S_t) \overline{q}(S_t)) \left( 1 - \frac{Q(S_t)}{\lambda(S_t)} \right)
\]  

(20)

\[
aim(S_t) = Q(S_t)^{-1} q(S_t)
\]  

(21)
4.4 Aim portfolios

Using the estimated model coefficients, we first study the aim portfolios across states in the presence and absence of transaction costs. Figure 5 illustrates the aim portfolios for the optimal policy in these cases. We also compare this optimal policy with a simple unconditional mean-variance benchmark, in which the portfolio rule holds a constant dollar amount equal to $\frac{\mu_{\text{avg}}}{\sigma_{\text{avg}}^2}$ in the risky asset. Here, $\mu_{\text{avg}}$ and $\sigma_{\text{avg}}^2$ are the sample mean and variance of the market returns between 1967 Q3 and 2017 Q2.

In the top panel, the red solid line illustrates the aim portfolios in the absence of transaction costs. Without transaction costs aim portfolios are simply the conditional mean-variance optimal Markowitz portfolios. Compared to the unconditional mean-variance constant benchmark portfolio, the optimal aim portfolio is very aggressive in Regime 1 and holds a smaller amount than the constant portfolio in all other states. In Regime 3, the holdings are very close to a risk-free position.

In the bottom panel, we plot the aim portfolios taking into account estimated transaction costs in the different states. Surprisingly, Regime 4 has the smallest aim portfolio whereas Regime 3, the lowest Sharpe ratio state, has slightly higher holdings. This is due to differences in trading costs, as well as to the transition probabilities, across states. For example, trading costs are largest in Regime 3, thus the optimal aim portfolio, which will determine trading in that state, should depend on the average positions expected in states that it will transition from, essentially Regime 4 (probability of $\sim 6\%$) and Regime 2 (probability of $\sim 1\%$), as well as from states it will transition too, again Regime 2 (probability of $\sim 2\%$) and Regime 4 (probability of $\sim 1\%$). These considerations make the desired holdings in Regime 3 higher.

4.5 In-sample Analysis

In this section, we evaluate the performance of the optimal policy using the in-sample estimates from the earlier sections. We compare it to a few benchmark policies both in the presence and absence of transaction costs.

In order to evaluate the performance of the policies, we need to assign each trading day to a regime state so that we can determine the appropriate values of $\sigma^2(S_t)$ and $\lambda(S_t)$. For this purpose, we use the smoothed probabilities from the regime switching model and assign the regime of each trading day to that with the highest probability. We also skip a day to implement the optimal and myopic policies without any forward-looking bias. That is to say, to determine the position on day $t$, we use the smoothed probabilities from day $t - 1$.

Let $x_t^{\text{opt}}$ be the optimal policy as computed from Equation (18) and the above implementation methodology. We break down the realized objective function into two terms, wealth and risk.
penalties:

\[
W_{T}^{\text{opt}} = \sum_{t=1967}^{T=2017} \rho^{t-1967} \left[ x_{t}^{\text{opt}} - x_{t-1}^{\text{opt}} R_{t} + \frac{1}{2} \lambda(S_{t}) \left( x_{t}^{\text{opt}} - x_{t-1}^{\text{opt}} R_{t} \right) \right]^{2} \quad (22)
\]

\[
RF_{T}^{\text{opt}} = \sum_{t=1967}^{T=2017} \rho^{t-1967} \left[ x_{t}^{\text{opt}} \sigma(S_{t}) \right]^{2} \quad (23)
\]

As described earlier, the first benchmark policy is the constant-dollar rule in which the investor chooses \( x_{t}^{\text{con}} = \frac{c \mu_{\text{avg}}}{\gamma \sigma_{\text{avg}}} \). The parameters of this policy are also obtained using the full in-sample data. We choose \( c \) so that the policy has the same risk exposure as the optimal policy, i.e., the discounted sum of risk penalties from this policy equals \( RF_{T}^{\text{opt}} \).

