The (Time-Varying) Importance of Disaster Risk

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Abstract

The biggest finance questions are (1) How plausible are black swans in the stock market? (2) How calamitous could black swans be? and (3) How much of the annual premium that investors earn was compensation for black swans? We can provide reasonable answers to all three. First, under diffuse priors on the probability of disasters, the posterior probability that one-or-more true dark events have simply not yet been observed is 37%. Second, since 1983 (when CME options were introduced), protecting a self-financed stock market position with rolling one-month 15%-below-the-money put options at prevailing put prices would have cost about 1-2% drag per year. If the maximum loss in a disaster had been much worse than about –70%, even risk-neutral investors would have been better off not purchasing unprotected stock positions even over as long an interval as 1983 to 2012. Third, regardless of prior and disaster magnitude, put option prices limit the plausible disaster-risk component in the historically-observed equity premium to about 1-2% per year. Such “disaster-insurance” was almost free in some years (and stocks delivered about 6% more than T-bills in these specific years) and expensive in others. As of mid-2015, disaster insurance costs less than 0.5% per year.
The geometric rate of return of U.S. equity premium over short-term Treasury bills from July 1926 to December 2014 was 6.3% per annum (0.51% per month) with a standard deviation of about 18% (5.2%). Mehra and Prescott (1985) pointed out that this average rate of return was higher than what investors with ordinary levels of risk-aversion should have demanded. There are a number of possible reasons. The most prominent are:

1. Investors may be more reluctant to hold stocks than suggested by standard models (e.g., due to habit formation, labor income, non-tradeable wealth, market imperfections, and so on).

2. There was ordinary sampling variation. A true underlying equity premium of 3-4% would be compatible both with ordinary risk-aversion and with a 90% confidence interval in the 1926 to 2014 data. It is also compatible with the view that the 20th century was the U.S. century Jorion and Goetzmann (1999), in which investors were regularly surprised by better than expected performance, and/or a regime-shift in the equity premium (in which we sampled from a higher-mean distribution in the past than what we face now). These explanations can be viewed as wholesale shifts of mass in the historical sampling distribution relative to the prevailing population distribution.

3. There was extraordinary sampling variation: investors were concerned about large catastrophic left-tail events that just happened not to have occurred (Rietz (1988), Taleb (2001), Barro (2006)).

As in the case of ordinary sampling variation, the true equity premium would then be much lower than the historically realized equity premium. However, the spirit of the disaster-risk argument is different in that it assumes that these events were rare but sharp drops that happened simply not to have been observed yet.

My paper looks at 323 months (about 27 years) from 1983 to 2012. In this sample, the geometric equity premium was an even higher 7.2% per annum (0.58% per month) with a standard deviation of about 15% (4.3%).

My first point is that we know that the maximum disaster component in the equity premium could not have been more than 2% per annum from 1983 to 2012, because index put option protection against disasters would have cost less than this. An upper limit of

\[^1\] There is also an active macro-finance economics literature about disasters. See, e.g., Chen, Dou and Kogan (2013), Gao and Song (2013), Gourio (2012), Kelley and Jiang (forthcoming), Kozeniauskas, Orlik and Veldkamp (2014), Orlik and Veldkamp (2014), and Wachter (2013). However, none of them use put-option prices to bound disaster risk.
2%/year is economically large, but not sufficient to explain the entire equity premium. To put this *maximum* 2% disaster risk in perspective, it is less than the 2-3% standard error that is just standard-normal sampling uncertainty.

Here is a more detailed outline of the reasoning.

An investor could have purchased 323 one-month CME index puts that had strike prices of about 15% below-the-money. (I could not have gone lower than 15%, because put options with lower strike prices did not trade often enough to inspire much confidence in the accuracy of their reported prices.) Such puts could not have protected stock positions against consecutive monthly –14% rates of return, but they could have protected quite well against sporadic disasters. Fortunately, there is good reason to believe and good evidence that stock returns are not serially correlated.

A put-protected stock portfolio would have been much less affected even by the most extreme possible disaster of –100% than a naked stock position, losing at most –15% in any one month. From 1983 to 2012, if exactly one disaster had occurred, the observed 323-month geometric mean of 0.58% would have instead been about

\[
\left[ (1 + 0.58\%)^{323} \cdot (1 - 15\%) \right]^{1/324} - 1 \approx 0.53\% .
\]

One more 15% loss would have hurt but not greatly altered the geometric mean performance of the put-protected position. The reduction attributable to one crash would have been 5 bp per month. Because the probability of more than one or two terrible disasters *when not even one has in fact been observed* is low, the expected realized return contribution due to disasters would likely be modest.

The cost of disaster protection would then be the requirement that one purchase the puts every month. How expensive is this right now? For example, On April 16, 2015, the S&P 500 stood at 2,105. A put option with a strike of 1,800 expiring on May 8, 2015 (3 weeks), had an ask quote of $0.35 (bid of $0.25), for an implied volatility under 20%. This put cost $0.00017 per $1 of protection. With 17 three-week periods per annum, if the pricing of puts were to remain at this level, an investor could purchase monthly protection for about 17 \times 0.35/2, 105 \approx 0.2\% per annum. For example, on April 24, 2015, the S&P stood at 2,118, and a June 2016 option with a strike price of 35% below the money (1,375) cost $22.80, or about 1% of the index. Such a put would protect even against repeated slower and gradual stock market declines.²

²The protection cost in this example is overstated. The put would also have protected and paid off on rare occasion in the sample itself. The put would however not protect against a different new regime, in which the expected rate of return would be very low for years to come.
My second point is that disaster insurance was remarkably expensive at some times and remarkable cheap at other times (as it is in mid-2015). The maximum possible 2% disaster component in the equity premium was for an investor who would have indiscriminately purchased protection for whatever price the put sold for and ended up paying the (average) price. By switching out of the market altogether when the put option protection was very expensive, an investor could have earned 5.8% (losing some good months but also paying less upfront in many months), further limiting the maximum disaster component of the equity premium to about 1.5%. Again, the conclusion is that disaster risk could very plausibly have been a contributing but not the primary cause for the 7.2% equity premia from 1983 to 2012.

Note that I cannot distinguish between a sudden (rational or irrational) increase in disaster fear and a sudden increase in disaster probability. Thus, I can also not say whether a truly risk-neutral marginal investor should have been holding naked unprotected stock when puts were very expensive but not otherwise. I can only discuss the disaster-risk implications of strategies in which a risk-neutral investor would have shifted between put-protected stock and bills, both strategies that are immune to the association of put prices and subsequent unobserved disasters.

