Risks For the Long Run: Estimation and Inference*

Ravi Bansal†
Dana Kiku‡
Amir Yaron§

First Draft: June 2006
Current Draft: November 2010

*We thank Andy Abel, George Constantinides, Lars Hansen, John Heaton, Tom Sargent, Martin Schneider, Annette Vissing-Jorgensen, seminar participants at Boston University, Copenhagen School of Business, Carnegie-Mellon University, Duke University, Harvard University, LBS, LSE, Norwegian School of Management, NYU, Stockholm School of Economics, Tel-Aviv University, University of California-Berkeley, University of Chicago, University of Texas-Austin, University of Washington-St. Louis, University of Wisconsin, Wharton, Yale, and conference participants at the Summer Econometric Society Meetings, the NBER Asset-Pricing meeting, Nemmers Prize Conference (2007), Frontiers of Monetary policy—St. Louis FRB, and CIRANO Financial Econometrics Conference in Montreal for useful comments. Yaron thanks support from the Rodney White Center at the Wharton School.

†Fuqua School of Business, Duke University and NBER, ravi.bansal@duke.edu.
‡The Wharton School, University of Pennsylvania, kiku@wharton.upenn.edu.
§The Wharton School, University of Pennsylvania and NBER, yaron@wharton.upenn.edu.
Abstract

The long-run risks (LRR) asset pricing model emphasizes the role of low-frequency movements in expected growth and economic uncertainty, along with investor-preferences for early resolution of uncertainty, as an important economic-channel that determines asset prices. In this paper, we empirically evaluate the LRR model and find considerable support for it. We develop a method that allows us to estimate models with recursive preferences, latent state variables, and time-averaged data. We document that time-averaging can significantly affect parameter estimates and statistical inference. Imposing the pricing restrictions and explicitly accounting for time-averaging, we show that the estimated LRR model can account for the joint dynamics of aggregate consumption, asset cash flows and prices, including the equity premia, risk-free rate and volatility puzzles.
1 Introduction

The long-run risks (LRR) model developed by Bansal and Yaron (2004) is shown to be able to account for many asset pricing puzzles. The model captures the intuition that risks embodied in low-frequency movements in the expected growth and conditional volatility of consumption are important for understanding asset prices. In this paper, we use macro and financial data to estimate and empirically evaluate the LRR model. Estimation of the model has to confront three challenges. First, the return on the aggregate consumption asset, a key input of the LRR model based on Epstein and Zin (1989) preferences, is not observable. A second challenge is in extracting low-frequency expected growth and volatility movements in the observed consumption data. A third challenge is dealing with time-averaging that emanates from a potential mismatch between the decision interval of the agent and the sampling frequency of the data, which could distort inference and parameter estimates. We develop an estimation method that addresses these challenges and find considerable support for the LRR model. Our evidence suggests that long-run growth risks and time-variation in economic uncertainty are important risk channels in financial markets.

To make estimation of the model feasible, we exploit consumption dynamics and the model pricing restrictions to derive the unobservable return on the consumption claim in terms of the state variables and underlying parameters. We extract the latent state variables – the long-run growth component and conditional volatility of consumption growth – from the observed price-dividend ratio and the risk-free rate by imposing the model-implied cross-equation restrictions. We show how to account for a discrepancy between the frequency of the model and the sampling interval of the data and derive the dynamics of time-averaged cash-flows and prices. Incorporating all these pieces together, we estimate the model using a GMM framework along the lines of Hansen (1982) by exploiting a set of moment restrictions of the joint dynamics of time-averaged consumption, dividends and asset prices.

Our estimation evidence is based on annually sampled data and the assumption that the decision interval of the agent is monthly. We use a long span of data over the 1929-2008 period that covers a wide range of macroeconomic events which potentially contain important information about variation in expected consumption growth and its volatility. To highlight the importance of long-run risk channels, we estimate two nested models. Our encompassing model is the LRR model with both low-frequency movements in expected growth and time-
varying consumption volatility. The other specification assumes \textit{i.i.d.} growth rate dynamics allowing for only short-run consumption risks. The two models are estimated by imposing restrictions of temporal aggregation.

Overall, our empirical findings provide considerable support for the LRR model. Our estimation results suggest that: (i) investors have a preference for early resolution of uncertainty, (ii) shocks to the expected growth component of consumption have a long-run effect that persists beyond typical business cycle frequencies, (iii) although variation in consumption volatility is relatively small, the effect of volatility shocks is long-lasting, (iv) the model is not rejected by the overidentifying restrictions and can account for the observed risk premia and volatility of equity returns, the risk-free rate dynamics and other stylized features of macro and asset market data.

More specifically, the estimated values of risk aversion and intertemporal elasticity of substitution (IES) in the LRR model, which we denote LRR(h12) to describe the degree of time-averaging, are 6.3 and 2.3 respectively. Both estimates have relatively tight standard errors – 3.1 for risk aversion and 1.2 for the IES. We find that the long-run growth and volatility components of consumption are highly persistent, with implied annual autocorrelations of 0.94 and 0.87, respectively.\footnote{Annual persistence is computed by raising the estimate of monthly autocorrelation to the 12-th power.} These estimates underscore the long-run nature of expected growth and volatility shocks that manifest into high equity premia and high volatility of asset prices. The estimated model implies a market risk premium of 4.8\% and a 17.4\% volatility of stock market returns, and generates a low risk-free rate of about 1.3\%. Importantly, the LRR(h12) specification is not rejected by the overidentifying moments of the joint dynamics of consumption, market dividends and returns – the p-value associated with the J-test statistic is 12\%.

In contrast, the IID specification produces an extremely large estimate of risk aversion of almost 30, which is the only mechanism that allows it to match the equity premium. Even then, the IID model which shuts off any variation in expected growth and discount rates, fails to account for variation in asset prices and, therefore, is sharply rejected by the data with an effectively zero p-value. We also find that opening up variation in the low-frequency growth component lowers the estimate of risk aversion considerably, compared to the IID model since long-run growth risks contribute substantially to return volatility, risk premia, and persistence of the price-dividend ratio. While offering significant improvements over the
IID model, the specification that allows for long-run growth risks but excludes variation in the conditional volatility is still formally rejected, largely by the moments related to return and consumption growth predictability.

To evaluate the effect of time-averaging, we also estimate an annual version of the LRR model, referred to as LRR(h1), that assumes that the decision interval of the agent and the data sampling frequency are both annual and, therefore, ignores restrictions of temporal aggregation. We find this model specification to be strongly rejected in the data. Similar to the LRR(h12) case, the estimates of the LRR(h1) model imply high persistence in the expected consumption growth. However, the contribution of long-run risks to the volatility of consumption growth in the two specifications is very different – it is much higher in the LRR(h12) model than in the LRR(h1) specification. These differences are driven by time-averaging effects. In the LRR(h1) specification, the entire shock to annual consumption growth is identified as a short-run risk, while under the null of the LRR(h12) model, a portion of this shock comes from long-run risk fluctuations. Thus, the LRR(h1) model is misspecified, which leads to distortions in parameter estimates and, in particular, a much larger estimate of risk aversion of about 19. Using simulations, we further document that when time-averaging of monthly dynamics is ignored, the model is overly rejected, the risk aversion estimate rises, and the contribution of long-run growth risks diminishes, all of which is consistent with our empirical findings. Our evidence suggests that when the restrictions of time-averaging are not imposed in the estimation, a sizable portion of the low-frequency growth shock tends to be attributed to the short-run shock, which lowers the role of long-run risks and makes it hard for the model to match the volatility of asset returns and prices. Note that in the i.i.d. specification which assumes away low-frequency shocks, time-averaging has virtually no effect in estimation. Overall, our evidence suggests that accounting for temporal aggregation in estimating the model and measuring the contribution of different risk sources is extremely important, particularly in the presence of low-frequency fluctuations in consumption.

In addition to time-series dynamics, we evaluate the cross-sectional implications of our LRR(h12) model for size and book-to-market sorted portfolios. We show that assets with large mean returns, such as value and small market capitalization, are more sensitive to long-run and volatility risks. Similar to the implications for the market portfolio, we find that low-frequency growth risks are the key source of risk premia in the cross section. Importantly, we show that the LRR model is also able to replicate the failure of the CAPM — our LRR(h12)
specification generates low market betas and high CAPM alphas of the value-minus-growth and small-minus-large strategies, of the same magnitudes as in the data.

Earlier work by Epstein and Zin (1989) relies on the GMM of Hansen and Singleton (1982) to estimate a recursive preference based model. However, they replace the return on the consumption asset with the value-weighted market return. We show that this approach may bias parameter estimates and lead to false rejections of the model since the market return may be fairly different than the return on the consumption claim. Our approach, as discussed above, allows us to infer the dynamics of the wealth return from the observed data and obviates the need to substitute it with the stock market return. More recently, a series of papers explore the ability of long-run growth risks to account for asset market data. Bansal, Dittmar, and Lundblad (2005), Hansen, Heaton, and Li (2008) show that long-run risks in cash flows are important in understanding cross-sectional variation in risk premia. Bekaert, Engstrom, and Xing (2005), Bansal, Gallant, and Tauchen (2007), Kiku (2006), Malloy, Moskowitz, and Vissing-Jorgensen (2009), Lettau and Ludvigson (2005), Parker and Julliard (2005), Jagannathan and Wang (2010), and Constantinides and Gosh (2008) exploit features of the recursive preferences and/or of long-run risks to account for various features of asset returns. Distinct from these papers, we estimate and evaluate the LRR model in the GMM framework while imposing the model restrictions on the joint dynamics of consumption, dividends, and prices that appropriately account for temporal aggregation.

The paper continues as follows. Section 2 presents the model and its testable restrictions. Section 3 provides details of our estimation methodology. Section 4 describes the data. We report and discuss results of our empirical analysis in Section 5. Section 6 provides concluding remarks.

2 Model

In this section we specify the long-run risks model based on Bansal and Yaron (2004). The underlying environment is one with complete markets and a representative agent that has Epstein and Zin (1989) type preferences, which allow for a separation of risk aversion and the elasticity of intertemporal substitution. Specifically, the agent maximizes her life-time
utility, which is defined recursively as,

\[ V_t = \left[ (1 - \delta)C_t^{\frac{1-\gamma}{\theta}} + \delta \left( E_t[V_{t+1}^{1-\gamma}] \right)^{\frac{1}{\theta}} \right]^{\frac{\theta}{1-\gamma}}, \]  

(1)

where \( C_t \) is consumption at time \( t \), \( 0 < \delta < 1 \) reflects the agent’s time preferences, \( \gamma \) is the coefficient of risk aversion, \( \theta = \frac{1-\gamma}{1-\psi} \), and \( \psi \) is the elasticity of intertemporal substitution (IES). Utility maximization is subject to the budget constraint,

\[ W_{t+1} = (W_t - C_t)R_{c,t+1}, \]  

(2)

where \( W_t \) is the wealth of the agent, and \( R_{c,t} \) is the return on all invested wealth.

Consumption growth has the following dynamics:

\[ \Delta c_{t+1} = \mu_c + x_t + \sigma_t \eta_{t+1} \]
\[ x_{t+1} = \rho x_t + \varphi_e \sigma_t e_{t+1} \]
\[ \sigma^2_{t+1} = \sigma^2_0 + \nu(\sigma^2_t - \sigma^2_0) + \sigma_w w_{t+1}, \]  

(3)

where \( \Delta c_{t+1} \) is the growth rate of log consumption, and the three shocks, \( \eta, e, \) and \( w \) are assumed to be i.i.d Normal and uncorrelated. The conditional expectation of consumption growth is given by \( \mu_c + x_t \), where \( x_t \) is a small but persistent component that captures long-run risks in consumption growth. The parameter \( \rho \) determines the persistence in the conditional mean of consumption growth. For parsimony, as in Bansal and Yaron (2004), we have a common time-varying volatility in consumption, which, as shown in their paper, leads to time-varying risk premia. The unconditional variance of consumption is \( \sigma^2_0 \) and \( \nu \) governs the persistence of the volatility process.

### 2.1 The Long-Run Risks Model’s IMRS

For these preferences, the log of the IMRS, \( m_{t+1} = \log(M_{t+1}) \), is

\[ m_{t+1} = \theta \log \delta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1)r_{c,t+1}, \]  

(4)
where \( r_{c,t+1} \) is the continuous return on the consumption asset, which is endogenous to the model. Thus, in order to characterize the intertemporal marginal rate of substitution, one needs to solve for the unobservable return on the consumption claim. To solve for \( r_{c,t+1} \), we use the dynamics of the consumption growth and the log-linear approximation of the continuous return, namely,

\[
r_{c,t+1} = \kappa_0 + \kappa_1 z_{t+1} + \Delta c_{t+1} - z_t ,
\]

where \( z_t = \log(P_t/C_t) \) is the log price-consumption ratio (i.e., the valuation ratio corresponding to a claim that pays aggregate consumption), and \( \kappa \)'s are constants of log-linearization,

\[
\kappa_1 = \frac{\exp(\bar{z})}{1 + \exp(\bar{z})} \quad \quad (6)
\]

\[
\kappa_0 = \log(1 + \exp(\bar{z})) - \kappa_1 \bar{z}, \quad \quad (7)
\]

where \( \bar{z} \) denotes the mean of the log price-consumption ratio.

