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OPTIMAL INATTENTION TO THE STOCK MARKET WITH INFORMATION COSTS AND TRANSACTIONS COSTS

BY ANDREW B. ABEL, JANICE C. EBERLY, AND STAVROS PANAGEAS

Information costs, which comprise costs of gathering and processing information about stock values and costs of deciding how to respond to this information, induce a consumer to remain inattentive to the stock market for finite intervals of time. Whether, and how much, a consumer transfers assets between accounts depends on the costs of undertaking such transactions. In general, optimal behavior by a consumer facing both information costs and transactions costs is state-dependent, with the timing of observations and the timing and size of transactions depending on the state. Surprisingly, if the fixed component of the transactions cost is sufficiently small, then eventually, with probability 1, a time-dependent rule emerges: the interval between observations is constant and on each observation date, the consumer converts enough assets to liquid assets to finance consumption until the next observation. If the fixed component of transactions costs is large, the optimal rule remains state-dependent indefinitely.

KEYWORDS: Inattention, information costs, transactions costs, consumption, portfolio choice.

A pervasive finding in studies of microeconomic choice is that adjustment to economic news tends to be sluggish and infrequent. Investors rebalance their portfolios and revisit their spending behavior at discrete and potentially infrequent points of time. Between these times, inaction is the rule. If individuals take several months or even years to adjust their portfolios and their spending plans, the standard predictions of the consumption smoothing and portfolio choice theories might fail, and the standard intertemporal Euler equation that relates asset returns and consumption growth may not hold. Similar sorts of inaction also characterize the financing, investment, and pricing behavior of firms. These observations have led economists to formulate models that are consistent with infrequent adjustment.

Formal models of infrequent adjustment are often described as either time dependent or state dependent. In time-dependent models, adjustment is triggered simply by calendar time. In state-dependent models, adjustment takes place only when a particular state variable reaches some trigger value, so the timing of adjustments depends on factors other than, or in addition to, calen-

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2See, for example, Lynch (1996) and Gabaix and Laibson (2002).

3Stokey (2009) presented a comprehensive analysis of issues related to inaction and infrequent adjustment.

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A classic example of state-dependent adjustment is the \((S, s)\) model. The distinction between time-dependent and state-dependent models can have crucial implications for important economic questions. For instance, monetary policy has substantial real effects that persist for several quarters if firms change their prices according to a time-dependent rule. However, if firms adjust their prices according to a state-dependent rule, then monetary policy may have little or no effect on the real economy. (See, e.g., Caplin and Spulber (1987) and Golosov and Lucas (2007).)

In this paper, we develop and analyze an optimizing model that can generate both time-dependent adjustment and state-dependent adjustment. The economic context is an infinite-horizon continuous-time model of consumption and portfolio choice that builds on the framework of Merton (1971). We augment Merton’s model by requiring consumption to be purchased with the liquid asset and by introducing two sorts of costs: (i) an information cost that comprises the costs of observing the consumer’s wealth and the costs of processing this information and making decisions about consumption and portfolio allocation, and (ii) a cost of transferring assets between a transactions account consisting of liquid assets and an investment portfolio consisting of risky equity and riskless bonds. Specifically, we model the cost of transferring assets as the sum of a component that is proportional to the amount of assets transferred and a component that is a homogeneous linear function of the balances in the transactions account and in the investment portfolio. Since the second component is independent of the amount of assets transferred, we refer to it as the fixed component of transaction costs.

Because it is costly to observe the value of wealth and to process this information, the consumer chooses to observe this value only at discretely spaced points in time. At these observation times, the consumer chooses when next to observe the value of wealth, executes any transfers between the investment portfolio and the transactions account, chooses the risky share of the investment portfolio, and chooses the path of consumption until the next observation date. During intervals of time between consecutive observations, the consumer remains inattentive to the value of equities in her portfolio and thus follows a consumption path that is unresponsive to any news about the value of equities.

In the absence of any transactions costs, optimal behavior of a consumer with a homogeneous utility function would be time-dependent as described in Abel, Eberly, and Panageas (2007). The timing of observations (and transactions, which would be perfectly synchronized with observations) would be independent of the value of stocks or any other state variable, and the time between consecutive observations would be constant. In addition, the consumer would run down the transactions balance to zero on each observation date and then would transfer a constant fraction of the investment portfolio to the transactions account immediately after observing the value of equities.

In our current framework with transactions costs in addition to information costs, optimal behavior, including the timing of observations and transactions
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is, in general, state-dependent. The relevant state of the consumer’s balance sheet at time \( t \) is \( x_t \), which is defined as the ratio of the balance in the transactions account to the contemporaneous value of the investment portfolio. When the transactions account is large relative to the investment portfolio on observation date \( t_j \), so that \( x_{t_j} \) is high, the consumer will transfer some assets from the transactions account to the investment portfolio. Alternatively, when the transactions account is small relative to the investment portfolio on observation date \( t_j \), so that \( x_{t_j} \) is low, the consumer will sell some assets from the investment portfolio to replenish the transactions account so as to finance consumption until the next observation date. However, when \( x_{t_j} \) has an intermediate value on an observation date, the consumer will not find it worthwhile to pay the costs associated with transferring assets between the investment portfolio and the transactions account.

Because the timing, direction, and size of asset transfers depend on the value of \( x_{t_j} \), these transfers are state-dependent. A surprising result of our analysis, however, is that if the fixed component of the cost of transferring assets is not large, the timing of an optimally inattentive consumer’s observations and asset transfers will eventually—by which we mean with probability 1, throughout the paper—become time-dependent, with a constant length of time between consecutive observations and a transfer from the investment portfolio to the transactions account on every observation date. We demonstrate this finding by showing that eventually optimal behavior by a consumer facing information costs leads to a low value of \( x_{t_j} \) on an observation date with probability 1. Once a low value of \( x_{t_j} \) is realized on an observation date, the consumer transfers only enough assets to the transactions account to finance consumption until the next observation date, provided that the fixed component of the cost of transferring assets is not too large. This behavior is optimal because it is costly to transfer each additional dollar of assets, and the liquid asset in the transactions account earns a lower rate of return than does the riskless bond in the investment portfolio. In this case, the consumer plans to hold a zero balance in the transactions account on the next observation date, so that \( x_{t_j} \) will equal zero on the next observation date and on all subsequent observation dates.

This paper is related to two strands of literature. The first strand is the large literature on transactions costs. In Baumol (1952) and Tobin (1956), who are the forerunners of the cash-in-advance model used in macroeconomics, consumers can hold two riskless assets that pay different rates of return: money, which pays zero interest, and a riskless bond, which pays a positive rate of interest. As in our paper, consumers are willing to hold money, despite the fact that its rate of return is dominated by the rate of return on riskless bonds, because money is necessary to purchase goods. That is, money offers liquidity services.

More recent contributions to this strand of the literature, including Constantinides (1986) and Davis and Norman (1990), model the cost of transferring
assets between stocks and bonds in the investment portfolio as proportional to the size of the transfers. Here we also include proportional transactions costs, but these costs apply only to transfers between the liquid asset in the transactions account, on the one hand, and the investment portfolio of stocks and bonds, on the other. We do not model the costs of reallocating stocks and bonds within the investment portfolio. For a retired consumer who finances consumption by withdrawing assets from a tax-deferred retirement account, the cost of withdrawing assets from the investment portfolio includes taxes paid at the time of withdrawal. For most consumers in this situation, the marginal tax rate, which is part of the cost of transferring assets from the investment portfolio to the transactions account, is likely to be far greater than any costs associated with reallocating stocks and bonds within the investment portfolio.\footnote{Bilias, Georgarakos, and Haliassos (2010) found panel data evidence of substantial inertia in household asset adjustments, particularly among retirement accounts. Brunnermeier and Nagel (2008, p. 715) also use panel data to show that risky asset holdings exhibit substantial inertia, which they determine to be “the dominant factor in determining changes in asset allocation.”}

