Endogenous Liquidity and Defaultable Bonds

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April 21, 2012

Abstract

This paper studies the interaction between fundamental and liquidity for defaultable corporate bonds that are traded in an over-the-counter secondary market with search frictions. Bargaining with dealers determines a bond’s endogenous liquidity, which depends on both the firm fundamental and the time-to-maturity of the bond. Corporate default decisions interact with the endogenous secondary market liquidity via the rollover channel. A default-liquidity loop arises: Earlier endogenous default worsens a bond’s secondary market liquidity, which amplifies equity holders’ rollover losses, which in turn leads to earlier endogenous default. Thus, our model characterizes the full inter-dependence between liquidity premium and default premium in understanding credit spreads for corporate bonds. We also study the optimal maturity implied by the model, and an extension where worsening secondary market liquidity feeds back to endogenous under-investment.

Keywords: Positive Feedback, Fundamental and Liquidity, Over-The-Counter Market, Secondary Bond Market, Structural Models, Transaction Cost for Corporate Bonds, Bid-Ask Spread

*For helpful comments, we thank Nittai Bergman (MIT), Bruce Carlin (UCLA), Hui Chen (MIT), Gustavo Manso (MIT), Holger Mueller (NYU), Barney Hartman-Glaser (Duke) and seminar participants of the MIT Sloan, Columbia GSB lunchtime workshop, ASU winter conference, Duke-UNC asset pricing conference, UNC, Boston University. We are especially grateful to Rui Cui for excellent research assistance.

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1 Introduction

The recent 2007-2008 financial crisis and the ongoing sovereign crisis have vividly demonstrated the intricate interaction between asset fundamental and asset liquidity in financial markets. Liquidity tends to dry up for assets with deteriorating fundamentals, reflected by soaring liquidity premia and/or prohibitive transaction costs in trading. In the mean time, asset fundamentals get even worse due to endogenous reactions of market participants (for instance, the default of Lehman) in response to worsening liquidity in financial markets.

This paper aims to deliver such a fundamental-liquidity spiral in the corporate bond market.\(^1\) It has been well documented that secondary corporate bond markets – which are mainly over-the-counter (OTC) markets – are much less liquid than equity markets.\(^2\) On the one hand, Edwards, Harris, and Piwowar (2007) and Bao, Pan, and Wang (2011) document a strong empirical pattern that the liquidity for corporate bonds (measured as the transaction cost) deteriorates dramatically for bonds with lower fundamental, i.e., bonds that are issued by firms closer to default (reflected by higher credit derivative swaps). On the other hand, the recent financial crisis of 2007-2008 illustrates that the deterioration of secondary market liquidity can exacerbate the incentives of equity holders to default by adversely affecting the refinancing operations of firms, which drives up the credit derivative swaps and credit spreads for corporate bonds (He and Xiong (2012b), hereafter HX12). Taken together, these two observations imply a positive feedback loop between the secondary market liquidity and asset fundamentals for corporate bonds.

To deliver such an default-liquidity spiral effect, we adopt two standard ingredients from the existing literature. First, we model the endogenous liquidity in the secondary corporate bond market as a search-based over-the-counter (OTC) market à la Duffie, Garlenau, and Pedersen (2005). Bond investors who are hit by liquidity shocks prefer early payments, and with a certain

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\(^1\) Corporate bond markets, for both financial and non-financial firms, make up a large part of the U.S. financial system. According to flow of funds, the values of corporate bonds reaches about 4.7 trillion in the first quarter of 2011, which consists of about one third of total liabilities of U.S. corporate businesses.

\(^2\) For instance, Edwards, Harris, and Piwowar (2007) study the U.S. OTC secondary trades in corporate bonds and estimate the transaction cost to be around 150 bps, and Bao, Pan, and Wang (2011) find an even larger number. Other empirical papers that investigate secondary bond market liquidity are Hong and Warga (2000), Schultz (2001), Harris and Piwowar (2006); Green, Hollifield, and Schurhoff (2007b,a).
matching technology they meet and trade with an intermediary dealer at an endogenous bid-ask spread. This endogenous liquidity for the secondary bond market depends on both the firm’s distance-to-default and the bond’s time-to-maturity.

The second important ingredient for the feedback between fundamental and liquidity is the endogenous default decision by equity holders. This mechanism is borrowed from the standard Leland-type corporate finance structural models, i.e., Leland (1994) and Leland and Toft (1996) (hereafter LT96). More specifically, a firm rolls over (refinances) maturing bonds by issuing new bonds. When firm fundamentals deteriorate, equity holders will face heavier rollover losses due to falling prices of newly issued bonds. Equity holders default optimally, at which point bond investors with defaulted bonds step in to recover part of the firm value due to dead-weight bankruptcy cost.

The secondary market liquidity of defaulted bonds, i.e., bonds of firms that have defaulted, is important in deriving the endogenous bond liquidity before the firm defaults. We model the (il)liquidity of defaulted bonds based on the fact that bankruptcy leads to a delay in the payout of any cash due to lengthy court proceedings, as for example in the Lehman Brothers bankruptcy.3 This serves as one of the boundary conditions needed to solve the system of Partial Differential Equations (PDEs) that describes the bond valuations.4 We solve this system of PDEs, as well as the equity valuation and the endogenous default boundary, in closed form in Section 3.

Consistent with empirical findings in Edwards, Harris, and Piwowar (2007) and Bao, Pan, and Wang (2011), we show in Section 4.1 that the endogenous bid-ask spread is decreasing with the firm’s distance-to-default, holding the time-to-maturity constant, and decreasing in the bond’s time-to-maturity, holding the distance-to-default constant. Moreover, our model produces a novel testable empirical prediction that the slope with respect to time-to-maturity of the bid-ask spread will be greater for bonds with higher distance-to-default. Intuitively, as the stated maturity of corporate bonds plays no role in bankruptcy procedures, the difference between bonds with different

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3 After much legal uncertainty, payouts to the Lehman debt holders only started trickling out after about three and a half years.

4 This arises because bond valuations depend on firm fundamental, the bond’s time-to-maturity, and the liquidity state of bond holders. Another possibility for the bankruptcy boundary condition is to assume some adverse selection with regard to the bankruptcy recovery value, a path we have do not pursue in this model due to the difficulties inherent in tracking persistent private information.
time-to-maturities vanishes if firms are close to default.

Thanks to the endogenous liquidity derived in this model, Section 4.2 illustrates the positive feedback loop between liquidity and default in the corporate bond market. Imagine a negative shock which pushes the firm closer to default and therefore lowers the bond’s fundamental value. More importantly, because bonds of defaulted firms suffer greater illiquidity, the outside option of bondholders when bargaining with the dealer declines. This worsens the secondary market liquidity and lowers the bond prices even further. The wider refinancing gap between the newly issued bond prices and promised principals gives rise to heavier rollover losses, which causes equity holders to default earlier and thus pushes the firm even closer to default. As a result, lower distance-to-default reduces the fundamental value of the corporate bonds even further, and so forth. This default-liquidity spiral for corporate bonds can be further generalized to the fundamental-liquidity spiral for firms; as an simple extension, in Section 5.1 we show that this feedback loop also applies to the interaction between firm real investment and its financing liquidity. The outcome of these spirals is a unique fixed point bankruptcy threshold.

The feedback loop between fundamental and liquidity implies a full inter-dependence between liquidity and default components in the credit spread for corporate bonds. The model contrasts with the widely-used reduced-form approach in the empirical research, where it is common to decompose firms’ credit spreads into independent liquidity-premium and default-premium components and then assess their quantitative contributions, e.g., Longstaff, Mithal, and Neis (2005), Beber, Brandt, and Kavajecz (2009), and Schwarz (2010). Our model also offers a potential resolution to the hitherto difficulty of structural models to explain the AAA credit spread as pointed out by Huang and Huang (2003), as an illiquid secondary market implies a positive spread in excess of treasuries even for default-free bonds. Moreover, our ongoing project suggests that the positive feedback effect may amplify small liquidity frictions into quantitatively significant liquidity and default premia. Overall, our fully solved structural model calls for more structural approaches in the future empirical study about the impact of liquidity factors upon the credit spread of corporate bonds.

Additionally, for the use of short-term debt with a higher rollover frequency, there exists a
trade-off between better liquidity provision and earlier inefficient default. Regarding the liquidity provision of short-term debt, bond investors hit by liquidity shocks can either sell to dealers or sit out shocks by waiting to receive the face value when the bond matures. Shorter maturity improves upon the waiting option, resulting in a lower rent extracted by dealers and thus a greater secondary market liquidity. On the other hand, equity holders are absorbing rollover gains/losses ex post. As show in LT96 and emphasized in HX12, shorter-term debt with a higher rollover frequency leads to heavier rollover losses in bad times, which pushes equity holders default earlier to incur greater dead-weight bankruptcy costs. This tradeoff allows us to endogenize the firm’s initial choice of debt maturity, and unlike traditional capital structure models an optimal finite maturity structure arises.

Our paper belongs to the recent literature on the role of secondary market trading frictions in structural models of corporate finance (Black and Cox (1976), Leland (1994) and LT96). Ericsson and Renault (2006) analyze the interaction between secondary liquidity and the bankruptcy-renegotiation in a LT96 framework numerically. HX12 take the simplified secondary market friction introduced in the classic article of Amihud and Mendelson (1986), i.e., bond investors hit by liquidity shocks are forced sell their holdings immediately at an exogenous and constant transaction cost. Through the same rollover channel as our paper, HX12 emphasize that default component and liquidity component are naturally intertwined in credit spreads, and exogenous shocks to secondary market liquidity may lead to a significant rise of the default component. Crucially, because in HX12 the bond market liquidity is modeled in an exogenous way, that paper can only speak to the one-way economic channel from exogenous liquidity to default. Our paper endogenizes the secondary market liquidity by micro-founding the bond trading in a search-based secondary market, and derives the equilibrium liquidity jointly with equilibrium asset prices.\footnote{Two other well-known endogenous market illiquidity models based on private information are Kyle (1985) and Glosten and Milgrom (1985). We deem that the search based framework is suitable for the secondary market for corporate bonds, and also has the advantage of being integrated seamlessly into the dynamic firm setting in LT96.} It is the endogenous liquidity that distinguishes our paper from HX12, which plays a crucial role for the positive feedback mechanism between fundamental and liquidity for corporate bonds.
Our paper also makes a contribution to the search based asset-pricing literature, as represented by Duffie, Garlenau, and Pedersen (2005, 2007); Weill (2007); Lagos and Rocheteau (2007, 2009); Biais and Weill (2009). To our knowledge, this literature with concentration on OTC markets has thus far focused on the determinants of contact intensities and behavior of intermediaries, while eschewing time-varying asset fundamentals and asset maturities. Undoubtedly, asset-specific dynamics are important for the corporate bonds market, and we fill this gap by incorporating the firm’s distance-to-default and the bond’s time-to-maturity in deriving the asset (bond) valuations.\textsuperscript{6} Moreover, our paper demonstrates that, via the rollover channel, the endogenous search-based secondary market liquidity can have a significant impact on the firms’ behavior on the real side. Feldhutter (2011) is a related empirical paper that treats both corporate bond maturity and default as exogenous intensity processes to focus on estimation of search parameters in the absence of stochastic firm fundamentals.

Positive feedback is an active research topic in different areas. For instance, strategic complementarity naturally gives rise to positive feedback effect in the global games literature (e.g., Morris and Shin 2009), and a similar effect emerges in He and Xiong (2012a) who study dynamic coordinations among creditors whose debt contracts mature at different times. Through the information channel, Goldstein, Ozdenoren, and Yuan (2011) show that market prices can feedback to firm’s investment decisions. Brunnermeier and Pedersen (2009) illustrate the positive feedback loop between funding liquidity and market liquidity. Cheng and Milbradt (2012) show how managerial risk-shifting feeds back on bondholders decision to run, which in turn feeds back on managerial incentives. Manso (2011) points out that credit ratings affect a firm’s default decision, which feeds back into the rating decision.

The paper is organized as follows. Section 2 lays out the model, and Section 3 solves the model in closed-form. Section 4 illustrates the positive feedback loop between fundamental and liquidity,\textsuperscript{6}

\textsuperscript{6}The existing literature often assumes constant asset payoffs and infinite maturity; for instance, focusing on a very different market, Vayanos and Weill (2008) use a search framework to explain the difference between off-the-run and on-the-run treasury yields. As far as we know, the only paper with deterministic time dynamics in a search framework is the contemporaneous Afonso and Lagos (2011), which introduces deterministic time dynamics via an end-of-day trading close in the federal funds market. Also, because corporate bond payoffs are highly nonlinear in firm fundamentals, our closed-form solution with stochastic fundamentals is nontrivial.
and Section 5 provides discussions and extensions. Section 6 concludes. All proofs can be found in the Appendix.

2 The Model

2.1 Firm Cash Flows and Debt Maturity Structure

We consider a continuous-time model where a firm has assets-in-place that generate (after-tax) cash flows at a rate of \( \delta_t > 0 \), where \( \{\delta_t : 0 \leq t < \infty\} \) follows a geometric Brownian motion in the risk-neutral probability measure:

\[
\frac{d\delta_t}{\delta_t} = \mu dt + \sigma dZ_t, \tag{1}
\]

where \( \mu \) is the constant growth rate of cash flow rate, \( \sigma \) is the constant asset volatility, and \( \{Z_t : 0 \leq t < \infty\} \) is a standard Brownian motion, representing random shocks to the firm fundamental. We assume the risk-free rate \( r \) to be constant in this economy.\(^7\)

We follow LT96 in assuming that the firm maintains a stationary debt structure. At each moment in time, the firm has a continuum of bonds outstanding with an aggregate principal of \( p \) and an aggregate coupon payment of \( c \), where \( p \) and \( c \) are constants that we take as exogenously given. We normalize the measure of bonds to 1, so that each bond has a principal face value of \( p \) and a coupon flow payment of \( c \). Each bond has a maturity \( T \), and expirations of the bonds are uniformly spread out across time. Here, \( \frac{1}{T} \) is the firm’s rollover frequency on its debt; that is, during a time interval \((t, t + dt)\), a fraction \( \frac{1}{T} dt \) of the bonds matures and needs to be rolled over.

These bonds differ only in the time-to-maturity \( \tau \in [0, T] \). Denote by \( D(\delta, \tau) \) the value of one unit of bond, which depends on firm fundamental \( \delta \) and its time-to-maturity \( \tau \). Following the LT96 framework, we assume that the firm commits to a stationary debt structure denoted \((c, p, T)\). In other words, when a bond matures, the firm will replace it by issuing a new bond with identical (initial) maturity \( T \), principal value \( p \), and coupon rate \( c \), in the primary market (to be modeled

\(^7\)This specification can easily account for systemic and idiosyncratic risk via an introduction of an exogenous pricing kernel to connect physical and pricing measures.
shortly). In the main analysis we take the firm’s debt maturity $T$ as given; Section 5.4 discusses the optimal ex-ante choice of debt maturity $T^*$ that maximizes the firm value.\footnote{One could also easily endogenize the firm initial leverage $(c,p)$ based on the trade-off between tax benefit and bankruptcy cost by following LT96. We leave this exercise for future research.}

### 2.2 Secondary Bond Market and Search-Based Liquidity

As in Duffie, Garlenau, and Pedersen (2005), individual bond investors are subject to idiosyncratic liquidity shocks, and once hit by shocks they need to search for market-makers/dealers to trade. More specifically, at any time with intensity $\xi$ an individual bond holder is hit by an idiosyncratic liquidity shock. We model this sudden need for liquidity as an upward jump in the discount rate from the common interest rate $r$ to a higher level $r > r$. For simplicity, this higher discount rate persists until either the agent manages to sell his debt-holdings, or the face value $p$ is paid out when the debt matures, after which the investor exits the market forever.\footnote{This assumption can easily be relaxed as shown in the Appendix.} We further assume an infinite mass of $H$ type buyers on the sidelines to simplify the calculations. Lastly, we simply assume that each bond investor only holds one unit of bond, and indicate the investor who has been hit by a liquidity shock by $L$ (i.e., liquidity state or low valuation agent).

In practice, secondary corporate bond markets are less liquid than equity or primary debt markets. Thus, we assume that the secondary debt markets are subject to the following trading friction. An $L$ bond investor who wants to sell his debt-holdings has to wait an exponential time with intensity $\lambda$ to meet a dealer. When they meet, bargaining occurs over the economic surplus generated. We follow Duffie, Garlenau, and Pedersen (2007) who show that it is sufficient to define Nash-bargaining weights $\beta$ of the investor and $1 - \beta$ of the dealer to model this bargaining.

The illiquidity of secondary bond markets give rise to wedges in bond valuations for different investor types. Define $D_H(\delta,\tau)$ and $D_L(\delta,\tau)$ to be the valuations of debt of the high (or normal) type and the low (or liquidity) type, respectively. Suppose that a contact between a type $L$ investor and a dealer occurs. We assume that the dealer faces a competitive inter-dealer market with a continuum of dealers, and at any time they can (collectively) contact $H$ type investors who are...
competitive as well. Thus, the particular dealer in question can turn around and instantaneously sell directly (or through another dealer) to $H$ type investors at a price of $D_H(\delta, \tau)$, which implies that the surplus from trade is

$$S(\delta, \tau) \equiv D_H(\delta, \tau) - D_L(\delta, \tau).$$

The transaction price at which $L$ types sell to the dealer, $X(\delta, \tau)$, thus implements the following splits of the surplus according to the bargaining weights,

$$D_H(\delta, \tau) - X(\delta, \tau) = (1 - \beta) S(\delta, \tau)$$
$$X(\delta, \tau) - D_L(\delta, \tau) = \beta \cdot S(\delta, \tau),$$

(2)

so that

$$X(\delta, \tau) = \beta D_H(\delta, \tau) + (1 - \beta) D_L(\delta, \tau).$$

(3)

Relating to the micro-structure literature, in our model, the ask price at which dealers sell to $H$ type investors is simply their valuation $D_H$, while the bid price at which $L$ type investors sell their bond holdings to dealers is $X$. This implies that $D_H - X = (1 - \beta)(D_H - D_L)$ is also the (dollar) bid-ask spread, as $X$ is the dealer’s purchase price while $D_H$ is his selling price to $H$ type investors. Thus, $H$ type investors are indifferent between buying and not buying the bond, whereas $L$ type investors strictly prefer selling the bond when they have the opportunity for any $\beta > 0$.

The endogenous transaction cost $(1 - \beta)(D_H - D_L)$ captures the liquidity of the secondary market for corporate bonds. In a preview of the solution, by the dynamic nature of the model, the difference at issuance for a say AAA bond, $D_H - D_L$, will be determined by the probability that the bond defaults before it matures interacted with the wedge that prevails at bankruptcy, and the probability that the bond matures before it defaults interacted with the necessarily zero wedge at maturity between $D_H$ and $D_L$. 

