Model Disagreement, Volatility, and Trading Volume*

Daniel Andrei™ Bruce Carlin‡ Michael Hasler§

October 31, 2013

Abstract

We study the impact of model disagreement on the dynamics of asset prices, return volatility, and trade in the market. In our continuous-time framework, two investors have homogenous preferences and equal access to information, but disagree about the length of the business cycle. We show that while the absolute level of volatility is driven primarily by long-run risk, the variation and persistence of volatility (i.e., volatility clustering) is driven by disagreement. Not only can disagreement amplify volatility in the market, but it is also the primary channel through which volatility affects trading volume. Compared to previous studies that consider model uncertainty with a representative agent or those which study heterogeneous beliefs with no model disagreement, our paper helps us to understand the evolution of the persistent, time-varying volatility process that we observe empirically.

*We would like to thank Tony Berrada, Mike Chernov, Julien Cujean, Jerome Detemple, Bernard Dumas, Julien Hugonnier, Arvind Krishnamurthy, Francis Longstaff, and Pascal St.-Amour for their useful advice. We would also like to acknowledge comments from participants at the SFI finance seminar in Gerzensee 2010 and the 4th International Forum on Long-Term Risks in Paris 2011. Financial support from the Swiss Finance Institute, NCCR FINRISK of the Swiss National Science Foundation, UCLA, and the University of Toronto is gratefully acknowledged.

Daniel Andrei™ UCLA, Anderson School of Management, 110 Westwood Plaza, Suite C420, Los Angeles, CA 90095, USA, daniel.andrei@anderson.ucla.edu, www.danielandrei.net

Bruce Carlin‡ UCLA, Anderson School of Management, 110 Westwood Plaza, Suite C413, Los Angeles, CA 90095, USA, bruce.carlin@anderson.ucla.edu, http://www.anderson.ucla.edu/faculty/finance/faculty/carlin

Michael Hasler§ University of Toronto, Rotman School of Management, 105 St. George Street, Toronto, ON, M5S 3E6, Canada, Michael.Hasler@rotman.utoronto.ca, www.rotman.utoronto.ca/Faculty/Hasler.aspx
1 Introduction

The field of finance is currently grappling with the fact that there are limits to applying the standard Bayesian paradigm to asset pricing. Specifically, in a standard Bayesian framework, beliefs are updated with a particular model in mind. However, as noted by Hansen and Sargent (2007), many economic models cannot be trusted completely, thereby introducing the notion of model uncertainty. Theoretically, though, as long as the potential set of models that all agents in an economy consider is the same ex ante, the Bayesian framework can still apply because agents can update their beliefs about which model explains the economy. However, if the agents consider different sets of models or they adhere to different paradigms, then disagreement will persist regarding which model is best to describe the world or predict the future. It is this notion of model disagreement that we focus on in this paper and characterize its effects on asset prices, return volatility, and trade in the market.

Empirically, model disagreement appears to be important. For example, in a recent paper by Carlin, Longstaff, and Matoba (2013), the authors study the effects of disagreement about prepayment speed forecasts in the mortgage-backed securities market on risk premia, volatility, and trading volume. Indeed, the prepayment models that traders use are often proprietary and differ from each other, while the inputs to these models are publicly observable (e.g., unemployment, interest rates, inflation). In that paper, the authors show that disagreement is associated with a positive risk premium and is the primary channel through which return volatility impacts trading volume.

In this paper, we analyze a continuous-time framework in which investors exhibit model disagreement and study the dynamics of asset prices and trading volume that results. We show that while the absolute level of volatility is driven primarily by long-run risk, the variation and persistence of volatility (i.e., volatility clustering) is driven by disagreement. Importantly, we show that model disagreement amplifies volatility in the market. Moreover, we show that model disagreement is the primary channel through which volatility affects trading volume, which is consistent with the findings in Carlin et al. (2013).