To derive the time series for \( r_{c,t+1} \), we require a solution for log price-consumption ratio, which we conjecture follows,

\[
z_t = A_0 + A_1 x_t + A_2 \sigma_t^2 . \quad \quad (8)
\]

The solution coefficients \( A \)'s depend on all the preference parameters and the parameters that govern the dynamics of consumption growth. For notational ease, let \( z_t = A' Y_t \), where \( Y_t' = [1 \ x_t \ \sigma_t^2] \) is the vector of state variables, and \( A' = [A_0 \ A_1 \ A_2] \), which is given by

\[
A' = \left[ A_0 \quad \frac{1 - \frac{1}{\kappa_1 \rho}}{1 - \kappa_1 \rho} \quad -\frac{(\gamma - 1)(1 - \frac{1}{\kappa_1})}{2 (1 - \kappa_1 \nu)} \left[ 1 + \left( \frac{\kappa_1 \phi e}{1 - \kappa_1 \rho} \right)^2 \right] \right] \quad \quad (9)
\]

be the corresponding vector of price-consumption elasticities.\(^2\) As discussed in Bansal and Yaron (2004), the elasticities of the price-consumption ratio with respect to the expected growth component, \( x_t \), and volatility, \( \sigma_t \), depend on the preference configuration. In particular, for the elasticity \( A_1 \) to be positive, the IES parameter has to be greater than one. Moreover, for the price-consumption ratio to exhibit a negative response to an increase

---

\(^2\) The expressions for \( A_0 \) and \( \Gamma_0 \) in equation (13) below, as well as the derivations of all other expressions, are given in Appendix A.1.
in economic uncertainty, the IES again has to be larger than one, given that risk aversion is greater than one.

Note that the derived solutions depend on the approximating constants, $\kappa_0$ and $\kappa_1$, which, in turn, depend on the endogenous mean of the price-consumption ratio, $\bar{z}$. In order to solve for $\bar{z}$, we first substitute expressions for $\kappa$’s (equations (6) and (7)) into the expressions for $A$’s and solve for the mean of the price-consumption ratio. Specifically, $\bar{z}$ can be found numerically by solving a fixed-point problem,

$$\bar{z} = A(\bar{z})'\bar{Y},$$  
(10)

where the dependence of the $A$’s on $\bar{z}$ is given above, and $\bar{Y}$ is the mean of the state vector $Y$. This is quite easy to implement in practice.

Given the solution for $z_t$, the IMRS can be stated in terms of the state variables and innovations,

$$m_{t+1} = \Gamma'Y_t - \Lambda'\zeta_{t+1},$$  
(11)

where the three sources of risks are

$$\zeta_{t+1}' = \left[ \sigma_t\eta_{t+1} \quad \sigma_t\epsilon_{t+1} \quad \sigma_w\omega_{t+1} \right],$$  
(12)

and the three dimensional vectors $\Gamma$ and $\Lambda$ are given by,

$$\Gamma' = \left[ \Gamma_0 \quad -\frac{1}{\psi} \quad -(\gamma - 1)(\gamma - \frac{1}{\psi})\frac{1}{2}[1 + \left(\frac{\kappa_1\phi_e}{1 - \kappa_1\rho}\right)^2] \right],$$  
(13)

$$\Lambda' = \left[ \Gamma \quad (\gamma - \frac{1}{\psi})\frac{\kappa_1\phi_e}{1 - \kappa_1\rho} \quad -(\gamma - 1)(\gamma - \frac{1}{\psi})\frac{\kappa_1}{2(1 - \kappa_1\rho)}\left[1 + \left(\frac{\kappa_1\phi_e}{1 - \kappa_1\rho}\right)^2\right] \right].$$  
(14)

Note that the stochastic discount factor in equation (11) is exact up to an approximation error emanating from the linearization around the theoretical value of average price-consumption ratio. We find that this approximation error is quite small and does not materially affect our empirical results that follow. Appendix A.5 provides a detailed discussion of the magnitude of the approximation error and a comparison of the above log-linear solution with a solution based on numerical methods.

Other assets can easily be priced using the IMRS given in equation (11). We assume
that the dividend dynamics for any other asset $j$ follow

$$
\Delta d_{j,t+1} = \mu_j + \phi_j x_t + \varphi_j \sigma_t u_{j,t+1}
$$

(15)

where $\phi_j$ and $\varphi_j$ determine asset $j$'s exposure to the long-run and volatility risks, respectively. Exposure to short-run consumption risks is determined by the correlation between dividend and consumption innovations, $u_{j,t+1}$ and $\eta_{t+1}$, which we denote by $\varrho_j$. We let $\Delta d_{t+1}$ denote the dividend growth rate of the aggregate market portfolio, and reserve $d$-subscript for various quantities of the stock market index. Specifically, we use $\mu_d$, $\phi_d$, and $\varrho_d$ for the aggregate market dividend and let $z_{d,t}$ and $r_{d,t+1}$ denote the price-dividend ratio and the return on the aggregate market portfolio.

The first-order condition yields the following asset pricing Euler condition,

$$
E_t [\exp (m_{t+1} + r_{j,t+1})] = 1,
$$

(16)

where $r_{j,t+1}$ is the log of the gross return on asset $j$. Similar to the claim to consumption, the price-dividend ratio for any asset $j$, $z_{j,t} = A_j^t Y_t$, with the solutions given in Appendix A.2. Furthermore, given the expression for the IMRS, it follows that the risk premium on asset $j$ is,

$$
E_t [r_{j,t+1} - r_{f,t} + 0.5 \sigma_{t,r_j}^2] = \beta_{\eta,j} \lambda_{\eta} \sigma_t^2 + \beta_{e,j} \lambda_e \sigma_t^2 + \beta_{w,j} \lambda_w \sigma_w^2,
$$

(17)

where $\beta_{i,j}$ is the return beta for asset $j$ with respect to the $i^{th}$ risk source where $i = \{\eta, e, w\}$, and $\lambda_i$ is the corresponding entry of the vector of market prices of risks, $\Lambda$. Under the structure of the model, the return $\beta$'s and market price of risks $\lambda$'s will be functions of the preference parameters and the underlying parameters of consumption and dividend dynamics, details of which are given in Appendix A.2. Finally, it is easy to verify that the risk free rate can be represented as,

$$
r_{f,t} = F_0 + F_1 x_t + F_2 \sigma_t^2 = F' Y_t
$$

(18)

where the loading coefficients are given in Appendix A.3.

Intertemporal elasticity of substitution is a critical parameter in the LRR model. Work by Giovannini and Weil (1989), Tallarini (2000), Hansen, Heaton, and Li (2008), and Hansen and Sargent (2006) considers the special case where the IES parameter is one. Our estimation
methodology nests this special case in a continuous fashion (details are given in Appendix A.4). Namely, the IMRS components as given in equation (11) adjust in a continuous way as one takes the limit of the IES parameter at one.³ That is,

\[
\lim_{\psi \to 1} \kappa_1 = \delta \quad \lim_{\psi \to 1} \Gamma' = \Gamma'(\psi = 1, \kappa_1 = \delta) \quad \lim_{\psi \to 1} \Lambda' = \Lambda'(\psi = 1, \kappa_1 = \delta).
\]

(19)

The discussion above highlights the fact that the generalized pricing kernel (11) does not confine an econometrician to a prespecified value of the IES. That is, in estimation the IES is a free parameter.

3 Estimation Method

The key point of this paper is to evaluate and test the LRR model’s ability to jointly match consumption, dividend, and asset price dynamics. Specifically, we are interested in the estimates of the model’s parameters, \( \Theta = \{\gamma, \psi, \delta, \mu_c, \rho, \varphi_c, \sigma_0^2, \nu, \sigma_w, \mu_d, \phi_d, \varphi_d, \theta_d\} \), and whether at these estimates the model can account for aggregate macro and asset price data. This would provide direct evidence about the importance and magnitude of long-run growth and volatility risks in consumption and dividends, and the magnitude of preference parameters (IES and risk aversion). Our empirical results, as discussed below, shed light on the broader questions regarding the role of long-run versus shorter horizon risks (e.g., business cycle risks), the duration of long-run shocks, and the interaction of these dynamics with preferences in understanding risks that drive financial markets.

Our estimation strategy is a moment-based, GMM approach. We provide analytical expressions for moments of consumption and dividend dynamics, asset prices, and joint dynamics between consumption and asset prices. More specifically, we recover the state-variables \( x_t \) and \( \sigma_t^2 \) from the observed data, then provide analytical expressions for the conditional moments in terms of the state-variables, and then derive unconditional moments purely in terms of the parameters of interest, \( \Theta \). An issue of considerable importance in our estimation is that of time-averaging. The longest and best quality (in the sense of measurement error) observed data is annual. However, it is natural to assume the agent’s

³Evaluating the pricing kernel (11) under the above restrictions gives exactly the same solution as in Giovannini and Weil (1989), Tallarini (2000), and Hansen, Heaton, and Li (2008).
decision interval is shorter, and we assume it is monthly. We account for this time-averaging and are still able to provide analytical moments in terms of the parameters of interest, $\Theta$. Earlier papers that account for time-averaging in estimation in an asset-pricing context include Hansen and Sargent (1983) and Heaton (1995). We show that time-averaging has important effects on model estimation and inference. In the presence of time-averaging, the shocks in IMRS (equation (11)) cannot be recovered and hence the standard Euler Equation-based estimation approach, as in Hansen and Singleton (1982), cannot be used.\footnote{Using equation (11), the log of the time-averaged IMRS follows:

$$m_{t+h,k} = \sum_{j=1}^{h} m_{t+j} = \hat{\Gamma}'Y_t - \sum_{j=1}^{h} \left[ \lambda_{\eta}\sigma_{t+j-1}\eta_{t+j} + \lambda_{e}\sigma_{t+j-1}e_{t+j} + \lambda_{w}\sigma_{w}w_{t+j} \right]$$

It is easily shown that long-run and volatility shocks, $e_t$ and $w_t$, can be extracted from the available high-frequency financial data. However, short-run consumption innovations, $\eta_t$, cannot be recovered unless consumption data are observed at a fine (monthly) frequency.} Our moments-based approach allows for estimation even when the shocks and the IMRS are not available to the econometrician.

### 3.1 Recovering the state variables

To recover the state variables $x_t$ and $\sigma_t^2$, we use the fact that the price dividend ratio and the risk free rate are affine in these state variables. That is, equations (8) and (18) constitute, for each date $t$, the following system

$$\mathcal{M}_t(\Theta) = \begin{bmatrix} z_{d,t} \\ r_{f,t} \end{bmatrix} - \begin{bmatrix} A_d(\Theta)' \\ F(\Theta)' \end{bmatrix} \begin{bmatrix} Y_t \end{bmatrix}.$$  \hfill (20)

Given the observed financial variables, the price-dividend ratio and the risk-free rate, it is possible to recover $x_t$ and $\sigma_t$. In practice we solve, for each date $t$ and given $\Theta$, the pair of state variables that minimizes the system given in (20) above, while ensuring positivity of the variance. This has the added virtue of minimizing any measurement errors in the observed price-dividend ratio and the ex-ante real risk-free rate. It is worth noting that we utilize here the monthly price-dividend ratio and risk free rate on an annual frequency corresponding to the appropriate beginning of year information. Moreover, it is important to recognize that this extraction step of state variables is ultimately done simultaneously with the overall GMM estimation of the moment conditions characterizing asset prices—see
further discussion on the estimation below.

### 3.2 Moments

Given the state variables, we focus on moments that capture several key features of the consumption, dividend, and asset data. First, we focus on the consumption and dividend growth transition moments which ensures that the consumption and dividend dynamics are consistent with the data. The second set of moments focuses on the level of returns, and the third set of moments focuses on predictability of asset returns and consumption. More specifically, the list of moments we use for consumption and dividend growth are their respective volatility, autocorrelation, and joint correlations. In terms of return-based moments, we utilize the level of equity and risk-free rates, and the price-dividend ratio. In addition, we use the volatility of the market return, which exposes our estimation to both the asset return level puzzles as well as the volatility puzzles. To account for predictability, we use as moments the correlations of the price-dividend ratio with future consumption growth and the market return respectively. We also use the contemporaneous correlation between the price-dividend ratio and consumption volatility as a moment capturing their negative relationship. Finally, we impose the model implications for the orthogonality between annual consumption growth innovations and $x$ – the expected consumption growth state, as well as the orthogonality between the innovation to the squared consumption growth innovation and the volatility state $\sigma^2$ as two additional moments. A detailed description of all the moment conditions is given in Table III, under the consumption & dividend moments, asset return moments, and predictability moments respectively. Our estimation approach, which uses first and second moments, allows us to impose model-restrictions and goes beyond the mean return restrictions that follow from using only $E(M_{t+1}R_{t+1}) = 1$. This is important as the second moment restrictions (such as the volatility puzzles) contain considerable information about model parameters of interest.\(^5\)

To derive the unconditional moments, we first provide an analytical expression for the conditional moments based on the state-variables. We can express such moments for general frequencies of time aggregation. We fix the decision interval to be monthly, and hence in our baseline configuration, time indexed by $t$, accumulates by monthly increments. Although

\[^5\text{E.g., with an i.i.d growth model, the price-dividend ratio is constant and has zero variance, which is clearly at odds with the data.}\]
in practice we will be using observable data that is at an annual frequency, our notation is
general in accommodating a sampling frequency that is lower (than the monthly decision
interval) with \( h \) months per year. That is, quarterly data corresponds to \( h = 3 \), and annual
data to \( h = 12 \). Specifically, we assume monthly consumption, \( C_t \), is unobserved and the
only observable consumption is total consumption over the year, namely, \( \sum_{i=1}^{h} C_{t+i} \). Thus, observable
data is available only for \( t = 0, h, 2h \ldots \) etc. To understand the role of time
aggregation, we also estimate a model whereby the econometrician (incorrectly) assumes the
agent has an annual decision interval and the data is sampled annually, in which case \( h = 1 \)
and time is interpreted to accumulate at yearly increments. Based on the notation above, it
can be shown (see Appendix B) that observed log annual consumption growth, \( \Delta c_{t+h,h} \), can
be well approximated as

\[
\Delta c_{t+h,h} \equiv \log \left( \frac{\sum_{i=1}^{h} C_{t+i}}{\sum_{i=1}^{h} C_{t-h+i}} \right) \approx \sum_{j=2}^{h} \frac{j-1}{h} \Delta c_{t-h+j} + \sum_{j=1}^{h} \frac{h-j+1}{h} \Delta c_{t+j}
\]

(21)

where the first subscript denotes the calendar time and the second one denotes the time
periods included in the aggregation. Note this structure for observed annual consumption
growth thwart the usual use of the Euler equation based IMRS approach as the true shocks
are not uniquely identified – this provides further motivation for the use of our estimation
approach.