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2.3 Primary bond market and Debt Rollover

As mentioned, at any time the firm replaces the maturing bonds with newly issued ones in the so-called primary market, where the firm hires a dealer who can place the new debt to $H$ type investors. As dealers are competitive in the primary market, the firm receives the full bond value of the high type $D_H$.\textsuperscript{10}

As a crucial part of our feedback mechanism, the $H$ type incorporates in his bond valuation $D_H$ the possibility that he will be hit by a liquidity shock in the future and thus has to use the illiquid secondary market to sell the bond. In other words, due to either fluctuating firm fundamental or changing secondary market illiquidity, the newly issued bond price $D_H$ might be higher or lower than the required principal repayments to the maturing bonds. Equity holders are the residual claimants of any rollover gains/losses. Again, following LT96, we assume that any gain will be immediately paid out to equity holders and any loss will be funded by issuing more equity at the market price. Thus, over a short time interval $(t, t+dt)$, the net cash flow to equity holders (omitting $dt$) is

$$NC_t = \left[ \delta_t \left( CF + \text{Coupon} - \left( 1 - \pi \right) c + \frac{1}{T} \left[ D_H(\delta_t, T) - p \right] \right) \right].$$ \hspace{1cm} (4)

The first term is the firm’s cash flow. The second term is the after-tax coupon payment to bond investors, where $\pi$ denotes the marginal tax benefit rate of debt.\textsuperscript{11} The third term captures the firm’s rollover gains/losses by issuing new bonds to replace maturing bonds. This term can be understood as \textit{repricing} the bonds at a rate of $\frac{1}{T}$. In this transaction, there is a $\frac{1}{T} dt$ fraction of bonds maturing, which requires a principal payment of $\frac{1}{T} pdt$; while the primary market value of the newly issued bonds is $\frac{1}{T} D_H(\delta_t, T) dt$. When the newly issued bond price $D_H(\delta_t, T)$ drops so that $D_H(\delta_t, T) < p$, equity holders have to absorb the negative cash-flow stemming from rollover $\frac{1}{T} [D_H(\delta_t, T) - p] dt$. Thus, the rollover frequency $\frac{1}{T}$ (or the inverse of debt maturity) affects the

\textsuperscript{10}Segura and Suarez (2011) present a banking model without secondary markets which concentrates on the effect of periodic shut-downs of the primary market for debt funding.

\textsuperscript{11}For each dollar received by bond investors, the government is subsidizing $\pi$ dollars so that equity holders only have to pay $1 - \pi$ dollars. The tax advantage of debt is a heritage from Leland models and does not play any role in our analysis.
extent of rollover losses/gains.

2.4 Bankruptcy

When the firm issues additional equity to fund these rollover losses, the equity issuance dilutes the value of existing shares.\textsuperscript{12} Equity holders are willing to buy more shares and bail out the maturing debt holders as long as the equity value is still positive (i.e. the option value of keeping the firm alive justifies absorbing the rollover losses). When the firm defaults, its equity value drops to zero. The default threshold $\delta_B$ is endogenously determined by equity holders, which is an important ingredient for the feedback loop between firm fundamentals and secondary market liquidity.\textsuperscript{13}

When the firm declares bankruptcy, we simply assume that creditors can only recover a fraction $\alpha$ of the firm’s unlevered value from liquidation, which is $\alpha \frac{\delta_B}{1-\alpha}$.\textsuperscript{14} As usual, the bankruptcy cost is ex post borne by debt holders but represents a deadweight loss to equity holders ex ante. Since the stated maturity for bonds per se does not matter in bankruptcy, for simplicity we assume equal seniority of all creditors.

Because one driving force of our model is that agents value receiving cash early, our bankruptcy treatment has to be careful in this regard. If bankruptcy leads investors to receive the proceeds immediately, $L$ type investors who are trying to sell their bonds could view default as a beneficial outcome.\textsuperscript{15} In other words, bankruptcy confers a benefit similar to maturity that may outweigh the deadweight loss stemming from the bankruptcy cost $1 - \alpha$. This “liquidity by default” runs

\textsuperscript{12}A simple example works as follows. Suppose a firm has 1 billion shares of equity outstanding, and each share is initially valued at $10. The firm has $10 billion of debt maturing now, but the firm’s new bonds with the same face value can only be sold for $9 billion. To cover the shortfall, the firm needs to issue more equity. As the proceeds from the share offering accrue to the maturing debt holders, the new shares dilute the existing shares and thus reduce the market value of each share. If the firm only needs to roll over its debt once, then the firm needs to issue 1/9 billion shares and each share is valued at $9. The $1 price drop reflects the rollover loss borne by each share.

\textsuperscript{13}To focus on the liquidity effect originating from the debt market, we ignore any additional frictions in the equity market such as transaction costs and asymmetric information. It is important to note that while we allow the firm to freely issue more equity, the equity value can be severely affected by the firm’s debt rollover losses. This feedback effect allows the model to capture difficulties faced by many firms in raising equity during a financial-market meltdown even in the absence of any friction in the equity market.

\textsuperscript{14}The bankruptcy cost is standard in the trade-off literature, and can be interpreted in different ways, such as loss from selling the firm’s real asset to second-best users, loss of customers because of anticipation of the bankruptcy, asset fire-sale losses, legal fees, etc.

\textsuperscript{15}This would be the case for example for a CDS contract written on the firm which features immediate payouts at the time of a bankruptcy/credit event.
counter to the fact that in practice bankruptcy leads to a much more illiquid secondary market, the freezing of assets within the company, and a delay in the payout of any cash depending on court proceeding.\textsuperscript{16}

Motivated by these facts, we make the following assumption for defaulted bonds. Suppose that after bankruptcy the cash flow stays constant at $\delta_B$ forever. To capture the uncertain timing of the court decision, the payout of cash $\alpha \frac{\delta_B}{r-\mu}$ occurs at a Poisson arrival time with intensity $\theta$. We focus on situations where $\alpha \frac{\delta_B}{r-\mu} < p$ (which holds for all our examples) so that the recovery rate to bond holders is below 100%. Also, the secondary market for defaulted bonds is illiquid with contact intensity $\lambda_B$. Then, the defaulted bond values $D^B_H$ and $D^B_L$ satisfy

$$r D^B_H = \theta \left( \alpha \frac{\delta_B}{r-\mu} - D^B_H \right) + \xi \left( D^B_L - D^B_H \right),$$

$$r D^B_L = \theta \left( \alpha \frac{\delta_B}{r-\mu} - D^B_L \right) + \lambda_B \left( X^B - D^B_L \right),$$

where as before $X^B = \beta D^B_L + (1 - \beta) D^B_H$ is the transaction price received by $L$ type investors. Plugging $X^B$ into the above equations, we can solve for $D^B_i = \alpha_i \frac{\delta_B}{r-\mu}$ for $i \in \{ H, L \}$ where

$$\alpha_H = \frac{\theta \alpha (r + \theta + \lambda_B \beta + \xi)}{\tau (\xi + \theta + r (\theta + \lambda_B \beta) + \theta (\xi + \theta + \lambda_B \beta))},$$

$$\alpha_L = \frac{\theta \alpha (r + \theta + \lambda_B \beta + \xi)}{\tau (\xi + \theta + r (\theta + \lambda_B \beta) + \theta (\xi + \theta + \lambda_B \beta))}. \quad (5)$$

Note that this establishes the boundary conditions at the bankruptcy boundary $\delta_B$, $D_i = \alpha_i \frac{\delta_B}{r-\mu}$ for $i \in \{ H, L \}$. One can easily see that $\alpha_H > \alpha_L$ as $\tau > r$. We denote the (bold face) vector $\alpha \equiv [\alpha_H, \alpha_L]^\top$ as the effective bankruptcy cost factors from the perspective of different bond holders. Clearly, the wedge $\alpha_H - \alpha_L$ characterizes the illiquidity of the defaulted bonds when the firm (i.e. equity holders) declares bankruptcy. Throughout the paper we focus on the situation where the illiquidity in the default state is sufficiently high, in order to conform our model to the regular empirical pattern that bonds closer to default are more illiquid (e.g., Edwards, Harris, and

\textsuperscript{16}The Lehman Brothers bankruptcy in September 2008 is a good case in point. After much legal uncertainty, payouts to the debt holders only started trickling out after about three and a half years.
3 Model Solutions

3.1 Debt Valuations

We first derive bond valuations by taking the firm’s default boundary $\delta_B$ as given. Recall that $D_H(\delta, \tau)$ and $D_L(\delta, \tau)$ are the value of one unit of bond with time-to-maturity $\tau \leq T$, an annual coupon payment of $c$, and a principal value of $p$ to a type $H$ and $L$ investor, respectively. We have the following system of partial differential equation (PDE) for the values of $D_H$ and $D_L$, where we omit the two-dimensional argument $(\delta, \tau)$ for both debt value functions:

\[
\begin{align*}
\tau D_H &= c - \frac{\partial D_H}{\partial \tau} + \mu \delta \cdot \frac{\partial D_H}{\partial \delta} + \frac{\sigma^2 \delta^2}{2} \frac{\partial^2 D_H}{\partial \delta^2} + \xi [D_L - D_H], \\
\tau D_L &= c - \frac{\partial D_L}{\partial \tau} + \mu \delta \cdot \frac{\partial D_L}{\partial \delta} + \frac{\sigma^2 \delta^2}{2} \frac{\partial^2 D_L}{\partial \delta^2} + \lambda \left[ X - D_L \right].
\end{align*}
\]

The boundary conditions are $D_H = D_L = p$ at $\tau = 0$ because of the principal payment at maturity, and $D_i = \alpha_i \frac{\delta B}{r - \mu}$ at $\delta = \delta_B$ as discussed in Section 2.4.

The first equation in (6) is the type $H$ bond valuation. The left-hand side $\tau D_H$ is the required (dollar) return from holding the bond for type $H$ investors. There are four terms on the right-hand side, capturing expected returns from holding the bond. The first term is the coupon payment. The next three terms capture the expected value change due to change in time-to-maturity $\tau$ (the second term) and fluctuation in the firm’s fundamental $\delta_t$ (the third and fourth terms). The last term is a loss $D_L - D_H$ caused by the liquidity shock that transforms $H$ investors into $L$ investors, multiplied by the intensity of the liquidity shock.

The second equation in (6), the type $L$ bond valuation, follows a similar explanation to the one above. The two differences are that the left hand side now has a higher required return $\tau > r$, and there is the value impact of the secondary market reflected in the last term of the right hand side.
A type $L$ investor meets a dealer with an intensity of $\lambda$ and is then able to sell his bond (with a private value $D_L$) at a price of $X = (1 - \beta) D_L + \beta D_H$. Plugging in equation (2) into equation (6), we see that there is a bargaining weighted intensity $\lambda\beta$ of 'transitioning' (via a sale) back from the $L$ state to the $H$ state. It is easy to show that when $\lambda \to \infty$, debt values converge to the LT96 case with perfectly liquid secondary markets. The surplus from intermediating trades vanishes because the outside option of meeting another dealer becomes very large.

We can now define the matrix $A$ that incorporates the discounting factors and the effective transition intensities $\xi$ and $\lambda\beta$ of the states. Then, the following decomposition holds:

$$A \equiv \begin{bmatrix} r + \xi & -\xi \\ -\lambda\beta & r + \lambda\beta \end{bmatrix} = \hat{P} \hat{D} \hat{P}^{-1}. $$

We let $\hat{D} \equiv \text{Diag} \left[ \hat{\tau}_1 \; \hat{\tau}_2 \right]$, where $\hat{\tau}_i = \frac{r + \xi + r + \lambda\beta \pm \sqrt{(r + \xi + r + \lambda\beta)^2 + 4\xi\lambda\beta}}{2}$ satisfying $\hat{\tau}_1 > \tau > \hat{\tau}_2 > r$, be the matrix of eigenvectors of $A$, and denote by $\hat{P}$ be the matrix of stacked eigenvalues. For a given $\delta_B$, we derive the closed-form solution for the bond values in the next proposition.$^{17}$

**Proposition 1** The debt values are given by

$$\begin{bmatrix} D_H (\delta, \tau) \\ D_L (\delta, \tau) \end{bmatrix} = \hat{P} \begin{bmatrix} A_1 + B_1 e^{-\hat{\tau}_1 \tau} \left[ 1 - F (\delta, \tau) \right] + C_1 G_1 (\delta, \tau) \\ A_2 + B_2 e^{-\hat{\tau}_2 \tau} \left[ 1 - F (\delta, \tau) \right] + C_2 G_2 (\delta, \tau) \end{bmatrix}. \quad (7)$$

Here, by defining $a \equiv \frac{\mu - \sigma^2}{\sigma^2}, \; \gamma_1 \equiv 0, \; \gamma_2 \equiv -2a, \; \eta_{j,1,2} \equiv -a \pm \sqrt{a^2 + \frac{2}{\sigma^2} \hat{\tau}_j},$ and $q (\delta, \chi, t) \equiv \frac{\log(\delta_B) - \log(\delta) - (\chi + \alpha) \sigma^2 t}{\sigma \sqrt{t}},$ the constants in (7) are given by:

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \equiv c \hat{D}^{-1} \hat{P}^{-1} 1, \quad \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \equiv \hat{\rho} \hat{P}^{-1} 1 - c \hat{D}^{-1} \hat{P}^{-1} 1, \quad \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \equiv \frac{\delta_B}{r - \mu} \hat{P}^{-1} \alpha - c \hat{D}^{-1} \hat{P}^{-1} 1, \quad$$

$^{17}$All derivations, because of the linear decomposition, would go through even if creditors would be subject to possibly different shock states, $\tau_1, \tau_2, \ldots$ and if the capital structure of the firm consisted of different issues of debt differing in $T$ or the like.
where $N(x)$ is the cumulative distribution function for a standard normal distribution.

A closer inspection of the solution reveals a linear combination (via the matrix $P$) of two sub-solutions each closely related to the original LT96 solution. The main difference to the LT96 solution is that each of these independent sub-solutions $i = \{1, 2\}$ has a distorted discount rate $\hat{r}_i > r$ (for $\xi > 0$), a distorted coupon rate $\hat{c}_i \equiv (cP^{-1}1)_i$ and a distorted recovery rate $\hat{\alpha}_i \equiv (P^{-1}\alpha)_i$.\(^{18}\) As in the LT96 solution, the first term $A_i$ gives the value of infinite risk-free debt, the term multiplied by $B_i$ encapsulates the probability that the bond will mature before default, and the term multiplied by $C_i$ encapsulates the probability that the bond will default before maturity.

### 3.2 Equity Valuation

The next key step is the equity holders’ decision to default, given that they receive the net cash flow in (4) every instant. Because equity is naturally an infinite maturity security and we are investigating a stationary (debt maturity structure) setting, the equity value $E(\delta; \delta_B)$ satisfies the following ordinary differential equation:

\[
    rE = \delta - (1 - \pi) c + \frac{1}{T} [D_H(\delta, T) - p] + \mu \delta E' + \frac{\sigma^2 \delta^2}{2} E'',
\]

where the left hand side is the required rate of return of equity holders. On the right hand side, the first three terms are the equity holders net cash flows, and the next two terms are capturing the instantaneous change of the firm fundamental. As mentioned earlier, the term involving square brackets is the cash-flow term that arises from rolling over debt (while keeping coupon, principal, and maturity stationary), with $\frac{1}{T}$ being the rollover frequency.

Similar to HX12, equity value in our model is no longer the difference between the levered firm value and debt value, a common calculation performed in Leland-type models. This is because part

\(^{18}\)Here, for any matrix or vector $M$, $(M)_i$ selects the $i$-th row and $(M)_{ij}$ selects the $i$-th row and $j$-th column of the matrix.
of the firm value goes to the dealers in the secondary bond market, and part vanishes because of inefficient holdings of debt by \( L \) types. Instead, we need to solve for \( E (\delta) \) directly via (8), which is non-trivial due to the highly-nonlinear form of \( D_H (\delta, T) \) given in (7). The next proposition gives the equity value.

**Proposition 2** Given a default boundary \( \delta_B \), the equity value is given by

\[
E (\delta; \delta_B) = K \left( \frac{\delta}{\delta_B} \right)^{\kappa_2} + \frac{\delta}{r - \mu} + K_0 - \frac{g_F (\delta)}{T} \sum_{j=1}^{2} P_{0j} B_j e^{-\delta_j T} + \frac{1}{T} \sum_{j=1}^{2} P_{0j} C_j g_{G_j} (\delta),
\]

where \( P_{01} = P_{11} \) and \( P_{02} = P_{12} \) and \( P_{ij} \) gives the element of \( \mathbf{P} \) in row \( i \) and column \( j \), \( \kappa_{1,2} = -a \pm \sqrt{a^2 + 2a\sigma^2} \), \( \Delta \kappa \equiv \kappa_1 - \kappa_2 \), and

\[
egin{align*}
K_0 &= \frac{1}{\tau} \left\{ - (1 - \pi) c + \frac{1}{\tau} \left[ \sum_{j=1}^{2} P_{0j} A_j + \sum_{j=1}^{2} P_{0j} B_j e^{-\delta_j T} - p \right] \right\}, \\
K &= \frac{1}{\tau} \left[ \delta_B + K_0 - \frac{1}{\tau} g_F (\delta_B) \sum_{j=1}^{2} P_{0j} B_j e^{-\delta_j T} + \frac{1}{\tau} \sum_{j=1}^{2} P_{0j} C_j g_{G_j} (\delta_B) \right], \\
g_F (x) &= \frac{1}{-\Delta \kappa / \sigma^2} \sum_{i=1}^{2} \left\{ \frac{x^2}{\delta_B^2} H (x, \gamma_i, \kappa_2, T) - \frac{x^2}{\delta_B^2} H (x, \gamma_i, \kappa_1, T) \right\}, \\
g_{G_j} (x) &= \frac{1}{-\Delta \kappa / \sigma^2} \sum_{i=1}^{2} \left\{ \frac{x^2}{\delta_B^2} H (x, \eta_{ij}, \kappa_2, T) - \frac{x^2}{\delta_B^2} H (x, \eta_{ij}, \kappa_1, T) \right\}, \\
H (\delta, \chi, \kappa, T) &= \frac{1}{\kappa - \chi} \left\{ \delta^{\kappa - \chi} N [q (\delta, \chi, T)] - \delta^{\kappa - \chi} e^{\frac{1}{2} [\kappa (\kappa + a)^2 - (\chi + a)^2] \sigma^2 T N [q (\delta, \kappa, T)]} \right\},
\end{align*}
\]

where \( q (\cdot, \cdot, \cdot) \) is given in Proposition 1.

Given these results, we can also calculate the total levered firm value at time zero with fundamental \( \delta_0 \), which is given in the Appendix.

### 3.3 Endogenous Default Boundary

So far we have taken the default boundary \( \delta_B \) as given. We now use the standard smooth pasting condition \( E_\delta (\delta_B; \delta_B) = 0 \) to determine the optimal \( \delta_B \) that is chosen by equity holders. The following proposition gives the closed-form solution for the endogenous default boundary \( \delta_B \).
<table>
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<td>$\sigma$</td>
<td>Volatility</td>
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<td></td>
<td>$\alpha_L$</td>
<td>Recovery value $L$ type</td>
<td>3/10</td>
</tr>
</tbody>
</table>

Table 1: Benchmark parameters.