In our setup, two investors have homogenous preferences and equal access to information, but disagree about the length of the business cycle. Each investor knows that the expected dividend growth rate mean-reverts, but uses a different parameter that governs the rate at which this fundamental returns to its long-term mean. We assume that each agent commonly knows each other’s reversion rate parameter, but adheres to his own model when deciding whether to trade. As such, we are considering the extreme case in which both agents have no overlap in the models they consider.¹ Further, they agree to disagree. Note that in our

¹In light of the previous discussion, the set of models that each agents consider is a distinct singleton. Unlike in the previous discussion, though, here each agent know exactly the model of the other agent, but each agent adheres to her own model—they agree to disagree.
analysis, we do not assume that one of the investors uses the correct parameter. Given this, there are potentially three probability measures: an objective measure that is unobservable and two that arise based on the perceptions of the agents.

Using disagreement about the length of the business cycle is natural and plausible. For example, Massa and Simonov (2005) show that forecasters strongly disagree on recession probabilities, which implies that they have different beliefs regarding the duration of recessionary and expansionary phases. The origin of this disagreement may arise from many sources. Indeed, there still remains much debate regarding the validity of long-run risk models (e.g., Beeler and Campbell 2012; Bansal, Kiku, and Yaron 2012). Additionally, in practice agents might use different time-series to estimate the mean-reversion parameter (e.g., use consumption versus production data). Likewise, their estimation methods may differ (e.g., fitting the model to past analyst forecast data versus a moving-average of output growth versus performing maximum-likelihood Kalman filter estimation). Finally, as Yu (2012) documents, least-squares and maximum-likelihood estimators of the mean-reversion speed of a continuous-time process are significantly biased. Some investors might be aware of the existence of this bias and would adjust their estimation accordingly, whereas other investors might ignore it.

In the equilibrium in our model, disagreement drives the volatility of the risk-adjusted discount factor and consequently also the volatility of stock returns. Persistent transmission from investors beliefs to stock market volatility via disagreement causes excess volatility, which is time-varying and persistent. We cleanly disentangle the impact of disagreement from the impact generated by the other driving forces by decomposing stock return volatility. We show that, indeed, disagreement is the main driving force of persistent fluctuations in stock market volatility. In contrast, the level of the volatility is mainly driven by long-run risk, as the long-run risk literature (Bansal and Yaron, 2004) suggests.

Our results help to explain three well-known characteristics about financial market volatility. First, volatility systematically exceeds that justified by fundamentals (Shiller, 1981; LeRoy and Porter, 1981). Indeed, we show that model disagreement amplifies volatility, over and above the usual effect of uncertainty. We perform a numerical analysis that compares an economy populated by a representative agent to that populated by two agents with model disagreement. Both settings are otherwise observationally equivalent in terms of their average expected growth rate and average uncertainty. We show that the volatility is higher with model disagreement than what an observationally equivalent representative agent economy generates.

Second, volatility is time-varying and counter-cyclical (Schwert, 1989; Mele, 2008). This arises naturally out of our model because disagreement is mean-reverting. When we isolate the impact of disagreement on volatility, we find that fluctuations in volatility are mainly and
mostly driven by disagreement, whereas the level of volatility is driven by long-run risk. Also, we decompose the volatility of stock returns by means of Malliavin calculus; this further helps us show exactly how long-run risk drives the level of the volatility and disagreement drives its fluctuations. Last, volatility is persistent (Engle, 1982; Bollerslev, 1986; Nelson, 1991), occurring in clusters. This persistence (or predictability) has been described extensively in the empirical literature, but there is a paucity of theoretical explanations. We show that model disagreement generates a new channel of persistence transmission from investors beliefs to stock market volatility and we fit a GARCH(1,1) model on simulated stock returns to show that volatility is indeed persistent. This result is particularly impressive given that both consumption and consumption growth are assumed to have constant volatility in our model.