To better understand the way the estimation method accounts for time-averaging it is
instructive to present several moment conditions explicitly. To do so for the consumption
growth moments, it is first useful to write the annual consumption growth rate in terms of
the state variables available at the beginning month of the base year as well as a sequence
of innovations which are mean zero conditional on that information set,

\[
\Delta c_{t+h,h} = h\mu_c + \frac{\rho(1-\rho^h)}{h(1-\rho)^2} x_{t-h} + \sum_{j=1}^{h-1} a_j \varphi e \sigma_{t-h-1+j} e_{t-h+j} + \sum_{j=1}^{h} b_j \varphi e \sigma_{t+1-j} e_{t+h-j}
\]

\[
+ \sum_{j=0}^{h-1} \frac{j+1}{h} \sigma_{t+h-1-j} \eta_{t+h-j} + \sum_{j=0}^{h-2} \frac{h-j-1}{h} \sigma_{t-1-j} \eta_{t-j}
\]

(22)

where \( a_j = \frac{1}{h\rho^j} \left( [1-\rho^h] - \frac{1}{1-\rho} - (j-1)\rho^{j-1} \right) \) and \( b_j = \frac{1}{h\rho^{j+1}} \left( j - \rho \right) \).

The relevant state variables for computing any conditional moment of consumption
growth (and other variables) are $x_{t-h}$ and $\sigma_{t-h}^2$. For example, the conditional mean of consumption growth is $h\mu_c + \frac{\rho(1-\rho^h)^2}{h(1-\rho)^2}x_{t-h}$. It follows immediately that $E[\Delta c_{t+h,h}] = h\mu_c$. In the case of annual data and a monthly decision interval, this is equal to $12\mu_c$ (where $\mu_c$ is interpreted as a monthly quantity). In the case of an annual sampling frequency and an annual decision interval this simply reduces to $\mu_c$, since $h = 1$ and $t$ moves at annual increments. Based on equation (22), it is also straightforward to compute the second moment of consumption growth,

$$Var[\Delta c_{t+h,h}] = \left[\frac{\rho(1-\rho^h)^2}{h(1-\rho)^2}\right]^2 \text{var}(x_{t-h})$$

$$+ \sum_{j=1}^{h-1} \left[\left(a_j\varphi e\right)^2 + \left(\frac{j}{h}\right)^2\right] \sigma_0^2 + \sum_{j=1}^{h} \left[\left(b_j\varphi e\right)^2 + \left(\frac{j}{h}\right)^2\right] \sigma_0^2.$$  

The $k$-th autocovariance of annual consumption growth can be readily computed in analogous fashion. In the estimation, we utilize the first two autocorrelations of consumption growth as moments. Appendix B provides more details for derivations of the moments used in our estimation. The dynamics for annual dividend growth can be written in a similar fashion as those given in equation (22) for consumption growth (that is dividend growth can be represented in terms of $x_{t-h}$ and $\sigma_{t-h}^2$ and a sequence of shocks orthogonal to time $t-h$ information). Hence, the variance, the first autocorrelation of dividend growth, as well as the covariation between annual dividend and consumption growth also serve as moments in our estimation.

To capture the volatility dynamics, conditional moments are constructed in a similar fashion to those for consumption and dividend growth. In particular, define $\eta_{t+h,h} \equiv \Delta c_{t+h,h} - \mu_c h - \frac{\rho(1-\rho^h)^2}{h(1-\rho)^2}x_{t-h}$. The variance of $\eta_{t+h,h}$ based on information available at time
\[ t - h, \text{is then given by,} \]

\[ \sigma_{t-h|h}^2 \equiv E_{t-h}[(\eta_{t+h|h})^2] = \sum_{j=1}^{h-1} \left( (a_j \varphi_e)^2 \left[ \sigma_0^2 (1 - \nu^{j-1}) + \nu^{j-1} \sigma_{t-h}^2 \right] \right) \]

\[ + \sum_{j=1}^{h-1} \left( (b_j \varphi_e)^2 \left[ \sigma_0^2 (1 - \nu^{2(j-1)}) + \nu^{2(j-1)+1} \sigma_{t-h}^2 \right] \right) \]

\[ + \sum_{j=0}^{h-1} \frac{j+1}{h} \left[ \sigma_0^2 (1 - \nu^{(2j-1)}) + \nu^{(2j-1)+1} \sigma_{t-h}^2 \right] \]  

\[ + \sum_{j=0}^{h-2} \frac{h-j-1}{h} \left[ \sigma_0^2 (1 - \nu^{(h-1)-j}) + \nu^{(h-1)-j} \sigma_{t-h}^2 \right] \]  

(23)

Based on this representation, we compute the mean, variance, and first order autocovariance of the squared annual residual.

Turning to moments that include asset prices, it is useful to consider first the annual price-dividend ratio \( z_{d,t|h} \equiv \log \frac{P_t}{\sum_{j=0}^{h-1} D_{t-j}} \) defined as the log of the end of year price over the twelve-month trailing sum of dividends. Recall that the solution to the monthly price-dividend ratio takes the form, \( z_{d,t} = A_{0,d} + A_{1,d} x_t + A_{2,d} \sigma_t^2 \) with the solutions for \( A_{d}s \) given in Appendix A.2. Using the definition of the annual price dividend ratio, it can be shown that its dynamics follow,

\[ z_{d,t|h} = A_{0,d} + A_{2,d} \sigma_0^2 (1 - \nu^h) - \log(h) + 0.5 \mu_d (h-1) \]

\[ + [\pi + A_{1,d} \rho]x_{t-h} + A_{2,d} \nu^h \sigma_t^2 + \sum_{j=1}^{h} (q_j + A_{1,d} \rho^j) \varphi_e \sigma_{t-h-1+j} e_{t-h+j} \]

\[ + \sum_{j=1}^{h-1} \frac{h-j}{h} \varphi_e \sigma_{t-j} u_{t-j+1} + \sum_{j=1}^{h} A_{2,d} \sigma_w \nu^{h-j} w_{t-h+j} \]  

(24)

where \( \pi \) and \( q_j \) are given in Appendix B. It follows that the mean of the annual price-dividend ratio is \( E[z_{d,t|h}] = A_{0,d} + A_{2,d} \sigma_0^2 - \log(h) - 0.5 \mu_d (h-1) \). Similarly, the variance of
the price-dividend ratio is,

\[
\text{Var}[z_{d,t,h}] = [\pi + A_{1,d}\rho^h]^2 \text{var}(x_t) + [A_{2,d} \nu^h]^2 \text{var}(\sigma_t^2) + \sum_{j=1}^{h} [q_j + A_{1,d} \rho^{h-j}]^2 (\varphi_e \sigma_0)^2 \\
+ \sum_{j=1}^{h-1} \left[ \frac{h-j}{h} \varphi_d \right]^2 \sigma_0^2 + \sum_{j=1}^{h} [A_{2,d} \sigma_w \nu^{h-j}]^2.
\]

The \(k\)-th order autocovariance of the price-dividend growth can be computed similarly (see details in Appendix B), and in the estimation we make use of the first and second autocorrelation of the price-dividend ratio.

The remaining moments involve the risk free rate and the market return. Given that both have a known representation in terms of \(\Theta\) and the state variables (e.g., see equation (18) for the risk free rate), we utilize the same methodology to compute their respective unconditional means as well as the volatility of the market return. Finally, these representation allow us also to compute the covariations between the price-dividend ratio and the market return, consumption growth, and consumption volatility respectively, which serve as our last set of moments.

### 3.3 Estimation

Let \(\mathcal{M}^h(\Theta_T; \{Data\})\) denote the difference between the model based moment conditions (evaluated at \(\Theta_T\)) and their data counterpart. That is, an element in this vector is one of the moments described above minus the same moment based purely on the data. We choose the parameter vector \(\Theta\) by evaluating the annual based moment conditions \(\mathcal{M}^h(\Theta; \{Data\})\) while *simultaneously* choosing the state variables \(x_t\) and \(\sigma_t^2\) as described above. Let the set of dates in a sample of length \(T\) that represent the beginning of each year’s information set be \(\mathcal{T} = \{0, h, 2h, ..., T - h\}\), then the parameter vector \(\Theta_T\) is estimated by minimizing the GMM criteria,

\[
\Theta_T = \text{argmin}_\Theta \mathcal{M}^h(\Theta_T; \{Data_{t+h}\}_{t \in \mathcal{T}}) W(\Theta_T) \mathcal{M}^h(\Theta_T; \{Data_{t+h}\}_{t \in \mathcal{T}})
\] (25)
where \( Data_{t+h} \) pertains to observed annual data used in evaluating the moment conditions.\(^6\) The weighting matrix \( W(\Theta_T) \) used in estimation is the diagonal inverse of the variance-covariance matrix of the moment conditions and is updated continuously, motivated by Hansen, Heaton, and Yaron (1996). To construct the chi-squared test for over-identifying restrictions, we compute \( J \)-statistic using Lemma 4.2 in Hansen (1982) which holds for a general weighting matrix. The variance-covariance matrix is computed using the Newey and West (1987) estimator.

4 Data

We use data on consumption and asset prices for the time period from 1930 till 2008. We take the view that this sample better represents the overall variation in asset and macroeconomic data. Importantly, the long span of the data helps in achieving more reliable statistical inference. We work with the data sampled on an annual frequency as they are less prone to errors that arise from seasonality and other measurement problems highlighted in Wilcox (1992).

To estimate the model, we exploit the dynamics of the observed aggregate consumption, the stock market portfolio, and the risk-free rate. Consumption data represent per-capita series of real consumption expenditure on non-durables and services from the NIPA tables available from the Bureau of Economic Analysis. Aggregate stock market data consist of annual observations of returns, dividends, and prices of the CRSP value-weighted portfolio of all stocks traded on the NYSE, AMEX, and NASDAQ. Price and dividend series are constructed on the per-share basis as in Campbell and Shiller (1988), Bansal, Dittmar,\(^6\)

\(^6\)In the estimation of (25) the extraction of state variables \( x_t \) and \( \sigma_t^2 \) follows: given the parameter vector \( \Theta \), a pair of state variables for each date \( t \) is extracted by minimizing a weighted sum of squared errors, i.e.,

\[
\{x_t, \sigma_t^2\} = \arg\min \mathcal{M}_t(\Theta_T) W_T \mathcal{M}_t(\Theta_T), \quad \forall \ t,
\]

where \( W_T \) is a diagonal matrix of second moments of the observed price-dividend ratio and the risk-free rate. To guarantee positivity of the variance component, we solve the minimization problem by searching over a two-dimensional grid in the \( \{x_t, \sigma_t^2\} \)-space. We allow for a grid of 4,000 possible pairs, and make sure that the permissible space is wide enough to ensure that solutions lie in the interior region. We find that refining the grid further or expanding the boundary of the state space does not affect the implied state dynamics and our GMM estimates. Note that, alternatively, one could extract the states by using a constrained least-squares solver that has the advantage of allowing for a continuum of values for \( x \) and \( \sigma^2 \). Given that our grid is quite fine, our estimation yields almost identical results as the one based on the standard constrained OLS.
and Lundblad (2005), and Hansen, Heaton, and Li (2008). Market data are converted to real using the consumer price index (CPI) from the Bureau of Labor Statistics. Growth rates of consumption and dividends are constructed by taking the first difference of the corresponding log series. Finally, the ex-ante real risk-free rate is constructed as a fitted value from a projection of the ex-post real rate on the current nominal yield and inflation over the previous year. To run the predictive regression, we use monthly observations on the three-month nominal yield from the CRSP Fama Risk Free Rate tapes and CPI series. The annual real risk-free rate is defined as the annualized predicted value as of the beginning of year. Table I provides key sample statistics for aggregate consumption growth, the stock market index, and the risk-free rate. As well known, the data feature a sizable equity premium of about 7%, high volatility of equity returns and low and relatively stable interest rates.

To explore the cross-sectional implications of the model, we employ portfolios with opposite size and book-to-market characteristics that are known to have provided investors with quite different premia over the years. The construction of portfolios is standard (see Fama and French (1993)). In particular, for the size sort, we allocate individual firms into 5 portfolios according to their market capitalization at the end of June of each year. Book-to-market quintiles are likewise re-sorted at the end of June by ranking all the firms based on their book-to-market ratios, defined as book equity at the last fiscal year end of the prior calendar year divided by market equity at the end of December of the previous year. NYSE breakpoints are used in both sorts. For each portfolio, we construct value-weighted monthly returns, as well as per-share price and dividend series. Monthly data are then time-aggregated to an annual frequency and converted to real using the consumer price index. Over the sample period, small stocks have outperformed large firms by about 7% and the spread in returns on value and growth firms has averaged almost 6% (see Table VII). Heterogeneity in risk premia across size and book-to-market portfolios is known to present a challenge for the standard CAPM since the market betas show almost no cross-sectional variation. Below, we will evaluate the ability of the LRR model to account for the observed size and value premia as well as the failure of the CAPM betas.
5 Empirical Findings

In this section, we present the estimates of the LRR model and discuss their implications for the joint dynamics of aggregate consumption, dividends and prices of the market portfolio. We also highlight the effect of time-averaging on preference parameter estimates and inference, and evaluate the cross-sectional implications of the model.