**Proposition 3** The endogenous default boundary $\delta^*_B$ is given by

$$
\delta^*_B(T) = (r - \mu) \left[ \kappa_2 - 1 + \frac{1}{T} \sum_{j=1}^{2} P_{0j} \hat{\alpha}_j h_{G_j} \right]^{-1} \left( -\kappa_2 K_0 + \frac{h_F}{T} \sum_{j=1}^{2} P_{0j} B_j e^{-\hat{r}_j T} + \frac{1}{T} \sum_{j=1}^{2} P_{0j} A_j h_{G_j} \right),
$$

where $\hat{\alpha} \equiv P^{-1} \alpha$, $P_{0j}$ is defined in the previous proposition and

$$
h_F \equiv -\frac{2}{\sigma^2} \sum_{i=1}^{2} \frac{1}{\kappa_1 - \gamma_i} \left\{ N \left[ -\left( \gamma_i + a \right) \sigma \sqrt{T} \right] - e^{T N \left[ -\left( \kappa_1 + a \right) \sigma \sqrt{T} \right]} \right\},
$$

$$
h_{G_j} \equiv -\frac{2}{\sigma^2} \sum_{i=1}^{2} \frac{1}{\kappa_1 - \eta_{ij}} \left\{ N \left[ -\left( \eta_{ij} + a \right) \sigma \sqrt{T} \right] - e^{(r - \hat{r}_j) T N \left[ -\left( \kappa_1 + a \right) \sigma \sqrt{T} \right]} \right\}.
$$

Relating to existing literature, in the absence of debt rollover, secondary market frictions cannot affect the equity holders’ default decision once debt is in place. Infinite debt maturity features no rollover, and thus we have $\lim_{T \to \infty} \delta^*_B = \lim_{T \to \infty} V^*_B = \frac{\kappa_2 (1 - \pi) c}{\kappa_2 - 1}$, which coincides with the bankruptcy boundary derived in Leland (1994).

### 4 Endogenous Liquidity, Feedback Effects and Credit Spreads

We discuss model implications in this section. Section 4.1 analyzes the endogenous liquidity that depends on both firm fundamental and time-to-maturity. Based on endogenous liquidity, Section 4.2 illustrates the positive feedback effect between fundamental and liquidity for corporate bonds. Section 4.3 discusses the model implications on the observed credit spreads. Table 1 gives the the baseline parameters that we use for illustration in this section.
4.1 Endogenous Liquidity

4.1.1 Endogenous bid-ask spread

As mentioned, the (dollar) bid-ask spread is simply the difference between the bid price $X(\delta, \tau)$ and the ask price $D_H(\delta, \tau)$:

$$(1 - \beta) S(\delta, \tau) = D_H(\delta, \tau) - X(\delta, \tau),$$

which is just a constant positive fraction of the surplus $S$. In the following proofs, we thus concentrate on the behavior of $S$. We plot the bid-ask spread in Figure 1 as a function of both distance-to-default (that is state dynamics) and time-to-maturity (that is time dynamics). Note that the highest time-to-maturity is just the maturity for newly issued bonds, which in the figure is $T = 2$. The distance-to-default is captured by the difference between the current firm fundamental $\delta$ and the endogenous bankruptcy boundary $\delta^*_B(2) \approx 0.064$.

**Time-to-maturity** First, let us study the effect of time-to-maturity by fixing firm fundamental. Figure 1 shows that the endogenous bid-ask spread is lower for shorter time-to-maturities. Formally, we have the following proposition.

**Proposition 4** Under sufficient conditions provided in the Appendix, we have $S_\tau(\delta, \tau) > 0$, i.e. the bid-ask spread is larger for bonds with longer time-to-maturity.

The intuition for this result is simple. Because a shorter time-to-maturity delivers the full principal back to $L$ type investors sooner, this enhances $L$ type investors’ outside option in the bargaining and reduces the rent extracted by dealers, thereby resulting in a smaller bid-ask spread. In fact, by the boundary conditions the surplus vanishes as time-to-maturity goes towards 0, i.e.,

$$\lim_{\tau \to 0} S(\delta, \tau) = 0.$$
Figure 1: **Dollar bid-ask spread** $D_H - X$ with $T = 2$ as a function of both time-to-maturity $\tau$ and fundamental cash-flow $\delta$. The bankruptcy boundary is $\delta_B^* (2) \approx .064$.

If the bond is almost immediately demandable from the firm, $L$ type investors gain little value from trade with dealers, and as a result the bid-ask spread vanishes. This indicates that short-term debt provides liquidity for bond investors, and we will discuss the role of liquidity provision in more detail in Section 5.4.

**Distance-to-default** Second, let us fix the time-to-maturity $\tau > 0$ and then investigate the bid-ask spread by varying the distance-to-default (i.e., $\delta - \delta_B$). As shown in Figure 1, the bid-ask spread rises when the firm fundamental deteriorates towards the bankruptcy boundary $\delta_B$, which is consistent with the empirical regularity in Edwards, Harris, and Piwowar (2007) and Bao, Pan, and Wang (2011). Formally, we have the following proposition.

**Proposition 5** Under sufficient conditions provided in the Appendix, we have $S_\delta (\delta, \tau) < 0$, i.e. the bid-ask spread is smaller for bonds with higher firm fundamental.

The sufficient conditions in Proposition 5 can be understood as follows. Recall that, in Section 2.4, motivated by the empirical facts, we assume that bond investors need to wait quite a long time before they receive the cash pay-out. It is easy to show that as the firm fundamental converges
towards $\delta_B$, for any bonds that still have time-to-maturity left, i.e. $\tau > 0$, we have

$$\lim_{\delta \to \delta_B} S(\delta, \tau) = (\alpha_H - \alpha_L) \frac{\delta_B}{r - \mu} > 0,$$

(11)

We focus on the situation where the post-default illiquidity $\alpha_H - \alpha_L$ derived in equation (5) is sufficiently high, especially relative to the bid-ask spread for default-free bonds. As a result, the endogenous illiquidity rises when the cash flow rate $\delta$ deteriorates and the firm is closer to bankruptcy.

Comparative statics are as expected. Fixing the effective bankruptcy recovery vector $\alpha$, a lower $\lambda$ or $\beta$ leads to higher surplus / more illiquidity, as such a parameter shift shrinks the investors outside option or bargaining power directly, respectively. For a fixed bankruptcy boundary $\delta_B$, a higher cash-flow volatility again leads to higher surplus / more illiquidity away from the boundaries, as for any point $(\delta, \tau)$ the probability of hitting $\delta_B$ before maturity has increased, which in turn decreases the outside option of investors in bargaining.

For analytical tractability we have focused on dollar bid-ask spread $S(\delta, \tau)$. Another commonly used illiquidity measure is the effective percentage bid-ask spread $\Delta(\delta, \tau)$, defined as the dollar bid-ask spread $S(\delta, \tau)$ divided by the mid point of transaction prices (bid price $X$ and ask price $D_H$):

$$\Delta(\delta, \tau) = \frac{1 - \beta}{2} \left[ D_H(\delta, \tau) - D_L(\delta, \tau) \right] = \frac{1}{2} X(\delta, \tau) + \frac{1}{2} D_H(\delta, \tau) = \frac{2 (1 - \beta) S(\delta, \tau)}{(1 + \beta) S(\delta, \tau) + D_L(\delta, \tau)}.$$

(12)

\footnote{The intuition is quite simple: When $\delta = \infty$, so that bonds are risk-free, we have

$$\begin{bmatrix} D_H(\infty, \tau) \\ D_L(\infty, \tau) \end{bmatrix} = A^{-1} c + \exp(-A\tau) \begin{bmatrix} p - A^{-1} c \end{bmatrix}$$

$$= \frac{c}{(r + \xi_1)(r + \lambda \beta) - \xi_1 \lambda \beta} \left[ \frac{\tau + \xi_1 + \lambda \beta}{r + \xi_1 + \lambda \beta} \right] + \exp(-A\tau) \left[ \frac{p - \left( \frac{c(\tau + \xi_1 + \lambda \beta)}{(r + \xi_1)(r + \lambda \beta) - \xi_1 \lambda \beta} \right)}{p - \left( \frac{c(\tau + \xi_1 + \lambda \beta)}{(r + \xi_1)(r + \lambda \beta) - \xi_1 \lambda \beta} \right)} \right].$$

Together with $S_r(\delta, \tau) < 0$, we know that $S$ reaches a maximum when $\tau = T$. The most important part of the proof is that $S(\delta_B, \tau) - \lim_{\delta \to \infty} S(\delta, \tau) < 0$. That is, a necessary condition is that the bid-ask spread of the default-free bond is below that of the defaulted bond. Unfortunately we are unable to show the sufficiency of this condition due to the complex nature of the functions involved, and in the proof of Proposition 5 we impose stronger sufficient conditions.}

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The percentage illiquidity $\Delta(\delta, \tau)$ shares the same qualitative properties as $S(\delta, \tau)$. In addition, since the bond value decreases as the firm gets closer to default and thus tends to amplify $\Delta(\delta, \tau)$, this negative force naturally strengthens our result of increasing illiquidity for bonds that are closer to default. To the extent that the percentage illiquidity is more empirically relevant, the sufficient conditions in Proposition 5 are much stronger than necessary, and our theoretical results should be more general than they appear.

**Interaction between time-to-maturity and distance-to-default and new empirical predictions** We now investigate the impact of the interaction between time-to-maturity and distance-to-default on the endogenous bid-ask spread. As our goal is to provide some novel empirical predictions, in this subsection we focus on the percentage bid-ask spread $\Delta(\delta, \tau)$ in (12) which is commonly used in the empirical literature.

Similar to $S(\delta, \tau)$, we find numerically that $\Delta(\delta, \tau)$ is increasing with $\tau$ for $\delta > \delta_B$ as shorter maturity provides better liquidity. However, we also know from (11) that, as we approach the bankruptcy boundary $\delta_B$, $\Delta(\delta, \tau)$ becomes independent of $\tau > 0$, i.e., we have the same liquidity across all maturities. Thus, when the firm edges closer and closer to default, the slope of $\Delta(\delta, \tau)$ with respect to time-to-maturity $\tau$ becomes flatter and flatter. This result that the slope on time-to-maturity increases in distance to default is illustrated in Figure 2. More specifically, for financially healthy firms, the difference between the bid-ask spreads of long-term bond and short-term bond is greater than that of firms in imminent danger of bankruptcy.

This property is intuitive. Default, by forcing firms to enter lengthy bankruptcy proceeding that puts all debt holders of equal seniority on equal footing, eliminates difference due to maturities. For financially healthy firms, default is remote, and therefore the time-to-maturity has a positive and significant impact on the bid-ask spread. However, when default is imminent, although the bid-ask spreads for both long-term and short-term bonds soar, their difference diminishes as it is more likely that the stated time-to-maturity eventually becomes irrelevant. This intuition is quite general, as it only relies on the fact that maturity plays no role in bankruptcy.
The above discussion suggests the following regression specification:

$$\Delta_{i,t} = b_0 + b_{Maturity} \cdot Maturity_{i,t} + b_{CDS} \cdot CDS_{i,t} + b_{Maturity \cdot CDS} \cdot Maturity_{i,t} \times CDS_{i,t}. \quad (13)$$

As shown, our model predicts a positive $b_{Maturity}$, i.e., bonds with longer time-to-maturity should have a higher bid-ask spread. Further, the model predicts a positive $b_{CDS}$, i.e., the bond that is closer to default should have a higher bid-ask spread as well. These two predictions conform with the empirical findings in Edwards, Harris, and Piwowar (2007), and Bao, Pan, and Wang (2011). Finally, Figure 2 implies that $b_{Maturity \cdot CDS} < 0$, i.e., the difference between the bid-ask spreads of long-term and short-term bonds in financially healthy firms is greater than that of financially distressed firms. As just explained, this new testable prediction is intuitive and awaiting future empirical research.

### 4.1.2 Delay to trade and instantaneous transaction costs

Our paper has two important distinctions in modeling liquidity relative to models with constant transaction cost such as Amihud and Mendelson (1986) for equity markets and HX12 for bond markets. First, in those papers there is no delay to trade for investors hit by liquidity shocks, because they are forced (and able) to sell their holdings to some dealers at a discount $k$ immediately.
Second, this constant proportional transaction cost $k$ is given exogenously which implies state-independence. In contrast, our model has a secondary market modeled as a search market, which features delay to trade and endogenous bargaining.

To compare our model to the exogenous transaction cost model of HX12, we define $k_{\text{implied}} (\delta, \tau)$ as the hypothetical constant transaction cost that equates the debt price derived in HX12 to the newly issued price $D_H (\delta, T)$ derived in our model, i.e.,

$$D_H (\delta, T) = D (\delta, T; k_{\text{implied}}, \text{HX12 Model}).$$

This measure will be used in Section (4.2.1) to emphasize role of endogenous liquidity. Clearly, $k_{\text{implied}}$ varies with the firm fundamental $\delta$, which is in sharp contrast with the assumption of constant transaction costs in HX12. The plot of the implied transaction costs $k_{\text{implied}} (\delta, \tau)$ is similar to the liquidity surface presented in Figure 1.

### 4.2 Feedback Loop between Fundamental and Liquidity

By linking the secondary market liquidity endogenously to firm fundamental, we now demonstrate the positive default-liquidity spiral in which the deterioration of firm fundamental, via worsening liquidity of the secondary bond market, edges the firm even closer to default, which in turn leads to further deterioration in secondary market liquidity.

#### 4.2.1 Endogenous liquidity, rollover losses, and endogenous default

The combination of the endogenous secondary market liquidity and endogenous default decision taken by equity holders are the building blocks for the positive feedback loop between fundamental and liquidity. For illustration, we assume counterfactually that we are in a world of an exogenous transaction cost $k$ that applies whenever $H$ investors are hit by liquidity shocks and have to sell the bond holding immediately. In both our paper and HX12, equity holders make endogenous default

\[20\] The reader can think of the derivation of $k_{\text{implied}}$ as an exercise similar to deriving the implied volatility w.r.t. the Black-Scholes formula for any given or observed option price.
Figure 3: Left panel: Rollover loss $\frac{1}{T} \left| D_H(\delta, T) - p \right|$ as a function of fundamental value $\delta$ for main model (solid line) and the HX12 model (dashed line) with $k_{HX12} = 0.74\%$. Right panel: Implied transaction cost $k_{implied}$ for HX12 model (dashed line), and for main model with fully optimal bankruptcy boundary (solid line).

decision; however, in our paper the bond market liquidity (bid-ask spread) endogenously worsens when the firm is closer to default. Relative to the HX12 benchmark, this endogenous pro-cyclical secondary bond market liquidity drives equity holders to default earlier.

To understand the mechanism, consider the rollover losses borne by equity holders as a function of firm cash flow rate $\delta$. The dashed line in the left panel of Figure 3 graphs the benchmark rollover losses implied by HX12 who assume a constant (proportional) transaction cost $k$ (with a value of $k_{HX12} = 0.74\%$ in this example). In LT96 and HX12, the (absolute value) of rollover losses $\frac{1}{T} \left| D(\delta, T) - p \right|$ rises when the firm fundamental deteriorates, simply because forward looking bond investors adjust the market price of newly issued bonds downward when the firm is closer to default.

In our model, the endogenous secondary market liquidity further amplifies the rollover losses. The right panel of Figure 3 graphs the implied endogenous transaction cost $k_{implied}$ (solid line) of our model against the transaction cost in HX12 (dashed line) under the assumption $k_{HX12} = 0.74\%$. As mentioned in Section (4.1.2), $k_{implied}$ is the hypothetical constant transaction cost that equates the price of newly issued bonds in both models. As expected, the implied transaction cost $k_{implied}$ goes up as $\delta$ decreases, indicating a worsening secondary market liquidity for firms with lower

\[21\] The HX12 benchmark has discount rate $r$, coupon $c$, principal $p$, recovery value $\alpha_H$, and liquidity shock intensity $\xi$ given in Table 1. Additionally, we match pick $k_{HX12}$ so that $\lim_{\delta \to \infty} D_H(\delta, T) = \lim_{\delta \to \infty} D(\delta, T; k_{implied}, HX12\_Model)$, which results in a transaction cost of $k_{HX12} = 0.74\%$ or 74bps.
fundamentals.

Therefore, relative to HX12 with constant secondary market liquidity, the endogenous search market depresses the bond market price $D_H(\delta,T)$ further for low fundamental states as the implied transaction costs rise. Consequently, rollover losses increase in bad times as shown in the solid line in the left panel in Figure 3; relative to HX12, equity holders’ rollover losses in our model are more sensitive to the firm cash flow state $\delta$. The pro-cyclical secondary market liquidity significantly reduces the equity holders’ option value of servicing the debt especially in bad times, and hence they default earlier.

### 4.2.2 Positive feedback between fundamental and liquidity

The above discussion implies an important positive feedback loop between firm fundamental and secondary market liquidity for corporate bonds, which is illustrated in Figure 4. For investors of corporate bonds, the bond fundamental can be measured as the firm’s distance to default, i.e., $\delta - \delta_B$. Imagine a negative shock to firm cash flow rate $\delta$. Since this negative shock brings the firm closer to default, this constitutes a pure-fundamental driven negative shock to bond investors and lowers the holding values of $D_H$ and $D_L$. This force is already present in LT96 and HX12.

Now relative to HX12 which has fixed secondary market liquidity (or transaction costs), in our model a negative $\delta$ shock not only lowers debt values, but also worsens the secondary market
liquidity. The lower distance to default worsens the $L$ types’ outside option when bargaining with a dealer, as default leads to drawn out bankruptcy court decisions. Consequently, bonds in the secondary market becomes more illiquid, as indicated by the left large arrow in Figure 4. This is the pro-cyclicality of liquidity we already discussed above.

Rational $H$ type bond investors will thus value bonds less, i.e., a lower $D_H$, because they expect to face a less liquid secondary market once hit by liquidity shocks. As shown in Figure 4, the worsening liquidity in the secondary market gives rise to a lower primary market bond issuing price $D_H$ relative to an environment with constant market liquidity.

The lower bond prices now feed back to the equity holders’ default decision via the rollover channel, indicated by the arrow on the right of Figure 4. This is because equity holders are absorbing heavier rollover losses (i.e. net cash flow $NC_t$ in (4) goes down), as suggested by the left panel of Figure 3. Equity holders hence default earlier at a higher threshold $\delta_B$, relative to an environment with a constant market liquidity.

The higher default threshold now translates into a shorter distance to default $\delta - \delta_B$. But just as discussed before, the search-based secondary market kicks in again: as shown on the left-hand side in Figure 4, the shorter distance to default further worsens market liquidity via the declining outside option of the $L$ type investors. The loop repeats as the lower liquidity now again lowers effective bond prices, and finally stops at the fixed point $\delta_B$ given in Proposition 3.

The positive feedback loop between a worse bond fundamental (i.e., a lower distance-to-default, $\delta - \delta_B$) and a worse secondary market liquidity is the novel economic channel present in our model. In equilibrium, equity holders default earlier in our model with endogenous secondary market liquidity, compared to the model in HX12 with exogenous constant liquidity. The loop is solved by the unique fixed point $\delta_B(T)$ derived in Proposition 3.