The second benchmark policy is the myopic policy with transaction cost multiplier, a widely used practitioner approach. This approach solves a myopic mean-variance problem, that is given some initial position \( x_{t-1} \) and the state \( S_{t}, r_{t} \), it solves \( \max_{u_{t}} x_{t} \mu(S_{t}) - \frac{1}{2} \gamma \sigma(S_{t}) x_{t}^{2} - \frac{1}{2} h u_{t}^{2} \lambda(S_{t}) \) subject to the dynamics \( x_{t} = x_{t-1}(1 + r_{t}) + u_{t} \). The myopic policy with transaction cost multiplier \( h \) is given by

\[
x_{t}^{\text{my}}(S_{t}) = (1 - \tau(S_{t}))(1 + r_{t}) x_{t-1}^{\text{my}} + \tau(S_{t}) \frac{\mu(S_{t})}{\gamma \sigma^{2}(S_{t})} \quad \forall S_{t} \in \{1, 2, 3, 4\} \quad (24)
\]

\[
\tau(S_{t}) = \frac{1}{1 + \frac{h \lambda(S_{t})}{\gamma \sigma^{2}(S_{t})}} \quad (25)
\]

Note that this policy, like the optimal one, trades partially towards an aim portfolio. However, since it takes the current state as given and ignores the implications of any future transitions in the state, the aim portfolio is the conditional mean-variance efficient Markowitz portfolio and the trading inertia \( 1 - \tau(S_{t}) \approx \frac{h \lambda(S_{t})}{\gamma \sigma^{2}(S_{t})} \) only depends on the current state size of transaction costs relative to risk. We choose \( h \) so that the myopic policy has the same total risk exposure as the optimal policy. Since in the absence of transaction costs, the myopic policy is optimal, we compare it to the optimal one only in the presence of transaction costs.

Figure 6 compares the optimal policy to the constant portfolio in the absence of trading costs. Both policies have the same risk penalty by construction (see bottom-right panel), thus the wealth dynamics are direct measures of performance. The top-left panel illustrates that the optimal policy has a much higher performance. We observe that this is achieved by trading more and timing the regimes of the return data. This confirms that there is predictability and that, at least in the absence of transaction costs, there is value to rebalancing across the estimated regimes.

Figure 7 compares the optimal policy to the constant portfolio in the presence of trading costs. Both policies again have the same risk penalty by construction. Top-left panel illustrates that the difference in performance is more pronounced in the presence of trading costs. One reason for this is the excessive trading of the constant portfolio policy as illustrated in the medium-left and bottom panel. Compared to the previous case, we note that optimal policy trades much more slowly as shown in medium-right panel. The constant policy trades a lot after large return shocks in order
to keep a constant dollar amount invested in the market portfolio. Therefore, the constant-dollar policy incurs much larger cumulative transaction costs than the optimal policy as we see in the bottom panel, which contributes a significant portion of the observed wealth difference between the two strategies.

Figure 8 compares the optimal policy to the myopic portfolio in the presence of trading costs. Compared to the constant policy, myopic policy trades very small amounts. The performance difference as illustrated by wealth dynamics in the top-left panel is again substantial. However, the myopic policy achieves better performance than the constant portfolio policy. The optimal policy performs better due to accounting for the future dynamics of the position. We note that the myopic policy is very slowly moving in building and getting out of position.

4.6 Out-of-sample Analysis

The in-sample analysis was useful in studying the expected properties and benefits of a fully dynamic portfolio policy, but to better assess the value of the regime switching model, we perform an out-of-sample analysis. We implement a two-state example in this section for faster estimation of the parameters as we need to estimate a regime switching model every day from 1967 to 2017.

First, we estimate the model parameters to determine the parameters of the objective function. We use all the available market return data from 1926 Q1 to 2017 Q2. Table 5 illustrates the estimated coefficients. We estimate the transaction cost regimes using the same methodology but now with two potential regimes. We again use the estimates from the liquid subset, i.e., 50 stocks with largest market capitalization. These parameters will be used to assess the performance of the policy and the true regimes of the model will be identified from the smoothed probabilities. The investment horizon is again from 1967 to 2017 and all policies cannot use any forward looking data.

The investor is not aware of the true parameters of the model and uses only information up to trading day \( t \) in order to make a trading decision for day \( t + 1 \).

We construct the optimal policy in the out-of-sample data as follows. First, we estimate a two-state regime switching model using the market return data from 1926 Q1 to 1967 Q2. We use these estimated parameters for the construction of the optimal trading policy. The only other necessary information is to predict the current regime. For this purpose, we rerun the estimation of the switching model everyday by augmenting the data with the previous day. We use the smoothed probabilities of the previous trading day to label the current regime.