We also analyze the implications of our model for trade in the market and for the joint dynamics of volatility and trading volume. In particular, we show that the form of disagreement studied here is the primary channel through which volatility affects trading volume. We again perform a numerical analysis and compare economies with different degrees of model disagreement to those in which there is a representative agent. As before, all of economies are otherwise observationally equivalent in terms of their average expected growth rate and average uncertainty. Since there is no trade in the representative agent economies, model disagreement is necessary for uncertainty to affect trading volume. Moreover, we show that the severity of model disagreement is positively correlated with trading volume. These results imply that model disagreement not only amplifies volatility, but also provides an important mechanism by which uncertainty affects trade.

Finally, we include a survival analysis. Indeed, in any model with heterogeneous agents, whether all types survive in the long-run is a reasonable concern. To address this, we perform simulations and show that all agents in our economy with model disagreement survive for long periods of time. Based on this, we posit that model disagreement can have long-lasting effects on asset prices without eliminating any players from the marketplace, which likely makes our analysis economically important.

Our approach contrasts with previous work and thus adds to the previous finance literature. Certainly, there are many forms of disagreement and other ways to tackle this research agenda. As already mentioned, Hansen and Sargent (2007) studies model misspecification and model uncertainty, but does so for a single investor. In contrast, our study investigates the consequence of disagreement about models in an economy with different investors. We assume that investors disagree about the model governing the economy. This makes our analysis different from other, more common, forms of disagreement considered in the literature in which investors agree on the model governing the economy but disagree on the interpretation of public signals (see, e.g., Scheinkman and Xiong 2003, Cao and Ou-Yang 2009, Dumas, Kurshev, and Uppal 2009, or Xiong and Yan 2010). These models are able
to generate excess volatility but they do not identify the cause of persistent fluctuations in volatility.

Two papers feature model disagreement and thus are closely related to our study. First, Zapatero (1998) investigates the effect of financial innovation on the volatility of the equilibrium interest rate. In his model, as in ours, agents disagree about the growth rate of the economy—some agents are pessimists and others are optimists. Our paper has a different emphasis. It seeks to understand the role of model disagreement in generating persistent fluctuations in stock market volatility and trading volume. We should also mention that in our setup agents are not labelled \textit{ex ante} pessimists or optimists, but they randomly alternate between these two different behaviors. Second, model disagreement is also present in David (2008), who studies a setup in which agents disagree on the parameters of the fundamental process of the economy. In his model, consumption volatility is highly related to stock volatility and thus the latter inherits the countercyclical properties of the former. It is not model disagreement which is responsible for fluctuations in stock volatility; these fluctuations would also arise in a single agent setup. By contrast, in our model both consumption volatility and consumption growth volatility are constant, yet stock volatility fluctuates precisely because of model disagreement.

The remainder of the paper is organized as follows. Section 2 describes the model and its solution, Sections 3 and 4 expose the results on volatility and trading volume, Section 5 discusses the survival of investors, and Section 6 concludes. Derivations and computational details can be found in Appendix A.

\section{The Model}

We consider an economy populated by two types of agents, A and B, with different perceptions of the underlying macro-economy but with same preferences. These agents consume the dividend stream provided by a single risky asset. There is one perishable consumption good, the numeraire, with price equal to unity. Agents can invest in the risky asset, which is in positive supply of one unit, and in a risk free asset, which is in zero net supply. The risky asset is defined as being a claim to the dividend process $\delta$ defined by

$$
\frac{d\delta_t}{\delta_t} = f_t dt + \sigma_\delta dW^\delta_t,
$$

$$
df_t = \lambda (\bar{f} - f_t) dt + \sigma_f dW^f_t,
$$

where $W^\delta$ and $W^f$ are two independent Brownian motions under the objective probability measure $\mathbb{P}$. The expected dividend growth rate $f$, henceforth called \textit{fundamental}, is unobservable and mean-reverts to its long-term mean $\bar{f}$ at speed $\lambda$. The parameters $\sigma_\delta$ and $\sigma_f$ are the volatilities of the dividend growth and of the fundamental, respectively.
Both agents understand that the fundamental mean-reverts but disagree on the length of the business cycle. That is, they disagree on the value of the true mean-reversion parameter $\lambda$. Agent A’s perception of the dividend and fundamental is

$$\frac{d\delta_t}{\delta_t} = f_{At} dt + \sigma_{\delta} dW_{At}^{\delta},$$
$$df_{At} = \lambda_{A} (\bar{f} - f_{At}) dt + \sigma_{f} dW_{At}^{f},$$

where $W_{A}^{\delta}$ and $W_{A}^{f}$ are two independent Brownian motions under agent A’s probability measure $\mathbb{P}^A$.