---

$^{22}$This mechanism is similar in spirit to Brunnermeier and Pedersen (2009). We can view the refinancing (rollover) operation as the firm’s funding operation, and the “funding liquidity” captures the ease of raising outside money against the promise of a fixed payment in the future. Then, the secondary market liquidity feeds back into the funding liquidity of the firm. Thus, “funding liquidity” in turn affects the “market liquidity” via its impact on the default decision of equity holders and the investors’ outside option in the ensuing bargaining.
4.3 Credit spreads

The positive liquidity-default spiral in the previous section can have significant impact on observed corporate bond spreads (and equivalently primary market and secondary market ask prices $D_H(\delta, \tau)$), as bond investors rationally anticipate pro-cyclical liquidity and the default decision of equity holders. Recall that the bond credit spread is the spread between the corporate bond yield and the risk-free rate $r$. We compute the bond yield as the equivalent return on a bond conditional on it being held to maturity without default or re-trading. Given a bond of value $D(\delta, \tau)$, the bond yield $y$ is defined as the solution to the following equation:\footnote{23}

$$D(\delta, \tau) = \frac{c}{y}(1 - e^{-y\tau}) + pe^{-y\tau}, \quad (14)$$

Here, we use the ask price $D_H(\delta, \tau)$ in Proposition 1 as our bond price for the left-hand side of equation (14). We set $\tau = T$ in the following discussion.

In Figure 5 we plot the credit spread $y - r$ as a function of $\delta$. The dashed line gives the credit spread under HX12 with exogenous liquidity. The credit spread increases as $\delta$ decreases because the firm edges closer to bankruptcy. We also observe that there is a non-negligible credit-spread for very high $\delta$ (and hence default-free) due to pure liquidity reasons. The solid line gives the credit spread under our model, which takes into account the full liquidity-default spiral discussed above. We observe that the difference between the HX12 yield and the one generated by our model could be quantitatively important, especially when the firm is not doing well (in this example, the divergence reaches about 400 basis points for the low range of $\delta$).

\footnote{23}The right-hand side is the present value of a bond (discounted by $y$) with a constant coupon payment $c$ and a principal payment $p$, conditional on no default or trading before maturity.
Figure 5: Credit spread $y - r$ as a function of fundamental cash-flow $\delta$ for main model (solid line), for the HX12 model with $k_{HX12} = 0.74\%$ (dashed line) for bond at issuance (i.e., $T = 2$).

5 Discussions and Extensions

5.1 Endogenous Investment

So far we focused on the feedback loop between liquidity and default, and the latter captures the fundamental value of corporate bonds. Generally, this mechanism, i.e., a feedback loop between the firm fundamental and the firm’s (debt) financing liquidity, should encompass a broader set of firm level decisions beyond default. Indeed, one may broadly interpret default as a form of disinvestment. Although in our model the cash flow rate $\delta$ evolves as an exogenous stochastic process in (1), the total firm value (including both equity and debt values) in (A.1) depends on the equity holders’ default decision. This subsection pushes this idea further to consider a simple extension where equity holders make an initial endogenous investment decision.\footnote{This simple extension is enough to illustrate the feedback between firm fundamental and its financing liquidity; a full-blown model with dynamic investment opportunities is treated in follow-up work and requires a modified model to remain tractable.}

Suppose that at date-0 the equity holders, by investing $I$, can improve the firm cash flow rate $\delta_0$ to $\delta_0 + I$ (and thus the progression of all $\delta$ thereafter via equation (1)). The investment cost $\frac{\phi}{2} I^2$ is quadratic, and for simplicity, we assume that equity holders bear the investment outlay $I$. 
Figure 6: **Optimal initial investment** $I$ as a function of initial cash-flow $\delta_0$ for $\phi = \frac{18}{(r-\mu)^2}$ for the LT96 model (dashed line) and for the main model (solid line).

Hence, at date 0 equity holders are solving

$$\max_{I>0} E(\delta_0 + I) - \frac{\phi}{2} I^2,$$

where $E(\delta_0 + I)$ is given in (9) with the optimal $\delta_B = \delta_B^*(T)$. The above problem gives the endogenous date-0 investment $I^*$, and we are interested in the impact of endogenous secondary market liquidity on the firm investment.

Similar to Diamond and He (2011), in this setting the equity holders’ investment decision suffers the classic debt overhang problem coined in Myers (1977). Here, the investment incentive of equity holders enters the mechanism in Figure 4. Consider a hypothetical negative shock to the date-0 firm fundamental cash flow $\delta_0$. The lower fundamental worsens the firm’s financing liquidity immediately, and as discussed above this pro-cyclical financing liquidity pushes equity holders to default earlier. Moreover, in this extension, earlier default leads to a greater truncation of investment benefit received by equity holders, and thus a lower endogenous initial investment (taken by equity holders). The lower fundamental adversely affects the firm’s financing liquidity, which lowers the initial investment even further, and so forth. The extra positive feedback effect is illustrated in the wedge between the investment implied by our model and that of LT96 in Figure
5.2 Liquidity Premium and Default Premium

It has been widely recognized that the credit spread of corporate bonds not only reflects a default premium determined by the firm’s credit risk, but also a liquidity premium due to the illiquidity of the secondary debt market, e.g., Longstaff, Mithal, and Neis (2005), and Chen, Lesmond, and Wei (2007). However, both academics and policy makers tend to treat the default premium and liquidity premium as independent, and thus ignore interactions between them. For instance, it is common practice to decompose firms’ credit spreads into independent liquidity-premium and default-premium components and then assessing their quantitative contributions, e.g., Longstaff, Mithal, and Neis (2005), Beber, Brandt, and Kavajecz (2009), and Schwarz (2010).

In our model, the endogenous inter-dependence between the liquidity and default premia for corporate bonds challenges this approach. In fact, based on reasonable calibrations, HX12 have demonstrated that an exogenous rise of the liquidity premium (say, bond investors become more likely to suffer liquidity shocks) will lead to a sizable increase in default premium, due to the endogenous earlier default by equity holders. Our paper goes further by endogenizing the secondary market liquidity. Importantly, beyond the fact that exogenous aggregate liquidity shocks may drive liquidity premia for corporate bonds, our paper demonstrates that the origin of shock to liquidity premia can be found in the deterioration of the firm fundamental itself. Thus, both default premium and liquidity premium are inter-dependent with each other, and the positive feedback loop further amplifies and reinforces both premia in a nontrivial way.

Another important implication of our model is in the structural credit risk literature. As Huang and Huang (2003) point out, structural credit models have difficulty in producing the quantitatively significant AAA credit spread observed in the data, once calibrated to historic default probabilities and asset prices. In our model, in Figure 5 there remains a non-negligible credit spread even for

\footnote{For instance, HX12 show that if an unexpected shock causes liquidity premium to increase by 100 bps, default premium of a firm with speculative grade B rating and 1 year debt maturity (a financial firm) would rise by 70 bps, which contributes to 41% of the total credit spread increase.}
large $\delta$ (hence default-free bonds). This is because in our setting the liquidity risk is uninsurable on the agent level and thus does not affect the translation of the physical probabilities to the risk-neutral probabilities, which is consistent with the idea that the triple-A spread can be explained by liquidity reasons. Perhaps more interestingly, the positive liquidity-default spiral emphasized in this paper has the potential to amplify the relatively small liquidity shocks to quantitatively significant liquidity and default premia. A careful calibration exercise therefore is a priority for future work.

5.3 Discussion of Asymmetric Information

In our model, the important driving force behind the spiking illiquidity near default is that there is a significant valuation wedge between $H$ and $L$ type investors for defaulted bonds, as summarized by the individual recovery values $\alpha_H$ and $\alpha_L$. In the literature as well as in practice, an equally compelling explanation for the deteriorating liquidity of corporate bonds near default is a possibly worsening adverse selection problem due to information asymmetry. More specifically, one can imagine that some bond investors have private information regarding the bond’s recovery value in default. As the firm edges closer to default, the informed agent’s information becomes more valuable and he is more likely to attempt to sell his bonds. Thus, to guard against such adversely selected investors, a market maker in the Glosten and Milgrom (1985) tradition would raise the bid-ask spread.

Modeling such persistent adverse selection with long-lived bond investors, however, requires a lot more technical apparatus. To the extent that a model based on adverse selection could conceivably lead to a similar result if asymmetric information is concentrated in the bond’s recovery value, our model has the advantage of incorporating standard structural bond valuation models in a simpler setting but still delivering the first-order empirical patterns.

\footnote{If, instead, the adverse selection might not necessarily worsen when the firm goes closer to the default boundary, we would not expect, absent the bargaining frictions presented in this article, a monotonically increasing pattern of illiquidity towards the boundary.}
5.4 Optimal Debt Maturity

Beyond the feedback loop between fundamental and liquidity, our model features a natural trade-off between liquidity provision and earlier inefficient default. This natural trade-off allows us to derive the optimal debt maturity (given the stationary maturity structure).

5.4.1 Liquidity provision: the bright side of short maturity

Section 4.1 has shown that bonds with shorter maturity have a more liquid secondary market, suggesting the role of liquidity provision for short-term debt. The efficiency gain due to short-term maturity arises from two channels.

First, debt holders hit by liquidity shocks become inefficient holders of bonds, and due to trading frictions the inefficient holding lasts for a while. As detailed in the Appendix, the steady-state proportion of $L$ types if the firm is able to issue to only $H$ types is

$$\mu_L(T) = \frac{\xi}{\lambda + \xi} - \frac{\xi [1 - e^{-T(\lambda + \xi)}]}{T (\lambda + \xi)^2},$$

with $\mu'_L(T) > 0$, $\lim_{T \to \infty} \mu_L(T) = \frac{\xi}{\lambda + \xi}$ and $\lim_{T \to 0} \mu_L(T) = 0$. Hence, the second term in (15) is the allocative efficiency gain of shortening the bond maturity $T$. Intuitively, shortening maturity alleviates this inefficiency because of the firm’s superior primary market liquidity: whenever debt matures, the firm moves debt from inefficient $L$ investors to efficient $H$ investors via new bond issuance.\(^{27}\)

Second, a shorter maturity reduces the rent extracted by dealers in the secondary market, thus leading to a bargaining efficiency gain. Intuitively, a shorter maturity, by allowing $L$ investors

\(^{27}\)The firm could, instead of providing liquidity via maturity, allow bondholders with liquidity shocks to put back their bonds at the face value $p$. This seemingly perfect solution suffers two important drawbacks. First, if the firm cannot distinguish who was hit by a liquidity shock, whenever $D_H < p$ everyone will put back their debt at the same time. In fact, the put provision is akin to making bonds demand deposits and we are at traditional models of bank runs with potential bad run equilibrium. Second, even if the liquidity shock is observable, there will be an additional flow term $\xi [D_H - p] dt$ as $L$ investors are putting back their bonds to the firm every instant. This additional refinancing losses may influence the bankruptcy boundary in an adverse way and destroy the liquidity thus provided. The full implications of expanded bond contract terms (beyond the choice of initial maturity $T$ covered in this paper) is left for future work.
to receive principal payment earlier, raises their outside option of waiting and in turn lowers the
dealer’s rent.\footnote{Given that $\lambda$ was assumed as an exogenous parameter, this effect is independent of the first effect discussed above.}

5.4.2 Earlier default: the dark side of short maturity

On the other hand, as first shown in LT96 (and formally proven in HX12), shorter debt maturity in
an LT96 style model always leads to earlier default and thus greater dead-weight bankruptcy cost.
In other words, the optimal maturity in LT96 and HX12 is $T^* = \infty$, so that debt should always
take the form of an infinitely lived consol bond. As discussed, the equity holders’ rollover losses
are $\frac{1}{T} [D_H(\delta, T) − P]$. In bad times (low fundamental $\delta$), notwithstanding the fact that short-term
debt has a greater market price $D_H(\delta, T)$, the effect of a higher rollover frequency $\frac{1}{T}$ dominates,
leading to heavier rollover losses. As a result, equity holders default earlier if the firm is using
shorter maturity debt.

5.4.3 Optimal Debt Maturity

The above trade-off naturally leads to an endogenous optimal maturity structure. In Figure 7 we
plot in the left-hand panel the ex ante levered firm value $TV(\delta_0)$ given in (A.1) for both our model
(solid line) and the LT96 benchmark model\footnote{The LT96 benchmark has discount rate $r$, coupon $c$, principal $p$ and recovery value $\alpha_H$.} (dashed line) as a function of the debt maturity $T$
for an initial unlevered value $V(\delta_0) = 4$. The hump shape of levered firm value suggests that we
can find an interior solution for the optimal maturity structure, which is just less than 1 year in
this case. In contrast, the total firm value in the LT96 case is monotonically increasing in debt
maturity.

As explained, we can loosely interpret the initial leverage as the distance of the initial unlevered
firm value $V_0 \equiv \frac{\delta_0}{r-\mu}$ to the face value $p$. In the right panel of Figure 7 we draw the optimal maturity
$T^*$ as a function of $V_0 = \frac{\delta_0}{r-\mu}$ (for different levels of $\delta_0$, holding all other parameters constant),
which is inversely related to “initial leverage.” The solid line depicts the optimal maturity for
Figure 7: Left panel: **Total firm value** in the main model (solid line) and without frictions in the LT 96 model (dashed line). Right panel: **Optimal maturity** $T^*$ in the main model as a function of unlevered firm value $V_0 = \frac{\delta_0}{r-\mu}$ for different levels of search frictions, $\lambda = 0.1$ (solid line) and $\lambda = 1$ (dashed line). Both lines at some point jump to $T^* = \infty$ for small enough finite $V_0$.

a secondary market with low intermediation, i.e., $\lambda = 1/10$, whereas the dashed line depicts the optimal maturity for a secondary market with high intermediation, i.e., $\lambda = 1$. For low (high) initial leverage, bankruptcy becomes more (less) remote, and the effect of liquidity provision (bankruptcy cost) dominates, resulting in a shorter (longer) optimal debt maturity. Additionally, for poorly intermediated markets with $\lambda = 0.1$, the firm provides liquidity to its debt holders through shorter maturity. In contrast, for a better intermediated market with $\lambda = 1$, the optimal maturity shifts out uniformly, and jumps to infinity for firms with relatively low initial unlevered firm values $V_0$.

In other words, a better functioning secondary market reduces the need to provide liquidity via shorter maturity and thus alleviates the bankruptcy pressure generated by the short debt structure.

### 6 Conclusion

We investigate the liquidity-fundamental spiral in the corporate bond market, by studying the endogenous liquidity of defaultable bonds in a search-based OTC markets together with the endogenous default decision by equity holders from the firm side. By solving a system of PDEs, we derive the endogenous secondary market liquidity jointly with the debt valuations, equity valuations, and endogenous default policy, in closed-form. The fundamentals of corporate bonds, which
is mainly driven by the firm’s distance-to-default, affects the endogenous liquidity of corporate bonds. And, through the rollover channel in which equity holders are absorbing refinancing losses, worsening liquidity of corporate bonds significantly hurts the equity holders’ option value of keeping the firm alive. As a result, illiquidity of secondary corporate bond market feeds back to the fundamental of corporate bonds by edging the firm closer to bankruptcy. With the aid of recent empirical techniques, we hope our fully solved structural model can pave the way of bringing more structural approach in the empirical study of the impact of liquidity on corporate bonds.
References


Appendix

First, let us call \( r_H \equiv r, r_L \equiv r, \xi_H \equiv \xi \) and \( \xi_L \equiv \lambda \), and \( \tilde{\mu} = \mu - \frac{\sigma^2}{2} \). Second, define the log-transform \( \hat{\delta} = \log(\delta) \) so that \( d\hat{\delta} = \tilde{\mu} dt + \sigma dZ \). Third, for brevity we use the notation \( D' = \frac{\partial D}{\partial v} \) and \( \tilde{D} = \frac{\partial \tilde{D}}{\partial v} \). We will, with abuse of notation, write \( \hat{q}(\hat{\delta}, \ldots) \) to mean \( \frac{\delta_{B-\hat{\delta}+\ldots}}{\delta_{B-\hat{\delta}+\ldots}} \).

A.1 2x2 matrix formulas

As the 2x2 specification is frequently used in the text, we present the results here in compact form. Suppose

\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},
\]

then \( A = PD\tilde{P}^{-1} \) where

\[
A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},
\]

\[
P = \begin{bmatrix} \frac{1}{r_1-d} & b \\ \frac{b-a}{r_2-a} & 1 \end{bmatrix},
\]

\[
\tilde{D} = \begin{bmatrix} \hat{r}_1 & 0 \\ 0 & \hat{r}_2 \end{bmatrix},
\]

where of course alternative versions of \( P \) can be chosen. However, to show convergence to frictionless markets we chose this form of \( P \) as it allows convergence to an upper triangular form. The roots

\[
\hat{r}_{1/2} = \frac{a + d \pm \sqrt{(a + d)^2 - 4(ad - bc)}}{2}
\]

solve \( \det [A - \rho I] = 0 \), i.e. \( \hat{r}_{1/2} \) are the roots of the characteristic polynomial

\[
g(\hat{\rho}) = (a - \hat{\rho})(d - \hat{\rho}) - bc = \hat{\rho}^2 - (a + d)\hat{\rho} + (ad - bc).\]

If \( a > 0 \) and \( d > 0 \) and \( b < 0 \) and \( c < 0 \) as well as \( (ad - bc) > 0 \), then both roots \( \hat{r}_{1/2} > 0 \).

Identifying \( a = r_H + \xi_H, b = -\xi_H, c = -\xi_L, d = r_L + \xi_L \), we have

\[
\hat{r}_i = \frac{r_H + r_L + \xi_H + \xi_L - (-1)^i \sqrt{(r_H + \xi_H) - (r_L + \xi_L)}]}{2}.
\]

We can also derive bounds on \( \hat{r}_i \) by noting the following results:

\[
g(r_H) = \xi_H (r_L - r_H) > 0
\]

\[
g(r_L) = -\xi_L (r_L - r_H) < 0
\]

\[
g(r_H + \xi_H) = -\xi_H \xi_L < 0
\]

\[
g(r_L + \xi_L) = -\xi_H \xi_L < 0
\]

\[
g(r_H + \xi_H + \xi_L) = -\xi_L (r_L - r_H) < 0
\]

\[
g(r_L + \xi_H + \xi_L) = \xi_H (r_L - r_H) > 0
\]

so that we know that

\[
r_H < \hat{r}_1 < \min\{r + \xi_H, r_L\}
\]

\[
\max\{r_H + \xi_H + \xi_L, r_L + \xi_L\} < \hat{r}_2 < r_L + \xi_H + \xi_L.
\]

It is easy to show that as \( \xi_H \to 0, \hat{r}_1 = r_L + \xi_L \) and \( \hat{r}_2 = r_H \), and \( \lim_{b \to 0} P = \begin{bmatrix} 0 & 1 \\ \cdot & \cdot \end{bmatrix} \), so that \( D_H \) converges towards the LT96 solution.

Next, consider \( \lambda \to \infty \) such that \( \xi_L \to \infty \), that is, what happens when the market becomes very liquid. Note
that we can rewrite the characteristic polynomial as

\[ g(\hat{r}) = \xi_L \left[ (r_H + \xi_H - \hat{r}) \left( \frac{r_L}{\xi_L} + 1 - \frac{\hat{r}}{\xi_L} \right) \right] - \xi_H \]

Suppose now that \( \hat{r} \) is finite. Then we know that the square bracket, as \( \xi_L \to \infty \), becomes

\[ (r_H + \xi_H - \hat{r}) - \xi_H = 0 \]

so that \( \hat{r}_2 = r_H > 0 \). Thus, as both roots are positive, we must have that the second root \( \hat{r}_1 \to \infty \). The diagonal decomposition becomes unstable, in that \( \lim_{\lambda \to \infty} P = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \).