A second strand of the literature analyzes optimally inattentive behavior by consumers or firms. Two distinct approaches to modeling inattention appear in this strand of literature. One approach, introduced by Sims (2003), and used by Moscarini (2004), Woodford (2009), and Mackowiak and Wiederholt (2009), uses the information-theoretic concept of entropy to model rational inattention as the outcome of the limited ability of people to infer the true values of decision-relevant variables. In those papers, the decisionmaker generally receives noisy information, and can choose the timing and information content of signals about these variables. The other approach specifies the costs of observing decision-relevant variables, processing this information, and formulating decisions. In this approach, which we call the information-cost approach for brevity, the decisionmaker optimally conserves on information costs by observing these variables only at discretely spaced points of time. Two considerations led us to pursue the information-cost approach rather than the entropy-based approach. The first consideration is tractability. Specifically, the nonconvex transaction costs we analyze would be particularly difficult to analyze in the entropy-based approach. However, by pursuing the information-cost-based approach, we develop a tractable framework that easily accommodates nonconvex transactions costs. More importantly, whether the optimal state-dependent rule evolves to a time-dependent rule depends on a comparison of the sizes of transactions costs and information costs. This comparison is readily apparent in the information-cost-based approach, and would appear to be strained, at best, in the entropy-based approach.
The two closest antecedents to our current paper\(^5\) are Duffie and Sun (1990) and Abel, Eberly, and Panageas (2007).\(^6\)\(^7\) These papers, as well as the current paper, require consumption to be purchased with a liquid asset, such as cash. In addition, because these papers include an information cost, the consumer will not continuously observe the value of the stock market. In Abel, Eberly, and Panageas (2007),\(^8\) which includes explicit information costs, the consumer transfers assets from the investment portfolio to the transactions account on every observation date, because, in contrast to the current paper, there are no transactions costs incurred after the consumer incurs the information cost. In Duffie and Sun (1990, p. 35), the transactions dates and observation dates are perfectly synchronized by the assumption that “the agent observes his or her current wealth only when making a transaction.” In both of these papers, the synchronization of observations and transactions follows directly from the assumptions that underlie the respective framework, but in our model, synchronization of observations and transactions emerges endogenously—and only under particular conditions. That is, initially (and unlike Duffie and Sun (1990) and Abel, Eberly, and Panageas (2007)), transactions will occur on some observation dates but not on others. However, if the fixed component of the transactions costs is sufficiently small, then, with probability 1, optimal behavior will evolve to a time-dependent rule with perfect synchronization of observations and transactions, and with a constant interval of time between observations.

Existing models of infrequent adjustment—including both transactions cost models and inattention models—are not capable of addressing the larger

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\(^5\) Reis (2006) developed and analyzed a model of optimal inattention for a consumer with constant absolute risk aversion who faces a cost of observing additive income, such as labor income. In that model, the consumer can hold only a single riskless asset so there is no asset allocation problem.

\(^6\) Gabaix and Laibson (2002) is very similar to Abel, Eberly, and Panageas (2007). An important difference, however, is that (unlike our formulation in Abel, Eberly, and Panageas (2007) and in the current paper) the formulation of the information cost in Gabaix and Laibson does not preserve homogeneity of the value function. Therefore, Gabaix and Laibson computed an approximate solution.

\(^7\) Huang and Liu (2007) applied the concept of rational inattention to study the optimal portfolio decision of an investor who can obtain costly noisy signals about a state variable that governs the expected growth rate of stock prices. Huang and Liu did not include any costs of trading assets and they allowed continuous observation of stock prices so that the investor continuously trades assets within the investment portfolio. However, our modeling of transfer costs and infrequent observation of stock prices leads to infrequent transfers of assets. Finally, and more importantly, Huang and Liu imposed a time-dependent rule for what they call periodic news because they assumed a constant interval of time between the acquisition of periodic news. Thus they cannot address the distinction between state-dependent and time-dependent behavior.

\(^8\) In Abel, Eberly, and Panageas (2007), the information cost reduces the value of wealth and thus, indirectly, reduces utility. In the current paper, the information cost directly reduces utility without reducing wealth. The major results of the paper do not depend on whether information costs are utility costs or resource costs, and we have adopted a utility cost because it seems to capture the effort and hassle of gathering and interpreting relevant information, and using this information to make decisions.
question of whether optimal behavior is time-dependent or state-dependent. Specifically, models that include transactions costs (such as Constantinides (1986), Davis and Norman (1990)9), but not inattention, will generate infrequent adjustment that is state-dependent. On the other hand, models of inattention based on information frictions (such as Moscarini (2004), Reis (2006), Huang and Liu (2007), and Abel, Eberly, and Panageas (2007)) generate optimal behavior that is time-dependent. By including separate information costs and transactions costs10 in our model, we can determine endogenously whether the optimal timing of adjustment is time-dependent or state-dependent, as well as whether observations and transactions are synchronized. While the ultimate emergence of a time-dependent rule occurs with probability 1 if the fixed component of the transactions costs is sufficiently small, optimal behavior can remain state-dependent, and transactions and observations may not be synchronized, if the fixed component of transactions costs is large.

In a recent paper, Alvarez, Guiso, and Lippi (2012) developed a model to study the synchronization of observations and transactions. In their model, as well as ours, synchronization arises if transactions costs, appropriately defined, are sufficiently small. With synchronization, of course, all observations are accompanied by transactions, that is, there are no instances of “inaction” on observation dates. In Alvarez, Guiso, and Lippi, the “inaction region” disappears when transactions costs are sufficiently small, so there are no instances of inaction. In our paper, the inaction region remains intact when the fixed component of transactions costs is sufficiently small, but, in the long run, the consumer is never in this region on an observation date, so the mechanisms leading to synchronization are different in the two models.

Section 1 sets up the consumer’s decision problem. Section 2 characterizes the optimal trigger and return values for the state variable $x_t$. In addition, this section contains a detailed discussion of a typical indifference curve of the value function to illustrate various aspects of optimal adjustment behavior. The dynamic evolution of $x_t$ is analyzed in Section 3, which also characterizes


10We emphasize that the information costs and transactions costs are separate, so that, in principle, costly observations can occur at times without transactions, and costly transactions can occur at times without observation. In contrast, as we mentioned, Duffie and Sun (1990) assumed that transactions and observations are synchronized. Similarly, in the context of a pricing problem, Woodford (2009, p. S104) assumed that “the menu cost is also the fixed cost of obtaining new (complete) information about the state of the economy.” Furthermore, the setup in Woodford (2009, p. S106) precluded a study of the distinction between time- and state-dependent adjustment since “[t]he assumption that memory is (at least) as costly as information about current conditions external to the firm implies that under an optimal policy, the timing of price reviews is (stochastically) state-dependent, but not time-dependent, just as in full-information menu-cost models…. If, instead, memory were costless, the optimal hazard under a stationary optimal plan would also depend on the number of periods $n$ since the last price review....”
the long-run situation that is attained with probability 1 if the fixed component of transactions costs is sufficiently small. Section 4 presents a numerical illustration of the constant length of time between consecutive observations in the long run, followed by a discussion of the Euler equation. Section 5 concludes. The Appendix, which is provided in the Supplemental Material (Abel, Eberly, and Panageas (2013)), contains proofs of all lemmas and propositions, along with the precise statements of a few ancillary lemmas and propositions that are not included in the text.

1. CONSUMER’S DECISION PROBLEM

Consider an infinitely lived consumer who does not earn any labor income, but has wealth that consists of risky equity, riskless bonds, and a riskless liquid asset. Risky equity and riskless bonds are held in an investment portfolio, and the consumer is not permitted to take either a leveraged or a negative position in equity. Consumption must be purchased with the liquid asset, which the consumer holds in a transactions account separate from the investment portfolio.

1.1. Asset Returns

Equity is a non-dividend-paying stock with a price \( P_t \) that evolves according to a geometric Brownian motion

\[
\frac{dP_t}{P_t} = \mu dt + \sigma dz,
\]

where \( \mu > 0 \) is the mean rate of return and \( \sigma \) is the instantaneous standard deviation. The riskless bond in the investment portfolio has a constant rate of return \( r_f < \mu - \frac{\sigma^2}{2} \). The total value of the investment portfolio, consisting of equity and riskless bonds, is \( S_t \) at time \( t \). At time \( t \), the consumer holds \( X_t \) in the liquid asset in the transactions account, which pays a riskless rate of return \( r_L \), where \( r_L < r_f \) because the liquid asset provides transactions services not provided by the bond in the investment portfolio.

Suppose the consumer observes the value of the investment portfolio at time \( t_j \) and next observes its value at time \( t_{j+1} = t_j + \tau_j \). On observing the value of \( S_{t_j} \), the consumer may transfer assets between the investment portfolio and the transactions account (at a cost described below) so that at time \( t_{j+1} \), the

\[11\]The assumption that \( r_f < \mu - \frac{\sigma^2}{2} \) implies that the expected equity premium expressed in logarithms, \( \frac{1}{2}E[\ln P_{t_{j+1}} - \ln P_j] - r_f \), as well as the expected equity premium expressed in levels, \( \mu - r_f \), are positive.

\[12\]Because the transactions account does not include any risky assets, the consumer continuously knows the value of \( X_j \).
The value of the investment portfolio is $S_{t_j^+}$. The consumer chooses to hold a fraction $\phi_j$ of $S_{t_j^+}$ in risky equity and a fraction $1 - \phi_j$ in riskless bonds, and does not rebalance the investment portfolio before the next observation.\(^{13}\) Since the consumer cannot take a negative position or a leveraged position in equity, $0 \leq \phi_j \leq 1$. When the consumer next observes the value of the investment portfolio, at time $t_{j+1} = t_j + \tau_j$, its value is

\[
S_{t_{j+1}} = R(t_j, \tau_j) S_{t_j^+},
\]

where

\[
R(t_j, \tau_j) \equiv \phi_j \frac{P_{t_{j+1}}}{P_{t_j}} + (1 - \phi_j) e^{r\tau_j}.
\]