On the other hand, Agent B believes that

$$\frac{d\delta_t}{\delta_t} = f_{Bt} dt + \sigma_{\delta} dW_{Bt}^{\delta},$$
$$df_{Bt} = \lambda_{B} (\bar{f} - f_{Bt}) dt + \sigma_{f} dW_{Bt}^{f},$$

where $W_{B}^{\delta}$ and $W_{B}^{f}$ are two independent Brownian motions under Agent B’s probability measure $\mathbb{P}^B$. Note that both agents agree on the long-term mean of the fundamental. Therefore, the disagreement among agents disappears in the long-run.

Note that we do not have to assume that Agent A or Agent B has the right length of the business cycle (the right parameter $\lambda$) in mind. Instead, the true length of the business cycle is assumed to lie somewhere in between the lengths perceived by Agents A and B. To summarize, there are 3 probability measures: the objective probability measure $\mathbb{P}$ which is not observable to any of the agents, and the two probability measures $\mathbb{P}^A$ and $\mathbb{P}^B$ as perceived by agents A and B respectively.

The economic theory acknowledges by now that there is uncertainty beyond the Bayesian paradigm. The work of Hansen and Sargent (2007) starts from the assumption that economic models cannot be trusted completely, introducing the notion of model uncertainty. We consider a related question—model disagreement. That is, instead of assuming that a single agent cannot trust the economic model at hand, we assume that different agents have different models in mind. Our aim is to analyze the impact of that sort of disagreement on the dynamics of asset prices. More precisely, we analyze how the disagreement generated by different perceptions of the length of the business cycle affects stock return volatility.

The existence of a disagreement about the length of the business cycle is motivated by multiple facts. First, there is a big debate regarding the validity of long-run risk models (e.g., Bansal and Yaron, 2004). On the one hand, Beeler and Campbell (2012) show that the high persistence of the fundamental assumed in the long-run risk literature doesn’t reflect the observed dynamics of consumption. On the other hand, Bansal et al. (2012) argue that there is ample evidence sustaining time-varying expected consumption growth and that the
assumed persistence of the latter generates consistent asset price features. To summarize, this debate proves that people disagree on the length of the business cycle.

Second, Agents A and B might use different time-series to estimate the mean-reversion parameter of the fundamental. Indeed, investors could either use production or consumption data for instance. Moreover, time-series considered by each investor are unlikely to be of the same length and of the same frequency, implying estimates that differ from one agent to another.

Third, estimation methods used by investors might very well differ. One could fit his estimated model (the filtered dynamics) directly to past analyst forecast data, another could fit his unestimated model to a moving-average of past output growth, whereas a more sophisticated investor could do the leaning exercise using past output data and estimate the parameters by performing a maximum-likelihood Kalman filter estimation (see Hamilton, 1994). This small set of examples shows how likely it is that investors come up with different estimations of the length of the business cycle.

Fourth, Yu (2012) shows that least-squares and maximum-likelihood estimators of the mean-reversion speed of a continuous-time process are significantly biased. Some investors might be aware of the existence of this bias and would adjust their estimation accordingly, whereas other investors might completely ignore it. This would again generate the type of disagreement we consider in this paper.

For the aforementioned reasons, we believe that the form of disagreement we consider is perfectly plausible and is often observed among analysts. Indeed, as shown in Massa and Simonov (2005) forecasters significantly disagree regarding recession probabilities, meaning that they disagree on the length of the business cycle.