Finally, for \( r = r_H = r_L \) we can show that \( P^{-1} \mathbf{1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \) so that \( \check{c} = \begin{bmatrix} c \\ 0 \end{bmatrix} \), and for \( \alpha = \alpha_H = \alpha_L \) we have \( \check{\alpha} = \begin{bmatrix} \alpha \\ 0 \end{bmatrix} \).

### A.2 Proofs of Section 3

#### A.2.1 Debt

**Proof of Proposition 1.**

Applying the log transform \( \check{\delta} = \log (\delta) \) to the system of PDEs we are left with a linear system of PDEs:

\[ \begin{bmatrix} r_H + \xi_H & -\xi_H \\ -\xi_L & r_L + \xi_L \end{bmatrix} \begin{bmatrix} d_H \\ d_L \end{bmatrix} = \begin{bmatrix} c \\ \rho c \end{bmatrix} + \check{\mu} \begin{bmatrix} d_H \\ d_L \end{bmatrix} \}

\[ + \frac{\sigma^2}{2} \begin{bmatrix} d_H \\ d_L \end{bmatrix}'' - \begin{bmatrix} d_H \\ d_L \end{bmatrix} \]

\[ \iff \ A \cdot \mathbf{d} = c + \check{\mu} \mathbf{d}' + \frac{\sigma^2}{2} \mathbf{d}'' - \dot{\mathbf{d}} \]

Here we allow for general changes to the coupon payment \( c \) by premultiplying by a parameter \( \rho \leq 1 \) to acknowledge that there might be linear holding costs above and beyond the higher discount rate. In the paper, we have \( \rho = 1 \). Let us decompose \( A = \hat{D} P^{-1} \) where \( \hat{D} \) is a diagonal matrix with its diagonal elements the eigenvalues of \( A \) and \( P \) is a matrix of the respective stacked eigenvectors. The resulting eigenvalues are defined

\[ g(\hat{r}) = (r_H + \xi_H - \hat{r})(r_L + \xi_L - \hat{r}) - \xi_L \xi_H = 0 \]

and \( g(r_H) = \xi_H (r_L - r_H) > 0 \) and \( g(r_L) = -\xi_L (r_L - r_H) < 0 \). We thus have \( \hat{r}_i = \frac{r_H + \xi_H + \sqrt{[r_L + \xi_L - (r_H + \xi_H)]^2 + 4 \xi_L \xi_H}}{2} \).

Premultiplying the system by \( P^{-1} \) and noting that \( P^{-1} A = \hat{D} P^{-1} \) we have a delinked system PDEs with a common bankruptcy boundary \( \check{\delta}_B \equiv \log (\delta_B) \) and payout boundary \( t = 0 \)

\[ \hat{D} P^{-1} \mathbf{d} = P^{-1} c + \check{\mu} P^{-1} \mathbf{d}' + \frac{\sigma^2}{2} P^{-1} \mathbf{d}'' - P^{-1} \mathbf{d} \]

\[ \iff \ \hat{D} \mathbf{y} = \check{\mathbf{c}} + \check{\mathbf{y}}' + \frac{\sigma^2}{2} \mathbf{y}'' - \hat{\mathbf{y}} \]

where \( \mathbf{y} = P^{-1} \mathbf{d} \) and \( \check{\mathbf{c}} = P^{-1} c \). The rows of the system are now delinked, and we are left with two PDEs of the form

\[ \hat{r}_i \mathbf{y}_i = \check{c}_i + \check{\mu}_i \mathbf{y}_i' + \frac{\sigma^2}{2} \mathbf{y}_i'' - \hat{\mathbf{y}}_i \]

with given boundary conditions at \( t = 0 \) and \( \check{\delta} = \check{\delta}_B \), whose solutions are known from LT96. The decomposition works because the boundaries are the same across rows. The solution takes the form

\[ y_i = A_i + B_i e^{-\check{r}_i t} (1 - F_i) + C_i G_i \]

\[ F_j (\check{\delta}, t) = \sum_{i=1}^{2} e^{(\check{\delta} - \check{\delta}_B) \gamma_{ij}} N \left[ q \left( \check{\delta}, \gamma_{ij}, t \right) \right] \]

\[ G_j (\check{\delta}, t) = \sum_{i=1}^{2} e^{(\check{\delta} - \check{\delta}_B) \eta_{ij}} N \left[ q \left( \check{\delta}, \eta_{ij}, t \right) \right] \]
where
\[ q (\delta, \chi, t) = \frac{\delta_B - \delta - (\chi + a) \cdot \sigma^2 \cdot t}{\sigma \cdot \sqrt{t}} \]
and constants
\[ A_i = \frac{\hat{c}_i}{\hat{r}_i} \]
\[ B_i = \left( \hat{p}_i - \frac{\hat{c}_i}{\hat{r}_i} \right) \]
\[ C_i = \left( \hat{\alpha}_i e^{\delta_B} \frac{\delta - \hat{\chi}}{r - \mu} - \frac{\hat{c}_i}{\hat{r}_i} \right) \]
and some yet to be determined parameters \( \gamma_{ij}, \eta_{ij} \). Note that \( \lim_{t \to 0} q (\delta, \chi, t) = \lim_{t \to 0} \frac{\delta_B - \delta}{\sigma \cdot \sqrt{t}} = -\infty \) as \( \delta_B < \delta \), so \( N \left[ q (\delta, \chi, 0) \right] = 0 \) for all \( i \) and \( \delta > \delta_B \). Further note that \( \lim_{\delta \to -\infty} q (\delta, \chi, t) = -\infty \), so \( \lim_{\delta \to -\infty} N [ q (\delta, \chi, t) ] = 0 \).

Substituting the candidate solution \( y_i \) into the PDE with \( A_i = \frac{\hat{c}_i}{\hat{r}_i}, B_i = \hat{p}_i - \frac{\hat{c}_i}{\hat{r}_i}, C_i = \hat{\alpha}_i \exp (\delta_B) \frac{\delta - \hat{\chi}}{r - \mu} - \frac{\hat{c}_i}{\hat{r}_i} \), we see that
\[
\begin{align*}
&b_i e^{-\gamma_{t,i}} \left[ \hat{r}_i (1 - F_i) + \hat{\mu} F''_i + \frac{\sigma^2}{2} F'''_i - \left( \hat{r}_i (1 - F_i) + \hat{F}_i \right) \right] \\
&\quad + c_i \left[ \hat{r}_i G_i - \hat{\mu} G''_i - \frac{\sigma^2}{2} G'''_i + \hat{G}_i \right] = 0
\end{align*}
\]

\[
\iff\quad b_i e^{-\gamma_{t,i}} \left[ \hat{\mu} F'' + \frac{\sigma^2}{2} F''' - \hat{F} \right] \\
&\quad + c_i \left[ \hat{r}_i G_i - \hat{\mu} G''_i - \frac{\sigma^2}{2} G'''_i + \hat{G}_i \right] = 0
\]

We see that both \( \hat{F}_i \) and \( \hat{G}_i \) have no term \( N (\cdot) \). As \( q \) is linear in \( \delta \), we have \( q'' = 0 \) (where \( q' = \delta \) and \( \hat{q} = \hat{q}_t \)). We thus have, for \( F_i \),

\[
N \left[ q (\delta, \gamma, t) \right] \left[ \hat{\mu} \gamma + \frac{\sigma^2}{2} \gamma^2 \right] \\
+ \phi [q (v, \gamma, t)] \left[ \hat{\mu} q' + \frac{\sigma^2}{2} \left[ 2 \gamma q' - q \left( q' \right)^2 \right] - \hat{q} \right] = 0
\]

So the roots for \( F_i \) are \( \gamma_1 = 0 = -a + \alpha \) and \( \gamma_2 = -\frac{2a}{\hat{\alpha}} = -a - \alpha \) where \( a \equiv \frac{\hat{b}_2}{2} \). We see that this is independent of \( i \), that is, it is independent of what row of \( y \) we picked, as \( \hat{r}_i \) is cancelled out. Further, for \( G_i \), we have

\[
N \left[ q (v, \eta, t) \right] \left[ \hat{\mu} \eta + \frac{\sigma^2}{2} \eta^2 - \hat{r}_i \right] \\
+ \phi [q (v, \eta, t)] \left[ \hat{\mu} q' + \frac{\sigma^2}{2} \left[ 2 \eta q' - q \left( q' \right)^2 \right] - \hat{q} \right] = 0
\]

so the roots for \( G_i \) are \( \eta_{i1} = -\hat{b}_2 + \sqrt{\hat{b}_2^2 + 2 \sigma^2 \hat{r}_i} = -a + \sqrt{\alpha^2 + 2 \sigma^2 \hat{r}_i} \) and \( \eta_{i2} = -a - \sqrt{\alpha^2 + 2 \sigma^2 \hat{r}_i} \). Simply plugging in the functional form of \( q \) results in the term in square brackets in the second row to vanish.

For the boundary condition, we have

\[
y (\delta, 0) = P^{-1} 1 \cdot p = \hat{p} \]
\[
y (\delta_B, t) = P^{-1} \exp (\delta_B) \frac{\delta - \hat{\chi}}{r - \mu} \]

which defines the remaining parameters of the solution.

As a last step, we retranslate the system back into the original debt functions by premultiplying by \( P \) and noting that \( F (v, t) = F_i (v, t) = F_{-i} (v, t) \) by the symmetry of the \( \gamma \)'s, and by rewriting it in terms of \( \delta \) instead of \( \hat{\delta} \).
A.2.2 Equity

Proof of Proposition 2.

Equity has the following ODE where for notational ease we define \( m = \frac{1}{\gamma} \)

\[
rE = \exp(\delta) - (1 - \pi) c + \tilde{\mu} E' + \frac{\sigma^2}{2} E'' + m [D_H(\delta, T) - p]
\]

The term in square brackets is the cash-flow term that arises out of rollover of debt (while keeping coupon, principal and maturity stationary), a term first pointed out by LT96. We will establish the (closed-form) solution in several steps.

First, the homogenous solutions to the ODE are \( M(\delta) = e^{\kappa_1 \delta} \) and \( U(\delta) = e^{\kappa_2 \delta} \) where

\[
\frac{\sigma^2}{2} \kappa^2 + \tilde{\mu} \kappa - r = 0
\]

so that

\[
\kappa_{1/2} = -\tilde{\mu} \pm \sqrt{\tilde{\mu}^2 + 2\sigma^2 r} = -a \pm \sqrt{\tilde{\mu}^2 + 2\sigma^2 r}
\]

and \( \kappa_1 > 0 > \kappa_2 \). As the debt term \( D_H \) is bounded, to impose the condition that equity does not grow orders of magnitude faster than the unlevered value of the firm \( V = \frac{e^{\delta}}{1 - m} \) we need \( \lim_{\delta \to \infty} |K_1 e^{\kappa_1 \delta} c^{\kappa_2 \delta}| < \infty \). The only solution is then \( K_1 = 0 \). We are thus left with \( \kappa_2 \) and the coefficient \( K \) on \( e^{\kappa_2 \delta} \).

Next, let us establish the Wronskian

\[
Wr(s) = M(s) U'(s) - M'(s) U(s) = - (\kappa_1 - \kappa_2) \exp\{(\kappa_1 + \kappa_2) s\} = -\Delta \kappa \cdot M(s) U(s)
\]

Then, by the variation of coefficient solutions to linear ODEs, a technique described in most textbooks on differential equations or, e.g., in more detail in Milbradt (2012), we have

\[
g_p(x|l) = \frac{2}{\sigma^2} \int_x^l \frac{M(s) U(x) - M(x) U(s)}{Wr(s)} ds
\]

\[
g'_p(x|l) = \frac{2}{\sigma^2} \int_x^l \frac{M(s) U'(x) - M'(x) U(s)}{Wr(s)} ds
\]

\[
g''_p(x|l) = \frac{2}{\sigma^2} \int_x^l \frac{\kappa_2 M(s) U(x) - \kappa_1 M(x) U(s)}{Wr(s)} ds
\]

for an arbitrary limit \( l \in (v_B, \infty) \). Let us take \( l \to \infty \) and \( g_p(x) \equiv g_p(x|\infty) \). We see that \( g_p(x) \) and \( g'_p(x) \) (and so forth) consists of a finite sum of integrals of the form \( \int_{-\infty}^{\infty} e^{cst} N[g(s, \chi, T)] ds \) where \( cst \) is a constant.

Second, let us briefly establish two auxiliary results. First, let us note that for \( aa > 0 \) we have

\[
aa \int_{x}^{\infty} \phi(-aa \cdot s + bb) ds = \int_{-\infty}^{-aa x + bb} \phi(y) dy = N[-aa \cdot x + bb]
\]

by simple change of variables. Second, note that

\[
e^{cst \cdot x} \phi(-aa \cdot x + bb) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2} \left[(-aa \cdot x + bb)^2 - 2cst \cdot x\right]\right\}
\]

\[
= \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2} \left[\left(-aa \cdot x + bb + \frac{cst \cdot x}{aa}\right)^2 + bb^2 - \left(bb + \frac{cst \cdot x}{aa}\right)^2\right]\right\}
\]

\[
= \phi\left(-aa \cdot x + bb + \frac{cst \cdot x}{aa}\right) e^{\frac{cst \cdot x}{aa^2} \left(bb + \frac{cst \cdot x}{aa}\right)}
\]
Here, we used only with \( H \) are centered around \( \delta \). Now, we can solve the integral in question via integration by parts:

\[
\int_x^\infty e^{cst \cdot s} N[-aa \cdot s + bb] \, ds = \frac{e^{cst \cdot x}}{cst} N[-aa \cdot x + bb] + \frac{1}{cst} \int_x^\infty e^{cst \cdot s} \phi (-aa \cdot s + bb) \, ds
\]

\[
= -\frac{e^{cst \cdot x}}{cst} N[-aa \cdot x + bb] + \frac{1}{cst} \int_x^\infty \phi (-aa \cdot s + bb + \frac{cst}{aa}) \, ds \cdot e^{\frac{cst}{aa} (bb + \frac{cst}{aa})}
\]

where we used the fact that \( \lim_{x \to \infty} N[-aa \cdot x + bb] e^{cst \cdot x} = 0 \) for any constant \( cst \).

Next, note that \( D_1 (\delta, t) = \ldots + c e^{(\delta - \delta B) x} N[q(\delta, \chi, t)] + \ldots \) for some \( \chi \), so that we are essentially facing integrals

\[
\frac{2}{\sigma} \int_x^\infty e^{(\sigma - \delta_B) x} N[q(x, \chi, t)] \frac{M(s) U(x)}{W_T(s)} \, ds
\]

\[
= \frac{2}{\sigma - \Delta \kappa} e^{\kappa x} e^{-\delta_B x} \int_x^\infty e^{(\chi - \kappa_2) x} N[q(x, \chi, t)] \, ds
\]

\[
= \frac{2}{\sigma - \Delta \kappa} e^{\kappa_2 x} e^{-\delta_B x} \chi - \kappa_2
\]

\[
\times [e^{(\chi - \kappa_2) x} N[q(x, \chi, t)] + N[q(x, \kappa_2, t)] e^{(\chi - \kappa_2) \left( \delta_B \cdot \frac{1}{2} [(\kappa + a)^2 - (\chi + a)^2] \sigma^2 T \right)}]
\]

Here, we used \( cst = (\chi - \kappa_2), aa = \frac{1}{\sqrt{T}}, b = \frac{\delta_B - (\chi + a)^2}{\sigma \sqrt{T}}, q(x, \chi, t) + (\chi - \kappa) \sigma \sqrt{T} = q(x, \kappa, t) \) and the fact that

\[
(\chi - \kappa) (-) \left[ \chi + a - \frac{1}{2} (\chi - \kappa) \right] = (\chi - \kappa) (-) \left[ \frac{1}{2} \chi + \frac{1}{2} a + \frac{1}{2} \kappa + \frac{1}{2} a \right]
\]

\[
= \frac{1}{2} \left[ (\kappa + a)^2 - (\chi + a)^2 \right]
\]

where we note that the last term is independent of if we pick the larger or smaller root, as both \( \kappa \) and all possible \( \chi \) are centered around \( -a \). Lastly, we note that \( \int_x^\infty e^{(\sigma - \delta_B) x} N[q(x, \chi, t)] \frac{M(s) U(x)}{W_T(s)} \, ds \) has the same form of solution only with \( \kappa_1 \) replacing \( \kappa_2 \). Define

\[
H (x, \chi, \kappa, T) \equiv \int_x^\infty e^{(\chi - \kappa) x} N[q(x, \chi, T)] \, ds
\]

\[
= -\frac{1}{cst} \left\{ e^{cst \cdot x} N[q(x, \chi, T)] - e^{cst \cdot \delta_B} \exp \left\{ -cst \left( \chi + a - \frac{1}{2} cst \right) \sigma^2 T \right\} N[q(x, \chi, T) + cst \cdot \sigma \sqrt{T}] \right\}
\]

\[
= \frac{1}{\kappa - \chi} \left\{ e^{(\chi - \kappa) x} N[q(x, \chi, T)] - e^{(\chi - \kappa) \delta_B} e^{\frac{1}{2} [(\kappa + a)^2 - (\chi + a)^2] \sigma^2 T} N[q(x, \chi, T)] \right\}
\]

\(^{30}\)If \( cst > 0 \), a simple application of L'Hopital's rule is sufficient to establish the result:

\[
\lim_{x \to \infty} \frac{N[-aa \cdot x + b]}{e^{cst \cdot x}} = \frac{"0"}{"0"} = \lim_{x \to \infty} \frac{\phi (-aa \cdot x + b)}{-cst \cdot e^{-cst \cdot x}} = 0
\]

as \( \phi \) is of negative exponential quadratic form. However, for numerical purposes, we observe that this function can become very large before converging to zero. This observation also allows us to note that the integrals \( \int_x^\infty e^{cst \cdot x} N[-aa \cdot s + b] \, ds \) are nowhere bounded for \( x \geq 0 \), justifying our result that \( K_1 = 0 \).
The solution to the particular part for \( F \) then is

\[
g_p(x) = \frac{2}{\sigma^2} \int_x^\infty F(s) \frac{M(s) U(x) - M(x) U(s)}{Wr(s)} ds = \frac{1}{-\Delta\kappa \sigma^2} \sum_{i=1}^2 \left\{ e^{\kappa x} e^{-\gamma_i \delta_B} H(x, \gamma_i, \kappa_2, T) - e^{\kappa x} e^{-\gamma_i \delta_B} H(x, \gamma_i, \kappa_1, T) \right\}
\]

and the solution to the particular part for \( G_j \) is

\[
g_{G_j}(x) = \frac{2}{\sigma^2} \int_x^\infty G_j(s) \frac{M(s) U(x) - M(x) U(s)}{Wr(s)} ds = \frac{1}{-\Delta\kappa \sigma^2} \sum_{i=1}^2 \left\{ e^{\kappa x} e^{-\eta_i \delta_B} H(x, \eta_j, \kappa_2, T) - e^{\kappa x} e^{-\eta_i \delta_B} H(x, \eta_j, \kappa_1, T) \right\}
\]

Plugging in \( x = \delta_B \), and noting that \( q(\delta_B, \chi, t) = -(\chi + a) \sigma \sqrt{T} \), we make the important observation that

\[
e^{\kappa \delta_B} e^{-\gamma \delta_B} H(\delta_B, \chi, \kappa, T) = \frac{1}{\kappa - \chi} \left\{ N \left[ -(\chi + a) \sigma \sqrt{T} \right] - e^{\frac{1}{2} \left( (\kappa + a)^2 - (\chi + a)^2 \right) \sigma^2 T} N \left[ -(\kappa + a) \sigma \sqrt{T} \right] \right\}
\]

is independent of \( \delta_B \). We thus conclude that for any particular part \( g_p(\delta_B) \), of the form given above, and its derivative \( g'_p(\delta_B) \) are independent of \( \delta_B \) besides \( C(\delta_B) \) containing \( e^{\delta_B} \). Also note that for \( \chi = \{\gamma_1, \gamma_2\} \) we have

\[
e^{\frac{1}{2} \left( (\kappa + a)^2 - (\chi + a)^2 \right) \sigma^2 T} = e^{\sigma^2 T}
\]

and for \( \chi = \{\eta_1, \eta_2\} \) we have

\[
e^{\frac{1}{2} \left( (\kappa + a)^2 - (\eta_j + a)^2 \right) \sigma^2 T} = e^{(\kappa - \eta_j) \sigma^2}
\]

Total equity is now easily written out to be

\[
E(\delta) = K e^{\kappa_2(\delta - \delta_B)} + \frac{e^{\delta}}{r - \mu} + K_0 + g_p(\delta)
\]

\[
= K e^{\kappa_2(\delta - \delta_B)} + \frac{e^{\delta}}{r - \mu} + K_0 - m \left( P_{11} B_1 e^{-\tau_1 T} + P_{12} B_2 e^{-\tau_2 T} \right) g_F(\delta) + P_{11} mC_1(\delta B) g_{C_1}(\delta) + P_{12} mC_2(\delta B) g_{C_2}(\delta)
\]

where we scaled \( K \) by \( e^{-\kappa \delta_B} \). The constant term \( K_0 \) is

\[
K_0 = \frac{1}{r} \left\{ -(1 - \pi) c + m \left[ A_1 + A_2 + \sum_j P_{ij} B_i e^{-\tau_j T} - p \right] \right\}
\]
The constant $K$ is derived by setting

$$0 = E\left(\delta_B\right) = K + \frac{\delta_B}{r - \mu} + K_0 - m \left(\sum_j P_{ij} B_i e^{-\epsilon_j T}\right) g_F \left(\delta_B\right) + m \sum_{j=1}^{2} C_j \left(\delta_B\right) g_{C_j} \left(\delta_B\right)$$

\[\iff K \left(\delta_B\right) = - \left[\frac{\delta_B}{r - \mu} + K_0 - m \left(\sum_j P_{ij} B_i e^{-\epsilon_j T}\right) g_F \left(\delta_B\right) + m \sum_{j=1}^{2} C_j \left(\delta_B\right) g_{C_j} \left(\delta_B\right)\right]\]

The term in brackets only features linear combinations of constants independent of $\delta_B$. 