### 1.2. Costs of Transferring Assets

The consumer can transfer assets between the investment portfolio and the transactions account by incurring a resource cost that is proportional to the size of the transfer and a “fixed” resource cost that is independent of the size of the transfer. Specifically, if the consumer sells $-y^s \geq 0$ dollars of assets from the investment portfolio, there is a proportional transfer cost of $-\psi_s y^s$ dollars, where $0 \leq \psi_s < 1$, so that a sale of $-y^s$ dollars from the investment portfolio is accompanied by an increase in $X$ of $-(1 - \psi_s)y^s$ dollars. For transfers in the other direction, an increase of $y^b \geq 0$ dollars in the investment portfolio is accompanied by a decrease in $X$ of $(1 + \psi_b)y^b$ dollars, where $\psi_b \geq 0$. Assume that $\psi_s + \psi_b > 0$ so that at least one of the proportional transfer cost parameters is positive. One interpretation of $\psi_s$ and $\psi_b$ is that they represent brokerage fees. Alternatively, if the investment portfolio is a tax-deferred account, such as a 401k account, the consumer must pay a tax on withdrawals from the investment portfolio, and $\psi_s$ would include the consumer’s income tax rate, which would be substantially higher than a brokerage fee.\(^{14}\)

The fixed component of the transactions cost is independent of the size of the asset transfer, but is a homogeneous linear function of $X_t$ and $S_t$. Specifically, the fixed component of the transactions cost is $\theta_X X_t + \theta_S S_t$, where

\(^{13}\)Not only is the consumer precluded from rebalancing the investment portfolio between observation dates, the consumer is also precluded from transferring assets between the investment portfolio and the transactions account between observation dates. Proposition 5 in the Appendix addresses the case in which the consumer can decide at time $t_j$ to transfer funds between the investment portfolio and the transactions account at some time(s) before $t_{j+1}$.

\(^{14}\)This interpretation of $\psi_s$ as a tax rate is most plausible if the consumer only withdraws money from the investment portfolio and never transfers assets into the investment portfolio. As we show in Section 3, the long run is characterized by precisely this situation if the fixed component of the transfer cost is sufficiently small.
0 \leq \theta_X < \theta_X^{-} < 1, with \theta_X^{-} as defined later in (27) and 0 \leq \theta_S < 1 - \psi_s.^{15} This formulation of the fixed component of the transactions cost scales the cost to the components of wealth; technically, it preserves the homogeneity of the value function in X and S, which makes possible a stationary distribution for \( \frac{X_t}{S_t} \).

We assume that \( \theta_X X_t \) of the fixed component of transactions cost is paid from the transactions account and \( \theta_S S_t \) is paid from the investment portfolio.\(^{16} \)

Therefore,

\[
X_{t_j}^{+} = \left[ 1 - (1_{y^b(t_j) > 0} + 1_{y^s(t_j) < 0}) \theta_X \right] X_{t_j} - (1 + \psi_s) y^b(t_j) - (1 - \psi_s) y^s(t_j)
\]

and

\[
S_{t_j}^{+} = \left[ 1 - (1_{y^b(t_j) > 0} + 1_{y^s(t_j) < 0}) \theta_S \right] S_{t_j} + y^b(t_j) + y^s(t_j),
\]

where \( 1_{y^b(t_j) > 0} \) is an indicator function that equals 1 if \( y^b(t_j) > 0 \) and equals 0 otherwise, and \( 1_{y^s(t_j) < 0} \) is an indicator function that equals 1 if \( y^s(t_j) < 0 \) and equals 0 otherwise.

### 1.3. The Utility Function

Suppose that the consumer observes the value of the investment portfolio only at discretely spaced points in time \( t_0, t_1, t_2, \ldots \). At observation date \( t_j \), after observing the value of the investment portfolio, lifetime utility is

\[
E_t \left\{ \int_{t_j}^{\infty} \frac{1}{1 - \alpha} e^{1 - \alpha} e^{-\rho(t - t_j)} d t - \sum_{i=j}^{\infty} A(t_i, \tau_i) e^{-\rho(t_i + \tau_i - t_j)} \right\},
\]

\(^{15}\)We assume that \( \theta_X \) is small enough so that if \( X > 0 \) and \( S = 0 \), the consumer will not be deterred from transferring at least some assets from the transactions account to the investment portfolio. We assume that \( \psi_s + \theta_S < 1 \) to prevent assets from becoming "trapped" in the investment portfolio. When the consumer transfers \( -y^s > 0 \) from the investment portfolio, the transactions cost would be \( -\psi_s y^s + \theta_S S \geq (\psi_s + \theta_S)(-y^s) \), where the inequality follows from the fact that the transfer \( -y^s \) must be less than or equal to the value of the investment portfolio \( S \). Thus, if \( \psi_s + \theta_S \geq 1 \), the transaction cost, \( -\psi_s y^s + \theta_S S \), would equal or exceed the size of the transfer, \( -y^s \), and the consumer would not receive any liquid assets as a result of this transaction.

\(^{16}\)Duffie and Sun (1990) assumed that on each observation date, the consumer pays a portfolio management fee that is proportional to total wealth. In their model, optimal behavior implies that \( X = 0 \) on each observation date, so the fixed transaction cost \( \theta_X X + \theta_S S \) is simply \( \theta_S S \); hence, they do not need to explicitly specify the value of \( \theta_X \).
where \( c_t \) is consumption at time \( t \), \( 0 < \alpha \neq 1 \) measures risk aversion, the rate of time preference, \( \rho > 0 \), is large enough so that

\[
\rho > (1 - \alpha) \left[ r_f + \frac{1}{2\alpha} \left( \frac{\mu - r_f}{\sigma} \right)^2 \right],
\]

and \( A(t_i, \tau_i) \) is the utility cost of observing the investment portfolio at time \( t_i + \tau_i \), given that the preceding observation was at date \( t_i \).

We scale the utility cost of an observation, and its associated information processing and decisionmaking, to be a stationary fraction of the consumer’s utility from consumption over the interval of time between observations. This property prevents the information cost from asymptotically becoming prohibitively large or vanishingly small when measured in consumption-equivalent units.\(^{17}\) In particular,

\[
A(t_i, \tau_i) = \tilde{\kappa} \tilde{b}(\tau_i) \int_{t_i}^{t_i+\tau_i} c_t^{1-\alpha} e^{-\rho(t-t_i)} \, dt,
\]

where \( \tilde{b}(\tau_i) > 0 \) for \( \tau_i > 0 \) and \( \kappa > 0 \). We want \( A(t_i, \tau_i) \) to capture the notion that it is costly to increase the frequency of observation and infinitely costly to observe continuously. We also want this function to be well behaved for arbitrarily short or long inattention intervals. Therefore, we require, for any path \( c_t > 0, t_i < t \leq t_i + \tau_i, \) and \( \int_{t_i}^{t_i+\tau_i} c_t^{1-\alpha} e^{-\rho(t-t_i)} \, dt < \infty \), that \( A(t_i, \tau_i) \) has the three properties

\[
(9a) \quad 0 < \lim_{\tau_i \to 0} A(t_i, \tau_i) < \infty,
\]

\[
(9b) \quad \lim_{\tau_i \to \infty} e^{-\rho \tau_i} A(t_i, \tau_i) = 0,
\]

\[
(9c) \quad e^{-\rho \tau_i} A(t_i, \tau_i) + e^{-\rho (\tau_i + \tau_{i+1})} A(t_{i+1}, \tau_{i+1}) > e^{-\rho (\tau_i + \tau_{i+1})} A(t_i, \tau_i + \tau_{i+1}).
\]

Equation (9a) states that as the interval of time between consecutive observations vanishes, the utility cost per observation approaches a finite positive value. Therefore, the cost of continuous observation is infinite and, hence, it is not optimal to observe the value of the investment portfolio continuously. Equation (9b) states that as the length of time until the next observation grows without bound, the discounted value of the utility cost of that observation goes to zero; equivalently, the information cost does not grow faster than the rate of time preference. Finally, the left hand side of (9c) is the discounted (to time \( t_i \)) utility cost of observing the investment portfolio twice during the interval

\(^{17}\)This property is reminiscent of the specification in King, Plosser, and Rebelo (1988) in which the disutility of labor is a stationary fraction of the utility from consumption, with the implication that hours of labor can be stationary even though consumption is nonstationary.
(t_i, t_i + \tau_i + \tau_{i+1}]: once at time \( t_i + \tau_i \) and once at time \( t_i + \tau_i + \tau_{i+1} \). The right hand side of (9c) is the discounted (to time \( t_i \)) utility cost of observing the investment portfolio only once during this interval, at the end of the interval. The inequality in (9c) states that for a given interval of time, two observations are more costly than one observation. Equations (9a), (9b), and (9c) imply restrictions on the function \( \tilde{b}(\tau_i) \). Rather than work directly with the function \( \tilde{b}(\tau_i) \), it is more convenient to work with the function \( b(\tau) \) defined as

\[
(10) \quad b(\tau) \equiv e^{-\rho \tau} \tilde{b}(\tau).
\]

Multiplying both sides of (8) by \( e^{-\rho \tau_i} \) and using the definition of \( b(\tau) \) from (10) yields

\[
(11) \quad e^{-\rho \tau_i} A(t_i, \tau_i) = k b(\tau_i) \int_{\tau_i}^{t_i + \tau_i} c_i^{1-a} e^{-\rho(t-t_i)} \, dt.
\]

The following lemma presents some necessary properties of \( b(\tau) \).