My third point is that disasters and compounding can interact in non-intuitive ways. For example, long-run investors can prefer to be out of the stock market even when short-term investors can prefer to be in it. For example, with a dogmatic belief that there is a 1-in-a-100 probability of a –100% disaster, and a 99-in-a-100 probability of a +10% equity premium, a one-period risk-neutral investor expects to receive about 9%. Yet, an infinite-term investor knows that she will eventually lose it all. The gambler’s ruin works against her. This is different from our accustomed perspective that the long-run favors equity investing.

My point is not that I believe this to be true or false. Instead, it is that it illustrates the intuition that the cost-benefit analysis of stock investments and of protecting put investments is not only disaster-belief dependent but also horizon-dependent. Thus, it is also the case that an infinite-lived risk-neutral investor who believes that there is a reasonable positive probability mass that there is no disaster would always prefer to participate in order not to lose out on infinite accumulation if there is no disaster: the average between infinity and –100% is better than zero. (The same does not apply to a sufficiently risk-averse investor, of course.)
My fourth point is that it is useful both for intuition and for reasonable modeling to introduce a diffuse prior over the probability that there are (as-of-yet-unsampled) disasters. Similar diffuse prior assumptions are used in practically every regression analysis—the opposite, a dogmatic prior, invalidates all classical regression coefficient estimates.

I suggest a diffuse disaster prior probability, in which investors start out with a flat equal probability of terrible disasters. Roughly speaking, for the 323 months in the sample, assume that investors started with a 1/323 probability that there were zero disasters, a 1/323 probability that there was one disaster (probability 1/323), a 1/323 probability that there were two disasters (probability 2/323), and so on. Note that the diffuse prior on unobserved events does not assume that the sampling and population distribution are identical. It does however assume that observed returns were drawn fairly from an underlying distribution, so that history allows us to learn more about hypotheticals.

This diffuse prior offers many appealing features. Given that no disaster has as-of-yet been observed over any reasonable sample, zero black swans are most likely (with a 63.2% probability). It is an important and useful feature of this distribution that there is a positive discrete probability mass that there is no disaster. (Otherwise, there is zero chance of not eventually facing it.) One black swan is unlikely, but plausible (23.3%). Five black swans are not (0.4%). Adding up, the probability that there could be one or more disasters lurking that was just not observed is 36.8%. The expected number of disasters is neither zero nor above one: It is 0.58.

Armed with any probability estimate and an investment horizon, it is possible to pin down the maximum magnitude of catastrophes. For example, for the dogmatic prior in which it is certain (100%) that there is a $p = 1/100$ chance of a disaster, the per-month rate of return for the unprotected stock holding would be

$$E(r) = 100\% \cdot [p \cdot (r_D) + (1-p) \cdot (r_G)]$$

where $r_D$ is the disaster return and $r_G$ is the non-disaster return. This is positive iff $r_D > r_G \cdot (1-1/p)$. For $p = 1/100$ and $r_G = .58\%$, the disaster magnitude must not be worse than $-57.4\%$.

For the more interesting diffuse prior, 323 months and $r_G = 0.58\%$ per month are long enough to have induced a risk-neutral investor stay in the stock market rather than exit altogether, even with a 1-month horizon. The disaster component should not have been more than 1-2% per annum. For risk-neutral 323-month investors, purchasing naked stock would have been acceptable if unrealized disasters had been no worse than about $-70\%$. 
In sum, regardless of one’s priors over catastrophes and regardless of the catastrophe magnitude, the price of put protection limits the maximum annual premium component that can be due to disaster risk. It can be measured (and it is different) at different points in time. At this moment (mid-April 2015), disaster protection costs only about 30-50 bp premium per year. With diffuse priors, unobserved disasters could not have been expected to be much worse than about –70%, or even a risk-neutral investor would have preferred not to hold riskier naked stock.

I Historical Equity Premia

My data for the excess rate of return on value-weighted stocks over Treasury bills is Ken French’s factor-premium data set.³

Table 1 shows that from 1926/07 to 2014/12, the mean rate of return \( x_t \) on his value-weighted stock market with dividends net of the 30-day Treasury yield was 65 bp per month (standard deviation of 5.4%), which translates into an annualized 7.8% arithmetic and a 6.3% geometric rate of return. Six out of every ten months had positive excess returns. There was no obvious pattern in how monthly excess return means changed over the sample, but the variance seems to have declined somewhat.

Although the distribution of excess returns was not skewed, it was highly leptokurtic relative to the normal distribution (with an excess-kurtosis of 8). There was one return of –29%, six returns less than –20%, one return of +39% and five returns above +20%.⁴

If we assume that the underlying population distribution of monthly excess returns was stable and the historical sampling distribution was identical to the population distribution, then we can redraw from the sample (with replacement) for inference about the population geometric mean rate of return, even if we abandon the normality assumption:

³Ken French constructs this excess rate of return and distributes it as “Rm-Rf” in F-F_Research_Data_Factors.zip. Working directly in excess returns is often simpler than working with separate equity and short-term Treasury rate of returns. The differences are economically unimportant. The put-option pricing is calculated with S&P 500 index futures, however.

⁴Similarly, daily excess returns also have little skewness and high kurtosis. The properties of stock returns first documented in Fama (1965) have held up remarkably well. A reasonable approximation is that the underlying returns were sampled from a Student-T with 5 degrees of freedom. The assumption that stock returns are log-normal similarly fails. Indeed, logged stock returns are not only similarly kurtotic (7), but also have negative skewness. Monthly returns also have ARCH features, with squared returns having an AR1 half-life of about two weeks. This makes little difference in the analysis here.
<table>
<thead>
<tr>
<th>Mean</th>
<th>1st</th>
<th>5th</th>
<th>Med</th>
<th>95th</th>
<th>99th</th>
<th>Positive</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.063</td>
<td>0.014</td>
<td>0.028</td>
<td>0.063</td>
<td>0.098</td>
<td>0.113</td>
<td>99.9%</td>
</tr>
</tbody>
</table>

With 99.9% confidence, the true population geometric equity premium was positive. However, we cannot not reject with 95% confidence that it was >2.8%.

In this paper, I assume that the sampled distribution is not identical to the true population distribution. A good way to think about this is that sampled returns consist of lit and dark returns, where the lit returns are the realized (sampled) monthly returns, described above, and the dark returns have not yet been seen.\(^5\) The lit returns already contain multiple stock market crashes, such as a \(-20\)% excess return in October 1929, and a \(-23\)% return in October 1987. The true geometric excess rate of return from 1926 to 2014 then was

\[
(1 + x_{\text{true}}^{t+s}) = \left(\prod_t (1 + x_t^{\text{lit}})\right) \times \left(\prod_s (1 + x_s^{\text{dark}})\right) \\
\approx (1 + 6.27\%)^{1062} \times D \approx 216.2 \times D,
\]

where the dark factor \(D\) contains the net effect of all positive and negative returns that would be needed to transform the sampling to the population distribution in order to calculate the geometric mean return. Our key concern are plausible magnitudes of \(D\).