**Proof of Proposition 3.**

The optimal $\delta_B = e^{\delta_B}$ is now easily derived. Plugging in $K \left(\delta_B\right)$ into the smooth pasting condition $E' \left(\delta_B\right) = 0$, we can derive $\delta_B = e^{\delta_B}$ in closed form:

$$0 = E' \left(\delta_B\right)$$

$$= K \left(\delta_B\right) \kappa_2 + \frac{\delta_B}{r - \mu} - m \left(\sum_j P_{ij} B_i e^{-\epsilon_j T}\right) g_F \left(\delta_B\right) + m \sum_{j=1}^{2} P_{ij} C_j \left(\delta_B\right) g_{C_j} \left(\delta_B\right)$$

$$= \kappa_2 \left[ - \frac{\delta_B}{r - \mu} - K_0 + m \left(\sum_j P_{ij} B_i e^{-\epsilon_j T}\right) g_F \left(\delta_B\right) - m \sum_{j=1}^{2} P_{ij} \left(\alpha_j \frac{\delta_B}{r - \mu} - A_j\right) g_{C_j} \left(\delta_B\right) \right]$$

$$+ \frac{\delta_B}{r - \mu} - m \left(\sum_j P_{ij} B_i e^{-\epsilon_j T}\right) g_F' \left(\delta_B\right) + m \sum_{j=1}^{2} P_{ij} \left(\alpha_j \frac{\delta_B}{r - \mu} - A_j\right) g_{C_j}' \left(\delta_B\right)$$

$$= - \frac{\delta_B}{r - \mu} \left[ \kappa_2 - 1 + m \sum_{j=1}^{2} P_{ij} \alpha_j \left\{ \kappa_2 g_{C_j} \left(\delta_B\right) - g_{C_j}' \left(\delta_B\right) \right\} \right]$$

$$- \kappa_2 K_0 + m \left(\sum_j P_{ij} B_i e^{-\epsilon_j T}\right) \left\{ \kappa_2 g_F \left(\delta_B\right) - g_F' \left(\delta_B\right) \right\} + m \sum_{j=1}^{2} P_{ij} A_j \left\{ \kappa_2 g_{C_j} \left(\delta_B\right) - g_{C_j}' \left(\delta_B\right) \right\}$$

which yields

$$\delta_B = e^{\delta_B} = \left(\frac{r - \mu}{r - \mu}\right) \times \left[ \kappa_2 - 1 + m \sum_{j=1}^{2} P_{ij} \alpha_j \left\{ \kappa_2 g_{C_j} \left(\delta_B\right) - g_{C_j}' \left(\delta_B\right) \right\} \right]^{-1}$$

$$\times \left[ -\kappa_2 K_0 + m \left(\sum_j P_{ij} B_i e^{-\epsilon_j T}\right) \left\{ \kappa_2 g_F \left(\delta_B\right) - g_F' \left(\delta_B\right) \right\} + m \sum_{j=1}^{2} P_{ij} A_j \left\{ \kappa_2 g_{C_j} \left(\delta_B\right) - g_{C_j}' \left(\delta_B\right) \right\} \right]$$

where we note that the right hand side is independent of $\delta_B$ by previous results. We can simplify further by noting that each of the terms in curly brackets can be written as

$$\kappa_2 g_F \left(\delta_B\right) - g_F' \left(\delta_B\right)$$

$$= \kappa_2 \frac{2}{\sigma^2} \int_{\delta_B}^{\infty} F\left(s\right) \frac{M\left(s\right) U\left(\delta_B\right) - M\left(\delta_B\right) U\left(s\right)}{W_r\left(\delta_B\right)} ds - \frac{2}{\sigma^2} \int_{\delta_B}^{\infty} F\left(s\right) \frac{-\kappa_1 M\left(s\right) U\left(\delta_B\right) - \kappa_1 M\left(\delta_B\right) U\left(s\right)}{W_r\left(\delta_B\right)} ds$$

$$= \frac{2}{\sigma^2} \int_{\delta_B}^{\infty} F\left(s\right) \frac{\left(\kappa_1 - \kappa_2\right) M\left(\delta_B\right) U\left(s\right)}{W_r\left(\delta_B\right)} ds$$

$$= - \frac{2}{\sigma^2} \sum_{i=1}^{2} e^{(\kappa_1 - \gamma_i) \delta_B} H\left(\delta_B, \gamma_i, \kappa_1, T\right)$$

$$= - \frac{2}{\sigma^2} \sum_{i=1}^{2} \kappa_1 - \gamma_i \left\{ N \left[-\left(\gamma_i + a\right) \sigma \sqrt{T}\right] - e^{\frac{1}{2} \left(\kappa_1 + \gamma_i\right)^2 - \left(\gamma_i + a\right)^2} \sigma^2 T \right\}$$
We thus established a closed form, albeit quite complex, for the optimal \( \delta_B \).

The limit \( \lim_{T \to \infty} \delta_B \) can be easily derived by noting that the normal distributions either converge to 0 or 1, so the only difficulty remaining is the term \( e^{\frac{1}{2}[(\kappa_1 + a)^2 - (\gamma_l + a)^2] \sigma^2 T} = e^{r_H T} \). Let us establish a series of results:

First, we note that in addition to \( e^{\frac{1}{2}[(\kappa_1 + a)^2 - (\gamma_l + a)^2] \sigma^2 T} = e^{r_H T} \), we have

\[
e^{\frac{1}{2}[(\kappa_1 + a)^2 - (\gamma_l + a)^2] \sigma^2 T} = e^{(r_H - \hat{r}_j) T}
\]

and since we established that \( \hat{r}_j > r_H \) we note that this term is converging to zero.

Second, we note that

\[
\lim_{T \to \infty} \frac{\kappa_2 K_0(T)}{r - \mu} = \lim_{T \to \infty} V_B = \lim_{T \to \infty} \frac{\kappa_2 K_0(T)}{\kappa_2 - 1} = \frac{\kappa_2 (1 - \pi) c}{\kappa_2 - 1}
\]

which is the same result as in Leland (1994) once we identify (in Leland’s notation) \( x = -\kappa_2 \), so that \( \lim_{T \to \infty} V_B = \frac{(1 - \pi) c \kappa_2}{\kappa_2 - 1} \). In the infinite maturity limit, the equity holders care about the illiquidity they impose on bondholders via the valuation spread between \( H \) and \( L \) only at the beginning when issuing bonds, but since there is no rollover their default decision is not affected by bond market illiquidity for a given level of aggregate face value and coupon.

Next, let us investigate \( T \to 0 \), which essentially renders the secondary bond market completely liquid. But of course there is a large effect of \( T \to 0 \) on the bankruptcy decision of the equity holders. Using L’Hospital’s rule, we need to investigate

\[
\lim_{T \to 0} \frac{1}{T} \left[ \kappa_2 g_F'(v_B) - g_F'(v_B) \right]
\]

We see that two terms that exactly give \( \kappa_1 - \chi \) explode at the rate \( \frac{1}{T} \), so that in the limit we have

\[
\lim_{T \to 0} \frac{\kappa_2 K_0(T)}{r - \mu} = \lim_{T \to 0} V_B = \frac{\sum_{j=1}^{2} P_{0j} (B_j + A_j)}{\sum_{j=1}^{2} P_{0j} \alpha_j} = p \left[ \begin{array}{c} P_{01} \\ P_{02} \end{array} \right] \left[ \begin{array}{c} P_{11} \\ P_{12} \end{array} \right]^{-1} \left[ \begin{array}{c} x \\ \alpha \end{array} \right]
\]

If \( \alpha = \alpha_H = \alpha_L \), we are back to the L96 solution of \( V_B = \frac{p}{\pi} \).

### A.3 Proofs of Section 4

Recall that debt values are given by

\[
\begin{bmatrix}
D_H (\delta, \tau) \\
D_L (\delta, \tau)
\end{bmatrix}
= P \begin{bmatrix}
A_1 + B_1 e^{-\delta \tau} \left[ 1 - F (\delta, \tau) \right] + C_1 G_1 (\delta, \tau) \\
A_2 + B_2 e^{-\delta \tau} \left[ 1 - F (\delta, \tau) \right] + C_2 G_2 (\delta, \tau)
\end{bmatrix}
\]

\[
= P \begin{bmatrix}
A_1 \\
A_2
\end{bmatrix}
+ \left[ 1 - F (\delta, \tau) \right] P \left[ G_1 (\delta, \tau) 0 \\
0 G_2 (\delta, \tau)
\right] \left[ \begin{array}{c} B_1 \\ B_2 \end{array} \right]
\]

\[
= P \begin{bmatrix}
A_1 \\
A_2
\end{bmatrix}
+ \left[ 1 - F (\delta, \tau) \right] P \left[ G_1 (\delta, \tau) 0 \\
0 G_2 (\delta, \tau)
\right] \left[ \begin{array}{c} B_1 \\ B_2 \end{array} \right]
\]

\[
= P \begin{bmatrix}
A_1 \\
A_2
\end{bmatrix}
+ \left[ 1 - F (\delta, \tau) \right] P \left[ G_1 (\delta, \tau) 0 \\
0 G_2 (\delta, \tau)
\right] \left[ \begin{array}{c} B_1 \\ B_2 \end{array} \right]
\]

\[
= P \begin{bmatrix}
G_1 (\delta, \tau) 0 \\
0 G_2 (\delta, \tau)
\end{bmatrix}
\]
Here, by defining $a \equiv \frac{-r^2 - x}{x}$, $\gamma_1 \equiv 0$, $\gamma_2 \equiv -2a$, $\eta_{1,2} \equiv -a \pm \frac{\sqrt{a^2 - 4x + 2x^2 r}}{\sigma^2}$, and $q(\delta, \chi, t) \equiv \log(\delta_B) - \log(\delta) - (x + a) \sigma^2 t$, the constants in (7) are given by:

$$
\begin{align*}
&\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \equiv cD^{-1}P^{-1}1, \\
&\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \equiv pP^{-1}1 - cD^{-1}P^{-1}1, \\
&\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \equiv \frac{\delta_B}{\tau - \mu}P^{-1}e - cD^{-1}P^{-1}1
\end{align*}
$$

and the functions $F$ and $G$ are given by

$$
F(\delta, \tau) \equiv \sum_{i=1}^2 \left( \frac{\delta}{\delta_B} \right)^{\gamma_i} \left[ N[q(\delta, \gamma_i, \tau)], G_j(\delta, \tau) \equiv \sum_{i=1}^2 \left( \frac{\delta}{\delta_B} \right)^{\eta_{ij}} N[q(\delta, \eta_{ij}, \tau)] \right],
$$

where $N(x)$ is the cumulative distribution function for a standard normal distribution.

Define $\omega \equiv [1, -1]A = \begin{bmatrix} (r_H + \xi_H + \xi_L) \\ - (r_L + \xi_H + \xi_L) \end{bmatrix}^T$ and $S \equiv D_H - D_L = [1, -1] \begin{bmatrix} D_H \\ D_L \end{bmatrix}$. We will also write the shorthand $\sqrt{\tau}$ for $\sqrt{[(\tau + \xi) - (\tau + \lambda \beta)]^2 + 4\xi \lambda \beta}$ and note that $\hat{r}_1 - \sqrt{\tau} = \hat{r}_2 > 0$.

### A.3.1 Time-to-maturity $\tau$ derivative

**Proof of Proposition 4.**

The derivative w.r.t. $\tau$ is easily established: First, we note that $q_r(\delta, \chi, \tau) = \frac{\log(\delta_B) - \log(\delta) - (x + a) \sigma^2 \tau}{\sigma^2 \tau},$ so $\delta$ and $\delta_B$ have reversed signs. Then, we have

$$
\begin{bmatrix} D_H(\delta, \tau) \\ D_L(\delta, \tau) \end{bmatrix} = P \begin{bmatrix} -\hat{r}_1 B_1 e^{-\hat{r}_1 \tau} [1 - F(\delta, \tau)] - B_2 e^{-\hat{r}_2 \tau} F(\delta, \tau) + C_1 \hat{G}_1(\delta, \tau) \\ -\hat{r}_2 B_2 e^{-\hat{r}_2 \tau} [1 - F(\delta, \tau)] - B_2 e^{-\hat{r}_2 \tau} F(\delta, \tau) + C_2 \hat{G}_2(\delta, \tau) \end{bmatrix}
$$

and the derivatives of the auxiliary functions are

$$
\begin{align*}
\hat{F}(\delta, \tau) &= \sum_{i=1}^2 \left( \frac{\delta}{\delta_B} \right)^{\gamma_i} \left[ q[\delta, \gamma_i, \tau] q_r(\delta, \gamma_i, \tau) \right] \\
&= \phi[q(\delta, 0, \tau)] \sum_{i=1}^2 q_r(\delta, \gamma_i, \tau) \\
&= \phi[q(\delta, 0, \tau)] \log\left( \frac{\delta}{\delta_B} \right) \tau > 0 \\
\hat{G}_j(\delta, \tau) &= \sum_{i=1}^2 \left( \frac{\delta}{\delta_B} \right)^{\eta_{ij}} \left[ q[\delta, \eta_{ij}, \tau] q_r(\delta, \eta_{ij}, \tau) \right] \\
&= \phi[q(\delta, 0, \tau)] e^{-\hat{r}_j \tau} \sum_{i=1}^2 q_r(\delta, \eta_{ij}, \tau) \\
&= \phi[q(\delta, 0, \tau)] e^{-\hat{r}_j \tau} \log\left( \frac{\delta}{\delta_B} \right) \tau > 0
\end{align*}
$$

where we used

$$
\begin{align*}
\left( \frac{\delta}{\delta_B} \right)^{\gamma_i} \phi[q(\delta, \gamma_i, \tau)] &= \phi[q(\delta, 0, \tau)] \\
\left( \frac{\delta}{\delta_B} \right)^{\eta_{ij}} \phi[q(\delta, \eta_{ij}, \tau)] &= \phi[q(\delta, 0, \tau)] e^{-\hat{r}_j \tau}
\end{align*}
$$
This is easily derived:

\[
\begin{align*}
\left( \frac{\delta}{\delta_B} \right)^\gamma_{\phi \left[ q \left( \delta, \gamma, \tau \right) \right]} &= e^{-\gamma_{\left( \delta_B - \delta \right)}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} \left[ \frac{\left( \delta_B - \delta \right)^2}{\sigma^2 t} \right]} \\
&= \exp \left\{ -\gamma_{\left( \delta_B - \delta \right)} \right\} \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{\left( \delta_B - \delta \right)^2}{2\sigma^2 t} - 2 \left( \gamma_{\delta + a} + a \right) \frac{\sigma^2 t}{2\sigma^2 t} + \frac{\left( \gamma_{\delta + a} + a \right)^2 \frac{\sigma^2 t}{2\sigma^2 t}}{2\sigma^2 t} \right\} \\
&= \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{\left( \delta_B - \delta \right)^2}{2\sigma^2 t} - 2 \left( \gamma_{\delta + a} + a \right) \frac{\sigma^2 t}{2\sigma^2 t} + \frac{\left( \gamma_{\delta + a} + a \right)^2 \frac{\sigma^2 t}{2\sigma^2 t}}{2\sigma^2 t} \right\} \\
&= \phi \left[ q \left( \delta, 0, \tau \right) \right] \exp \left\{ -\frac{2\gamma_{\delta + a} \frac{\gamma^2}{2\sigma^2 t}}{2} \right\}
\end{align*}
\]

and we finally note that \( \mu \gamma + \frac{\gamma^2}{2\gamma^2} = 0 \iff \frac{\gamma^2}{\gamma^2} = 0 \iff 2\gamma_{\delta + a} + \gamma^2 = 0 \) which gives the result in conjunction with the fact that \( (\gamma_{\delta + a} + \gamma_{\delta - a}) = 0 \) as they are complementary roots centered around \(-a\). Plugging in, we have

\[
\begin{bmatrix}
D_H \left( \delta, \tau \right) \\
D_L \left( \delta, \tau \right)
\end{bmatrix}
= \mathbf{P} \begin{bmatrix}
-\tilde{r}_1 B_1 e^{-r_1 \tau} \left[ 1 - F \left( \delta, \tau \right) \right] + (C_1 - B_1) e^{-r_1 \tau} \hat{F} \left( \delta, \tau \right) \\
-\tilde{r}_2 B_2 e^{-r_2 \tau} \left[ 1 - F \left( \delta, \tau \right) \right] + (C_2 - B_2) e^{-r_2 \tau} \hat{F} \left( \delta, \tau \right)
\end{bmatrix}