**Lemma 1:** Suppose that \( A(t_i, \tau_i) \) satisfies (11) and has the properties in (9a), (9b), and (9c). Then the following statements hold:

(i) \( b(\tau) \) is nonincreasing.

(ii) \( 0 < \lim_{\tau \to 0} \tau b(\tau) < \infty \), which implies \( \lim_{\tau \to 0} \frac{\tau b(\tau)}{b(\tau)} = -1 \).

(iii) \( \lim_{\tau \to \infty} b(\tau) = 0 \) if \( \lim_{\tau \to \infty} \int_{\tau_i}^{t_i + \tau} c_i^{1-a} e^{-\rho(t-t_i)} \, dt > 0 \) is finite.

Finally, we adopt the normalization \( b(1) = 1 \). As an illustration of the function \( b(\tau) \), suppose that \( A(t_i, \tau_i) \) is proportional to the average rate at which (discounted) utility from consumption is accrued over the interval \( [t_i, t_i + \tau_i] \). Thus, \( \tilde{b}(\tau_i) \) in (8) is proportional to \( \frac{1}{\tau_i} \), and normalizing \( b(\tau) \equiv e^{-\rho \tau} \tilde{b}(\tau) \) so that \( b(1) = 1 \), we have

\[
(12) \quad b(\tau) = e^{-\rho(\tau-1)} \frac{1}{\tau}.
\]

It is straightforward to verify that \( b(\tau) \) in (12) satisfies conditions (i)–(iii) in Lemma 1. In the numerical example in Section 4, we use the specification of \( b(\tau) \) in (12), but everywhere else in the paper, we allow any \( b(\tau) > 0 \) that satisfies the properties in statements (i)–(iii) in Lemma 1.

Substitute the discounted information cost from (11) into the lifetime utility function in expression (6) to obtain the expression for expected lifetime utility:

\[
(13) \quad \frac{1}{1 - \alpha} E_{t_i} \left\{ \sum_{i=j}^{\infty} e^{-\rho(t_i-t_j)} \left[ 1 - (1 - \alpha) k b(\tau_i) \right] \int_{\tau_i}^{t_i + \tau_i} c_i^{1-a} e^{-\rho(t-t_i)} \, dt \right\}.
\]
Since the consumer will not observe any new information between times $t_j$ and $t_{j+1}$, she can, at time $t_j$, plan the entire path of consumption from time $t_j^+$ to time $t_{j+1}$. Let $C(t_j, \tau_j)$ be the present value, discounted at rate $r_L$, of the (deterministic) flow of consumption over the interval of time from $t_j^+$ until the next observation date, $t_{j+1} = t_j + \tau_j$. Specifically,

\begin{equation}
C(t_j, \tau_j) = \int_{t_j^+}^{t_{j+1}} c_s e^{-r_L(s-t_j)} \, ds, \tag{14}
\end{equation}

where the path of consumption $c_s$, $t_j^+ \leq s \leq t_{j+1}$, is chosen to maximize the discounted value of utility over the interval from $t_j^+$ to $t_{j+1}$. Let

\begin{equation}
U(C(t_j, \tau_j)) \equiv \max_{(c_s)_{t_j^+}} \int_{t_j^+}^{t_{j+1}} \frac{1}{1-\alpha} c_s^{1-\alpha} e^{-\rho(s-t_j)} \, ds, \tag{15}
\end{equation}

subject to a given value of $C(t_j, \tau_j)$ in (14). It is straightforward to show that

\begin{equation}
U(C(t_j, \tau_j)) = \frac{1}{1-\alpha} \left[h(\tau_j)\right]^{\alpha} \left[C(t_j, \tau_j)\right]^{1-\alpha}, \tag{16}
\end{equation}

where

\begin{equation}
h(\tau_j) \equiv \int_0^{\tau_j} e^{-\chi s} \, ds = \frac{1 - e^{-\chi \tau_j}}{\chi}, \tag{17}
\end{equation}

and we assume that

\begin{equation}
\chi \equiv \frac{\rho - (1-\alpha)r_L}{\alpha} > 0. \tag{18}
\end{equation}

\footnote{During the interval of time from $t_j^+$ to $t_{j+1}$, the (deterministic) Euler equation implies that optimal values of consumption satisfy

\begin{equation}
c_s = e^{-((\rho-r_L)/\alpha)(s-t_j^+)} c_{t_j^+} \quad \text{for} \quad t_j^+ \leq s \leq t_{j+1}. \tag{*}
\end{equation}

Substituting $c_s$ from (*) into (14) yields

\begin{equation}
C(t_j, \tau_j) = h(\tau_j)c_{t_j^+}, \tag{**}
\end{equation}

where $h(\tau_j)$ is defined in (17). Equations (*) and (**) imply that

\begin{equation}
c_s = \left[h(\tau_j)\right]^{-1} e^{-((\rho-r_L)/\alpha)(s-t_j^+)} C(t_j, \tau_j) \quad \text{for} \quad t_j^+ \leq s \leq t_{j+1}. \tag{***}
\end{equation}

Substituting (***) into (15), and using the definition of $h(\tau_j)$ in (17) yields

\begin{equation}
U(C(t_j, \tau_j)) = \frac{1}{1-\alpha} \left[h(\tau_j)\right]^{\alpha} \left[C(t_j, \tau_j)\right]^{1-\alpha}, \tag{16}
\end{equation}

which, along with (**), implies that $U'(C(t_j, \tau_j)) = c_{t_j^+}^{\alpha}$.}
Since consumption during the interval of time from $t_j^+$ to $t_{j+1}$ is financed from the transactions account, which earns an instantaneous riskless rate of return $r_L$, we have

\begin{equation}
X_{t_{j+1}} = e^{r_L \tau_j} (X_{t_j^+} - C(t_j, \tau_j)).
\end{equation}

Use (16) and the expression for lifetime utility in (13) to obtain the value function\textsuperscript{19} at observation date $t_j$, immediately after observing the value of the investment portfolio at date $t_j$,

\begin{equation}
V(X_{t_j}, S_{t_j}) = \sup_{C(t_j, \tau_j), y^b(t_j), y^s(t_j), \phi_j, \tau_j} \left[ 1 - (1 - \alpha) \kappa b(\tau_j) \right] U(C(t_j, \tau_j))
+ e^{-\rho \tau_j} E_j \left\{ V(e^{r_L \tau_j} (X_{t_j^+} - C(t_j, \tau_j)), R(t_j, \tau_j) S_{t_j^+}) \right\},
\end{equation}

where the maximization in (20) is subject to (4) and (5), and the inequality constraints $C(t_j, \tau_j) \leq X_{t_j^+}, 0 \leq \phi_j \leq 1, y^b(t_j) \geq 0$, and $y^s(t_j) \leq 0$.

In Appendix B, we show that the value function satisfies the recursive relationship (20), is finite, and is continuous, and that there exist policies that attain the supremum on the right hand side of (20) and are optimal for the problem we consider.\textsuperscript{20} We also show that the value function is homogeneous of degree $1 - \alpha$ in $X_{t_j}$ and $S_{t_j}$. Consequently, it can be written as

\begin{equation}
V(X_{t_j}, S_{t_j}) = \frac{1}{1 - \alpha} S_{t_j}^{1 - \alpha} v(x_{t_j}),
\end{equation}

for $S_t > 0$, where $\frac{1}{1 - \alpha} v(x_{t_j})$ is strictly increasing in $x_t$ and

\begin{equation}
x_t = \frac{X_t}{S_t}
\end{equation}

is the ratio of the transactions account to the investment portfolio. The optimal length of time, $\tau_j$, between consecutive observation dates $t_j$ and $t_{j+1}$, is a function of $x_{t_j}$.

2. TRIGGER AND RETURN VALUES OF $x$

The value of $x_{t_j} = \frac{X_{t_j}}{S_{t_j}}$ on an observation date $t_j$ determines whether, in which direction, and what amounts of assets the consumer transfers between the in-

\textsuperscript{19}If $\alpha > 1$, then $[1 - (1 - \alpha) \kappa b(\tau_j)] > 0$ for all $\tau > 0$; as we show in the Appendix, optimality implies that $\tau$ will be large enough so that $[1 - (1 - \alpha) \kappa b(\tau_j)]$ is positive even when $\alpha < 1$. Equation (16) gives the maximized value of $\frac{1}{1 - \alpha} \int_{t_j}^{t_{j+1}} c_j^{1 - \alpha} e^{-\rho t} \, dt$ in (13) subject to (14). Since $[1 - (1 - \alpha) \kappa b(\tau_j)] > 0$, we can substitute (15) into the continuous-time optimization problem in (13) to obtain the discrete-time problem in (20).