As already mentioned in the introduction, Table 1 also shows summary statistics on the equity premium in which I am most interested in: the months in which I have enough CME data to construct good beginning-of-month put option price estimates. The equity premium was even higher in these months, reaching 0.58% per month, which translates to 7.2% per annum with a standard deviation of about 15% (4.3%).

\(^5\)Ross (2015) uses this language to point out the analogy to the well-known physics conundrum—we have evidence that dark matter exists, but we do not know exactly what it is.
II Historical Put-Protection Prices

Next, I want to assess the historical pricing of −15% put-option protection. I obtained CME index put option price data from 1983 to 2012. Matching dividends and exact intra-day time is important for accurate implied-volatility estimation, because far-below-the-money index options do not trade frequently. In many cases, the end-of-day reported below-the-money (BTM) put price were from a trade in the morning, while the end-of-day futures price were from the afternoon. Calculating an implied volatility from such an option against the afternoon futures would be misleading. (The appendix describes the data and construction in more detail.)

Figure 1 begins by plotting the time-series of prevailing historical volatility and prevailing implied volatility of at-the-money (ATM) index puts. The smoothed line in the lower figure (and in later figures) is a local regression (loess) smoother. The figure shows that the implied volatility of ATM puts was about 3% above the realized volatility over the preceding 3 months. This is as one would expect it to be in the presence of disaster risk.

Ideally, we would protect our stock position with puts that were far below-the-money—say, 30% below the money. Such options would offer good protection against −30% to −100% disasters for a very small annual yield drag. Unfortunately, such options were not actively traded. Figure 2 plots the availability of options, specifically with respect to the moneyness of the 5th lowest-strike-price and 5th-quantile-lowest strike-price option transaction on each trading day. After 1985, transactions on put options 15% below-the-money were at least occasionally available. They became reliably available after 1987. My analysis focuses on months in which solid end-of-prior-month data on put options existed.

Figure 3 plots the left side of the well-known volatility smirk. Put options that were more below-the-money traded for higher implied volatilities. This figure suggests that an average implied volatility estimate of 32.5% (15% above the prevailing volatility) would be on the high side. However, options that were much closer and farther from −15% moneyness influenced this estimate. Thus, the next figure focuses only on options that were closer to the −15% target.
To determine the typical historical pricing of –15%-BTM put options, Figure 4 shows the historical time-series of implied volatilities of puts with moneyness between –13% and –17%. Again, the assumption that index puts traded for about 32.5% implied volatility and/or 15% above the prevailing historical 3-month volatility is reasonably representative, even though there were periods in which the implied volatility was higher.

Figure 5 provides a different window on the same data. For each day, it plots the prevailing volatility on the x-axis and the average implied volatility of –13% to –17% moneyness options on the y-axis. There is a regression effect: implied volatilities suggest that actual volatility mean-reverted. Even when the annualized historical volatility was as low as 10%, put options that were 15% BTM still traded for about 20% implied volatility, i.e., 10% above prevailing. When historical volatility was high (e.g., 40%), this difference shrank to about 5%.

My final put-option price assessment is the most restrictive. It considers only options that specifically match both criteria: options that were very close to 15% below-the-money and that were very close to 30 calendar days in expiration. There were only about 1,200 put options that had strikes of between –15.5% and –14.5% below-the-money and that had between 26 and 34 calendar days left to expiration. For these options, Panel A in Table 2 shows that the implied volatility was about 0.32. The inference is similar if I restrict myself to medium-volatility time periods, i.e., periods in which the preceding 3-month observed annualized log-volatility was between 16% and 24%.

In sum, the empirical data suggests that an implied-volatility pricing of 30% (15% above the prevailing volatility) seems reasonably representative for the average cost of 15%-below-the-money 30-day index puts during the post-1985 period.
III The Maximum Disaster Component in the Equity Premium

I can now assess the actual performance of a put-protected strategy during the 27-year sample when out-of-the-money S&P 500 index put option price assessments were available on the CME.

To assess the prevailing implied volatility at the start of each month, I average the implied volatilities of index put options with a strike price of about –15% over all trading days after the 23rd of the preceding month. My first month of data is 1983/04, my last month is 2012/10, with some missing months in between. A real-world investor could have been more strategic about buying relatively cheaper options. Table 1 listed equity premium statistics in this sample, too. The excess rate of return was 0.58% geometric (0.68% arithmetic per month, 4.3% standard deviation), for a geometric annual rate of return of 7.2%. The implied volatilities ranged from about 13% to about 77%, with a median of 26.5%, a mean of 28.0%, and a standard deviation of 8%. The 30-day put price varied from $0 to $0.028, with a mean of $0.00125, a median of $0.0004, and a standard deviation of $0.0026.

However, caveats are still in order. I am still using put price estimates, not actual put purchases from a fund that undertook this protection. I am still assuming that one could have purchased 30-day put options at the beginning of the month—although index options did not expire at the end of the month, but at the end of the third week of the month. I still do not use the ask price on the option, nor do I have information on the hypothetical price impact of put purchases. (I can only report the marginal costs of protection, not the average cost of protection if a giant fund like CalPERS had attempted to buy insurance for its whole portfolio.) I am still presuming that one could have purchased the appropriate put options at an interpolated price of similar options in the last week of the preceding month. Despite these caveats, there is no reason to believe that these assumptions are misleading. On the other hand, I did not seek to take advantage of put protection at 25% below-the-money when available, nor did I attempt to be opportunistic in what below-the-money options I was using. Either could have been considerably cheaper for rolled-over protection, given the (admittedly scant) evidence in Section II.
A  Time Variation and Dynamic Strategies

Figure 6 plots the annualized cost of monthly puts and 12-month moving average. There are long periods in which disaster protection was almost free. On average, adding it all up, the cost of put-protection over the entire sample was less than 2% per year. Interestingly, there are also periods in which the put-protection was more expensive than the equity premium. For example, when the implied volatility was 50%, a one-month put with a strike price 15% BTM would have cost about 85 basis points—considerably higher than the average observed geometric equity premium.