= \mathbf{P} \begin{bmatrix}
e^{-r_1 \tau} & 0 \\
0 & e^{-r_2 \tau}
\end{bmatrix} \begin{bmatrix}
-\tilde{r}_1 B_1 \left[ 1 - F \left( \delta, \tau \right) \right] + (C_1 - B_1) \hat{F} \left( \delta, \tau \right) \\
-\tilde{r}_2 B_2 \left[ 1 - F \left( \delta, \tau \right) \right] + (C_2 - B_2) \hat{F} \left( \delta, \tau \right)
\end{bmatrix}

= \exp \left\{ -\mathbf{D} \tau \right\} \begin{bmatrix}
-\tilde{r}_1 B_1 \left[ 1 - F \left( \delta, \tau \right) \right] + (C_1 - B_1) \hat{F} \left( \delta, \tau \right) \\
-\tilde{r}_2 B_2 \left[ 1 - F \left( \delta, \tau \right) \right] + (C_2 - B_2) \hat{F} \left( \delta, \tau \right)
\end{bmatrix}

= \mathbf{P} \exp \left\{ -\mathbf{D} \tau \right\} \begin{bmatrix}
-\left[ 1 - F \left( \delta, \tau \right) \right] \hat{D} \begin{bmatrix}
B_1 \\
B_2
\end{bmatrix} + \hat{F} \left( \delta, \tau \right) \begin{bmatrix}
C_1 - B_1 \\
C_2 - B_2
\end{bmatrix}
\end{bmatrix}

= \exp \left\{ -\mathbf{A} \tau \right\} \begin{bmatrix}
-\left[ 1 - F \left( \delta, \tau \right) \right] \mathbf{A} \mathbf{P} \begin{bmatrix}
B_1 \\
B_2
\end{bmatrix} + \hat{F} \left( \delta, \tau \right) \mathbf{P} \begin{bmatrix}
C_1 - B_1 \\
C_2 - B_2
\end{bmatrix}
\end{bmatrix}

\]

where we used the fact that \( \mathbf{P} \exp \left\{ -\mathbf{D} \tau \right\} = \exp \left\{ -\mathbf{A} \tau \right\} \mathbf{P} \) and \( \mathbf{P} \hat{D} \mathbf{P} = \mathbf{A} \). Premultiplying by the difference vector \( [1, -1] \) and plugging in the definitions of \( \mathbf{A}, B_1, C_1 \), we have

\[
\tilde{s} \left( \delta, \tau \right) = [1, -1] \begin{bmatrix}
D_H \left( \delta, \tau \right) \\
D_L \left( \delta, \tau \right)
\end{bmatrix}

= [1, -1] \exp \left\{ -\mathbf{A} \tau \right\} \begin{bmatrix}
\frac{c - pr_H}{c - pr_L} \\
\frac{\delta_p \alpha_H - p}{\tau - p \alpha_L - p}
\end{bmatrix}

\]

Let us derive a formula for a general vector \( \begin{bmatrix} x \\ y \end{bmatrix} \):

\[
[1, -1] \exp \left\{ -\mathbf{A} \tau \right\} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{e^{r_1 \tau}}{2\sqrt{\gamma}} \times \left\{ e^{r_1 \tau - 1} \left[ x \left( r_L - r_H - \xi_L - \xi_L \right) - y \left( r_L - r_L - \xi_L - \xi_L \right) \right] + \sqrt{\left( 1 + e^{r_1 \tau} \right)} \left[ x - y \right] \right\}

= \frac{e^{r_1 \tau}}{2\sqrt{\gamma}} \times \left\{ e^{r_1 \tau} \left[ r_L, -r_H \right] \begin{bmatrix} x \\ y \end{bmatrix} - \omega \begin{bmatrix} x \\ y \end{bmatrix} \right\} + \sqrt{\left( 1 + e^{r_1 \tau} \right)} \begin{bmatrix} 1, -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}

= \frac{e^{r_1 \tau}}{2\sqrt{\gamma}} \times \left\{ e^{r_1 \tau} \left[ r_L, -r_H \right] - \omega + \sqrt{\left[ 1, -1 \right]} \begin{bmatrix} x \\ y \end{bmatrix} + 2\sqrt{\left[ 1, -1 \right]} \begin{bmatrix} x \\ y \end{bmatrix} \right\}

\]

When \( x > y \), it is clear that for \( \tau = 0 \), we have \( [1, -1] \exp \left\{ -\mathbf{A} \cdot 0 \right\} \begin{bmatrix} x \\ y \end{bmatrix} = (x - y) > 0 \). Further, if it is to hold for
any $\tau$, we need
\[
(\varepsilon^{\tau\sqrt{\cdot}} - 1) \left( \begin{bmatrix} r_L, -r_H \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} - \omega \begin{bmatrix} x \\ y \end{bmatrix} + \sqrt{\cdot} \begin{bmatrix} 1, -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right) \geq 0
\]
Our derivation of $\hat{S}$ has two terms of this form, multiplied by $[1 - F] > 0$ and $\hat{F} > 0$. To ensure positivity, this implies conditions on $p, c, r_H, r_L, \alpha_H, \alpha_L, \delta_B$ once we identify
\[
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c - pr_H \\ c - pr_L \end{bmatrix}
\]
and
\[
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{\delta_B}{r - p} \alpha_H - p \\ \frac{\delta_B}{r - p} \alpha_L - p \end{bmatrix}.
\]
Thus, we have the following two conditions for these two cases:
\[
\begin{bmatrix} c - pr_H \\ c - pr_L \end{bmatrix} : -(r_L - r_H) \left[ p \left( r_H + r_L + \xi_H + \xi_L - \sqrt{\cdot} \right) - 2c \right] > 0
\]
\[
\iff (r_L - r_H) 2 \left[ c - pr_H \right] > 0
\]
\[
\iff w_1 \equiv c - pr_H > 0
\]
\[
\begin{bmatrix} \frac{\delta_B}{r - p} \alpha_H - p \\ \frac{\delta_B}{r - p} \alpha_L - p \end{bmatrix} : V_B \left[ \alpha_L \left( r_L - r_H + \xi_H + \xi_L - \sqrt{\cdot} \right) - \alpha_H \left( r_H - r_L + \xi_H + \xi_L - \sqrt{\cdot} \right) \right] + 2p(r_H - r_L) > 0
\]
\[
\iff V_B \left[ \alpha_L \left( -2r_H + 2\hat{r}_2 \right) - \alpha_H \left( -2r_L + 2\hat{r}_2 \right) \right] + 2p(r_H - r_L) > 0
\]
\[
\iff w_2 \equiv V_B \left[ \alpha_L (\hat{r}_2 - r_H) + \alpha_H (r_L - \hat{r}_2) \right] - p(r_L - r_H) > 0
\]
where $V_B = \frac{\delta_B}{r - p}$ and we note that $r_H < \hat{r}_2 < r_L$. So we need sufficiently high $c > p\hat{r}_2$ and also sufficiently high $\alpha_L, \alpha_H$ in the face of a large discount differential $r_L - r_H$. We thus have proved the following proposition. Thus, under the sufficient conditions
\[
w_1 \equiv c - pr_H \geq 0
\]
\[
w_2 \equiv V_B \left[ \alpha_L (\hat{r}_2 - r_H) + \alpha_H (r_L - \hat{r}_2) \right] - p(r_L - r_H) \geq 0,
\]
we have $S_r (\delta, \tau) > 0$, i.e. the bid-ask spread $(1 - \beta) S(\delta, \tau)$ is larger for bonds with longer time-to-maturity. ■

If either of these conditions are not satisfied, then we can still find $\tau_{cp}$ and/or $\tau_{apr}$ such that
\[
[1, -1] \exp (-A_{\tau_{cp}}) \begin{bmatrix} c - pr_H \\ c - pr_L \end{bmatrix} = 0
\]
\[
[1, -1] \exp (-A_{\tau_{apr}}) \begin{bmatrix} V_B \alpha_H - p \\ V_B \alpha_L - p \end{bmatrix} = 0
\]
and where $\tau_{xy}$ is given by
\[
\tau_{xy} = \frac{1}{\sqrt{\cdot}} \log \left[ 1 - \frac{2\sqrt{\cdot}(x - y)}{\left( \begin{bmatrix} r_L, -r_H \end{bmatrix} - \omega - \sqrt{\cdot} \begin{bmatrix} 1, -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right) \begin{bmatrix} x \\ y \end{bmatrix} \right] \]

Then a sufficient (but of course not necessary) condition for $\hat{S} > 0$ is $\tau \leq \min \{ \tau_{cp}, \tau_{apr} \}$, where $\tau_{xy} = \infty$ if the positivity condition holds.

We conclude with the observation that
\[
\hat{S}(\delta, 0) = [1 - F(\delta, 0)] p (r_L - r_H) + \lim_{r \to 0} \hat{F}(\delta, 0) \frac{\delta_B}{r - \mu} (\alpha_H - \alpha_L)
\]
\[
= p (r_L - r_H) > 0
\]

\[A.3.2 \text{ Proof of } S' < 0 \text{ via the system of PDEs and LHS}\]

Proof of Proposition 5.
First, note that when we subtract the second line from the first line of the differential equation we have

\[
\begin{bmatrix} 1, -1 \end{bmatrix} \begin{bmatrix} r_H + \xi_H & \xi_L \\ -\xi_L & r_L + \xi_L \end{bmatrix} \begin{bmatrix} D_H \\ D_L \end{bmatrix} = \begin{bmatrix} 1, -1 \end{bmatrix} \begin{bmatrix} c \\ c \end{bmatrix} + \tilde{\mu} \delta \begin{bmatrix} D_H \\ D_L \end{bmatrix} + \frac{\sigma^2}{2} \delta^2 \begin{bmatrix} D_H \\ D_L \end{bmatrix}'' - \begin{bmatrix} D_H \\ D_L \end{bmatrix}'
\]

\[
\Leftrightarrow \omega \begin{bmatrix} D_H \\ D_L \end{bmatrix} + \dot{S} = \tilde{\mu} \dot{S} + \frac{\sigma^2}{2} S''
\]

\[
\Leftrightarrow \text{LHS} = \tilde{\mu} \dot{S} + \frac{\sigma^2}{2} S''
\]

where

\[
\omega \equiv [r_H + \xi_H + \xi_L, -(r_L + \xi_L + \xi_H)].
\]

Let us first establish a limit of \( \text{LHS}(\delta, \tau) \):

\[
\lim_{\tau \to 0} \text{LHS}(\delta, \tau) = \omega \begin{bmatrix} D_H(\delta, 0) \\ D_L(\delta, 0) \end{bmatrix} + \lim_{\tau \to 0} \dot{S}(\delta, \tau)
\]

\[
= -p(r_L - r_H) + p(r_L - r_H)
\]

\[
= 0
\]

**Outline of the proof:**

1. Show that \( \text{LHS} \) as a function of \( \tau \) only changes sign once.
2. Show, when \( \tau \) is small, that \( \text{LHS} \) increases, that is

\[
\text{LHS}(\delta, \tau) > 0
\]

3. Show that \( \text{LHS}(\delta, \infty) \geq 0 \).
4. Show that

\[
S(\delta_B, \tau) - \lim_{\delta \to \infty} S(\delta, \tau) > 0
\]

Then we are done: (1.) implies that the can at most be one local extrema. By (2.), we know that there is a local maximum in \( \text{LHS} \) in terms of \( \tau \), i.e., \( \text{LHS} \) has to go up and then down again to approach from above the value in (3.), which is zero or something positive. Finally, (4.) gives us a contradiction if ever \( S' > 0 \). First, by continuity of the expectation, we have that \( S' < 0 \) for some part of the state space \( (\delta_B, \infty) \), as otherwise the surplus couldn’t be less at \( \infty \) than at 0. Suppose now that there is an interval on which \( S' < 0 \). This means that there exist a local maximum with \( S' = 0 > S'' \). But this would imply \( \text{LHS} = \tilde{\mu} \dot{S} + \frac{\sigma^2}{2} S'' < 0 \), a contradiction. Thus, \( S' > 0 \) everywhere.

**Step 1:** Recall that

\[
\begin{bmatrix} D_H(\delta, \tau) \\ D_L(\delta, \tau) \end{bmatrix} = \exp(-A\tau) P \begin{bmatrix} -\hat{\tau}_1 B_1 [1 - F(\delta, \tau)] + (C_1 - B_1) \bar{F}(\delta, \tau) \\ -\hat{\tau}_2 B_2 [1 - F(\delta, \tau)] + (C_2 - B_2) \bar{F}(\delta, \tau) \end{bmatrix}
\]

\[
= \exp(-A\tau) P \left( -[1 - F(\delta, \tau)] D \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + \bar{F}(\delta, \tau) \begin{bmatrix} C_1 - B_1 \\ C_2 - B_2 \end{bmatrix} \right)
\]
Thus, we have
\[
\begin{aligned}
\begin{bmatrix} D_H & D_L \end{bmatrix} \cdot (\delta, \tau) &= \exp(-A \tau) (-A) \mathbf{P} \left( -[1 - F(\delta, \tau)] \mathbf{D} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + \mathcal{F}(\delta, \tau) \begin{bmatrix} C_1 - B_1 \\ C_2 - B_2 \end{bmatrix} \right) \\
&+ \exp(-A \tau) \mathbf{P} \left( \mathcal{F}(\delta, \tau) \mathbf{D} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + \mathcal{F}(\delta, \tau) \begin{bmatrix} C_1 - B_1 \\ C_2 - B_2 \end{bmatrix} \right) \\
&= \exp(-A \tau) \left( [1 - F(\delta, \tau)] A^2 \mathbf{P} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} - \mathcal{F}(\delta, \tau) \mathbf{A} \mathbf{P} \begin{bmatrix} C_1 - B_1 \\ C_2 - B_2 \end{bmatrix} \right) \\
&+ \exp(-A \tau) \left( \mathcal{F}(\delta, \tau) \mathbf{A} \mathbf{P} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + \mathcal{F}(\delta, \tau) (...) \mathbf{P} \begin{bmatrix} C_1 - B_1 \\ C_2 - B_2 \end{bmatrix} \right)
\end{aligned}
\]

where we used the fact that \( \mathbf{A} \mathbf{P} = \mathbf{P} \hat{\mathbf{D}} \) and \( \mathbf{A} \exp(-A \tau) = \mathbf{P} \hat{\mathbf{D}} \mathbf{P}^{-1} \mathbf{P} \exp(-\hat{\mathbf{D}} \tau) \mathbf{P}^{-1} = \mathbf{P} \exp(-\hat{\mathbf{D}} \tau) \hat{\mathbf{D}} \mathbf{P}^{-1} = \exp(-A \tau) \mathbf{A} \) as diagonal matrices of the same order commute.

Thus, if we can show that \( \text{LHS} > 0 \) for any \( \delta > \delta_B \) we are done. Note that

\[
\frac{\partial^2 S(\delta, \tau)}{\partial \tau^2} = \hat{S} = [1, -1] (-A) \begin{bmatrix} \frac{D_H}{D_L} (\delta, \tau) \end{bmatrix} + [1, -1] \exp(-A \tau) \left\{ -\hat{F}(\delta, \tau) \begin{bmatrix} c - p_{rH} \\ c - p_{rL} \end{bmatrix} + \hat{F}(\delta, \tau) \begin{bmatrix} \frac{\delta_B}{\sigma_{rH}} \alpha_H - p \\ \frac{\delta_B}{\sigma_{rL}} \alpha_L - p \end{bmatrix} \right\}
\]

\[
= [1, -1] \exp(-A \tau) \left\{ -A \left[ 1 - F(\delta, \tau) \begin{bmatrix} c - p_{rH} \\ c - p_{rL} \end{bmatrix} + \hat{F}(\delta, \tau) \begin{bmatrix} \frac{\delta_B}{\sigma_{rH}} \alpha_H - p \\ \frac{\delta_B}{\sigma_{rL}} \alpha_L - p \end{bmatrix} \right] - \hat{F}(\delta, \tau) \begin{bmatrix} c - p_{rH} \\ c - p_{rL} \end{bmatrix} + \hat{F}(\delta, \tau) \right\}
\]

where we used the fact that \( \mathbf{A} \exp(-A \tau) = \hat{\mathbf{D}} \mathbf{P}^{-1} \mathbf{P} \exp(-\hat{\mathbf{D}} \tau) \mathbf{P}^{-1} = \mathbf{P} \exp(-\hat{\mathbf{D}} \tau) \hat{\mathbf{D}} \mathbf{P}^{-1} = \exp(-A \tau) \mathbf{A} \) as diagonal matrices of the same order commute.

We realize that the \( \omega \begin{bmatrix} \frac{D_H}{D_L} (\delta, \tau) \end{bmatrix} \) parts cancel out in \( \text{LHS} \), and we are left with

\[
\text{LHS}(\delta, \tau) = [1, -1] \exp(-A \tau) \left\{ -\hat{F}(\delta, \tau) \begin{bmatrix} c - p_{rH} \\ c - p_{rL} \end{bmatrix} + \hat{F}(\delta, \tau) \begin{bmatrix} \frac{\delta_B}{\sigma_{rH}} \alpha_H - p \\ \frac{\delta_B}{\sigma_{rL}} \alpha_L - p \end{bmatrix} \right\}
\]

Further note that with \( \hat{F}(\delta, \tau) = \phi[q(\delta, 0, \tau), q_\tau(\delta, 0, \tau)] \frac{\log \left( \frac{\delta}{\sigma} \right)}{\sigma^{3/2}} \), \( q_\tau(\delta, 0, \tau) = \frac{\log \left( \frac{\delta}{\sigma} \right)}{\sigma^{3/2}} - \frac{a^2 \sigma \tau}{2} \), and \( \phi'(x) = -x \phi(x) \), we have

\[
\hat{F}(\delta, \tau) = \phi[q(\delta, 0, \tau), q_\tau(\delta, 0, \tau)] \log \left( \frac{\delta}{\sigma} \right) + \phi[q(\delta, 0, \tau)] \frac{\log \left( \frac{\delta}{\sigma} \right)}{\sigma^{3/2}} \left( -\frac{3}{2} \right)
\]

\[
= \hat{F}(\delta, \tau) \left[ -q(\delta, 0, \tau) q_\tau(\delta, 0, \tau) - q_\tau(\delta, 0, \tau) \frac{3}{2} \right]
\]

\[
= \hat{F}(\delta, \tau) \left[ -\log \left( \frac{\delta}{\sigma} \right) - a^2 \sigma \tau \log \left( \frac{\delta}{\sigma} \right) - a^2 \sigma \tau - \frac{3}{2} \right]
\]

\[
= \hat{F}(\delta, \tau) \left[ -\frac{\log \left( \frac{\delta}{\sigma} \right)^2}{2 \sigma^2 \tau^2} - \frac{a^2 \sigma \tau}{2} - \frac{3}{2} \right]
\]

\[
= \hat{F}(\delta, \tau) \left[ \frac{\log \left( \frac{\delta}{\sigma} \right)^2}{2 \sigma^2 \tau^2} - \frac{a^2 \sigma \tau}{2} - \frac{3}{2} \right]
\]
so that

\[ LHS(\delta, \tau) = \tilde{F}(\delta, \tau) \left[ 1, -1 \right] \exp(-A\tau) \left\{ \left( \frac{(\delta^2\tau^2)^2}{\alpha^2} - \frac{a^2\sigma^2}{2} - \frac{3}{2\tau} \right) \left[ \frac{a^2\alpha_H - p}{c - pr_L} - \left[ \frac{a^2\alpha_H - p}{c - pr_L} \right] \right] \right\} \]

Let us now write out this term in more detail. First, note that

\[ [1, -1] \exp(-A\tau) \left[ \frac{V_B \alpha_H - p}{V_B \alpha_L - p} \right] = \frac{e^{-r_1\tau}}{2\sqrt{\tau}} \times \left\{ \left( e^{r_1} - 1 \right) w_2 + 2\sqrt{V_B} (\alpha_H - \alpha_L) \right\} \]

\[ [1, -1] \exp(-A\tau) \left[ \frac{c - pr_H}{c - pr_L} \right] = \frac{e^{-r_1\tau}}{2\sqrt{\tau}} \times \left\{ \left( e^{r_1} - 1 \right) w_1 + 2\sqrt{p} (r_L - r_H) \right\} \]

Then, let \( x \equiv \log \left( \frac{V_B}{\alpha_L} \right)^{2} \in (0, \infty) \), to simplify to

\[ LHS = \tilde{F} \times \frac{e^{-r_1\tau}}{2\sqrt{\tau}} \left[ \left( \frac{x}{\alpha^2} - \frac{a^2\sigma^2}{2} - \frac{3}{2\tau} \right) \left\{ \left( e^{r_1} - 1 \right) w_2 + 2\sqrt{V_B} (\alpha_H - \alpha_L) \right\} - \left\{ \left( e^{r_1} - 1 \right) w_1 + 2\sqrt{p} (r_L - r_H) \right\} \right] \]

As \( \tilde{F} \times \frac{e^{-r_1\tau}}{2\sqrt{\tau}} > 0 \), we know that the term \([\cdot]\) determines the sign of \( LHS \). Writing it out, we have

\[ \left\{ \left( e^{r_1} - 1 \right) w_2 + 2\sqrt{V_B} (\alpha_H - \alpha_L) \right\} - \left\{ \left( e^{r_1} - 1 \right) w_1 + 2\sqrt{p} (r_L - r_H) \right\} \]

We note that \( \lim_{\tau \to 0} \frac{e^{r_1} - 1}{\tau} = \frac{e^{r_1} - 1}{\tau} = \sqrt{\tau} > 0 \), so that \( \lim_{\tau \to \infty} \frac{e^{r_1} - 1}{\tau} = \infty \). Thus, at \( \tau \) in the vicinity of 0, the sign of the term is determined by \( w_2 \). Next, when \( \tau \to \infty \), we have the sign being determined by \( -\frac{2a^2}{2} w_2 - w_1 < 0 \).