\textsuperscript{20}When $\alpha > 1$, these statements hold for any $X_{t_j}$ and $S_{t_j}$ provided that $X_{t_j} + S_{t_j} > 0$.\n
vestment portfolio and the transactions account. There are two trigger values of \( x, \omega_1 \) and \( \omega_2 \), that determine whether the consumer transfers assets, and there are two values of \( x, \pi_1 \) and \( \pi_2 \), that help characterize the return value of \( x_t \) immediately after a transfer.

To define and characterize the trigger values, \( \omega_1 \) and \( \omega_2 \), first define the restricted value function \( \tilde{V}(X_t, S_t) \) at observation date \( t \) as the maximized expected value of utility over the infinite future, subject to the restriction that the consumer does not transfer any assets between the transactions account and the investment portfolio at time \( t \) (but optimally transfers assets between the transactions account and the investment portfolio at all future observation dates). Formally,

\[
\tilde{V}(X_t, S_t) = \sup_{C(t, \tau_j), \phi_j, \tau_j} \left\{ 1 - (1 - \alpha) \kappa b(\tau_j) \right\} U(C(t, \tau_j)) + e^{-\rho \tau_j} E_t \{ V(e^{L \tau_j} (X_t - C(t, \tau_j)), R(t, \tau_j) S_t) \},
\]

subject to \( C(t, \tau_j) \leq X_t \) and \( 0 \leq \phi_j \leq 1 \). For the remainder of this section, we suppress the time subscripts, with the understanding that the results apply at any observation date. Like the value function, the restricted value function is homogeneous of degree \( 1 - \alpha \) and, for \( S > 0 \), can be written as

\[
\tilde{V}(X, S) = \frac{1}{1 - \alpha} S^{1 - \alpha} \tilde{v}(x),
\]

where \( \frac{1}{1 - \alpha} \tilde{v}(x) \) is strictly increasing in \( x \). On any observation date, \( \tilde{V}(X, S) \leq V(X, S) \), with equality only if the optimal values of \( y_b \) and \( y_s \) are both zero.

Define

\[
\omega_1 \equiv \inf x > 0: \tilde{v}(x) = v(x)
\]

and

\[
\omega_2 \equiv \sup x > 0: \tilde{v}(x) = v(x).
\]

Proposition 1 below shows that \( \omega_1 \) and \( \omega_2 \) are trigger values for \( x \) in the sense that if \( x \) is less than \( \omega_1 \) on an observation date, the consumer will transfer assets to the transactions account, and if \( x \) exceeds \( \omega_2 \) on an observation date, the consumer will transfer assets to the investment portfolio. To ensure that \( \omega_2 \) is finite, we assume that \( \kappa \) and \( \theta_X \) are small enough that a consumer who holds all of her wealth in the transactions account on an observation date will not be deterred from transferring some assets from the transactions account to the investment portfolio. Specifically, we assume

\[
\theta_X < \bar{\theta}_X \equiv \left( 1 - \theta_S \right) \left( 1 - \frac{\psi_s}{1 + \psi_b} \right) \frac{X}{r_f - r_L} < 1
\]
and

\[ (28) \quad \kappa < \bar{\kappa} = \frac{\vartheta_X}{1 - \alpha} b\left(\hat{T}\right)(\exp(\chi \hat{T}) - 1), \]

where \( \hat{T} \equiv -\frac{1}{\chi} \ln(1 + \frac{\chi}{r_j - r_L}) \theta_X > 0 \). We also define

\[ (29) \quad \pi_1 \equiv \sup\left\{ x \geq 0 : \forall z \in \left(0, \frac{xS}{1 - \psi_s}\right], \right. \\
\left. (1) \quad V(xS, S) \geq V(xS - (1 - \psi_s)z, S + z) \right. \text{ and} \\
\left. (2) \quad V(xS, S) > \tilde{V}(xS - (1 - \psi_s)z, S + z) \right\} \]

and

\[ (30) \quad \pi_2 \equiv \inf\{ x \geq 0 : \forall z \in (0, S], \]

\[ (1) \quad V(xS, S) \geq V(xS + (1 + \psi_b)z, S - z) \quad \text{and} \]

\[ (2) \quad V(xS, S) > \tilde{V}(xS + (1 + \psi_b)z, S - z) \} . \]

Proposition 1 below shows that if \( x \leq \omega_1 \), the consumer will transfer enough assets from the investment portfolio to the transactions account to increase \( x \) to at least \( \pi_1 \). Alternatively, if \( x \geq \omega_2 \), the consumer will use the transactions account to buy enough assets in the investment portfolio to decrease \( x \) to a value no larger than \( \pi_2 \).

**PROPOSITION 1:** Assume that \( \kappa < \bar{\kappa} \) and \( \vartheta_X < \overline{\vartheta_X} \). Then the following statements hold:

(i) \( 0 < \omega_1 \leq \overset{\cdot}{\pi}_1 \leq \overset{\cdot}{\pi}_2 \leq \omega_2 < \infty \).

(ii) If \( x_{ij} < \omega_1 \), then (a) \( y^i(t_j) < 0 \), (b) \( x_{ij}^+ \geq \pi_1 \), (c) \( m(x_{ij}) \equiv \frac{V_S(X_{ij}, S_{ij})}{X_S(X_{ij}, S_{ij})} = \frac{1 - \psi_s}{1 - \vartheta_X} \), and (d) \( v(x_{ij}) = \left[\frac{(1 - \vartheta_X)x_{ij} + (1 - \theta)\vartheta_s}{1 - \vartheta_X\omega_1 + (1 - \theta)\vartheta_s}\right]^{1 - \alpha} v(\omega_1) \).

(iii) If \( x_{ij} > \omega_2 \), then (a) \( y^b(t_j) > 0 \), (b) \( x_{ij}^+ \leq \pi_2 \), (c) \( m(x_{ij}) \equiv \frac{V_S(X_{ij}, S_{ij})}{X_S(X_{ij}, S_{ij})} = \frac{1 + \psi_b}{1 + \vartheta_X} \), and (d) \( v(x_{ij}) = \left[\frac{(1 - \vartheta_X)x_{ij} + (1 - \theta)\vartheta_s}{1 - \vartheta_X\omega_2 + (1 - \theta)\vartheta_s}\right]^{1 - \alpha} v(\omega_2) \).

Proposition 1 is proved in the Appendix. Here we use the indifference curves in Figure 1 to illustrate this proposition and the properties of the trigger and return points. For simplicity, Figure 1 is drawn for the case in which \( \vartheta_X = \vartheta_s \). The indifference curve of the value function \( V(X, S) \) passes through points \( A \),
B, C, D, E, and F, and the indifference curve of the restricted value function \( \tilde{V}(X, S) \) passes through points K, B, C, D, E, and J. In Regions II, III, and IV, the two indifference curves are identical, reflecting the fact that \( V(X, S) = \tilde{V}(X, S) \). Therefore, Regions II, III, and IV represent the "inaction region" in which the consumer can attain \( V(X, S) \) without transferring any assets between the investment portfolio and the transactions account.

The consumer will transfer assets if \( V(X, S) > \tilde{V}(X, S) \), which is the case in Regions I and V. For instance, in Region I, the indifference curve of the restricted value function passes through point B and lies above the indifference curve of the value function that also passes through point B, thereby implying that \( V(X, S) > \tilde{V}(X, S) \) in this region.\(^{21}\) To attain the maximized value of expected lifetime utility, the consumer must transfer assets between the investment portfolio and the transactions account. As shown in statement (ii)(a) of Proposition 1, optimal \( y^i < 0 \), so the consumer sells assets from the investment portfolio to increase the amount of liquid assets in the transactions account. Similarly, according to statement (iii)(a), if the consumer is in Region V on an

\(^{21}\)To see that \( V(X, S) > \tilde{V}(X, S) \) in Region I, use the fact that \( V(X, S) \) is strictly increasing in \( X \) and \( S \) to obtain \( V^K > V^A = V^B = \tilde{V}^B = \tilde{V}^K \), where \( V^i \) is the value of \( V(X, S) \) at point \( i \) and \( \tilde{V}^j \) is the value of \( \tilde{V}(X, S) \) at point \( j \) in the figure.
observation date, the optimal policy is to use some of the liquid assets in the transactions account to purchase additional assets in the investment portfolio.

Now consider the return value of $x_j^t$, which is equal to $\pi_1$ in Figure 1. We proceed in two steps. First, assume that the consumer has already paid the fixed component of the transfer cost $\theta(X + S)$, where $\theta$ is the common value of $\theta_X = \theta_S$, and that the consumer is choosing the size of the asset transfer from the investment portfolio to the transactions account. In the second step, we consider the impact of the fixed component, $\theta(X + S)$, of the transactions cost on the optimal transfer.