2.1 The Learning Problem

We denote by \( \hat{f}_A \) and \( \hat{f}_B \) Agent A’s and Agent B’s estimations of the unobservable fundamental \( f \), respectively. Estimated fundamentals—filters—are defined by

\[
\hat{f}_{it} = \mathbb{E}_t^{pt} \left( f_{it} \right), \quad \text{for } i \in \{A, B\}
\]

and are computed using Bayesian updating techniques. More precisely, Agents A and B observe the dividend stream \( \delta \) and use it to estimate the fundamental \( f \) under their respective probability measures. This inference is implemented via Kalman filtering.

Applying Theorem 12.7 in Liptser and Shiryaev (2001) yields\(^2\)

\[
d\hat{f}_{it} = \lambda_i \left( \hat{f}_i - \hat{f}_{it} \right) dt + \frac{\gamma_{it}}{\sigma_\delta} d\tilde{W}_{it}, \quad \text{for } i \in \{A, B\}
\]

\(^2\)Refer to Appendix A.1 for computational details.
where $\gamma_i$ denotes the posterior variance perceived by Agent $i$. The posterior variance, also called *Bayesian uncertainty*, is defined by

$$
\gamma_{it} \equiv \text{Var}_P(f_{it}), \quad \text{for } i \in \{A, B\}
$$

and measures the precision of the estimated fundamental. Following Theorem 12.7 in *Liptser and Shiryaev (2001)*, this posterior variance follows a deterministic process and converges to a steady-state value. As in *Scheinkman and Xiong (2003)* or *Dumas et al. (2009)*, we will assume below that this posterior variance has already converged to a constant.

The innovation processes $\hat{W}_A^{\delta}$ and $\hat{W}_B^{\delta}$ are Brownian motions under $\mathbb{P}^A$ and $\mathbb{P}^B$, respectively. They are such that Agent $i$ has the following system in mind

$$
d\delta_t = \delta_t (f_{it} dt + \sigma_\delta d\hat{W}_it),
$$

$$
d\hat{f}_{it} = \lambda_i (\hat{f} - \hat{f}_{it}) dt + \frac{\gamma_i}{\sigma_\delta} d\hat{W}_it, \quad i \in \{A, B\}.
$$

Note that in the above equation $\gamma_i$ is assumed to be constant. That is, we have assumed that the prior variance is equal to the steady-state variance.\(^3\)

From now on, and without loss of generality, we choose to work under Agent $B$’s probability measure. Consequently, Agent $A$’s perception of the economy has to be converted to Agent $B$’s perception. To that end, let us define the change of measure between Agent $A$’s probability measure $\mathbb{P}^A$ and Agent $B$’s probability measure $\mathbb{P}^B$. This change of measure $\eta$ is written

$$
\eta_t \equiv \frac{d\mathbb{P}^A}{d\mathbb{P}^B}_{\Omega_t} = e^{-\frac{1}{2} \int_0^t \nu_s^2 ds - \int_0^t \nu_s d\hat{W}_B^\delta_s},
$$

where $\Omega_t$ is the observation filtration at time $t$ and $\nu$ some process to be determined.

For example, if agent $B$ assigns higher probabilities than agent $A$ to states where $\hat{f}$ is high, then $B$ believes the expected growth rate is larger and is then more optimistic than $A$. Consequently, the change of measure expresses the difference in *sentiment* between agents. For this reason *Dumas et al. (2009)* call $\eta$ the sentiment variable. Applying Girsanov’s Theorem yields

$$
\frac{d\eta_t}{\eta_t} = -\frac{1}{\sigma_\delta} \hat{g}_t d\hat{W}_B^\delta,
$$

$$
\eta_t = e^{-\frac{1}{2} \int_0^t \nu_s^2 ds - \int_0^t \nu_s d\hat{W}_B^\delta_s},
$$

\(1\)

where $\hat{g} \equiv \hat{f}_B - \hat{f}_A = \sigma_\delta \nu$ represents the disagreement among agents. The dynamics of the