This raises the concern that put protection may have been cheap only when there was no equity premium to be had. Although we cannot measure the frequency of (unrealized) disasters in these periods, we can measure the realized ex-post average mean return:

<table>
<thead>
<tr>
<th>Put Protection Cost (Per Annum)</th>
<th>–0.5%</th>
<th>–1%</th>
<th>1 – 2%</th>
<th>2 – 4%</th>
<th>4%–</th>
</tr>
</thead>
<tbody>
<tr>
<td>(7.2%) Realized Geometric Equity Premium</td>
<td>10.5%</td>
<td>5.3%</td>
<td>5.1%</td>
<td>18%</td>
<td>13%</td>
</tr>
<tr>
<td>N</td>
<td>161</td>
<td>203</td>
<td>55</td>
<td>35</td>
<td>30</td>
</tr>
</tbody>
</table>

The data suggests that there were times in which disaster probabilities or fears were high enough that insuring against them would have been too expensive. But there were also times in which disaster protection was trivially cheap.

Returning to my original concern, this suggests that any marginal investor could have been in the market, disaster-protected, earning a premium of 5-10% when put protection was cheap, and exiting the market when it was too expensive. This informs us about reasonable equity premia in months in which naked disaster protection would have been affordable. There are however also months in which disaster protection was so expensive that we can say little about what a reasonable risk-tolerant investor should have done.

We will consider such dynamic strategies that protect against disasters only when the put is cheap. The principle of exiting the stock market altogether when put protection became too expensive is sound, and did not require a lot of sophistication. We can determine the disaster component from the difference in return between protected and unprotected stock:

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6Future work modeling disasters from distributional observations (Orlik and Veldkamp (2014)) could see whether it is the implied disaster probabilities that are priced.

7One could also consider strategies that protect the stock only if puts are cheap and remain in stock otherwise. Such strategies are still safer than the always-in-stock strategy. For example, an investment strategy that protected against disasters only when the put’s implied-vol price was less than 0.2 and remained unprotected in the market was not only safer, but would also have yielded average rates of return of just
Exit Market+Put at IVOL | 30% | 35% | 40% | 45% | 50% | ≥ 55% | Never
---|---|---|---|---|---|---|---
(7.2%) Average Geometric Return | 3.0% | 3.9% | 5.2% | 4.7% | 5.8% | 5.6% | 5.1%

Given that a dynamic strategy that exited the market at 50% would have yielded 5.8% per annum (rather than 7.2% in the same time interval), a 1.5% maximum disaster risk premium is defensible if transaction costs were not too high.

B Mid-2015

My main data set above ended in October 2012. Thus, I could not include the prevailing cost of protection in the figures. Some basic quotes, available on common option pricing websites, can illustrate it:

- On April 16, 2015, the S&P 500 stood at 2,105. A put option with a strike of 1,800 expiring on May 8, 2015 (3 weeks), had an ask quote of $0.35 (bid of $0.25), for an implied volatility under 20%. This put cost $0.00017 per $1 of protection. With 17 three-week periods per annum, the prevailing put price implies a cost drag of under 0.5% per annum for this protection.

- On April 20, 2015, the S&P 500 stood at 2,086.20. A put option with a strike of 1,600 and expiring on July 15, 2015 (85 days), had a last transaction of $0.70, for an implied volatility of 33%. Rolling over 85-day contracts 4.3 times covers the year. Thus, the cost of protection at a strike of 67% of the stock was $3 per year, about 0.15% per annum.

In mid-2015, disaster protection is very cheap.

about 6%. Unfortunately, my key assumption that disasters were iid draws would no longer be reasonable: it could be that disaster probabilities were higher when the implied volatilities were higher. Therefore, a dynamic put exit strategy cannot deliver a convincing bound. Until we start observing some disaster events, we cannot assess whether dark-event disasters are more frequent when put options are priced at higher than 30% volatility. Thus, we cannot determine whether 1% or 2% drag is more plausible.
IV A Diffuse-Prior Probability Distribution of Dark Disasters

Consideration about disasters can be enhanced if we can obtain reasonable magnitude assessments. Should we think of disasters as 1-in-a-1000 probability events or as 1-in-a-million probability events? Is it sensible to assume an epsilon probability of a –100% disaster, so that all long-run investors eventually lose all? How could the existing historical data reasonably influence our assessments, if at all?\(^8\)

A Conditional Probability of Observing A Disaster

Consider a discretized form of the population distribution, with \(T\) equally likely probability areas at different locations (specific equity premia magnitudes). To help with the intuition, here are drawings of hypothetical population distributions, in which each slice has the same probability mass:

Within each area, the location of the mass can be indeterminate. The entire probability mass in the left-most segment could actually be on one discrete outlier—say, \(-\infty\). Now sample \(T\) times to assess the distribution of the population. As \(T\) increases, the expected minimum draw value decreases and the probability mass within each segment decreases.

Our interest is the probability of having sampled the single left-most range in \(T\) draws. Each segment has a probability of \(p = 1/T\). Say we work with 10 draws. If we knew a

\(^8\)It may be preferably without reference to a distribution which already strongly assumes a left tail. For example, in Orlik and Veldkamp (2014), even small local shifts near the mean can easily have large global consequences for the left tail. This is because it assumes a particular global functional distributional form.
priori that the probability that there was exactly one \( \frac{1}{T} \) mass to the left of \(-1.3\), we would observe it with

\[
\text{prob} \left[ \text{zero (specific/dark) events are sampled in } T \text{ draws} \right] = 1 - \left(1 - \frac{K}{T}\right)^T \approx 1 - (1 - 1/10)^{10} \approx 65\% .
\]

If we knew a priori that there was no probability mass left of \(-1.3\), our

\[
\text{prob} \left[ \text{zero (specific/dark) events are sampled in } T \text{ draws} \right] = 1 - \left(1 - 0/T\right)^T = 1 .
\]

If we thought the probability mass below \(-1.3\) was \(2/10\) (or, analogously, if instead of the \(-1.3\) or below range, we looked at the \(-0.9\) or below range), we would expect to sample it

\[
\text{prob} \left[ \text{zero (specific/dark) events are sampled in } T \text{ draws} \right] = 1 - \left(1 - 2/T\right)^T \approx 1 - (1 - 1/10)^{10} \approx 89\% .
\]

It turns out that these probabilities are roughly invariant to the number of samples drawn because \( (1 - K/T)^T \approx e^{-K} \) for \( T/K \gg 5 \). Given whatever (lowest) realizations were seen in the sample so far, the probability that there is \( K = 1 \) more probability lump (to the left of this lowest realization) that was not drawn is the same, regardless of whether ten or a ten-thousand realizations have been observed:

\[
\binom{10}{0} \cdot \left(\frac{1}{10}\right)^0 \cdot \left(1 - \frac{1}{10}\right)^{10} \approx \binom{100}{0} \cdot \left(\frac{1}{100}\right)^0 \cdot \left(1 - \frac{1}{100}\right)^{100} = (1 - 1/T)^T \approx e^{-1} \approx 36.8\% .
\]

If we had seen ten draws, and there are ten possible (equally likely) population areas we could have drawn from, then the probability of not having seen any particular one interval (and specifically the worst) would be 36.8\%. If we had one-hundred draws, and we view the population distribution as containing one-hundred possible equally likely population areas we could have drawn from, then the probability of not having seen any particular one interval (and specifically the worst) would be 36.8\%. It is somewhat counterintuitive that this probability does not depend on \( T \), i.e., on whether 10 samples or 100 samples have been drawn, but it is a direct consequence of fair sampling.