Suppose that after paying the fixed cost $\theta(X + S)$, the consumer is at point $A'$. Having already paid the fixed cost, the consumer can move instantaneously to any point up and to the left of point $A'$ along the dashed line with slope $-1 - \psi_s$ by reducing $S$ by $-y^s > 0$ dollars and increasing $X$ by $(1 - \psi_s)(-y^s)$ dollars. The consumer will sell assets from the investment portfolio until $(X, S)$ reaches point $C$, where the dashed line with slope $-(1 - \psi_s)$ is tangent to the indifference curve, which is essentially a smooth-pasting condition. At point $C$, the ratio of $X$ to $S$, that is, $x$, is equal to $\pi_1$, as indicated by the line through points $O$, $C$, and $G$, which has slope equal to $\pi_1$.

Now consider the impact of the fixed cost $\theta(X + S)$ on the optimal transfer of assets. If $\theta > 0$, the consumer cannot move from point $A'$ to point $C$. To see the impact of $\theta > 0$, consider the line through points $G$, $B$, and $A$, which is parallel to the line through points $C$, $B'$, and $A'$, and hence has slope $-1 - \psi_s$. Point $G$ lies on the half-line through the origin $\pi_1$ and is located so that the length of $\overline{OC}$ is $1 - \theta$ times the length of $\overline{OG}$. The properties of similar triangles imply that the length of $\overline{OB'}$ is $1 - \theta$ times the length of $\overline{OB}$ and that the length of $\overline{OA'}$ is $1 - \theta$ times the length of $\overline{OA}$.

Now suppose that the consumer starts at point $A$ and transfers $-y^s > 0$ dollars from the investment portfolio, thereby incurring a cost of $\theta(X + S) - \psi_s y^s$ dollars. The fixed cost of $\theta(X + S)$ dollars reduces both $X$ and $S$ by the fraction $\theta$, and can be represented by the movement from point $A$ to point $A'$; the transfer of $-y^s > 0$ dollars from the investment portfolio can be represented by a movement from point $A'$ upward and leftward along the dashed line through points $C$, $B'$, and $A'$. The consumer will be willing to move from $A$ to point $C$ only if doing so increases (or at least does not lower) the value of the value function. That is, the gain in value from moving to an improved allocation between $X$ and $S$ must outweigh the fixed cost $\theta(X + S)$ represented by the movement downward and leftward from the line through points $G$, $B$, and $A$ to the line through points $C$, $B'$, and $A'$. For a large change in the ratio $x$, such as the change in moving from point $A$ to point $C$, the net gain in value is positive. For a small change in $x$, the change is not worthwhile. At point $B$, the gain from the improved allocation between $X$ and $S$ is exactly offset by the
cost of moving from the line through points \( G, B, \) and \( A \) to the line through points \( C, B', \) and \( A' \).

For points along the segment \( \overline{GB} \), the change in the value of \( x \) is small enough that the improved allocation between \( X \) and \( S \) is outweighed by the fixed cost \( \theta(X + S) \). Therefore, the consumer will not transfer assets from any points along this segment. The fact that the consumer will not move from points along segment \( \overline{GB} \) to point \( C \) is illustrated by the fact that these points lie above the indifference curve of the value function that passes through point \( C \). Alternatively, for points below and to the right of point \( B \) along the line through points \( A \) and \( B \), the improved asset allocation made possible by moving to point \( C \) and the associated increase in value are large enough to compensate for the fixed transfer cost, and the consumer will move from any of these points to \( C \) (statements (ii)(a) and (ii)(b)). Since the consumer ends up at the same point, namely point \( C \), from any point below and to the right of point \( B \), all of these points have the same value. Thus, all of these points lie on the same indifference curve (statement (ii)(d)), so that indifference curve has slope equal to \(- (1 - \psi_s) \) below and to the right of point \( B \), which is statement (ii)(c) in Proposition 1.\(^{22}\)

We have used Figure 1 to illustrate the trigger point \( \omega_1 \) and the return point \( \pi_1 \) when the consumer chooses to transfer assets from the investment portfolio to the transactions account. A similar set of arguments can explain the trigger point \( \omega_2 \) and the return point \( \pi_2 \) when the consumer chooses to transfer assets from the transactions account to the investment portfolio.

We conclude this section with the following corollary to Proposition 1.

**Corollary 1:** The optimal value of \( x_i \) immediately following observation date \( t_j \) satisfies \( \omega_1 \leq x_i^{\ast} \leq \omega_2 \).

The value of \( x_i \), immediately following any observation date \( t_j \) (and following any optimal asset transfers at that date) is confined to the closed interval

\(^{22}\)If we relax the assumption that \( \theta_X = \theta_S \), then statement (ii)(c) of Proposition 1 implies that the slope of the linear portion of the indifference curve through points \( B \) and \( A \) is \(- (1 - \psi_s) \frac{1 + \theta_s}{1 - \theta_s} \), while the slope of the dashed line through points \( C, B', \) and \( A' \) remains \(- (1 - \psi_s) \). The horizontal intercept of the indifference curve, \( S \), is \( \frac{1}{1 - \psi_s} \geq 1 \) times as large as \( \overline{S} \), the horizontal intercept of the dashed line through points \( C, B', \) and \( A' \). Because starting from \((X, S) = (0, \overline{S})\), the fixed transaction cost moves the allocation \((X, S)\) to \((0, (1 - \theta_s) \overline{S}) = (0, \overline{S})\). Therefore, even if \( \theta_X > \theta_S \), so that the linear portion of the indifference curve slopes downward more steeply than the dashed line, the linear portion of the indifference curve will not cross the dashed line for any nonnegative values of \( X \). Also, statement (iii)(c) of Proposition 1 implies that the slope of the indifference curve through points \( E \) and \( F \) is \(- (1 + \psi_s) \frac{1 - \theta_s}{1 + \theta_s} \). The vertical intercept of the indifference curve is \( \frac{1}{1 - \theta_s} \geq 1 \) times as large as the vertical intercept of the dashed line through point \( D \) and thus the indifference curve does not cross this dashed line for nonnegative values of \( S \).
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[ω₁, ω₂]. This result will be useful when we analyze the dynamic behavior of asset holdings in the next section.

3. DYNAMIC BEHAVIOR

We have shown that the direction of the optimal transfer on an observation date depends on the value of \( x_{t_j} \). In this section, we examine the dynamic behavior of the stochastic process for \( x_{t_j} \). If the value of \( X_{t_j} \) is positive on an observation date, then, depending on the outcome of the stochastic process for \( S \), the value of \( x_{t_j} \) could be in any of the five regions in Figure 1. However, the stochastic process for \( x_{t_j} \) will be absorbed with probability 1 at \( x_{t_j} = 0 \) provided that \( \theta_S \) is sufficiently small.

**Proposition 2:** There exists \( \theta_S > 0 \), such that for any nonnegative \( \theta_S < \theta_S^* \), if \( x_{t_j} < \omega_1 \) on observation date \( t_j \), then \( x_{t_k} = 0 \) on all subsequent observation dates \( t_k > t_j \).

The proof of Proposition 2 is in the Appendix. Here we provide an intuitive argument. First, consider the case in which \( \theta_X = \theta_S = 0 \). If \( x_{t_j} < \omega_1 \) on observation date \( t_j \), the optimal transfer is from the investment portfolio to the transactions account. Since each additional dollar that is transferred from the investment portfolio to the transactions account incurs a transactions cost \( \psi_s \), and since the transactions account earns a lower riskless rate of return than the riskless rate of return on bonds in the investment portfolio, the consumer would never transfer more assets from the investment portfolio than are needed to finance consumption until the next observation date. Thus, the consumer will arrive at the next observation date with zero liquid assets, so that \( x_{t_{j+1}} = 0 \), \( x_{t_{k+1}} = 0 \) on every observation date \( t_{k+1} > t_j \).

If \( \theta_S \) is positive, then we need to consider the possibility that the consumer would want to arrive at the next observation date with enough liquid assets in the transactions account to avoid transferring assets from the investment portfolio and thus avoid paying the fixed component of the transactions cost at that date. As the proof of Proposition 2 shows, if \( \theta_S \) is small enough, the consumer will still optimally choose to arrive at the next observation date with a zero balance in the transactions account, even though this action necessitates payment of the fixed component of the transaction cost at the next observation date. Alternatively, if \( \theta_S \) is large, the consumer may choose to arrive at observation dates with a positive balance in the transactions account; holding

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23 We do not need to be concerned that a positive value of \( \theta_X \) will induce the consumer to want to hold additional liquid assets on the next observation date to avoid having to make a transfer at that time. In fact, since \( \theta_X \) effectively acts as a tax on the transactions account if the consumer turns out to want to make a transfer on that date, a positive value of \( \theta_X \) provides an incentive to reduce the transactions account on the next observation date.
a positive transactions balance gives the consumer the option to avoid paying a transaction cost if $\omega_1 < x_{t_{j+1}} < \omega_2$ on observation date $t_{j+1}$ and this option becomes valuable when the fixed cost of transactions is large.