5.1 Estimation Evidence

Using the methodology outlined above, we estimate two models: the LRR model and the nested “IID” specification that restricts consumption and dividend dynamics to be \(i.i.d\). For both models the decision interval is assumed to be monthly while the sampling frequency of the data is annual.\(^7\) In the notation introduced in Section 3, this corresponds to \(h = 12\). We will, therefore, refer to the LRR specification as LRR(h12). The parameter estimates of the two models are presented in Table II. Table III provides the data estimates and the corresponding model (population) values for the moments utilized in estimation. These moments are useful in assessing which data dimensions the models have difficulty with and which ones the models fit well. Table III formalizes this by also providing t-statistics for the hypothesis that the model-implied moments are significantly different from the data-based moments.

The results in Table II show that the IID model is strongly rejected – the p-value associated with the J-test statistic is essentially zero. Furthermore, as is commonly found in the literature, when cashflow/consumption dynamics are \(i.i.d\), the risk aversion estimate is quite large. In our estimation, the parameter of risk aversion is almost 30. As a result, this specification does achieve a sizable risk premium. However, the volatility of the annual price-dividend ratio is too low relative to the data (0.11 compared to 0.44) and, in fact, is lower than that of the dividend growth rate.\(^8\) Further, the autocorrelation of the price-dividend ratio implied by the IID specification is zero. Both of these moments are significantly far from their data counterparts as reflected in the sizable t-statistics for the difference between

\(^7\)We also estimated our model using post-war quarterly data and assuming a monthly decision interval, and the results by and large are consistent with the findings reported below using annual data.

\(^8\)Note that although the monthly price-dividend ratio is constant in the IID model, time-averaging results in some variation in the annual price-dividend ratio.
the model-based population values and the data statistics.

The parameter estimates of the LLR(h12) model, reported in the right panel of Table II, provide strong evidence for a persistent component in expected consumption growth, as well as evidence for persistent time-varying uncertainty. The estimate of $\rho$, which governs the autocorrelation of the conditional mean of consumption growth, is 0.9949 and is significantly different from zero. The magnitude of long-run risks is quite small, $\varphi_e = 0.0139$, suggesting that this component contributes relatively little to the overall variation of consumption growth. These parameter values allude to the difficulty in detecting long-run risks solely from the consumption growth data. The persistence parameter of the conditional volatility process, $\nu$, is also large at 0.9892, but is driven by quantitatively small shocks. The estimated long-run risk component of consumption growth, along with realized values of annual growth rates, are plotted in Figure 1. Figure 2 shows the extracted volatility component along with the squared innovation in annual consumption. For expositional purposes, the series in both figures are standardized and the realized (predicted) values are given on the left (right) scale. The figures reveal a quite notable correlation between the predicted and realized series. Notice also that the conditional volatility exhibits a pronounced variation across time and a considerable decline in the 90's. As further shown in Table II, dividends of the market portfolios are significantly exposed to long-run risks, with $\phi_d$ estimated at 3.37. The short-run correlation between consumption growth and dividends, on the other hand, is not estimated precisely, with the point estimate and the standard error both around 30%.

The estimate of risk aversion in the LRR specification is about 6.3 and is quite low from the perspective of the asset pricing literature. The IES estimate is well above 1, which is an essential restriction in order to generate a negative price of volatility risks and a positive relationship between the price-consumption ratio and expected growth (see discussion in Bansal and Yaron (2004)). It is also important to note that the parameter estimates of the LRR(h12) model presented in Table II are largely within one standard error of the values used in many calibrated versions of the LRR model and, in particular, close to those in Bansal, Kiku, and Yaron (2009).

The parameter estimates governing the dynamics of expected consumption growth and volatility capture the economic mechanism highlighted in the LRR literature, first discussed in Bansal and Yaron (2004). They suggest that the expected growth and volatility shocks have an economically significant impact on growth expectations and future uncertainty and,
therefore, assets’ valuations. In addition, the estimates of the long-run risk dynamics indicate that the expected growth shocks have a long-lasting effect on future consumption growth, which is quite different from the implications of typical business-cycle risks.

Table III shows that the LRR(h12) model is able to account for the level and volatility of excess market returns. It also captures the low mean and volatility of the risk-free rate in the data. Further, the estimation puts a relatively large weight on matching the properties of the price-dividend ratio. The model, therefore, generates a persistent and volatile price-dividend ratio matching the dynamics of the observed series. As the table also shows, the model is successful in explaining many other moments of the joint dynamics of consumption, dividends and asset returns. This success manifests itself formally through the $J$-test for overidentifying restrictions that indicates that the LRR(h12) model is not rejected at the standard 5 percent significance level.

To visually evaluate the ability of the LRR model to track the observed price-dividend ratio, we utilize the LRR(h12) estimates of Table II and plot in Figure 3 the time-series of the annual log price-dividend ratio in the data and from inside the model. To construct the model-implied dynamics, we use a reprojection method. In particular, given the model solution we fit the relationship between the annual log price-dividend ratio and the state variables $x_t$ and $\sigma_t^2$. Then, we apply the model-based estimated relationship to the state variables extracted from the data to form a prediction of the price-dividend ratio. The line referenced “Reprojected” in Figure 3 is the resulting fitted log price-dividend ratio; the line labelled “Observed” is the realized series. The figure shows that the model-based price-dividend ratio tracks that of the data quite well, including the declines in 1930 and 2008. This suggests that movements in measured expected growth and consumption volatility indeed drive asset prices as implied by the LRR model.

One important feature of the LRR framework is predictability of consumption growth. Note that our estimation exploits moments related to predictability of consumption growth and equity returns by the price-dividend ratio. These moments, however, have relatively large variances and, therefore, do not receive too much weight in estimation. We have also experimented with augmenting our estimation-moments by including additional moments of longer-horizon return and consumption growth predictability by the price-dividend ratio; however, our results do not change in a significant manner as these moments have relatively large variances. In the data, Bansal, Kiku, and Yaron (2009), document that consumption
growth is quite predictable — the $R^2$'s for consumption growth implied by a VAR that includes consumption growth, the price-dividend ratio and risk-free rate are 26%, 22%, 17%, 11% for horizons of 1-, 5-, 10- and 20-years, respectively. These results are statistically significant both at short and long horizons, and underscore the presence of a persistent, predictable component in consumption growth, consistent with our GMM evidence.

In results that are not reported in tables, we also estimate a specification that allows for time-variation in the expected consumption growth but restricts the conditional volatility to be constant. This specification also indicates a significant presence of a small persistent component in consumption growth and delivers estimates of risk aversion and IES that are similar in magnitude to those in the fully specified LRR model. However, the preference parameters in this case are estimated much less precisely, especially the time-discount factor and IES. The large standard errors point out difficulties in separately identifying the rate of time preferences and the IES parameter without informative moments on time-variation of risk premia as afforded by the fully specified LRR model. Further, while this specification provides some major improvements over the IID model by generating a sizable risk premia, high volatility of equity returns and a persistent price-dividend ratio, it is still formally rejected. This result reveals the importance of time-variation in consumption volatility and the ensuing variation in risk premia.

We also evaluate the role of recursive preferences by estimating a model with the same dynamics as in the LRR(h12) model but with time-separable CRRA preferences. That is, we impose that risk aversion and IES are reciprocals of each other, $\gamma = 1/\psi$. The estimates of IES and risk aversion parameters in this specification are 0.6 and 1.7, respectively. Although the estimated consumption dynamics are quite similar to those reported for the LRR(h12) model, the CRRA specification implies a high risk-free rate with a mean of 4.2%, an essentially zero risk premia, and low volatilities of market returns and price-dividend ratio of 11% and 13%, respectively. All of these implications are sharply inconsistent with the sample moments, therefore, the power-utility specification is strongly rejected with a chi-square value of about 52. This evidence highlights the importance of recursive preferences in enhancing risks embedded in consumption and dividend dynamics.
5.2 The Effect of Ignoring Time-Averaging

To assess the effect of time-averaging, we re-estimate the model using annual data, assuming that the decision interval is annual. We will refer to this specification as LRR(h1) since it corresponds to the case of \( h = 1 \) in our notation above. Note that this specification effectively ignores time-averaging since the decision frequency of the agent and the sampling frequency of the data in this case are both annual. Table IV provides the parameter estimates of the LRR(h1) model based on the same set of moment conditions that we exploited earlier. The columns under the heading of Empirical Data report the estimates and their standard errors based on the observed data. These entries are the counterparts to the LRR(h12) results in Table II where the decision interval is monthly and time-averaging is appropriately taken into account. A comparison of the two sets of estimates reveals that ignoring time-averaging due to the misspecification of the decision frequency leads to three significant differences. First, it results in a substantially higher risk aversion. The estimate of risk aversion in the LRR(h1) specification is 19, which is about three times higher than that in the time-averaged monthly model. Second, the contribution of long-run risks to the overall variation of annual consumption growth is much smaller in the LRR(h1) specification relative to that in the LRR(h12) model. Third, volatility dynamics are estimated with substantially larger standard errors. Consistent with Drost and Nijman (1993), we find that it is much harder to detect the role of the volatility channel using low-frequency data and disregarding restrictions of temporal aggregation. Importantly, Table IV also shows that the LRR(h1) specification is strongly rejected in the data.

To understand the rejection of this specification, in the left panel of Table V we report population moments of the joint dynamics of consumption, dividends and asset prices implied by the LRR(h1) estimates. Notice first that this specification is able to generate a sizable equity premium. This is achieved simply due to a relatively high degree of risk aversion. It also matches the level of the risk-free rate since the estimate of the intertemporal elasticity of substitution is greater than one. However, the LRR(h1) specification ultimately fails to explain excess volatility of asset prices — it significantly underestimates variation in the price-dividend ratio and implies only 10% volatility of equity returns. In this respect, this specification is similar to the i.i.d. case discussed above.

By and large, the failure of the LRR(h1) specification is driven by its inability to identify shocks to the long-run risk component. If the decision interval of the agent is shorter
than annual, the LRR(h1)-based moment restrictions that ignore time-averaging are severely misspecified. This misspecification shifts the emphasis of the model from long-run risks to short-run innovations in consumption. To highlight the intuition, we refer to equation (22) that describes the dynamics of the time-averaged consumption growth. For concreteness, we maintain the assumption that the true model is monthly while the data are annual. Note that the innovation in annual consumption growth is a mixture of the underlying long- and short-run monthly shocks. While our LRR(h12) model appropriately accounts for this composite innovation structure, the LRR(h1) specification attributes the whole innovation to short-run fluctuations in consumption growth. In other words, it amplifies the contribution of short-run risks at the expense of low-frequency movements in consumption. It is, therefore, not surprising that the LRR(h1) specification duplicates various features of the IID model.

To further evaluate the effect of time-averaging, we simulate from the LRR(h12) model using the parameter estimates reported in Table II. We aggregate the monthly simulated data to construct annual consumption, dividends and prices and use those to estimate the LRR(h1) specification. This experiment is designed to illustrate what happens if the true model frequency is monthly but an econometrician, equipped with annual data, assumes a yearly decision interval and, therefore, ignores restrictions of temporal aggregation. The output of this simulation exercise is reported in the right panel of Table IV. We present finite-sample distributions of the parameter estimates as well as population values computed using a long sample of simulated data. Overall, we find that the estimated parameters in these simulations are close to the LRR(h1) estimates based on the observed annual data. In particular, the misspecification of the model frequency suppresses the contribution of long-run risks and magnifies risk aversion. As in the data, the LRR(h1) specification fails to account for high volatility of prices and returns in the simulated data (see the last column of Table V). Note that this simulation evidence suggests that our LRR(h12) model can account for the estimation output of the LRR(h1) specification, supporting, albeit indirectly, our choice of the monthly decision interval. This evidence also reveals difficulties in detecting long-run and volatility risks if one neglects time-averaging and ignores how the underlying monthly dynamics are integrated into the dynamics of the annual data used in estimation.
5.3 Substituting the Market Return for the Return on Wealth

Epstein and Zin (1991) also pursue a GMM estimation approach but in evaluating the pricing kernel in equation (4) they replace the return on the consumption claim, $r_c$, with the observed value-weighted NYSE stock market return. The market return, however, may not reflect appropriately the three risks sources of the LRR model and, therefore, may lead to significant distortions in the IMRS and biases in parameter estimates. To evaluate the effect of potential misspecifications, we simulate the data using our LRR(h12) estimates and re-estimate the model with an IMRS that replaces the return on consumption claim with the return on the aggregate market portfolio. The resulting estimates in population (i.e., long sample) as well as percentiles of finite-sample distributions are reported in Table VI. We find that the estimate of risk aversion in this case is quite low. This is mostly due to the large volatility induced into the pricing kernel by the volatile market return. For comparison, the volatility of the market return implied by the LRR(h12) estimates is 17.4% while the volatility of the return on wealth is only 4.1%. As the table also shows, this specification is vastly rejected in finite samples. This experiment shows how critical it is to utilize the correct return on consumption for assessing the implications of the LRR model and avoiding biases in parameter estimates.\(^9\)

5.4 Cross-Sectional Implications

One of the important dimensions of financial data is the cross-sectional heterogeneity in mean returns, in particular along size and book-to-market dimensions. Table VII shows that the average return of the high book-to-market portfolio is higher than that of the low book-to-market portfolio by about 6% per annum. This is the well-known value premium. Similarly, the portfolio of small market-capitalization firms outperforms the large-firm portfolio by about 7%, on average. The observed dispersion in mean returns on size and book-to-market sorts is known to present a challenge for the standard CAPM. In the data, the market betas for the value-minus-growth and small-minus-large portfolios are quite small, while the market-beta-adjusted returns (i.e., CAPM $\alpha$’s) are large and significant.