In contrast to the usual binomial-formula application, \( p \equiv K/T \) decreases with the number of samples drawn. This is because the number of sample draws \( T \) also inform us about \( p \). Even as we obtain more and more sample draws, we are interested not in the
probability of not having seen the lowest k% quantile realizations, but in the probability of not having seen the lowest K realizations.

We can expand \((1 - i/T)^T \approx e^{-i}\) into a simple table,

<table>
<thead>
<tr>
<th>Number of (Dark) Events i in a T-Discretized Population</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>prob that the specific dark event is not sampled in T draws, (p(ND</td>
<td>p_i = i/T))</td>
<td>100.0%</td>
<td>36.8%</td>
<td>13.5%</td>
<td>5.0%</td>
<td>1.8%</td>
</tr>
</tbody>
</table>

As already noted, the number of dark events in the population can be mapped into a probability, given the number of draws. Consider \(T = 100\). If \(K = 0\), there is no area to the left of the drawn sample. In this case, we could never have seen anything worse. If \(K = 2\), two specific bins (presumably the two most left ones), containing probability mass of \(2/100\), would not be seen in 13.5% of histories.

### B  A Diffuse Prior Distribution

Next, we need to assume a reasonable prior about the frequency of disasters in the population before data becomes available. Here opinions can play a role and my choice of diffuse prior is controversial.

One prior would be to assume certainty, so that there would be no learning from data. For example, if your prior is that there is an 0.0001% probability of a \(-\infty\) disaster, then no amount of data can alter your probability assessment of future disasters.

Another prior would be to assume that the full true distribution is of a particular shape. For example, if your prior is that the full true distribution is log-normal with only a mean and variance to update (and no probability to the left of the main distribution), then the data will alter the probability assessment of future disasters only to the extent that data shifts the mean and variance of the inferred population distribution in a way that shifts a certain amount of probability mass left of the cumulative distribution function.
The diffuse-prior approach assumes something different. It considers that the number of left-tailed disasters that have not yet been observed is equally likely: it is equally likely that there are zero disasters, one disaster, two disasters, and so on.\(^9\)

Prior on \(p_i = 1/T\)

This can be translated into a prior of \(1/T\) that the probability of a disaster is 0 per period, a prior of \(1/T\) that the probability of a disaster is \(1/T\) per period, a prior of \(1/T\) that the probability of a disaster is \(2/T\) per period, up to a \(1/T\) probability that the probability of a disaster is 1 per period.

C The Posterior Distribution Given Historical Data

Inverting this probability by Bayes rule gives the probability that there are \(i\) dark events lurking in the population. The marginal and cumulative probability density functions are

\[
\text{Posterior on } D: \quad f(i) = \frac{e^{-1}}{e^{i+1}}, \quad F(i) = 1 - e^{-(1+i)} \quad \forall i \in \{0, 1, \ldots\} \quad (1)
\]

where \(D\) is the number of disasters in the population from which \(T\) samples were drawn. Specifically, given a draw of any number of samples from a population, the probability that there is a left-tailed disaster that has not yet been observed is

<table>
<thead>
<tr>
<th>Number of (D) Dark Events (in (T) Draws From Full Population), Worse Than Worst Observed</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Probability</td>
<td>63.2%</td>
<td>23.3%</td>
<td>8.6%</td>
<td>3.1%</td>
<td>1.2%</td>
<td>0.4%</td>
</tr>
</tbody>
</table>

Again, these probabilities have assumed drawing from a stable underlying distributions, fair sampling, and my diffuse priors over unobserved dark events.\(^{10}\)

In normal representative sampling theory, one relies on the fact that the sampled distribution is the maximum-likelihood estimate of the population distribution. This is the case here, too. With this \textit{a priori} imposed diffuse distributional assumptions, the probability

\(^9\)Note that this diffuse prior specifically prohibits “certainty of knowledge,” e.g., of a complete log-normal distribution. In this case, the probability of a \(-1\) outcome is assumed to be zero. Instead, the thought experiment here (the prior) is that there is an equal prior probability of a log-normal distribution, an equal prior probability of a log-normal distribution plus one left-tail outcome (say, \(-1\)), an equal prior probability of a normal distribution plus two left-tail outcomes, and so on. We need to learn about the distribution from draws. Assuming a distribution without tail-risk eliminates learning about disasters in the distribution.

\(^{10}\)I assume a discrete distribution. It makes no sense to generalize this to a uniform continuous prior on the probability, because there needs to be a probability mass on zero disasters.
that we have indeed sampled any specific equally likely (e.g., the darkest) population mass in our sample is about 63.2%, which is about 2.7 times as high as the 23.3% probability that a specific (one dark) event has not yet observed.

The table shows that the diffuse prior offers some appealing features. It is outright implausible to assume that five or more specific dark returns (on the left) have not yet been sampled.\textsuperscript{11} But it is quite plausible that there are some such dark unobserved events. There is about a 2-in-3 chance that there was no dark rate of return lurking that we just did not observe (i.e., that we have already sampled from a particular probability mass, such as the one most left), and a 1-in-3 chance that there were. The probability of one or more dark events is \(1 - f(0) \approx 36.8\%\). The expected number of dark events that have not been observed is not 0, but about \(\sum f(i) i \approx 0.58\).

V Long-Term vs. Short-Term Investing

Of course, our concern is not that there are a few unsampled dark realizations in the center of the population distribution. Instead, our concern is that there are a few large-magnitude left-tail realizations that happened not to have been observed yet. It is easiest to assume that our concern is with dark returns of just one specific magnitude, \(r_\text{D}\). (If the magnitudes of dark returns are heterogeneous, most of the discussion still applies to the geometric mean of these dark returns.) Most bounds emerge if we consider the worst case, an unrecoverable rate of return of \(r_\text{D} = (-1)\).\textsuperscript{12}

Interestingly, when there is a disaster probability, even risk-neutral investors cannot ignore their investment horizon. That is, an investor who plans to be one year in the market would have a different tradeoff than an investor who plans to be in the market for ten years, even if she can exit after one year, too. In the presence of disasters, the gambler’s ruin and compounding together can work such that long-term investors prefer bills and short-term investors prefer stock.