We construct the constant portfolio policy in the out-of-sample data similarly. We estimate \( \mu_{avg} \) and \( \sigma_{avg} \) using the market return data from 1926 Q1 to 1967 Q2. These parameters are held fixed throughout the investment horizon. The investor then constructs the following constant portfolio:

\[
x_{t}^{con} = \frac{c\mu_{avg}}{\sigma_{avg}^2}
\]

We choose \( c \) so that the policy has the same risk exposure as the optimal policy.

The myopic policy with transaction cost multiplier is constructed similarly using the estimates from the out-of-sample period only. The current day’s regime is predicted by using the state with the highest smoothed probability obtained from estimates of the regime switching model that uses data as of the previous trading day. We again set the multiplier \( h \) so as to equalize the risk exposure.
Figure 9 compares the optimal policy to the constant portfolio in the absence of trading costs in the out-of-sample data. Top-left panel illustrates that the optimal policy has higher performance in terms of terminal wealth. The outperformance is smaller than for the in-sample analysis, but the results show that the regime-switching model captures predictability out-of sample and that it is valuable, absent transaction costs, to rebalance to time these regimes.

Figure 10 compares the optimal policy to the constant portfolio in the presence of trading costs in the out-of-sample data. Top-left panel illustrates that the difference in performance is more pronounced in the presence of trading costs. Constant policy again trades a lot after large return shocks which reduces its overall performance. We can see that the difference in cumulative transaction costs paid by both strategies is very large and that this difference contributes substantially to the difference in wealth generated by both strategies. This hints to an important insight we confirm below. Even if expected return regimes are difficult to measure leading to smaller out-of-sample performance absent transaction costs, if transaction cost regimes are more accurately measured, which is plausible since t-costs vary with second moments, then optimally accounting for the variation in transaction costs leads to very sizable improvement in performance.

Figure 11 compares the optimal policy to the myopic portfolio in the presence of trading costs in the out-of-sample data. We find that the outperformance of the optimal policy is again substantial. Here the myopic policy again strains too little early on and achieves too large a position later at the end of the sample.

Overall, this out-of-sample analysis illustrates that the outperformance of the optimal policy is robust to parameter uncertainty of the regime switching model.

4.7 Which parameter should you time?

In this section, we investigate the value of timing each switching parameter of the general model. The switching parameters are $\mu$, $\sigma$ and $\lambda$. It is well-known at least since Merton (1980) that expected returns are estimated less precisely than volatilities. One might thus expect that out-of-sample the benefits of timing changes in volatility would be larger than timing changes in expected returns. We will show some evidence to that effect below. Further, since transaction costs vary with volatilities, we also provide quantitative evidence about the value of timing transaction cost regimes.

We use the same methodology as for our out-of-sample analysis to account for the potential bias introduced by imprecisely estimated parameters. First, we study the value of timing the switches in either volatility or expected returns in the absence of trading costs. In this analysis, if the investor times volatility, he takes into account that the volatility is time-varying between two states but assumes that expected return is constant throughout the investment horizon and is given by $\mu_{\text{avg}}$ (as in the case of the constant portfolio rule). Similarly, if the investor times expected returns, he models them as time-varying between high and low states and internalizes the potential switches in the expected return in his trading rule but he assumes that the volatility stays constant at a level of $\sigma_{\text{avg}}$ (as in the case of the constant portfolio rule).
Figure 12 compares these two timing approaches in the absence of trading costs using an out-of-sample trading approach. We scale both policies so that they both have the same risk exposure as the optimal policy that times both parameters. We find that timing volatility provides much higher performance. The terminal wealth of the policy that only times volatility is actually higher than the terminal wealth of the optimal policy that times both parameters as shown in Figure 9. This illustrates that trying to time expected returns may be actually detrimental in an out-of-sample trading strategy. The top-right panel shows that the $\mu$-timing policy has a wider range of position compared to the range observed in the $\sigma$-timing policy. In the absence of t-costs the strategies switch to their conditional mean-variance Markowitz portolios in every state. Recall that the estimated mean in the state 2 is negative and the volatility is high. This implies that the $\mu$-timing strategy, which underestimates the volatility in that regime, takes a very large short position in the risky asset. This hurts the out-of-sample performance of the strategy relative to the volatility timing strategy, probably because the negative expected return in those states is very imprecisely estimated.