\(^3\)The steady-state variance is such that $\frac{d\gamma_i}{dt} = 0$. Solving yields $\gamma_i = \sqrt{\sigma_\delta^2 (\sigma_\delta^2 \lambda_i^2 + \sigma_\delta^2) - \lambda_i \sigma_\delta^2}, \quad i \in \{A, B\}.$
disagreement are

\[ d\hat{g}_t = \left( (\lambda_A - \lambda_B)(\hat{f}_B - \bar{f}) - \left( \frac{\gamma_A}{\sigma^2} + \lambda_A \right) \hat{g}_t \right) dt + \frac{\gamma_B - \gamma_A}{\sigma^2} d\tilde{W}^\delta_{Bt}. \]  

(2)

Equation (2) shows that disagreement is mean-reverting, as in Dumas et al. (2009). In our model, however, the disagreement \( \hat{g} \) mean-reverts to the stochastic “long-term mean” \( \hat{f}_B \), whereas in Dumas et al. (2009) it mean-reverts to zero. More important, Equation (2) shows that the estimated fundamental \( \hat{f}_B \) is multiplied by \( (\lambda_A - \lambda_B) \). Therefore, if agents disagree on the length of the business cycle, in other words on the mean-reversion speed of the fundamental, the persistence of the estimated fundamental \( \hat{f}_B \) partially dictates the persistence of the disagreement \( \hat{g} \). Otherwise, \( \hat{f}_B \) disappears from the drift of \( \hat{g} \) and disagreement does not persist.

Equation (2) also shows that it is sufficient to have one agent believing in long-run risk to get a persistent disagreement. Indeed, if Agent B believes the fundamental is persistent, then the disagreement \( \hat{g} \) becomes persistent because the first term of the drift is persistent (see Equation (2)). If, on the contrary, Agent A believes the fundamental is persistent, then the disagreement \( \hat{g} \) becomes persistent because the second term of the drift is persistent.\(^5\)

To summarize, if at least one of the two agents believes in long-run risk (persistent fundamental), then the disagreement generated among agents is persistent. We would like to emphasize that this channel of persistence creation in the disagreement among agents is new to our knowledge. Our aim to investigate its asset pricing implications, especially on stock return volatility.

As it will be shown in Section 3, this new channel is extremely powerful in explaining the time-varying behavior of volatility. It turns out that the dynamics of the disagreement \( \hat{g} \) translate into a GARCH-type stock return volatility process. Hence we provide a foundation for the observed volatility clustering effect, which is based on the disagreement arising when agents assess differently the length of the business cycle. Although very unlikely, it is worth mentioning that if agents agreed on the length of the business cycle \( (\lambda_A = \lambda_B) \), then there would be no disagreement and volatility wouldn’t cluster. Note that the latter assertion applies for any value of the mean-reversion speed, as small as it can be.

Since we decided to work under Agent B’s probability measure \( \mathbb{P}^B \), let us write from now on and for notational ease the conditional expectation operator

\[ \mathbb{E}_t(\cdot) \equiv \mathbb{E}^{\mathbb{P}^B}_t(\cdot | \mathcal{F}_t). \]

\(^4\) Computational details are provided in Appendix A.2.
\(^5\) Note that \( \frac{\gamma_B - \gamma_A}{\sigma^2} \) is very small. Hence, in this case, the persistence of the disagreement \( \hat{g} \) is principally dictated by the value of \( \lambda_A \).
2.2 Equilibrium Pricing

This section is devoted to the computation of the equilibrium quantities and in particular the equilibrium asset price. To that end, we first write the optimization problem of each agent under Agent $B$’s probability measure $\mathbb{P}^B$. Then, we solve for the agents’ optimal consumption policies separately. Summing up agents’ optimal consumption policies and imposing market clearing yields the state-price density perceived by Agent $B$. The equilibrium asset price is then defined as the expectation under $\mathbb{P}^B$ of the sum of future dividend payments discounted using Agent $B$’s state price density.