For example, if your prior is such that you knew that a disaster had a dogmatic positive probability (100%), say a 1/323 probability of a \(r_\text{D} = -1\) disaster and a 322/323 probability of \(r_\text{G} = 0.0058\) for one month, then a single-month naked-stock investment would have an expected rate of return of \(100\% \cdot [(322/323) \cdot 0.0058 + 1/323 \cdot (-1)] \approx 0.27\%\) over

\textsuperscript{11}In the context of swans, each should be considered to represent a color of a swan not yet observed—a yellow swan, a green swan, a blue swan, a red swan, and a black swan. Representative sampling in this case means that it is not applicable to finding an entire subspecies of black swans.

\textsuperscript{12}This “worst admissible magnitude” analysis is akin to Hansen and Jagannathan (1991) bounds.
this one month. With a positive expected rate of return, if you are sufficiently risk-tolerant, you would prefer to be in the stock market. However, as \( T \) increases, the probability of falling into ruin quickly increases from \( 1/323 \) to 1. Thus, with a dogmatic prior, a short-run risk-tolerant investor may prefer to be in the stock market, while a long-run risk-tolerant investor may not.

### VI Magnitudes of Unsampled Events

This investment-horizon dependence issue applies to diffuse priors, too. The tradeoffs for a long-term investor are different from those of a short-term investor. Consequently, we can only answer the question of what the maximum disaster magnitude can be to keep a risk-neutral investor in the market within the context of a specific investment-horizon. We assume our diffuse prior, in which \( f(i, Th) \approx f(i) \) on \( Th \) historical data points, now for a fixed \( Tf \) future periods of investment. The posterior expected rate of return is

\[
E(r) = \sum_{i=0}^{Th} f(i[, Th]) \cdot [p(i, Tf) \cdot r_D(Tf) + [1 - p(i, Tf)] \cdot r_G(Tf)] .
\]

where \( p(i, Tf) \) is the probability of drawing \( i \) disasters within \( Tf \) investment periods. For a one-month investment, \( p(i, Tf) = 1/Th \). For an infinitely-lived investment, \( p(i, Tf) = 1 \). Note that both \( r_D \) and \( r_G \) also depend on \( Tf \), unless they are –1 or 0. Consider a case in which we have 323 months of historical data, the no-disaster return is \( r_G(Tf) = [(1 + r_G)^{Tf} - 1] \approx 1.0058^{Tf} - 1 \) and the disaster is worst case, \( r_B(Tf) = -1 \).

- The one-month investment expects to yield

\[
E(r) \approx 0.632 \cdot 0.58\% \\
+ 0.232 \cdot \left[ \frac{322}{323} \cdot 0.58\% + \frac{1}{323} \cdot (-1) \right] \\
+ 0.086 \cdot \left[ \frac{321}{323} \cdot 0.58\% + \frac{2}{323} \cdot (-1) \right] \\
+ ... \approx 0.4\% .
\]
• The infinitely-lived investment will increase indefinitely if there is no disaster possible \( (p_G(1) = 0) \). If there is a disaster possible \( (p_G(1) > 0) \), then we will eventually draw it and lose it all. Thus,

\[
E(r) \approx f(0) \cdot +[1 - f(0)] \cdot (-1) \approx 0.632 \cdot [1 + r_G]^{-1} + 0.368 \cdot (-1) \to \infty.
\]

This shows that, for our diffuse priors, the gambler’s ruin does not apply. The probability of zero disasters is positive, and the expected geometric rate of return grows with the investment horizon. For a risk-neutral long-term investor, as long as the compound rate of return \( r_G \) is such that she expects to earn more than 58% (e.g., 1.5% per annum for a 30-year investor), even a worst-case –100% possibility will not deter her from market participation. (Of course, a risk-averse investor may instead decline.)

A The 1982 To 2012 Sample

We can assess the expected rate of return as a function of the horizon for an investor who had planned to remain in the market from 1982 to 2012. In Figure 7, we consider the historically observed 323 months (with CME put option pricing data BTM), and ask what an investor with such an investment horizon would have preferred. This sets not only the historical sample \( (Th) \) to 323 months, but also the investment horizon \( (Tf) \).

[Insert Figure 7 here: Dark-Return Magnitude, Safer Strategies, and Equity Premia: ]

Figure 7 plots the performance of different investment strategies in the 27-year sample. We resample the observed no-disaster rates of return (together with historical beginning-of-month put pricing), and “inject” a disaster of a given magnitude (x axis), according to the diffuse posterior in random months. That is, we assume we started out with a .632 probability that there was no possible disaster, a .232 probability that we have a \( p = \frac{1}{323} \) disaster probability each month, an .086 probability that we have a \( p = \frac{2}{323} \) disaster probability, and so on. On the y-axis, we plot the expected net equity return if the investment were for 323 months. The blue line draws a resampled naked equity premium without disasters, which would have accumulated to about $8.80 with random resampling.\(^{13}\) The true expected rate of return in the naked-stock position is determined

\(^{13}\)The expected compound simulated expected payoff is higher in the simulation in the data. This is because of compounding with resampled monthly equity premium draws. The effect is analogous to the difference between the arithmetic and the geometric mean rate of return.
by the disaster probability assessment (here the diffuse prior) and the forward-looking investment horizon (here 323 months). This creates the black solid line. For example, if the unknown disaster were a 1-month realization of –80%, then the true expected payoff would have been about $6, or $2.80 less than the $8.80 resampled return.

The red line shows what expected rate of return we would have calculated if one disaster of magnitude $r_D$ had occurred. The graph shows that if one –85% disaster had occurred, we would have calculated a zero return. Not shown in the graph, we would probably be confused even if we had seen a disaster much less bad than –85%; We would then fail to reject the null hypothesis of risk-neutrality, even if the disaster was “only” –50%. Disaster-risk premium models operate in a very narrow band between an ability to lower the equity premium from the historically observed seemingly large geometric 6-7% per annum to the more model-pleasing 3-4% per annum on the one hand, and a non-rejectable zero equity premium on the other. This is because if no disaster is observed, the return seems too high. But if just one disaster is observed seems too low. It can then no longer be distinguished from zero at conventional statistical significance levels. If ever observed, the FAJ would then likely publish papers in which the tests will be designed to test whether the world is truly risk-neutral or not.

The figure illustrates nicely why the curvature of the true equity premium (which here is determined by the diffuse prior) does not affect the disaster component calculation of the equity premium: the component is the distance between the put-protected and the naked stock positions. The static put-protected portfolio would have ended up with about $5.60 rather than $8.80, for a geometric drag of about 1.4% per annum. The dynamic put-protected portfolio, exiting both the stock and the put if the implied-volatility pricing of the put was above 50%/annum, would have ended up with about $6.50, for a geometric drag of about 0.8% per annum. Regardless of where on the x-axis the average rate of return on the true naked equity premium intersects the put-protected equity premium, the distance between the lines is the same.