If there are trading costs in the model, then $\lambda$ will be switching through time between high and low transaction cost regimes. Now we consider combined timing strategies: Timing $\sigma$ and $\mu$, timing $\sigma$ and $\lambda$ or timing $\mu$ and $\lambda$. We consider the comparison across these policies in two different assumptions of $\gamma$: high risk-aversion and low risk-aversion. Figure 13 compares these three policies in the presence of transaction costs in the high risk-aversion case. We observe that the top performing policy times $\sigma$ and $\lambda$ and the worst performing policy times $\sigma$ and $\mu$. Overall these findings point to the value of timing transaction cost regimes.

Figure 14 compares these three policies in the low risk-aversion case. We again observe that the worst performing policy times $\sigma$ and $\mu$ but the underperformance is economically smaller. Thus, the cost of timing $\sigma$ and $\mu$ may be lower if the investor’s risk aversion is low when compared to the alternative timing strategies.

5 Conclusion

---

Note that in the absence of trading costs, changing risk aversion would not matter, as the wealth values will just be scaled by the ratio of the risk-aversion parameters.
References


Litterman, Robert, 2005, Multi-period portfolio optimization, working paper.


### Table 5: Estimates of the Model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Estimate</th>
<th>Parameter</th>
<th>Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_1$</td>
<td>0.0841%</td>
<td>$\sigma_1$</td>
<td>0.6110%</td>
</tr>
<tr>
<td>$\mu_2$</td>
<td>-0.0955%</td>
<td>$\sigma_2$</td>
<td>1.8886%</td>
</tr>
<tr>
<td>$P_{11}$</td>
<td>0.9866</td>
<td>$P_{12}$</td>
<td>0.0134</td>
</tr>
<tr>
<td>$P_{21}$</td>
<td>0.0431</td>
<td>$P_{22}$</td>
<td>0.9569</td>
</tr>
</tbody>
</table>

Table 6: Estimated values are multiplied by $10^{10}$. (Double-clustered at the stock and calendar day level.)

<table>
<thead>
<tr>
<th></th>
<th>Dependent variable: TC</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>All Stocks</td>
</tr>
<tr>
<td>$\lambda_1$</td>
<td>1.772***</td>
</tr>
<tr>
<td></td>
<td>(0.255)</td>
</tr>
<tr>
<td>$\lambda_2$</td>
<td>2.299***</td>
</tr>
<tr>
<td></td>
<td>(0.311)</td>
</tr>
</tbody>
</table>

Note: *p<0.1; **p<0.05; ***p<0.01
Figure 4: Regimes. Green=1, Blue: 2, Yellow: 3, Red: 4.
Figure 5: Aim portfolios when risk aversion is given by $\gamma = 1 \times 10^{-10}$ (we can think of as corresponding to a relative risk aversion of 1 for an agent with $10$ billion dollars under management.).
Figure 6: This compares the performance of the optimal policy with a constant-dollar portfolio in the absence of TC and in-sample model estimates. $\gamma = 1 \times 10^{-10}$ and trading is costless.
Figure 7: This compares the performance of the optimal policy with a constant-dollar portfolio in the presence of TC and in-sample model estimates. $\gamma = 1 \times 10^{-10}$ and trading is costly.
Figure 8: This compares the performance of the optimal policy with the myopic policy in the presence of TC and in-sample model estimates. $\gamma = 1 \times 10^{-10}$ and trading is costly.
Figure 9: This compares the performance of the optimal policy with a constant-dollar portfolio in the absence of TC and out-of-sample model estimates. $\gamma = 1 \times 10^{-10}$ and trading is costless. Fully out-of-sample.
Figure 10: This compares the performance of the optimal policy with a constant-dollar portfolio in the presence of TC and out-of-sample model estimates. $\gamma = 1 \times 10^{-10}$ and trading is costly. Fully out-of-sample.
Figure 11: This compares the performance of the optimal policy with the static policy in the presence of TC and out-of-sample model estimates. $\gamma = 1 \times 10^{-10}$ and trading is costly. Fully out-of-sample.
Figure 12: This compares the performance of dynamic policies with different timing strategies in the absence of TC and out-of-sample model estimates. $\gamma = 1 \times 10^{-10}$ and trading is costless. Fully out-of-sample.
Figure 13: This compares the performance of dynamic policies with different timing strategies in the presence of TC and out-of-sample model estimates. $\gamma = 1 \times 10^{-9}$ and trading is costly. Fully out-of-sample.
Figure 14: This compares the performance of dynamic policies with different timing strategies in the presence of TC and out-of-sample model estimates. $\gamma = 1 \times 10^{-10}$ and trading is costly. Fully out-of-sample.