In this section, we evaluate the cross-sectional implications of the LRR(h12) model and

---

\(^9\)Recent work focusing on the return on wealth also suggests that its dynamics are significantly different from those of the market (e.g., Lustig, Nieuwerburgh, and Verdelhan (2010)).
its ability to account for the value- and size-premia puzzles. To proceed with this analysis we estimate the dynamics of dividend growth rates of four portfolios, small and large, and value and growth. We use the estimated dividend exposure and market prices of the three sources of risks to compute the implied risk premia, CAPM betas, and alphas for each of the portfolios. To keep the estimation problem manageable, cross-sectional dynamics are estimated using the extracted state variables from the LRR(h12) model, and holding preferences, consumption, and market-dividend parameters fixed at the point estimates reported in Table II. The cross-sectional parameters are estimated using the following set of moments for each portfolio: the mean and volatility of the portfolio’s dividend growth rates, its correlation with consumption growth, dividend exposure to the expected consumption growth, $x$, the portfolio’s risk premium, the volatility of its return, the mean of the price-dividend ratio and the market beta. The cross-sectional moment conditions are evaluated using annual data by incorporating restrictions of temporal aggregation.

Panel A of Table VII presents the cross-sectional estimates and, in particular, dividends’ exposure to short-run, long-run and volatility risks for the four portfolios we work with. First, note that the long-run risk exposure of value firms is much larger than that of the growth portfolio (7.11 versus 3.96). Similarly, small firms feature higher exposure to long-run consumption risks of 6.3 relative to large firms that have the long-run risk loading of about 4. These estimates are consistent with empirical evidence in Bansal, Dittmar, and Lundblad (2005), and Hansen, Heaton, and Li (2008). The estimate of $\phi$, governs asset exposure to volatility risks, is higher for small and growth firms compared to portfolios with opposite size and book-to-market characteristics. The short-run correlation between dividend and consumption growth rates is higher for value than growth firms, and is quite uniform across size-sorted portfolios. The bottom line of Panel A presents the model-based risk premia for each of the four portfolios computed using the cross-sectional estimates and the estimated preferences and consumption dynamics reported in the LRR(h12)-model column of Table II. The model implies quite sizable value and size premia of about 5% each, similar to the magnitudes observed in the data. The risk-premium decomposition reveals that, across portfolios, about 70-80% of the premium comes as a compensation for long-run risks, about 20% is accounted by volatility risks, and the remaining fraction comes from asset exposure to short-run consumption risks.

Panel B of Table VII presents the CAPM implications of the long-run risk model by reporting the analytical market betas and alphas for the small-minus-large and value-minus-
growth portfolios. As in the data, the model-implied CAPM betas of the spread portfolios are quite low, both around 0.5. Consequently, the LRR model is able to replicate the failure of the CAPM by generating quantitatively large alphas of the arbitrage portfolios. The alphas of the small–large and value–growth portfolios are respectively 3% and 3.7%, in the data, and 2.2% and 2.4%, in the model. The ability of the LRR model to account for a significant portion of the value and size premium puzzles comes from the fact that in the model, the market beta is not a sufficient risk statistics since the market return does not span long-run, short-run and volatility risks. In particular, the market exposure to long-run risks is significantly lower than that of the underlying pricing kernel. Therefore, the model features high market-adjusted alphas of the small–large and value–growth portfolios since those are highly exposed to long-run risks in consumption.

5.5 Long-Run versus Business-Cycle Risks

In this section we provide some intuition regarding the role of long-run versus shorter horizon risks. In Table VIII, we report risk premia on the consumption claim, the dividend claim, and the risk-free rate for different values of the persistence parameter, $\rho$, and IES, fixing all other parameters at the estimated values of the LRR(h12) model.

A negative-$\rho$ specification can be interpreted as a business-cycle specification similar to the one used in Mehra and Prescott (1985). To see this, consider a generalized version of the Bansal and Yaron (2004) model by allowing for cyclical variations in aggregate consumption and dividends. Specifically, let the the level of log consumption $c_t$ consists of a deterministic trend, a stochastic trend $y_t$, and an AR(1) cyclical component $s_t$, with an autocorrelation parameter $\rho_s$. That is, $c_t = \mu_c t + y_t + s_t$. The growth rate of the stochastic trend is assumed to follow $\Delta y_{t+1} = x_t + \sigma_t \eta_{t+1}$, where $x_t$ is the long-run risk component that follows the same dynamics as earlier. The evolution of consumption growth, therefore, is given by:

$$\Delta c_{t+1} = \mu_c + x_t + \Delta s_{t+1} + \sigma_t \eta_{t+1}$$

In this case, the log of the price to consumption ratio is given by $z_t = A_0 + A_1 x_t + A_2 \sigma_t^2 + A_3 s_t$, where $A_1$ and $A_2$ are the same as in expression (9), and $A_3 = \frac{(1-\frac{1}{2})}{\frac{1}{1-\kappa_1 \rho_s}}$. Note that $\rho_s - 1$ is negative. Hence, when IES is greater than one, $A_3$ is negative — business-cycle shocks in $s_t$ lower the price-consumption ratio since the consumption level is expected to revert to
its mean. This implication is very different from the effect of trend shocks in $x_t$ that raise valuation ratios. A small negative value of $\rho$ in Table VIII is meant to capture the effect of $\rho_s - 1$ above, and implies business-cycle shocks in the level of consumption. In contrast, the long-run risks specification ($\rho = 0.995$) allows for trend shocks in consumption. See also Bansal, Kiku, and Yaron (2010) for a discussion of asset pricing implications of cyclical and trend shocks in the presence of jump risks.

As the table shows, when $\rho$ is negative, the risk premium is small. In fact, business-cycle risks receive close to zero risk premia. Moreover, if in addition the IES is less than one, the level of the risk-free rate is too high relative to the data. Only in the case in which long-run risks are present ($\rho = 0.995$) and IES is greater than one, a configuration that is close to our estimates, does the model match the risk premium, risk-free rate, and volatility moments. This evidence emphasizes the role of long-run risks (trend shocks) in explaining asset prices and is consistent with Lucas (1987)’s arguments that welfare costs of business-cycle fluctuations are relatively small, while costs associated with trend changes are significant.

6 Conclusions

This paper develops a method for estimating asset pricing models with recursive preferences and generalized consumption and cashflow dynamics that accounts for time-averaging. Utilizing this method, the long-run risks model of Bansal and Yaron (2004) is empirically evaluated. The paper shows how to estimate the short-run, long-run and volatility risk components in aggregate consumption and utilize these to construct the unobservable return to aggregate wealth — a key input in estimating models with Kreps and Porteus (1978), Epstein and Zin (1989)-Weil (1989) preferences.

Empirically we find that the long-run risks model is able to successfully capture the time-series and cross-sectional variation in returns. The estimation identifies a persistent long run risk component as well as a process for time variation in uncertainty that is long lasting. The model is not rejected by the overidentifying restrictions test and results in relatively low risk aversion of 6 and an IES estimate that is larger than one. We provide evidence that time-averaging can result in substantially biased estimates for risk aversion. Ignoring
time-averaging leads to rejections of the model and to large estimates of risk aversion.

At the estimated values for the preference parameters, the market price of long-run risks is high relative to that of the short-run and volatility risks. Overall, the model accounts for the low risk free rate, and the level of the market, value, and size premia, as well as the volatility of the market return, the risk free rate, and the price-dividend ratio. In all, this evidence provides empirical support for the economic risk channels highlighted by the LRR model.
Appendix

A.1 Consumption Claim

To derive asset prices we use the IMRS together with consumption and dividend dynamics given in (3) and (15). The Euler condition in equation (16) implies that any asset \( j \) in this economy should satisfy the following pricing restriction,

\[
E_t \left[ \exp \left( \theta \ln \delta - \frac{\theta}{\psi} \Delta c_{t+1} + (\theta - 1)r_{c,t+1} + r_{j,t+1} \right) \right] = 1 ,
\]

(A-1)

where \( r_{j,t+1} \equiv \log(R_{j,t+1}) \) and \( r_{c,t+1} \) is the log return on wealth. Notice that the solution to (A-1) depends on time-series properties of the unobservable return \( r_c \). Therefore, we first substitute \( r_{j,t+1} = r_{c,t+1} \) and solve for the return on the aggregate consumption claim; after that, we present the solution for the return on a dividend-paying asset.

We start by conjecturing that the logarithm of the price to consumption ratio follows, \( z_t = A_0 + A_1 x_t + A_2 \sigma_t^2 \). Armed with the endogenous variable \( z_t \), we plug the approximation \( r_{c,t+1} = \kappa_0 + \Delta c_{t+1} + \kappa_1 z_{t+1} - z_t \) into the Euler equation above. The solution coefficients, \( A \)'s, can now be easily derived by collecting the terms on the corresponding state variables. In particular,

\[
A_0 = \frac{1}{1 - \kappa_1} \left[ \log \delta + \kappa_0 + \left( 1 - \frac{1}{\psi} \right) \mu_c + \kappa_1 A_2 (1 - \nu) \sigma_0^2 + \frac{\theta}{2} \left( \kappa_1 A_2 \sigma_w \right)^2 \right]
\]

\[
A_1 = \frac{1 - \frac{1}{\psi}}{1 - \kappa_1 \rho}
\]

\[
A_2 = -\frac{(\gamma - 1)(1 - \frac{1}{\psi})}{2 (1 - \kappa_1 \nu)} \left[ 1 + \left( \frac{\kappa_1 \varphi_w}{1 - \kappa_1 \rho} \right)^2 \right]
\]

(A-2)

For more details, see the the appendix in Bansal and Yaron (2004).

Notice that the derived solutions depend on the approximating constants, \( \kappa_0 \) and \( \kappa_1 \), which, in their turn, depend on the unknown mean of the price to consumption ratio, \( \bar{z} \). In order to solve for the price of the consumption asset, we first substitute expressions for \( \kappa \)'s
(equations (6) and (7)) into the expressions for $A$’s and solve for the mean of the price to consumption ratio. Specifically, $\bar{z}$ can be found by numerically solving a fixed-point problem:

$$\bar{z} = A_0(\bar{z}) + A_2(\bar{z})\sigma_0^2,$$

where the dependence of $A$’s on $\bar{z}$ is given above.

The solution for the price-consumption ratio, $z_t$, allows us to write the pricing kernel as a function of the state variables and the model parameters,

$$m_{t+1} = \Gamma_0 + \Gamma_1 x_t + \Gamma_2 \sigma_t^2 - \lambda_\eta \sigma_t \eta_{t+1} - \lambda_e \sigma_t e_{t+1} - \lambda_w \sigma_w w_{t+1}, \tag{A-3}$$

where

$$\begin{align*}
\Gamma_0 &= \log \delta - \frac{1}{\psi} \mu_c - 0.5 \theta(\theta - 1)(\kappa_1 A_2 \sigma_w)^2 \\
\Gamma_1 &= -\frac{1}{\psi} \\
\Gamma_2 &= (\theta - 1)(\kappa_1 \nu - 1) A_2
\end{align*} \tag{A-4}$$

and

$$\begin{align*}
\lambda_\eta &= \gamma \\
\lambda_e &= (1 - \theta) \kappa_1 A_1 \varphi_e = \left(\gamma - \frac{1}{\psi}\right) \frac{\kappa_1 \varphi_e}{1 - \kappa_1 \rho} \\
\lambda_w &= (1 - \theta) \kappa_1 A_2 = -(\gamma - 1) \left(\gamma - \frac{1}{\psi}\right) \frac{0.5 \kappa_1}{1 - \kappa_1 \rho} \left[1 + \left(\frac{\kappa_1 \varphi_e}{1 - \kappa_1 \rho}\right)^2\right]
\end{align*} \tag{A-5}$$

Note that $\lambda$’s represent market prices of transient ($\eta_{t+1}$), long-run ($e_{t+1}$) and volatility ($w_{t+1}$) risks respectively. For more detailed discussion see Bansal and Yaron (2004).