The point of intersection only fixes the maximum disaster magnitude. If an investor had protected the stock with dynamic puts, the maximum disaster could not have been worse than about –65%. If it had been worse, then even a risk-neutral 323-month investor would have been better off protecting the stock with puts instead of holding naked stock.\footnote{An alternative way to calculate this is to compare the expected rate of return on the put (probability that a put will pay off in a disaster times the disaster magnitude) against its monthly cost of rolling over.}
VII  Conclusion

Historically, far-below-the-money put options were cheap enough that dark returns cannot explain more than a maximum 1-2% component in the historically observed equity premium (of 6-7% per annum above T-bills, geometric). This was a non-trivial amount, but it was also not as large as often presumed. The cost of protection was time-varying. As of mid-2015, the cost of disaster isolation was under 0.5% per annum. At other times, it could have been more than 6% per annum itself.

The assumption of a diffuse prior over disasters and that the historical sample was obtained by representative and fair draws from a stable underlying population distribution allows modeling disasters in an intelligent fashion, even in the absence of observed disasters. Although zero disasters is the maximum-likelihood scenario, given our history, with diffuse priors, there is still a 37% probability that there are black swans, i.e., that there were one or more unobserved dark disaster events in the data that happen never to have been observed.

Why have dark swans have received so much attention, both in the popular and in the academic press? One answer is time variation. Disaster fears and probabilities are evidently important at some times—though not at the moment. A more cynical answer is that disasters that do not occur are often viewed as too difficult to measure, giving authors a lot of freedom in their treatment. One can explain a lot of phenomena by assuming that investors fear real or imaginary disasters one day but not the next day. Who can say ex-post whether investors’ fears were rational or not? Seemingly impossible to reject conclusively, the deus-ex-machina of disasters can easily become near-theology—the divine can explain the mysteries that we cannot understand by other means. (Of course, there may well be the divine—or there may not.) Price data on far-below-the-money options to measure disaster insurance can help impose more discipline. It is true that we can still not observe (many) disasters and thus the disaster frequency, and so we cannot disentangle time-varying fear of disasters from time-varying disaster probabilities. But we can use below-the-money option price data to see if disaster models can explain at least some of the time-variation in the price of far-below-the-money puts. Disaster models that cannot explain the cost of existing disaster insurance seem possible, but not plausible.

In some periods, resorting to explanations based on divine intervention is not needed: disaster protection is so cheap that disasters cannot possibly be an important issue. In other periods, either the fear of or the frequency of disasters must have increased. I leave the reader with the deeper question: if disasters are still important, then why is disaster insurance so cheap today?
References


### Table 1: Averages and Standard Deviations for the Equity Premium

<table>
<thead>
<tr>
<th></th>
<th>Full Sample</th>
<th>Last Half</th>
<th>Last Quarter</th>
<th>Last 10 years</th>
<th>w/CME data</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean Monthly Returns</td>
<td>0.65%</td>
<td>0.57%</td>
<td>0.65%</td>
<td>0.63%</td>
<td>0.67%</td>
</tr>
<tr>
<td>Geo-Mean Monthly Returns</td>
<td>0.51%</td>
<td>0.46%</td>
<td>0.56%</td>
<td>0.56%</td>
<td>0.58%</td>
</tr>
<tr>
<td>Mean Monthly Log-Returns</td>
<td>0.51%</td>
<td>0.46%</td>
<td>0.55%</td>
<td>0.55%</td>
<td>0.58%</td>
</tr>
<tr>
<td>Std. Dev. (either)</td>
<td>5.4%</td>
<td>4.6%</td>
<td>4.4%</td>
<td>4.4%</td>
<td>4.3%</td>
</tr>
<tr>
<td>N</td>
<td>1,062</td>
<td>532</td>
<td>267</td>
<td>121</td>
<td>323</td>
</tr>
</tbody>
</table>

**Explanations:** The data is Rm-Rf from Ken French’s factor premium data set. The “w/CME data” column lists month used only months in which I could calculate an implied volatility for put options that were about 15% below-the-money.
**Figure 1: Historical and Implied ATM Log-Volatility**

Explanations: The black line is the historically realized log-volatility of the excess return over the preceding 3 months (64 trading days). The red lines are the average implied volatilities obtained from ATM puts, the blue lines from ATM calls. The lower figure plots the difference between the implied volatilities and the historically prevailing volatilities.

Interpretation: Implied volatilities of ATM options are about 3% above their historical equivalents.
Figure 2: Availability of BTM Index Options

Explanations: These lines plot the 5th-lowest and 5th-quantile-lowest option moneyness, respectively. The blue line are calls, the gray lines are puts.

Interpretation: Put options that were about 15% below-the-money were irregularly available after 1983 and became generally available around 1988. Call options were generally not available that far below-the-money.
**Figure 3: Implied Volatility Smile of Index Put Options**

Explanations: The x-axis is the moneyness, the y-axis is the implied B-S volatility. Unlike other figures, in this figure, each option (and not each transaction day) was one observation. The lower figure subtracts the prevailing 3-months historical volatility from the B-S implied volatility.

Interpretation: This is the left side of the volatility smirk. 15% BTM puts were priced at about 30% implied volatility.
Figure 4: Implied Volatilities of Index Put Options 15% Below-the-Money

Explanations: This is the time-series of average daily volatility for all index puts with moneyness between −13% and −17%.

Interpretation: There was strong time variation in put costs, both high-frequency and low-frequency.
**Figure 5:** Implied Volatilities of Index Put Options 15% Below-the-Money vs. Historical Volatility

*Explanations:* This is the average daily volatility for all index puts with moneyness between −13% and −17%, plotted against the prevailing historical volatility.