Multiplying out \( w_2 \left( e^{r_1} - 1 \right) > 0 \), and defining \( Q_1(x, \tau) = \left( \frac{x}{\alpha^2} - \frac{a^2\sigma^2}{2} - \frac{3}{2\tau} \right) \), we have

\[ Q(x, \tau) = Q_1(x, \tau) - \frac{w_1}{w_2} + \frac{2\sqrt{V_B} (\alpha_H - \alpha_L) - p (r_L - r_H)}{\left( e^{r_1} - 1 \right) w_2} \]

\[ = Q_1(x, \tau) - \frac{\left( e^{r_1} - 1 \right) w_1}{\left( e^{r_1} - 1 \right) w_2} + \frac{2\sqrt{V_B} (\alpha_H - \alpha_L) - p (r_L - r_H)}{\left( e^{r_1} - 1 \right) w_2} \]

\[ = Q_1(x, \tau) - \frac{\left( e^{r_1} - 1 \right) w_1}{\left( e^{r_1} - 1 \right) w_2} + \frac{2\sqrt{V_B} (\alpha_H - \alpha_L) - p (r_L - r_H)}{\left( e^{r_1} - 1 \right) w_2} \]

\[ = Q_2(x, \tau) - Q_2(\tau) \]

where

\[ w_3 = 2\sqrt{V_B} (\alpha_H - \alpha_L) - p (r_L - r_H) \].

Note that \( Q_1(x, \tau) \) changes sign only once. Then, we know that

\[ Q_2(\tau) = \sqrt{e^{r_1} w_1 \left( e^{r_1} - 1 \right) w_2} \left[ \left( e^{r_1} - 1 \right) w_1 - w_3 \right] \sqrt{e^{r_1} w_2} \]

\[ = \frac{w_2 w_3 \sqrt{e^{r_1}}}{\left( e^{r_1} - 1 \right) w_2} \]

Thus, if \( w_2 w_3 > 0 \), then \( Q_2(\tau) > 0 \) and we know that \( Q(x, \tau) \) is composed of a part that crosses from positive to negative as \( \tau \) increase \( (Q_1(x, \tau)) \) and of a part that is monotonically decreasing as \( \tau \) increases \( (-Q_2(\tau)) \).

**Step 2:** From the derivation above, we know that for \( \tau \) in the vicinity of 0, the sign of the \( LHS \) is determined by \( w_2 \). Next, when \( \tau \to \infty \), we have the sign being determined by \( -\frac{2a^2}{2} w_2 - w_1 < 0 \).
Step 3: Note that
\[
LHS(\delta, \infty) = \omega P \left[ \left( \frac{\delta}{\delta_B} \right)^{\eta_{12}} \begin{bmatrix} 0 \\ \eta_{22} \end{bmatrix} \right] P^{-1} P \left[ \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \right]
\]
with \(\eta_{12} < \eta_{22} < 0\), so that \(0 < X_1 = \left( \frac{\delta}{\delta_B} \right)^{\eta_{12}} \left( \frac{\delta}{\delta_B} \right)^{\eta_{22}} = X_2\). Note that for \(\delta \to \delta_B\), the LHS becomes
\[
\lim_{\delta \to \delta_B} LHS(\delta, \infty) = -\alpha_L (r_L - r_H) + (\alpha_H - \alpha_L) \frac{r_H + \xi_H + \xi_L}{2} > 0,
\]
so that we impose sufficient conditions on parameters for the above term to be positive.

First, let us note the following results:
\[
\begin{align*}
\omega P \left[ \begin{bmatrix} X_1 \\ 0 \\ X_2 \end{bmatrix} \right] P^{-1} \alpha &= \omega P \left[ \begin{bmatrix} X_1 \\ 0 \\ X_2 \end{bmatrix} \right] P^{-1} \begin{bmatrix} \alpha_H \\ \alpha_L \end{bmatrix} \\
&= \frac{\delta_B}{r - \mu} \alpha - cA^{-1} \mathbf{1}
\end{align*}
\]

Combining these results, we have that
\[
LHS(\delta, \infty) = \frac{\delta_B}{r - \mu} \left\{ \frac{\alpha_L (r_L - r_H)}{\sqrt{\gamma}} \left[ \hat{r}_1 (X_2 - X_1) - X_2 \sqrt{\gamma} \right] + \frac{(\alpha_H - \alpha_L) (X_2 - X_1)}{\sqrt{\gamma}} \left[ \frac{(r_H + \xi_H)(r_L - r_H - \xi_H) - \xi_L (r_L + \xi_L + 2\xi_H)}{2 \sqrt{\gamma}} \right] + \frac{(r_H - r_H)(X_2 - X_1)}{\sqrt{\gamma}} \right\}
\]

Because \(0 < X_1 < X_2\), the sufficient conditions for \(LHS(\delta, \infty) > 0\) are
\[
\frac{\delta_B}{r - \mu} \left( \alpha (r_L - r_H) \hat{r}_1 + (\alpha_H - \alpha_L) \frac{[(r_H + \xi_H)(r_L - r_H - \xi_H) - \xi_L (r_L + \xi_L + 2\xi_H)]}{2 \sqrt{\gamma}} \right) - \frac{c (r_L - r_H)}{\sqrt{\gamma}} > 0,
\]
\[
\frac{r_H + \xi_H + \xi_L}{2} (\alpha_H - \alpha_L) - \alpha_L (r_L - r_H) > 0.
\]

Step 4: We have
\[
S(\delta_B, \tau) = \frac{\delta_B}{r - \mu} (\alpha_H - \alpha_L)
\]
\[
\lim_{\delta \to \infty} S(\delta, \tau) = [1, -1] \left[ cA^{-1} \mathbf{1} + \exp (-A \tau) (p1 - cA^{-1} \mathbf{1}) \right]
\]
Under our assumption that \( S_\tau (\delta, \tau) > 0 \), we know that the highest \( S (\delta, \tau) \) is at \( \tau = \infty \). Noting

\[
[1, -1] A^{-1} 1 = \frac{r_L - r_H}{(r_H + \xi_H) (r_L + \xi_L) - \xi_H \xi_L}
\]

\[
[1, -1] \exp (-A \tau) 1 = \frac{(r_L - r_H) e^{-r_1 \tau} \left( e^{r_1 \tau} - 1 \right)}{\sqrt{\tau}}
\]

\[
[1, -1] \exp (-A \tau) A^{-1} 1 = -\frac{(r_L - r_H) e^{-r_1 \tau} \left[ \xi_1 \left( e^{r_1 \tau} - 1 \right) + \sqrt{\tau} \right]}{\sqrt{\tau} (r_H + \xi_H) (r_L + \xi_L) - \xi_H \xi_L}
\]

we have

\[
S (\delta_B, \tau) - \lim_{\delta \to \infty} S (\delta, \tau) > \lim_{\tau \to \infty} \left\{ S (\delta_B, \tau) - \lim_{\delta \to \infty} S (\delta, \tau) \right\} = (\alpha_H - \alpha_L) \frac{\delta_B}{r - \mu} - (r_L - r_H) \frac{c}{(r_H + \xi_H) (r_L + \xi_L) - \xi_H \xi_L} > 0
\]

for appropriate parameter restrictions.

Taken together, we established parameter restrictions that result in \( S_\delta (\delta, \tau) < 0 \).

Looser sufficiency conditions can be established for \( S_\delta (\delta, \tau) \) in the vicinity of \( \tau = 0 \) or \( \delta = \delta_B \). We omit these proofs for brevity.

### A.4 The steady-state distribution of types, trading volume

We now derive the cross-sectional (w.r.t. \( \tau \)) steady-state distribution of L types. Let \( p_H (t, \tau) \) be the proportion at time \( t \) of H types of maturity \( \tau \). Then we have

\[
\frac{\partial p_H (t, \tau)}{\partial t} - \frac{\partial p_H (t, \tau)}{\partial \tau} = \lambda p_L (t, \tau) - \xi p_H (t, \tau)
\]

as when time advances, maturity shrinks. To impose a steady-state, we note that \( \frac{\partial p_H (t, \tau)}{\partial t} = 0 \) and that \( p_H (t, T) = 1 \), i.e., at any time \( t \), due to the firm being able to issue to only H types, the proportion of H types with the longest maturity \( T \) is always 1. Further note that \( p_H + p_L = 1 \), so that in the end we have

\[
-\frac{\partial p_H (\tau)}{\partial \tau} = \lambda p_L (t, \tau) - \xi p_H (t, \tau)
\]

\[
p_H (\tau) = \frac{\lambda + \xi e^{(r - T)(\lambda + \xi)}}{\lambda + \xi}
\]

\[
p_L (\tau) = \frac{\xi}{\lambda + \xi} [1 - e^{(r - T)(\lambda + \xi)}]
\]

We of course have to adjust by the density of bonds \( \frac{1}{T} \) when looking at the steady state mass of H and L types. Then, the mass can be written as

\[
\mu_H (T) = \frac{1}{T} \int_0^T p_H (\tau) d\tau = \frac{\lambda}{\lambda + \xi} + \frac{\xi (1 - e^{-T(\lambda + \xi)})}{T(\lambda + \xi)^2}
\]

\[
\mu_L (T) = \frac{\xi}{\lambda + \xi} - \frac{\xi (1 - e^{-T(\lambda + \xi)})}{T(\lambda + \xi)^2}
\]

and we note that \( \mu'_L (T) > 0 > \mu'_H (T) \) (note that \( \frac{p_H (\tau)}{\partial \tau} \neq 0 \)), \( \lim_{T \to 0} \mu_H (T) = 1 \) and \( \lim_{T \to 0} \mu_L (T) = 0 \), as well as \( \lim_{T \to \infty} \mu_H (T) = \frac{\lambda}{\lambda + \xi} \) and \( \lim_{T \to \infty} \mu_L (T) = \frac{\xi}{\lambda + \xi} \).

Trade volume is now easily derived. It is simply the mass of agents that are in state \((L, \tau)\) times the intensity with which they meet a market maker and execute trades, \( \lambda \). Thus, trade volume (scaled by total bonds outstanding) for maturity \( \tau \) will be

\[
Volume (\tau) = \frac{\lambda}{T} p_L (\tau) = \frac{1}{T} \frac{\lambda \xi}{\lambda + \xi} [1 - e^{(r - T)(\lambda + \xi)}]
\]
A.5 Firm Value

We can calculate two measure of firm value. First, following LT96, we assume that at time 0 the firm is issuing new bonds to H type investors only with a uniform distribution of maturities on \([0, T]\). Given the results established above, the levered firm value \(TV_0(\delta_0, T; \delta_B)\) is

\[
TV_0(\delta_0, T; \delta_B) = E(\delta_0; \delta_B) + \frac{1}{T} \int_0^T D_H(\delta_0, \tau; \delta_B) \, d\tau
\]

\[
= E(\delta_0; \delta_B) + [P_{01}, P_{02}] \left( \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + \begin{bmatrix} B_1 \left( \frac{1-e^{-r_1 \tau}}{r_1} \right) - I_1(\delta_0, T) \\ B_2 \left( \frac{1-e^{-r_2 \tau}}{r_2} \right) - I_2(\delta_0, T) \end{bmatrix} \right) \right) + \begin{bmatrix} C_1 J_1(\delta_0, T) \\ C_2 J_2(\delta_0, T) \end{bmatrix}
\]

(A.1)

where

\[
I_j(\delta, T) = \frac{1}{r_j T} \left[ G_j(\delta, T) - e^{-r_j T} F(\delta, T) \right],
\]

\[
J_j(\delta, T) = \frac{1}{(\eta_j + a) \sigma \sqrt{T}} \sum_{i=1}^{2} (-1)^i \left( \frac{\delta}{\delta p} \right)^{\eta_{ij}} N[q(\delta, \eta_j, T)] q(\delta, \eta_j, T).
\]

Second, we can calculate the steady-state total value of the firm, given by the simple sum of the equity holders and creditors value functions, is thus

\[
TV_{ss}(\delta_0, T; \delta_B) = E(\delta_0; \delta_B) + \frac{1}{T} \int_0^T \left[ \lambda D_H(\delta_0, \tau; \delta_B) + \lambda D_L(\delta_0, \tau; \delta_B) \right] \, d\tau.
\]

A.6 Expected dealer profit

We can also derived expected dealer profit. First, note that expected dealer profit will be

\[
\mathbb{E} \left[ \int_0^T e^{-rt} \left( \int_0^T p_L(\tau) \lambda [D_H(\tau, \delta) - X(\tau, \delta)] \, d\tau \right) \, dt + e^{-rB} R \right]
\]

for some value \(R\), which for brevity we assume \(R = 0\). Changing the order of integration, let us concentrate on the total expected payoff to the market-maker from intermediating only \(\tau\) maturity bonds, i.e.

\[
m(\delta, \tau) = \mathbb{E} \left[ \int_0^T e^{-rt} \lambda [D_H(\tau, \delta) - X(\tau, \delta)] \, dt \right]
\]

Writing out the differential equation for \(m\), we see that it is very similar to the equity equation, i.e.

\[
m = \tilde{m} m' + \sigma^2 m'' + \lambda (1 - \beta) [1, -1] \begin{bmatrix} D_H \\ D_L \end{bmatrix}
\]

which has solution

\[
m(\delta, \delta_B|\tau) = K^m \left( \frac{\delta}{\delta_B} \right)^{\eta_2} + \frac{1}{r} \left( K^m_0 - g_F(\delta) [1, -1] P \begin{bmatrix} B_1 e^{-r_1 \tau} \\ B_2 e^{-r_2 \tau} \end{bmatrix} \right) + [1, -1] \begin{bmatrix} C_1 g_{C_1}(\delta) \\ C_2 g_{C_2}(\delta) \end{bmatrix}
\]

where \(K^m_0 = \frac{1}{r} [1, -1] P \begin{bmatrix} A_1 + B_1 e^{-r_1 \tau} \\ A_2 + B_2 e^{-r_2 \tau} \end{bmatrix} \), \(K^m\) such that \(m(\delta_B, \delta_B|\tau) = \frac{1}{\mu_L(T) T} R\) and \(g_F(\delta)\) and \(g_{C_1}(\delta)\) are given in Proposition 2 (with the appropriate \(\tau\) instead of \(T\) and with \(\lambda (1 - \beta)\) substituted for \(\lambda \beta)\). We then integrate w.r.t. to the steady state density to get

\[
\frac{1}{T} \int_0^T p_L(\tau) m(\delta, \delta_B|\tau) \, d\tau
\]
which is the expected intermediation profit of the dealer from intermediating bonds of a specific firm.

A.7 Steady state in the search market

So far, we have assumed that the intermediation intensity $\lambda$ is exogenously determined by the dealers. This assumption hinges on the fact that we assume an infinite mass of H type buyers waiting on the sideline who do not hold the asset, but in order to buy have to go through a dealer. We can relax these assumptions in several ways without substantially changing the model.

However, the two assumptions we will not relax are that (i) orders are ‘batched’ in the sense that there is no difference in intermediation intensities between different maturities $\tau \in [0, T]$, and (ii) that dealers extract all the surplus when bargaining with H type buyers. Relaxing (i) would destroy our closed form solution without providing much more insight.

The problem with relaxing (ii) is more subtle: by removing dealers and allowing direct negotiation between H and L types we essentially (except in extreme circumstances) leave some surplus beyond their own valuation to the H type buyers. If we assume that we have an infinite mass of H type buyers on the sideline, this would not change their behavior as every single one of the buyers expects never to have the opportunity to be able to buy and make a surplus. However, if there is only a finite mass of H type buyers waiting on the sideline (as we will allow later on), then (a) the firm’s pricing at issuance will be affected as it will have to entice H type buyers to participate instead of waiting and buying for a discount in the secondary market, and (b) we now have to track the value function of the H and L types who do not hold the asset. Although possible, this would result in much more complicated solutions that would not add more insight.

First, we can easily relax the model to allow for random transitioning back from the L to the H state for bondholders, say with intensity $\zeta$. Then we can simply use our current valuation formulas, but with $\lambda \beta + \zeta$ taking the place of $\lambda \beta$. Also, the impact on trade-volume can be easily handled but for brevity is not shown here.

Second, with this switching back intensity, we are now able to close the model to have 4 different finite population measures — H types with and without the bond, and L types with and without the bond. This then allows us, in the tradition of most search models, to define the meeting intensity $\lambda$ as some function of the steady-state masses of these populations, especially of the mass H types without the bond trying to buy and the mass of L types with the bond trying to sell. This would, in steady-state, result in a meeting intensity $\lambda(T)$ that is a function of the maturity structure, which however for a given $T$ would be constant. The valuation equations would simply include $\lambda(T)$ as a constant and thus would not change. The only thing that would change is the optimal maturity calculations we analyzed in Section 5.4 — here, the firm will take into account the impact of its maturity choice $T$ on the liquidity of the secondary market, and thus indirectly on the valuation of its bonds.