**A.2 Dividend Paying Assets**

The solution coefficients for the valuation ratio of a dividend-paying asset $j$ can be derived in a similar fashion as for the consumption asset. In particular, the price-dividend ratio for
a claim to dividends dynamics, given in (15), \( z_{j,t} = A_{0,j} + A_{1,j} x_t + A_{2,j} \sigma_t^2 \), where

\[
A_{0,j} = \frac{1}{1 - \kappa_{1,j}} \left[ \Gamma_0 + \kappa_{0,j} + \mu_j + \kappa_{1,j} A_{2,j} (1 - \nu) \sigma_0^2 + \frac{1}{2} \left( \kappa_{1,j} A_{2,j} - \lambda_w \right)^2 \sigma_w^2 \right] \\
A_{1,j} = \frac{\phi_j - \frac{1}{\psi}}{1 - \kappa_{1,j} \rho} \\
A_{2,j} = \frac{1}{1 - \kappa_{1,j} \nu} \left[ \Gamma_2 + \frac{1}{2} \left( \varphi_j^2 + \lambda_{\eta}^2 - 2 \varphi_j \lambda_{\eta} + (\kappa_{1,j} A_{1,j} \varphi_e - \lambda_e)^2 \right) \right]
\]

It follows then that the innovation into the asset return is given by,

\[
r_{t+1} = E_t[r_{j,t+1}] = \beta_{u,j} \sigma_t u_{j,t+1} + \beta_{e,j} \sigma_t e_{t+1} + \beta_{w,j} \sigma_w w_{t+1}, \quad (A-7)
\]

where the asset’s betas are defined as,

\[
\beta_{u,j} = \varphi_j, \quad \beta_{e,j} = \kappa_{1,j} A_{1,j} \varphi_e, \quad \beta_{w,j} = \kappa_{1,j} A_{2,j}
\]

The risk premium for any asset is determined by the covariation of the return innovation with the innovation into the pricing kernel. Thus, the risk premium for \( r_{j,t+1} \) is equal to the product of the asset’s exposures to systematic risks and the corresponding risk prices,

\[
E_t[r_{j,t+1} - r_{f,t}] + 0.5 \sigma_{t,r_j}^2 = -Cov_t \left( m_{t+1} - E_t(m_{t+1}), r_{j,t+1} - E_t(r_{j,t+1}) \right) \\
= \lambda_{\eta} \sigma_t^2 \beta_{\eta,j} + \lambda_e \sigma_t^2 \beta_{e,j} + \lambda_w \sigma_w^2 \beta_{w,j},
\]

where the exposure of asset \( j \) return to short-run consumption innovation is \( \beta_{\eta,j} = \varphi_j \varrho_j \).
A.3 Risk Free Rate

The solution coefficients for the risk-free dynamics follow directly from the state-representation of the SDF. In particular,

\[ F_0 = -\Gamma_0 - 0.5 \left[ \lambda_w \sigma_w \right]^2 \]
\[ F_1 = -\Gamma_1 \]
\[ F_2 = -\Gamma_2 - 0.5 \left[ \lambda_\eta + \lambda_\varepsilon^2 \right] \]  

(A-8)

A.4 IES=1

When \( \psi = 1 \), the log of the IMRS is given in terms of the value function normalized by consumption, \( v_c = \log(V_t/C_t) \),

\[ m_{t+1} = \log \delta - \gamma \Delta c_{t+1} + (1 - \gamma) v_c_{t+1} - \frac{1 - \gamma}{\delta} v_c_t \]

Conjecturing that \( v_c = \tilde{A}_0 + \tilde{A}_1 x + \tilde{A}_2 \sigma^2_t \) and using the evolution of \( v_c_t \):

\[ v_c_t = \frac{\delta}{1 - \gamma} \log E_t \left[ \exp \left\{ (1 - \gamma) (v_{c_{t+1}} + \Delta c_{t+1}) \right\} \right], \]

the solution coefficients are given by,

\[ \tilde{A}_0 = \frac{\delta}{1 - \delta} \left[ \mu_c + \tilde{A}_2 (1 - \nu) \sigma_0^2 + \frac{1}{2} (1 - \gamma) (\tilde{A}_2 \sigma_w)^2 \right] \]
\[ \tilde{A}_1 = \frac{\delta}{1 - \delta \rho} \]
\[ \tilde{A}_2 = -(\gamma - 1) \frac{0.5 \delta}{1 - \delta \nu} \left[ 1 + \left( \frac{\delta \varphi_e}{1 - \delta \rho} \right)^2 \right] \]  

(A-9)

As above, the pricing kernel can be expressed in terms of underlying preference
parameters, state variables and systematic shocks,

\[ m_{t+1} = \Gamma_0 + \Gamma_1 x_t + \Gamma_2 \sigma_t^2 - \lambda_\eta \sigma_t \eta_{t+1} - \lambda_e \sigma_t \varepsilon_{t+1} - \lambda_w \sigma_w w_{t+1} \]  \hfill (A-10)

where:

\[ \Gamma_0 = \log \delta - \mu_c - 0.5 (1 - \gamma)^2 (\tilde{A}_2 \sigma_w)^2 \]

\[ \Gamma_1 = -1 \]  \hfill (A-11)

\[ \Gamma_2 = -\frac{(\gamma - 1)^2}{2} \left[ 1 + \left( \frac{\delta \varphi_e}{1 - \delta \rho} \right)^2 \right] \]

and

\[ \lambda_\eta = \gamma \]  \hfill (A-12)

\[ \lambda_e = (\gamma - 1) \frac{\delta \varphi_e}{1 - \delta \rho} \]

\[ \lambda_w = - (\gamma - 1)^2 \frac{0.5 \delta}{1 - \delta \nu} \left[ 1 + \left( \frac{\delta \varphi_e}{1 - \delta \rho} \right)^2 \right] \]  \hfill (A-13)

Finally, note that in the IES=1 case, the wealth to consumption ratio is constant, namely, \( \frac{W_t}{C_t} = \frac{1}{1 - \delta} \). The price to consumption ratio, therefore, is equal \( \frac{P_t}{C_t} = \exp(\bar{z}) = \frac{\delta}{1 - \delta} \). Consequently, the parameter of the log-approximation of the log-wealth return,

\[ \kappa_1 = \frac{\exp(\bar{z})}{1 + \exp(\bar{z})} = \frac{\delta}{1 - \delta} = \delta. \]

Plugging \( \kappa_1 = \delta \) and \( \psi = 1 \) into equations (A-3), (A-4) and (A-5), yields exactly equation (A-10), (A-11) and (A-12). It then follows that

\[ \lim_{\psi \to 1} \kappa_1 = \delta \quad \lim_{\psi \to 1} \Gamma' = \Gamma'(\psi = 1, \kappa_1 = \delta) \quad \lim_{\psi \to 1} \Lambda' = \Lambda'(\psi = 1, \kappa_1 = \delta) \]  \hfill (A-14)
A.5 Pricing Kernel Approximation Error

In our empirical work, we rely on the approximate analytical solutions of the model presented above. In this section, we evaluate the accuracy of the log-linear approximation by comparing the approximate analytical solution for the price to consumption ratio to its numerical counterpart. The magnitude of the approximation error allows us to assess the reliability of the log-linear solution for the stochastic discount factor, and consequently, model implications based on the log-linear approximation.

Notice that the value function in the Epstein-Zin preferences is given by,

\[ V_t = (1 - \delta)^{1/\psi} W_t (W_t/C_t)^{1/\psi-1}, \]  

(A-15)

i.e., the life-time utility of the agent, normalized by the level of either consumption or wealth, is proportional to the wealth to consumption ratio. Hence, the solution to the wealth-consumption ratio (or, alternatively, price to consumption) based on the log-linearization of the wealth return in equation (5) determines the dynamics of the value function. Recall also that the evolution of the IMRS (see equation (4)), through the return on wealth, depends on the valuation of the consumption claim. Thus, the log-linear solution for the IMRS, as well, hinges on the accuracy of the log-linear approximation of the price-consumption ratio.

Our numerical solutions are based on the approach proposed by Tauchen and Hussey (1991). This method relies on a discrete representation of the conditional density of the state variables, \( x \) and \( \sigma^2 \), which allows us to solve the pricing equation by approximating the integral in (16) with a finite sum using the Gauss-Hermite quadrature. Note that the resulting solutions are subject to a discretization error. In order to minimize the error and ensure the high quality of the benchmark numerical solutions, we use a sufficiently large number of grid points in the quadrature rule.\(^\text{10}\) In addition, in this exercise we shut-off the channel of time-varying consumption volatility. Aside from this restriction, we evaluate and compare numerical and log-linear analytical solutions using the parametrization of consumption growth dynamics given in caption of Table IX. The table presents the mean level of the price-consumption ratio and its volatility for various combinations of risk aversion.

\(^{10}\)Specifically, we discretize the dynamics of the expected growth component, \( x_t \), using a 100-point rule. We find that increasing the number of grid points leads to virtually identical numerical solutions.
and IES; the time-discount preference parameter $\delta$ is set at 0.9989.

Overall, we find approximate analytical and numerical solutions to be remarkably close to each other. In particular, for risk aversion of 10 and IES of 2, the mean and the volatility of the log price to consumption ratio implied by the log-linear approximation are 4.716 and 0.0321. Numerical solutions yield 4.724 and 0.0318, respectively.\(^{11}\) The approximation error, expressed as a percentage of the corresponding numerical value, is about 0.17% for the mean and 0.86% for the standard deviation of the log price-consumption ratio. As the elasticity of intertemporal substitution decreases to 0.5, the percentage error falls to about 0.02% for $\bar{z}$ and 0.42% for $\sigma_z$. Although the accuracy of the log-linearization slightly deteriorates as the magnitude of risk aversion increases, deviations between analytical and numerical solutions remain relatively small. For example, holding IES at 2 and varying risk aversion between 5 and 15 results in 0.03%–0.51% error band for the mean and 0.17%–2.17% for the standard deviation of the log price to consumption ratio.

As discussed above, the dynamics of the price to consumption ratio has a direct bearing on the time-series properties of the IMRS. The fairly small approximation error in the price-consumption ratio that we document guarantees the accuracy of the pricing implications based on the log-linear solutions. Indeed, we find that approximate analytical and numerical solutions deliver very similar quantitative implications along all dimensions of the model, including levels and variances of the risk-free rate, price-dividend ratios, returns on consumption and dividend claims, and the pricing kernel.\(^{12}\) This evidence confirms that empirical findings presented in the paper are robust to the log-linearization of the model.

B  Time Averaged Moments

The mean and variance for annual consumption growth are already given in the text. Based again on equation (22), the first and second autocovariances of annual consumption growth

\(^{11}\)All the numbers reported in this section are in monthly terms.

\(^{12}\)Available upon request, the detailed evidence is not reported here for brevity.
can be written as,

\[
AC1(\Delta c_{t+h,h}) = \rho^h \left[ \frac{\rho(1 - \rho^h)^2}{h(1 - \rho)^2} \right]^2 \text{var}(x_{t-2h}) \\
+ \sum_{j=1}^{h-1} \frac{h-1}{h} \left( \frac{h-j}{h} \right) \sigma_0^2 + \sum_{j=0}^{h-2} \left( \frac{h-j}{h} \right) \sigma_0^2 + \sum_{j=0}^{h} (\rho^j \varphi_e)^2 \sigma_0^2
\]

\[
AC2(\Delta c_{t+h,h}) = \rho^{2h} \left[ \frac{\rho(1 - \rho^h)^2}{h(1 - \rho)^2} \right]^2 \text{var}(x_{t-2h}) \\
+ \sum_{j=1}^{h-1} \frac{h-1}{h} \left( \frac{h-j}{h} \right) \sigma_0^2 + \sum_{j=0}^{h-2} \left( \frac{h-j}{h} \right) \sigma_0^2 + \sum_{j=0}^{h} (\rho^j \varphi_e)^2 \sigma_0^2
\]

Given the monthly dynamics for dividends, equation (15), the dynamics for annual dividend growth can be written in a similar fashion as those for annual consumption growth,

\[
\Delta d_{t+h,h} = h \mu_d + \phi_d \frac{\rho(1 - \rho^h)^2}{h(1 - \rho)^2} x_{t-h} + \phi_d \sum_{j=1}^{h-1} a_j \varphi_e \sigma_{t-h+1+j} e_{t-h+j} + \phi_d \sum_{j=1}^{h} b_j \varphi_e \sigma_{t+h-1-j} e_{t+h-j} \\
+ \phi_d \sum_{j=0}^{h-1} \frac{j+1}{h} \sigma_{t+h-1-j} u_{t+h-j} + \phi_d \sum_{j=0}^{h-2} \frac{h-j-1}{h} \sigma_{t-1-j} u_{t-j}.
\]

(Hence, the mean, variance, and first autocovariance of annual dividend growth are easily computed as for consumption growth. Finally, the unconditional covariation between dividend and consumption growth is given by

\[
cov(\Delta c_{t+h,h}, \Delta d_{t+h,h}) = \left[ \frac{\rho(1 - \rho^h)^2}{h(1 - \rho)^2} \right]^2 \phi_d \text{var}(x_{t-h}) \\
+ \sum_{j=1}^{h-1} \left[ \phi_d (a_j \varphi_e)^2 + \varphi_d g_d \left( \frac{h-j}{h} \right)^2 \right] \sigma_0^2 + \sum_{j=1}^{h} \left[ \phi_d (b_j \varphi_e)^2 + \varphi_d g_d \left( \frac{j}{h} \right)^2 \right] \sigma_0^2
\]

The volatility dynamics moments are based on equation (23).

The text discusses the first and second moment of the log annualized price-dividend ratio
\( z_{d,t,h} \). Based on equation (25), the \( k \)-th autocovariance of \( z_{d,t,h} \) is,

\[
\text{cov}(z_{d,t+kh,h}, z_{d,t,h}) = [(A_{1,d}^2 + \pi^2)\rho^{kh} + A_{1,d}\pi\rho^{-1}(1 + \rho^{2h})]\text{var}(x_t) + A_{2,d}\rho^{kh}\text{var}(\sigma_t^2) + \sum_{j=1}^{h-1} q_j(\varphi_j\sigma_0)^2[\pi\rho^{kh-j} + A_{1,d}\rho^{(k+1)h-j}]
\]

(A-18)

where \( \pi = \frac{\phi_d}{h(1-\rho)}[\rho^{h-1}(1-\rho) - (h-1)\rho^h] \) and \( q_j = \frac{\phi_d}{h\rho^{j-1}(1-\rho)}[\frac{1}{1-\rho} - (h-1)\rho^{h-1} - \frac{1-\rho^{j-1}}{1-\rho} + (j - 1)\rho^{j-1}] \).