*Interpretation:* Implied Volatilities Indicate Volatility Mean-Reversion.
### Table 2: Most-Applicable Monthly –15% Put Options

**Panel A:** 26-34 Days to Expiration and –15.5% to –14.5% Strike Price

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Sd</th>
<th>25th</th>
<th>Med</th>
<th>75th</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annualized 3-Month Prevailing Log-Vol</td>
<td>0.19</td>
<td>0.09</td>
<td>0.12</td>
<td>0.17</td>
<td>0.22</td>
</tr>
<tr>
<td>Option Implied Log-Volatility</td>
<td>0.33</td>
<td>0.07</td>
<td>0.28</td>
<td>0.32</td>
<td>0.36</td>
</tr>
<tr>
<td>Difference</td>
<td>0.14</td>
<td>0.06</td>
<td>0.12</td>
<td>0.15</td>
<td>0.17</td>
</tr>
<tr>
<td>Date</td>
<td>1997</td>
<td>2000</td>
<td>2006</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Panel B:** 26-34 Days to Expiration and –15.5% to –14.5% Strike Price, Exclude Outliers

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Sd</th>
<th>25th</th>
<th>Med</th>
<th>75th</th>
</tr>
</thead>
<tbody>
<tr>
<td>Annualized 3-Month Prevailing Log-Vol</td>
<td>0.19</td>
<td>0.02</td>
<td>0.17</td>
<td>0.19</td>
<td>0.21</td>
</tr>
<tr>
<td>Option Implied Log-Volatility</td>
<td>0.34</td>
<td>0.05</td>
<td>0.31</td>
<td>0.33</td>
<td>0.36</td>
</tr>
<tr>
<td>Difference</td>
<td>0.15</td>
<td>0.05</td>
<td>0.11</td>
<td>0.15</td>
<td>0.18</td>
</tr>
<tr>
<td>Date</td>
<td>1999</td>
<td>2000</td>
<td>2007</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Figure 6: Annual Average Cost of Monthly –15% Put Protection

Explanations: The black line are the costs of monthly –15% put protection, quoted in annualized terms. The blue line is the 12-month rolling sum.

Interpretation: There were long stretches when disaster protection was almost free, but also long stretches when it was fairly expensive.
Figure 7: Dark Return Magnitude, Safer Strategies, and Equity Premia

Explanations: Given diffuse priors of the presence of any number of dark events (from zero up), this shows the observed and true equity premium as a function of the magnitude of this dark return, and adds the performance of the static put-protected stock positions. The put is assumed to have had a price based on the prevailing implied volatility in the last week of the previous month.

Interpretation: If the dark-event return had been less than −65%, the true expected rate of return on stocks would have been lower than the expected rate of return on the put-protected stock.
A Appendix: Option Pricing Data

This appendix describes the construction of the option pricing data. The original source data were 42 million transaction of quotes of S&P500\(^{15}\) index futures and 1 million put-option transactions from inception in 1983 to December 2012. The CME data is difficult to work with, partly because the CME sometimes included and sometimes did not include quote data. The CME data also contained a good number of other obviously incorrect transactions. In the end, there were about 400 option days per trading day. I excluded options closer than 3 days to expiration (when closing-out transactions could create some anomalous prices) and options with more than 200 days to expiration (rarely traded). The median number of days to expiration in the remaining raw data was 35 days, the mean was 45 days. I also excluded all options that traded before 8am or after 4pm. This left 822,260 call options and 984,987 put options.

The time and dividends were carefully matched to translate index and option prices into reasonable implied B-S volatilities. It is important to work with

A The Original Intraday Data

The original sample of CME S&P500 index intra-day data consisted of about 42 million records, both quotes and transactions, from inception in 1983 to December 2012. The quote data proved unreliable, because the CME sometimes does and sometimes does not include them. After removing these and other obviously incorrect records, there were about 3 million option actual transactions (about 400 option trades per trading-day) and 28 million futures transactions.

<table>
<thead>
<tr>
<th></th>
<th>Options</th>
<th>Futures</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Electronic</td>
<td>Floor</td>
</tr>
<tr>
<td>started with</td>
<td>1,710,150</td>
<td>4,883,571</td>
</tr>
<tr>
<td>remove cancelled trades</td>
<td>1,710,085</td>
<td>4,732,868</td>
</tr>
<tr>
<td>remove cabinet trades</td>
<td>1,710,085</td>
<td>4,719,214</td>
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<tr>
<td>remove zero-prices</td>
<td>1,710,085</td>
<td>4,719,214</td>
</tr>
<tr>
<td>remove quotes, pct of data</td>
<td>99%</td>
<td>38%</td>
</tr>
<tr>
<td>actual trades left*</td>
<td>4,388</td>
<td>2,908,457</td>
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</table>

I excluded all option transaction that were closer than 3 days to expiration (when closing-out transactions could create some anomalous prices) and options that had more than 200 calendar days to expiration (that were rarely traded).

\(^{15}\) The S&P500 and the value-weighted stock return have an excess return correlation of 99.0%. It is almost impossible to distinguish the two. Any differences are due to dividends having been paid.
It requires interpolation to determine the value of the market index that matched each option. (This is also why it is not possible to use the end-of-day data: the end-of-day stock market price may have moved between the option trade and the end of day, and more so for more sparsely-traded below-the-money options.) Thus, I matched each option to a prevailing-at-this-second index price. Unfortunately, even the index future did not trade every second. Moreover, index futures and index options also often had different expiration dates. Thus, for each option transactions, I interpolated a prevailing index futures price from the surrounding nearest-in-transaction-time and nearest-in-expiration index futures. If more than 5 minutes had elapsed since the last transactions or more than 5 minutes were to elapse to the next transaction, I discarded the option transaction. In the final data, the median time from the option transaction to the last futures trade was 5 seconds, the median time to the next trade was 31 seconds. We limited the extrapolation in expiration timing to –30% and +130%. The median option expired on the same day as the near S&P future.

For a risk-free rate that was applicable to the price of the option, I interpolated a prevailing risk-free rate from the 3-month T-Bill rate and the 1-year Treasury rate. This was done by finding the matching convex combination using the 3-month Treasury Bill (DTB3) and the 1-Year Treasury Bill (DTB1YR), both obtained from the St Louis Fed FRED data base, and again limited to a range of –30% to 130% off the convex combination.

The S&P500 index does not include dividends, which creates a complication with respect to the dividends paid by the index. In brief, although market-based methods show greater volatility than the historically prevailing dividend yields that are often used, it makes little difference which is used. In detail, I inferred a dividend yield from the spot-vs-future price differential. The put itself pays off based on the stock price net of dividends, which means that I needed to translate between index futures and index spot values. I started with an estimate of the prevailing index dividend yield net of the risk-free rate. One choice common in this context is to use the historical dividend yield in Shiller (2005), which is readily available online (e.g., at www.multpl.com or www.econ.yale.edu/~shiller/data.htm). It is instead possible to use a market-implied estimate instead. I extracted a prevailing implied dividend yield net of the risk-free rate from the difference in the widely available S&P500 NYSE end-of-day index value (4pm EST) and all available index futures that traded between 2:55pm and 3:05pm PST. These are related by the arbitrage condition $S_0 = I_0 \cdot \exp(\delta - r_f)$. On occasion, the futures-spot difference suggested a negative dividend yield. Because investors cannot be forced to pay in, in such cases, I winsorized the net dividend yield at zero.

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16Thus, if I had hypothetically had an option that had 10 days to expiration, but the futures were 100 days and 200 days to expiration, I would have used a 70-days-to-expiration extrapolated futures price.