The following lemma together with Proposition 2 allows us to prove that the stochastic process for $x_{t_j}$ is absorbed at zero with probability 1 if $\theta_S$ is sufficiently small.

**Lemma 2:** The optimal value of $x_i$ satisfies $x_{t_j} < \omega_1$ for some observation date $t_j$ with probability 1.

The proof of Lemma 2 is in the Appendix. Here we provide an intuitive argument. Because the expected rate of return on equity, $\mu$, exceeds the riskless rate of return, $r_f$, on bonds in the investment portfolio, the optimal share of equity, $\phi_j$, is positive. Therefore, during any given inattention interval, there is a chance that $R(t_j, \tau_j)$ will be sufficiently high that $x_{t_{j+1}} = \frac{e^{\mu \tau_j} (X_{t_j} - C(t_j, \tau_j))}{R(t_j, \tau_j) S_{t_j}}$ will be less than $\omega_1$. After sufficiently many spells of inattention, this event will occur with probability 1.

**Proposition 3:** There exists $\theta_S > 0$ such that for any nonnegative $\theta_S < \theta_S$, the stochastic process for $x_{t_j}$ is absorbed with probability 1 at zero for some observation date $t_j$; thereafter, the time between consecutive observations becomes constant.

Proposition 3 implies that, eventually, with probability 1, optimal asset holdings have a Baumol–Tobin flavor if $\theta_S \geq 0$ is sufficiently small. Specifically, the consumer will arrive at each observation date having just exhausted the liquid assets in the transactions account and will liquidate just enough assets from the investment portfolio to finance consumption until the next observation date. Observations and transfers are perfectly synchronized and a constant amount of time elapses between asset transfers.\textsuperscript{24} We refer to this situation as the long run.

Up to this point, we have assumed that transfers between the investment portfolio and the transactions account can occur only on observation dates. For the remainder of this section only, we consider the impact of allowing transactions to take place between observation dates.\textsuperscript{25} The essence of inattention is that between observation dates, the consumer does not observe the realization of random returns and does not change consumption in response to information that was not available at the time of the most recent observation. Formally,

\textsuperscript{24}The model in Duffie and Sun (1990) shares this property because it assumes that the consumer starts with $x_i = 0$.

\textsuperscript{25}In a price-setting framework, Bonomo, Carvalho, and Garcia (2010) analyzed “uninformed adjustments,” which are price adjustments that occur between observation dates. These uninformed adjustments are analogous to our automatic transactions (subsequently defined) in the consumer’s allocation of assets.
consumption between observation dates \( t_j \) and \( t_{j+1} \) must be \( F_t \)-measurable. Because the consumer would not know in advance the proceeds of any transfer that depends on the stock price at some time after the most recent observation, she would not be able to use the proceeds of such a transfer to finance consumption between \( t_j \) and \( t_{j+1} \). Accordingly, there would be no reason for the consumer to transfer assets during this interval of time from stocks to the transactions account, which pays a lower riskless rate than the riskless rate paid on bonds in the investment portfolio. In general, the size of any optimal transfer from the investment portfolio to the transactions account between \( t_j \) and \( t_{j+1} \) must be \( F_t \)-measurable, and thus must be a transfer from the riskless bond in the investment portfolio to the transactions account. Specifically, the consumer may consider asset transfers at times between observation dates \( t_j \) and \( t_{j+1} \) as long as (i) the amounts and timing of the transfers are \( F_t \)-measurable and (ii) \( X_t \geq 0 \) and \( S_t \geq 0 \) for all \( t \). Because these transfers are determined at time \( t_j \) and are executed after that date, we refer to them as automatic transfers.

We will show that the major result of this paper—that for sufficiently small \( \theta_S \geq 0 \), optimal behavior with probability 1 endogenously evolves to a time-dependent rule, with a constant interval of time between observations—can arise even in the presence of automatic transfers. To keep the argument uncluttered, we confine attention to the case with \( \theta_X = \theta_S = 0 \). In this case, it will never be optimal to transfer assets from the investment portfolio to the transactions account when the transactions account has a positive balance, because the consumer can earn more interest by keeping assets in the riskless bond earning \( r_f \) than in the transactions account earning \( r_L \) (Lemma 7 in the Appendix). However, once the transactions account reaches a zero balance at some date \( \tilde{t} \), it will remain zero forever (Lemma 8 in the Appendix), and the consumer will use continuous automatic transfers of assets from the investment portfolio to the transactions account between consecutive observation dates \( t_j \) and \( t_{j+1} \) at a rate just sufficient to purchase the contemporaneous flow associated with the consumption plan made at time \( t_j \). With \( X_t = 0 \) for all \( t \geq \tilde{t} \), we have \( x_{t_k} = 0 \) for all \( t_k \geq \tilde{t} \). Since optimal \( \tau_k \) is simply a function of \( x_{t_k} \), the optimal time between observations will be constant for all \( t_k \geq \tilde{t} \). Proposition 5 in the Appendix states that \( X_t \) will reach zero with probability 1 so that the time-dependent rule, characterized by a constant interval of time between observations, will emerge. Even though \( x_t \) will be absorbed at 0 with probability 1, which leads to a time-dependent rule, that absorption need not take place immediately (Lemma 10 in the Appendix) and so the time-dependent rule need not emerge immediately.

4. LONG-RUN BEHAVIOR

Table I presents the optimal time between consecutive observation dates in the long run for the case in which \( \theta_X = \theta_S = \theta \), there are no automatic transfers, and the parameter values are given in the table footnote. For these numerical exercises, we specify \( b(\tau) \) as in (12), so that the utility cost \( A(t_i, \tau_i) \) is
proportional to the average discounted utility of consumption accrued over the inattention interval. This formulation allows us to present both the information cost and the fixed component of the transactions cost in terms of dollars.\footnote{To obtain the equivalent dollar cost, we use the fact that the utility cost of an observation is \( A(t_j, \tau_j) = \kappa e^{\alpha} \times \frac{1}{\tau_j} \int_{t_j}^{t_j + \tau_j} c_i^{1 - \alpha} e^{-\rho(t_i - t_j)} dt = (1 - \alpha) \kappa e^{\alpha} \frac{1}{\tau_j} U(C(t_j, \tau_j)) \). In the long run, \( C(t_j, \tau_j) = X_{i_{\tau_j}} \), so the utility cost of an observation is \((1 - \alpha) \kappa e^{\alpha} \frac{1}{\tau_j} U(X_{i_{\tau_j}})\). We want to compute the reduction in the transactions balance at time \( t_j \) that would cause the same loss in utility over the interval \((t_j, t_j + \tau_j]\) as would the observation cost. Writing the reduction in the transactions balance as \( \lambda X_{i_{\tau_j}} \), we find the value of \( \lambda \) such that \( U(X_{i_{\tau_j}}) - U((1 - \lambda)X_{i_{\tau_j}}) = (1 - \alpha) \kappa e^{\alpha} \frac{1}{\tau_j} U(X_{i_{\tau_j}}) \). Since \( U() \) is homogeneous of degree \( 1 - \alpha \), we have \( 1 - (1 - \lambda)^{1 - \alpha} = (1 - \alpha) \kappa e^{\alpha} \frac{1}{\tau_j} \), which implies \( \lambda = 1 - [1 - (1 - \alpha) \kappa e^{\alpha} \frac{1}{\tau_j}]^{1/(1 - \alpha)} \). On any observation date in the long run, \( X_{i_{\tau_j}} = 0 \). Let \( \pi^* \equiv \frac{X_{i_{\tau_j}}}{\kappa} \) be the return value for \( x_{t_{j+1}} \). Equations (4) and (5), using the fact that \( X_{i_{\tau_j}} = 0 \) and \( y^b(t_j) = 0 \), imply \( S_{i_{\tau_j}} = \frac{1 - \theta_b}{1 - \theta_b + \sigma} (1 - \theta_b) S_{i_{\tau_j}} \) so that we have \( X_{i_{\tau_j}} = \pi^* \frac{1 - \theta_b}{1 - \theta_b + \sigma} (1 - \theta_b) S_{i_{\tau_j}} \). Therefore, for a consumer who has wealth of \( 10^6 \) dollars on an observation date, the observation cost is \( \lambda \pi^* \frac{1 - \theta_b}{1 - \theta_b + \sigma} (1 - \theta_b) 10^6 \) dollars. (Although the length of the optimal inattention interval is invariant to \( \psi_s \), the dollar-equivalent observation cost depends on \( \psi_s \). For this calculation, we have set \( \psi_s = 0.01 \.)}