To solve for the annualized return on the dividend paying asset and risk free rate, start with the monthly dynamics for these assets,

\[
\begin{align*}
r_{f,t} &= F_0 + F_1x_t + F_2\sigma_t^2 \\
r_{d,t+1} &= B_{0,d} + B_{1,d}x_t + B_{2,d}\sigma_t^2 + \beta_{e,d}\sigma_t e_{t+1} + \beta_{u,d}\sigma_t u_{t+1} + \beta_{w,d}\sigma_w w_{t+1}
\end{align*}
\]

where \( F \)'s are given in equations (A-9) and \( B \)'s follow directly from the solution for the price-dividend ratio. The moments for the annual risk free rate, \( r_{f,t,h} \equiv \sum_{j=0}^{h-1} r_{f,t+j} \), can now be easily derived,

\[
\begin{align*}
E[r_{f,t,h}] &= h[F_0 + F_2\sigma_0] \\
\text{var}[r_{f,t,h}] &= [F_1\frac{1-\rho^h}{1-\rho}]^2\text{var}(x_t) + [F_2\frac{1-\nu^h}{1-\nu}]^2\text{var}(\sigma_t^2) \\
&+ \sum_{j=1}^{h-1} [F_1\varphi_j \frac{1-\rho^{h-j}}{1-\rho}]^2\sigma_0^2 + \sum_{j=1}^{h-1} [F_2\frac{1-\nu^{h-j}}{1-\nu}]^2\sigma_w^2
\end{align*}
\]

Similarly the market return, \( r_{d,t+kh,h} \equiv \sum_{j=0}^{h-1} r_{d,t+1+j} \), can be written as,

\[
\begin{align*}
E[r_{d,t+kh,h}] &= h[B_{0,d} + B_{2,d}\sigma_0^2] \\
\text{var}[r_{d,t+kh,h}] &= [B_{1,d}\frac{1-\rho^h}{1-\rho}]^2\text{var}(x_t) + [B_{2,d}\frac{1-\nu^h}{1-\nu}]^2\text{var}(\sigma_t^2) \\
&+ \sum_{j=1}^{h} [B_{1,d}\varphi_j \frac{1-\rho^{h-j}}{1-\rho} + \beta_{e,d}]^2\sigma_0^2 + \sum_{j=1}^{h} [B_{2,d}\frac{1-\nu^{h-j}}{1-\nu} + \beta_{w,d}]^2\sigma_w^2 + h\beta_{u,d}^2\sigma_0^2
\end{align*}
\]

Finally, using the formulae for the annual price-dividend ratio, consumption growth, and
the market return, the moments characterizing their covariation are,

\[
cov(\Delta c_{t+h}, z_{d,t+h}, h) = \frac{\rho(1 - \rho^h)^2}{h(1 - \rho)^2} (\pi + A_1, d\rho^h) \text{var}(x_t) + \sum_{j=1}^{h-1} a_j (\varphi_e \sigma_0)^2 [q_j + A_1, d\rho^{h-j}]
\]

\[
= \frac{A_1, d}{h(1 - \rho)} [h - \rho (1 - \rho^h)] (\varphi_e \sigma_0)^2 + \sum_{j=1}^{h-1} [h - j] \varphi_d \varrho_d \sigma_0^2
\]

\[
cov(r_{d,t+h}, z_{d,t+h}, h) = B_{1, d} \frac{1 - \rho^h}{1 - \rho} (\pi + A_1, d\rho^h) \text{var}(x_t) + B_{2, d} A_2, d \frac{1 - \nu^h}{1 - \nu} \nu^{2h} \text{var}(\sigma_t^2)
\]

\[
+ \sum_{j=1}^{h} \rho^{h-j} B_{1, d} \frac{1 - \rho^h}{1 - \rho} (\varphi_e \sigma_0)^2 [q_j + A_1, d\rho^{h-j}] + A_2, d B_2, d \frac{1 - \nu^h}{1 - \nu} \sigma_w^2 \sum_{j=1}^{h} \nu^{2(h-j)}
\]
References


Table I
Summary Statistics

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>StdDev</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consumption Growth</td>
<td>1.93</td>
<td>2.16</td>
</tr>
<tr>
<td>Dividend Growth</td>
<td>1.15</td>
<td>11.02</td>
</tr>
<tr>
<td>Market Return</td>
<td>7.66</td>
<td>20.28</td>
</tr>
<tr>
<td>Log(P/D)</td>
<td>3.36</td>
<td>0.45</td>
</tr>
<tr>
<td>Risk-free Rate</td>
<td>0.60</td>
<td>2.86</td>
</tr>
</tbody>
</table>

Table I presents descriptive statistics for aggregate consumption growth, returns, dividend growth rates and the logarithm of the price-dividend ratio of the stock market portfolio, and the risk-free rate. Returns are value-weighted, dividends and price-dividend ratios are constructed on the per-share basis, growth rates are measured by taking the first difference of the logarithm of the corresponding series. Sample moments for returns and growth rates are expressed in percentages. All data are real, sampled on an annual frequency and cover the period from 1930 to 2008.
Table II
Model Estimates: Parameters

<table>
<thead>
<tr>
<th>Parameters</th>
<th>IID Model</th>
<th>LRR(h12) Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Estimate</td>
<td>S.E.</td>
</tr>
<tr>
<td>Preference</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\gamma$</td>
<td>28.62</td>
<td>27.41</td>
</tr>
<tr>
<td>$\psi$</td>
<td>1.75</td>
<td>2.28</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.9990</td>
<td>na</td>
</tr>
<tr>
<td>Cashflow</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho$</td>
<td></td>
<td>0.9949</td>
</tr>
<tr>
<td>$\varphi_e$</td>
<td></td>
<td>0.0139</td>
</tr>
<tr>
<td>$\sigma_0$</td>
<td>0.0081</td>
<td>0.0024</td>
</tr>
<tr>
<td>$\nu$</td>
<td></td>
<td>0.9892</td>
</tr>
<tr>
<td>$\sigma_w$</td>
<td></td>
<td>5.47e-6</td>
</tr>
<tr>
<td>$\phi_d$</td>
<td></td>
<td>3.37</td>
</tr>
<tr>
<td>$\varphi_d$</td>
<td>7.11</td>
<td>1.86</td>
</tr>
<tr>
<td>$\varrho_d$</td>
<td>0.38</td>
<td>0.24</td>
</tr>
<tr>
<td>J-stat</td>
<td>61.53</td>
<td></td>
</tr>
<tr>
<td>P-value</td>
<td>0.00</td>
<td></td>
</tr>
</tbody>
</table>

Table II presents parameter estimates and tests for overidentifying restrictions for two models. The IID Model assumes \textit{i.i.d.} dynamics for consumption and dividend growth rates. The LRR(h12) Model is the long-run risk model that incorporates both persistent expected growth and time-varying volatility in consumption and dividends. The two models assume monthly decision interval and are estimated using annual data and accounting for temporal aggregation. The set of moment conditions used in estimation is given in Table III.
Table III
Model Estimates: Moments

<table>
<thead>
<tr>
<th>Moments</th>
<th>Data</th>
<th>IID Model Estimate</th>
<th>T-stat</th>
<th>LRR(h12) Model Estimate</th>
<th>T-stat</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Consumption &amp; Dividends</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>vol((\Delta c_{t+h,h}))</td>
<td>0.019</td>
<td>0.023</td>
<td>-0.57</td>
<td>0.020</td>
<td>-0.10</td>
</tr>
<tr>
<td>AC1((\Delta c_{t+h,h}))</td>
<td>0.455</td>
<td>0.247</td>
<td>0.94</td>
<td>0.426</td>
<td>0.09</td>
</tr>
<tr>
<td>AC2((\Delta c_{t+h,h}))</td>
<td>0.152</td>
<td>0.000</td>
<td>0.68</td>
<td>0.226</td>
<td>-0.37</td>
</tr>
<tr>
<td>vol((\Delta d_{t+h,h}))</td>
<td>0.111</td>
<td>0.163</td>
<td>-1.48</td>
<td>0.106</td>
<td>0.08</td>
</tr>
<tr>
<td>AC1((\Delta d_{t+h,h}))</td>
<td>0.210</td>
<td>0.247</td>
<td>-0.23</td>
<td>0.321</td>
<td>-0.64</td>
</tr>
<tr>
<td>corr((\Delta c_{t+h,h}, \Delta d_{t+h,h}))</td>
<td>0.544</td>
<td>0.376</td>
<td>0.63</td>
<td>0.398</td>
<td>0.36</td>
</tr>
<tr>
<td>vol((\eta_{t+h,h}^2))</td>
<td>0.001</td>
<td>2.04e-4</td>
<td>0.99</td>
<td>0.001</td>
<td>0.93</td>
</tr>
<tr>
<td>AC1((\eta_{t+h,h}^2))</td>
<td>0.326</td>
<td>0.082</td>
<td>0.75</td>
<td>0.186</td>
<td>0.84</td>
</tr>
<tr>
<td>E((\eta_{t+h,h}x_{t-h}))</td>
<td>-2.3e-05</td>
<td>na</td>
<td>na</td>
<td>0.000</td>
<td>-2.13</td>
</tr>
<tr>
<td>E((\eta_{t+h,h}^2 - E_{t-2h}\eta_{t+h}^2</td>
<td>\sigma_{t-2h}^2))</td>
<td>-1.7e-07</td>
<td>na</td>
<td>na</td>
<td>0.000</td>
</tr>
<tr>
<td><strong>Asset Prices</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>E((z_{d,t+h,h}))</td>
<td>3.370</td>
<td>3.368</td>
<td>0.05</td>
<td>3.383</td>
<td>-0.12</td>
</tr>
<tr>
<td>vol((z_{d,t+h,h}))</td>
<td>0.441</td>
<td>0.108</td>
<td>4.15</td>
<td>0.354</td>
<td>0.62</td>
</tr>
<tr>
<td>AC1((z_{d,t+h,h}))</td>
<td>0.871</td>
<td>0.000</td>
<td>4.91</td>
<td>0.903</td>
<td>-0.10</td>
</tr>
<tr>
<td>AC2((z_{d,t+h,h}))</td>
<td>0.754</td>
<td>0.000</td>
<td>4.72</td>
<td>0.844</td>
<td>-0.27</td>
</tr>
<tr>
<td>E((R_{d,t+h,h} - R_{f,t-h,h}))</td>
<td>0.075</td>
<td>0.060</td>
<td>0.64</td>
<td>0.048</td>
<td>1.30</td>
</tr>
<tr>
<td>vol((r_{d,t+h,h}))</td>
<td>0.195</td>
<td>0.199</td>
<td>-0.12</td>
<td>0.174</td>
<td>0.57</td>
</tr>
<tr>
<td>E((r_{f,t+h,h}))</td>
<td>0.005</td>
<td>0.006</td>
<td>-0.03</td>
<td>0.012</td>
<td>-1.15</td>
</tr>
<tr>
<td><strong>Predictability</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>corr((\sigma_{t+h,h}^2, z_{d,t+h,h}))</td>
<td>-0.201</td>
<td>na</td>
<td>na</td>
<td>-0.306</td>
<td>0.58</td>
</tr>
<tr>
<td>corr((r_{d,t+h,h}, z_{d,t+h,h}))</td>
<td>-0.186</td>
<td>0.000</td>
<td>-1.58</td>
<td>-0.006</td>
<td>-1.72</td>
</tr>
<tr>
<td>corr((\Delta c_{t+h,h}, z_{d,t+h,h}))</td>
<td>0.232</td>
<td>0.249</td>
<td>-0.11</td>
<td>0.494</td>
<td>-1.18</td>
</tr>
</tbody>
</table>

Table III presents the models’ population moments at the estimated parameters reported in Table II and t-statistics for the hypothesis that population statistics are equal to the data-based moments. The IID Model assumes i.i.d. dynamics for consumption and dividend growths. The LRR(h12) Model is the long-run risk model that incorporates both persistent expected growth and time-varying volatility in consumption and dividends. The two models assume monthly decision interval and are estimated using annual data and accounting for temporal aggregation. E(\(\cdot\)), vol(\(\cdot\)), AC1(\(\cdot\)), AC2(\(\cdot\)), corr(\(\cdot, \cdot\)) denote the mean, standard deviation, first- and second-order autocorrelations, and correlation respectively. \(\Delta c_{t+h,h}\) and \(\Delta d_{t+h,h}\) denote time-averaged annual consumption and dividend growth rates, respectively. \(\eta_{t+h,h} = \Delta c_{t+h,h} - \mu_c - \frac{\rho(1-\rho^2)}{2(1-\rho^2)} \sigma_c \) corresponds to the innovation to annual consumption growth. The annual price-dividend ratio, \(z_{d,t+h}\), is defined as the log of the end of year price over the twelve-month trailing sum of dividends. \(r_{d,t+h,h} = \log(\frac{R_{d,t+h,h}}{R_{f,t-h,h}})\) is the continuous annual return on the aggregate market, and \(r_{f,t+h,h}\) is the annual risk-free rate. For the moments denoted with \(^\dagger\), the entries reported in Data column correspond to the LRR(h12) model.
Table IV
No Time-Averaging: Parameter Estimates of LRR(h1) Model