\begin{table}
<table>
<thead>
<tr>
<th>( \theta_X = \theta_S = \theta^* )</th>
<th>Information Cost</th>
<th>( \tau^* ), ( \theta = 0 )</th>
<th>( \theta^* \times 10^6 )</th>
<th>( \tau^* ), ( \theta = \theta^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baseline</td>
<td>2.3</td>
<td>0.097</td>
<td>6.5</td>
<td>0.190</td>
</tr>
<tr>
<td>( \kappa = 0.001 )</td>
<td>23.1</td>
<td>0.309</td>
<td>63.6</td>
<td>0.593</td>
</tr>
<tr>
<td>( \rho = 0.02 )</td>
<td>2.6</td>
<td>0.098</td>
<td>7.8</td>
<td>0.198</td>
</tr>
<tr>
<td>( \alpha = 3 )</td>
<td>2.4</td>
<td>0.092</td>
<td>5.9</td>
<td>0.174</td>
</tr>
<tr>
<td>( r_L = 0 )</td>
<td>2.3</td>
<td>0.080</td>
<td>11.3</td>
<td>0.194</td>
</tr>
<tr>
<td>( r_F = 0.03 )</td>
<td>2.8</td>
<td>0.084</td>
<td>27.3</td>
<td>0.281</td>
</tr>
<tr>
<td>( \mu = 0.07 )</td>
<td>2.7</td>
<td>0.089</td>
<td>6.1</td>
<td>0.161</td>
</tr>
<tr>
<td>( \sigma = 0.2 )</td>
<td>2.1</td>
<td>0.097</td>
<td>8.1</td>
<td>0.218</td>
</tr>
</tbody>
</table>

\( \theta^* \) is the largest value of \( \theta = \theta_X = \theta_S \) that leads to constant optimal inattention spans. Baseline parameters: \( \alpha = 4, \rho = 0.01, r_L = 0.01, r_F = 0.02, \mu = 0.06, \sigma = 0.16, \kappa = 0.0001 \).

\end{table}
$\theta^*$, which is the largest value of $\theta = \theta_X = \theta_S$ such that the time between consecutive observations eventually, with probability 1, becomes constant. For values of $\theta$ larger than $\theta^*$, the optimal rule remains state-dependent indefinitely and the frequency of observations will exceed the frequency of transactions indefinitely. The values reported in column 3 are actually $\theta^* \times S_j = \theta^* \times 10^6$ so that, for instance, in the baseline case, the fixed component of the transactions cost is $6.60 for a millionaire. Finally, column 4 reports the time between consecutive observations when $\theta = \theta^*$.

Table I allows us to draw two broad conclusions. First, even tiny information costs can lead to substantial inattention intervals. Column 2 shows that even when the fixed component of transactions costs is zero ($\theta = \theta_X = \theta_S = 0$), a consumer who has $1 million in her investment portfolio and incurs an information cost equivalent to about $2, will observe her portfolio at approximately a monthly frequency, which is the empirical frequency reported by Alvarez, Guiso, and Lippi (2012). Second, the fixed component of transaction costs can significantly magnify the effect of information costs to produce even larger inattention spans. The inattention spans in column 4 are about twice as large as the inattention spans in column 2. As we pointed out earlier, a consumer who transfers assets from the investment portfolio to the transactions account might consider transferring more assets into the transactions account than are needed to finance consumption until the next observation date so as to preserve the option not to transfer assets on that date. However, if the fixed component of the transactions costs is small, the value of this option is not large enough to overcome the lower interest earnings on assets held in the transactions account. Therefore, the consumer will optimally choose to arrive at the next observation date with a zero balance in the transactions account. Arriving at an observation date with a zero balance in the transactions account necessitates a transfer from the investment portfolio to the transactions account at that observation date, and hence observations and transactions are synchronized.

Because of this synchronization, the optimal inattention interval is determined as if the fixed component of transaction costs and information costs are bundled together, effectively magnifying the impact of the information cost. For instance, with an information cost of $2.30, the inclusion of a fixed component of transactions costs with $\theta = \theta^*$ approximately doubles the optimal time between observations to more than 2 months. The time between observations is invariant to the proportional transaction cost parameters $\psi_b$ and $\psi_s$. The irrelevance of $\psi_b$ results from the fact that in the long run, the consumer does not ever transfer any assets from the transactions account to the investment portfolio and thus never incurs any cost $\psi_b y^b$. On any observation date in the long run, all of the consumer’s wealth is in the investment portfolio. To consume any of this wealth, the consumer effectively must pay a tax at rate $\psi_s$ to transfer the wealth to the transactions account. Thus $\psi_s$ is a pure consumption tax and hence reduces the path of consumption by a fraction $\psi_s$, while leaving
the timing of transfers unchanged. This result is formalized in Proposition 6 in the Appendix.

Proposition 3 implies that in the long run, the consumer will transfer assets in the same direction (from the investment portfolio to the transactions account) on every observation date. Therefore, if the consumer is sufficiently risk averse\(^{27}\) so that optimal \(\phi_j\) is interior to \([0, 1]\), then an Euler equation, described in the following proposition, holds in the long run.\(^{28}\)

**Proposition 4:** There exists \(\theta_S > 0\) such that if \(0 \leq \theta_S < \theta_S\) and \(\alpha > \frac{\mu - r_f}{\sigma}\), then in the long run \(E_t\{c_{t+1}^{\pi} \left( \frac{P_{t+1}}{P_t} - e^{r_f \tau_j} \right) \} = 0\).

The Euler equation in Proposition 4, which is proved in the Appendix, resembles a standard Euler equation, but it is important to note that here the Euler equation applies only to intervals of time that begin and end on dates at which observations and transactions occur. This implication of the model is consistent with the evidence reported in Jagannathan and Wang (2007), who found that the consumption Euler equation is empirically more successful on dates and at frequencies where decisions are likely to be made.

5. CONCLUDING REMARKS

Rules that govern infrequent adjustment are typically categorized as time dependent or state dependent. Time-dependent rules depend only on calendar time and can optimally result from costs of gathering and processing information. State-dependent rules depend on the value of some state variable, typically reaching some trigger threshold, and can be the optimal response to a transactions cost. Our model combines costly information and costly transactions. In general, on any observation date, the consumer chooses the length of time until the next date at which to gather information and reoptimize, but that length of time may be state-dependent. Moreover, conditional on the information observed at that future date, the agent’s action (or lack thereof) may also be state-dependent. Thus, in general, the model has elements of both state- and time-dependent rules. If the fixed component of the transactions cost is sufficiently small, the optimal behavior evolves to a rule that is time-dependent. Once the consumer arrives at an observation date with a sufficiently small balance in the transactions account, she will optimally choose to arrive at all subsequent observation dates with zero liquid assets in the transactions account.

\(^{27}\)It is worth noting that “sufficiently risk averse” need not require a very high value of \(\alpha\). For instance, if the expected equity premium is \(\mu - r_f = 0.04\) and the standard deviation of the rate of return on equity is \(\sigma = 0.16\), then any value of \(\alpha\) greater than 1.5625 will be sufficiently risk averse.

\(^{28}\)Eberly (1994) showed that a version of the consumption Euler equation also holds in a model with a fixed cost of adjusting the stock of durables, by considering consumption at consecutive adjustment dates.
In our model, this behavior results from the facts that (i) the return on the transactions account is dominated by the return on riskless bonds in the investment portfolio and (ii) with a small fixed component of the transactions cost, there is not sufficient reason to arrive at the next observation date with assets in the transactions account to have the option not to transact at that time.

The endogenous emergence of a time-dependent rule is a novel feature of our model. However, there are forces that could prevent this situation from arising, even within the model. As we have pointed out, if the fixed component of the transactions cost is large, the consumer may choose to arrive at observation dates with a positive balance in the transactions account. And if the consumer arrives at an observation date with a positive amount of liquid assets, then the state variable $x_t$ could potentially take on any positive value, so that a time-dependent rule would not be optimal, even in the long run. Outside the model, one might consider allowing for the arrival of labor income in the transactions account or the occurrence of attention-grabbing events that occur when the consumer is not at a planned observation date.\footnote{Recent work by Yuan (2008) has documented that investors appear to react to news that the stock market has reached a new peak.}

We offer a more general view of time dependence by thinking of the distribution of the length of inattention intervals. With a sufficiently small fixed component of transactions costs, the long run is characterized by a constant length of inattention intervals and thus the distribution is degenerate. More generally, even if the model is configured or amended so that time dependence does not emerge, the value of $x_{ij}$ will frequently be below the lower trigger value. Whenever $x_{ij}$ is lower than the lower trigger value, the length of time until the next observation date will be the same regardless of the value of $x_{ij}$. Therefore, the distribution of inattention intervals will have a mass at that length of time.\footnote{A similar argument applies to the inattention interval associated with optimal behavior for $x_{ij}$ above the upper trigger value.} This mass point in the distribution of inattention intervals can be viewed as a generalization of the endogenous emergence of a time-dependent rule that we have analyzed in this paper.

REFERENCES


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