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Empirical Data</th>
<th>Simulated Data</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Estimate</td>
<td>S.E.</td>
</tr>
<tr>
<td>Preference</td>
<td></td>
<td></td>
</tr>
<tr>
<td>γ</td>
<td>19.10</td>
<td>17.50</td>
</tr>
<tr>
<td>ψ</td>
<td>1.93</td>
<td>1.57</td>
</tr>
<tr>
<td>δ</td>
<td>0.9930</td>
<td>0.0056</td>
</tr>
<tr>
<td>Cashflow</td>
<td></td>
<td></td>
</tr>
<tr>
<td>ρ</td>
<td>0.9422</td>
<td>0.1306</td>
</tr>
<tr>
<td>φ_e</td>
<td>0.0645</td>
<td>0.0482</td>
</tr>
<tr>
<td>σ_0</td>
<td>0.0235</td>
<td>0.0044</td>
</tr>
<tr>
<td>ν</td>
<td>0.9593</td>
<td>0.0748</td>
</tr>
<tr>
<td>σ_w</td>
<td>7.20e-6</td>
<td>2.59e-5</td>
</tr>
<tr>
<td>φ_d</td>
<td>2.25</td>
<td>2.36</td>
</tr>
<tr>
<td>φ_d</td>
<td>4.30</td>
<td>1.86</td>
</tr>
<tr>
<td>θ_d</td>
<td>0.49</td>
<td>0.18</td>
</tr>
<tr>
<td>J-stat</td>
<td>45.74</td>
<td></td>
</tr>
<tr>
<td>P-value</td>
<td>0.00</td>
<td></td>
</tr>
</tbody>
</table>

Table IV presents the estimated parameters and tests for overidentifying restriction for the LRR(h1) specification. The estimation does not account for time-averaging, and both the sampling frequency and the decision interval are assumed to be annual. The set of moments employed in estimation is given in Table V. The first set of columns, under the heading Empirical Data, provides the estimates and standard errors when the estimation is applied to the observed data. The second set of columns, under the heading Simulated Data, provides the estimates in Population (i.e., long sample) and the 5 and 95 percentiles of finite-sample distributions, for which the data are generated using the parameter estimates of the LRR(h12) specification reported in Table II.
Table V

No Time-Averaging: Moments of LRR(h1) Model

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Empirical Data</th>
<th>Simulated Data</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Estimate</td>
<td>T-stat</td>
</tr>
<tr>
<td>Consumption &amp; Dividends</td>
<td></td>
<td></td>
</tr>
<tr>
<td>vol(\Delta c_{t+h,h})</td>
<td>0.024</td>
<td>-0.57</td>
</tr>
<tr>
<td>AC1(\Delta c_{t+h,h})</td>
<td>0.034</td>
<td>1.45</td>
</tr>
<tr>
<td>AC2(\Delta c_{t+h,h})</td>
<td>0.032</td>
<td>0.59</td>
</tr>
<tr>
<td>vol(\Delta d_{t+h,h})</td>
<td>0.102</td>
<td>0.22</td>
</tr>
<tr>
<td>AC1(\Delta d_{t+h,h})</td>
<td>0.009</td>
<td>1.19</td>
</tr>
<tr>
<td>corr(\Delta c_{t+h,h}, \Delta d_{t+h,h})</td>
<td>0.500</td>
<td>0.13</td>
</tr>
<tr>
<td>vol(\eta_{t+h,h})</td>
<td>0.023</td>
<td>-3.36</td>
</tr>
<tr>
<td>AC1(\eta_{t+h,h}^2)</td>
<td>1.13e-6</td>
<td>1.58</td>
</tr>
<tr>
<td>E(\eta_{t+h,h}^2 x_{t-h})</td>
<td>0.000</td>
<td>-2.99</td>
</tr>
<tr>
<td>E(\eta_{t+h,h}^2 - E_{t-2h} \eta_{t+h}^2)</td>
<td>0.000</td>
<td>1.64</td>
</tr>
<tr>
<td>Asset Prices</td>
<td></td>
<td></td>
</tr>
<tr>
<td>E(z_{d,t+h,h})</td>
<td>3.629</td>
<td>-2.70</td>
</tr>
<tr>
<td>vol(z_{d,t+h,h})</td>
<td>0.096</td>
<td>2.70</td>
</tr>
<tr>
<td>AC1(z_{d,t+h,h})</td>
<td>0.943</td>
<td>-0.30</td>
</tr>
<tr>
<td>AC2(z_{d,t+h,h})</td>
<td>0.888</td>
<td>-0.56</td>
</tr>
<tr>
<td>E(R_{d,t+h,h} - R_{f,t-h,h})</td>
<td>0.037</td>
<td>1.71</td>
</tr>
<tr>
<td>vol(r_{d,t+h,h})</td>
<td>0.106</td>
<td>2.25</td>
</tr>
<tr>
<td>E(r_{f,t+h,h})</td>
<td>0.006</td>
<td>-0.19</td>
</tr>
<tr>
<td>Predictability</td>
<td></td>
<td></td>
</tr>
<tr>
<td>corr(\sigma_{t,h}^2, z_{d,t,h})</td>
<td>-0.150</td>
<td>-0.01</td>
</tr>
<tr>
<td>corr(\gamma_{d,t+h,h}, z_{d,t,h})</td>
<td>0.021</td>
<td>-1.67</td>
</tr>
<tr>
<td>corr(\Delta c_{t+h,h}, z_{d,t,h})</td>
<td>0.187</td>
<td>0.23</td>
</tr>
</tbody>
</table>

Table V presents the population moments at the estimated parameter values given in Table IV for the LRR(h1) specification. The estimation does not account for time-averaging, and both the sampling frequency and the decision interval are assumed to be annual. The first set of columns, under the heading Empirical Data, provides the population values and their respective t-statistics relative to the data when the estimation is applied to the observed data. The second set of columns, under the heading Simulated Data, provides the population moments when the data are generated using the parameter estimates of the LRR(h12) specification reported in Table II.
Table VI
Substituting Market Return for Return on Wealth

<table>
<thead>
<tr>
<th>Parameters</th>
<th>True Values</th>
<th>Using $R_m$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Pop</td>
</tr>
<tr>
<td>Preference</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\gamma$</td>
<td>6.26</td>
<td>2.12</td>
</tr>
<tr>
<td>$\psi$</td>
<td>2.28</td>
<td>1.72</td>
</tr>
<tr>
<td>$\delta$</td>
<td>0.9990</td>
<td>0.9995</td>
</tr>
<tr>
<td>Cashflow</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\rho$</td>
<td>0.9949</td>
<td>0.9942</td>
</tr>
<tr>
<td>$\varphi_e$</td>
<td>0.0139</td>
<td>0.0123</td>
</tr>
<tr>
<td>$\sigma_0$</td>
<td>0.0062</td>
<td>0.0064</td>
</tr>
<tr>
<td>$\phi$</td>
<td>0.9892</td>
<td>0.9742</td>
</tr>
<tr>
<td>$\sigma_w$</td>
<td>5.47e-6</td>
<td>2.07e-6</td>
</tr>
<tr>
<td>$\phi_d$</td>
<td>3.37</td>
<td>4.25</td>
</tr>
<tr>
<td>$\varphi_d$</td>
<td>5.74</td>
<td>5.88</td>
</tr>
<tr>
<td>$\phi_d$</td>
<td>0.29</td>
<td>0.30</td>
</tr>
</tbody>
</table>

Table VI presents estimates of the LRR model in which the market return $R_d$ is substituted for the return on wealth $R_c$ in the IMRS given in equation (11). The decision interval is assumed to be monthly and the model is estimated using annual data and accounting for temporal aggregation. The set of moment conditions used in estimation is the same as in Table III. The data are simulated using the parameter values given in True Values column. The estimates in Population column are based on a long sample of simulated data; the 5 and 95 percentiles correspond to the percentiles of finite-sample distributions.
Panel A: Cross-Sectional Estimates and Risk Premia

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Small (%)</th>
<th>Large (%)</th>
<th>Growth (%)</th>
<th>Value (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mu_j$</td>
<td>0.44 (0.06)</td>
<td>0.17 (0.04)</td>
<td>0.17 (0.07)</td>
<td>0.36 (0.04)</td>
</tr>
<tr>
<td>$\phi_j$</td>
<td>6.30 (1.40)</td>
<td>4.04 (0.65)</td>
<td>3.96 (0.77)</td>
<td>7.11 (1.33)</td>
</tr>
<tr>
<td>$\varphi_j$</td>
<td>6.18 (5.30)</td>
<td>4.64 (1.61)</td>
<td>6.58 (1.68)</td>
<td>4.20 (4.78)</td>
</tr>
<tr>
<td>$\psi_j$</td>
<td>0.40 (1.17)</td>
<td>0.47 (0.83)</td>
<td>0.07 (0.22)</td>
<td>0.45 (1.05)</td>
</tr>
</tbody>
</table>

Risk Premia (%)

<table>
<thead>
<tr>
<th></th>
<th>Data</th>
<th>Model</th>
<th>Data</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>13.49</td>
<td>6.62</td>
<td>6.30</td>
<td>12.15</td>
</tr>
<tr>
<td></td>
<td>10.99</td>
<td>6.05</td>
<td>5.56</td>
<td>10.58</td>
</tr>
</tbody>
</table>

Panel B: CAPM Implications

<table>
<thead>
<tr>
<th></th>
<th>Small–Large</th>
<th>Value–Growth</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta^{CAPM}$</td>
<td>Data</td>
<td>Model</td>
</tr>
<tr>
<td></td>
<td>0.55</td>
<td>0.57</td>
</tr>
<tr>
<td>$\alpha^{CAPM}$ (%)</td>
<td>2.97</td>
<td>2.24</td>
</tr>
</tbody>
</table>

Panel A of Table VII presents estimated parameters of dividend dynamics for the extreme quintile portfolios sorted by size (large and small) and book-to-market characteristic (value and growth), as well as the risk premia implied by the model and in the data. Panel B provides the CAPM betas and alphas for the value-minus-growth and small-minus-large strategies.
Table VIII

Trend and Business Cycle Shocks

<table>
<thead>
<tr>
<th>Moments</th>
<th>$\rho = -0.25$</th>
<th>$\rho = 0$</th>
<th>$\rho = 0.995$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\psi = 0.25$</td>
<td>$\psi = 2.3$</td>
<td>$\psi = 0.25$</td>
</tr>
<tr>
<td>$E(r_{c,t+h,h} - r_{f,t-h,h})$</td>
<td>0.28</td>
<td>0.29</td>
<td>0.28</td>
</tr>
<tr>
<td>vol($r_{c,t+h,h}$)</td>
<td>2.34</td>
<td>2.16</td>
<td>2.34</td>
</tr>
<tr>
<td>$E(r_{d,t+h,h} - r_{f,t-h,h})$</td>
<td>0.46</td>
<td>0.43</td>
<td>0.46</td>
</tr>
<tr>
<td>vol($r_{d,t+h,h}$)</td>
<td>12.52</td>
<td>12.44</td>
<td>12.52</td>
</tr>
<tr>
<td>$E(r_{f,t+h,h})$</td>
<td>8.28</td>
<td>1.84</td>
<td>8.28</td>
</tr>
<tr>
<td>vol($r_{f,t+h,h}$)</td>
<td>2.39</td>
<td>0.53</td>
<td>2.39</td>
</tr>
</tbody>
</table>

Table VIII presents asset moments for the claim to consumption, dividend-paying asset, and the risk-free rate for the LRR(h12) model with alternative specifications for persistence, $\rho$, and the IES parameter, $\psi$. A configuration with $\rho < 0$ captures business cycle risks in consumption level, while $\rho > 0$ case incorporates shocks to the trend in aggregate consumption.
Table IX
Approximation Error

Panel A: Approximate Analytical Solutions

<table>
<thead>
<tr>
<th>IES</th>
<th>Mean log(P/C)</th>
<th>Vol log(P/C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>5</td>
<td>0.059</td>
</tr>
<tr>
<td>1.5</td>
<td>1.5</td>
<td>0.021</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.032</td>
</tr>
</tbody>
</table>

Panel B: Numerical Solutions

<table>
<thead>
<tr>
<th>IES</th>
<th>Mean log(P/C)</th>
<th>Vol log(P/C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>5</td>
<td>0.059</td>
</tr>
<tr>
<td>1.5</td>
<td>1.5</td>
<td>0.021</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.032</td>
</tr>
</tbody>
</table>

Panel C: Approximation Error (as a % of numerical values)

<table>
<thead>
<tr>
<th>IES</th>
<th>Mean log(P/C)</th>
<th>Vol log(P/C)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>5</td>
<td>0.04</td>
</tr>
<tr>
<td>1.5</td>
<td>1.0</td>
<td>-0.16</td>
</tr>
<tr>
<td>2</td>
<td>1.7</td>
<td>-0.17</td>
</tr>
</tbody>
</table>
Figure 1. Realized and Expected Growth of Consumption

Figure 1 plots time series of realized (solid line) and expected (dash line) growth in consumption. Consumption is defined as the per-capita expenditure on non-durables and services. The data are real, sampled on an annual frequency and cover the period from 1930 to 2008.
Figure 2. Conditional Volatility of Consumption Growth

Figure 2 plots time series of the extracted conditional volatility component of consumption growth (dash line) and the squared innovation of annual consumption growth (solid line). Consumption is defined as the per-capita expenditure on non-durables and services; data are real, sampled on an annual frequency and cover the period from 1930 to 2008.
Figure 3. Realized and Model Based Price-Dividend Ratio

Figure 3 plots the observed price-dividend ratio (solid line) and the reprojected price-dividend ratio (dash line).