We assume that both agents have the same power utility over life-time consumption. The utility function is written

$$U(t, c) = e^{-\rho t} \frac{c^{1-\alpha}}{1 - \alpha},$$

where $\alpha$ is the risk aversion coefficient and $\rho$ the subjective discount rate. Note that, in equilibrium, the market is complete since there is one risky asset (stock) and a single source or risk. Therefore, the equilibrium can be solved by applying the martingale approach of Karatzas, Lehoczky, and Shreve (1987) and Cox and Huang (1989). All technical details are left in Appendix A.3 for the reader’s convenience.

Remember that we first consider each agent’s consumption problem separately and solve for the respective optimal consumption policies. Then, imposing market clearing yields the state-price density perceived by Agent $B$. This state-price density, $\xi$, satisfies

$$\xi_t = e^{-\rho t} \delta_t^{-\alpha} \left[ \left( \frac{\eta_t}{\kappa_A} \right)^{1/\alpha} + \left( \frac{1}{\kappa_B} \right)^{1/\alpha} \right]^\alpha,$$

where $\kappa_A$ and $\kappa_B$ are the Lagrange multipliers associated to the budget constraints of Agents $A$ and $B$, respectively.

Equation (3) shows that the state-price density $\xi$ depends on the sentiment variable $\eta$. Since $\eta$ is purely driven by the disagreement $\hat{g}$ (see Equation 1), a persistent disagreement generates persistence in the volatility of the state-price density and consequently in the stock return volatility. Detailed explanations of this effect will be presented in Section 3.

Substituting the state-price density $\xi$ in the optimal consumption policies yields the following consumption sharing rules

$$c_{At} = \omega(\eta_t) \delta_t,$$
$$c_{Bt} = [1 - \omega(\eta_t)] \delta_t,$$

where $\omega(\eta)$ denotes Agent $A$’s share of consumption. Agent $A$’s share of consumption $\omega(\eta)$
satisfies

\[ \omega(\eta_t) = \left( \frac{\eta_t}{\kappa_A} \right)^{1/\alpha} \left( \frac{1}{\kappa_B} \right)^{1/\alpha} + \left( 1 - \frac{1}{\kappa_B} \right)^{1/\alpha}. \]

Equation (4) shows that the optimal share of consumption is driven by the sentiment variable \( \eta \). If \( \eta \) tends to infinity, which means that Agent A’s perception of the world is more likely than Agent B’s perception, then Agent A’s share of consumption tends to one. Conversely, if \( \eta \) tends to zero, then \( \omega(\eta) \) converges to zero too. Unsurprisingly, Agent A’s consumption share increases with the likelihood of Agent A’s probability measure being true.

As in Dumas et al. (2009), we assume that the coefficient of relative risk aversion \( \alpha \) is an integer. This specification allows us to characterize the price of the single-dividend paying stock \( S_T \) as follows

\[ S_T^T = \mathbb{E}_t \left[ \frac{\xi_T}{\xi_t} \delta_T \right] = e^{-\rho(T-t)} (1 - \omega(\eta_t))^{\alpha} \delta_t^{1/\alpha} \left( \frac{1}{\eta_t} \right)^{1/\alpha} \left( \frac{\omega(\eta_t)}{1 - \omega(\eta_t)} \right)^{1/\alpha} \mathbb{E}_t \left( \eta_T^{2} \delta_T^{1-\alpha} \right). \]

Details on the latter expression and its derivation is provided in Appendix A.4.

Equation (5) shows that the single-dividend paying stock consists in a weighted sum of expectations with weights characterized by the consumption share \( \omega(.) \) and the sentiment variable \( \eta \). On the one hand, the term obtained by setting \( j = 0 \) represents the price of a single-dividend paying stock in the case of a single agent of type B. On the other hand, setting \( j = \alpha \) yields a term representing the price of a single-dividend paying stock in the case of a single agent of type A. Intermediary terms are called adjustment terms and are therefore hard to interpret.

The stock price \( S \) is then simply given by the sum of the single-dividend paying stocks:

\[ S_t = \int_t^\infty S^u_t du. \]

Since the last expectation in Equation (5) is the moment-generating function of the vector \( (\zeta \equiv \ln \delta, \mu \equiv \ln \eta)^\top \), we can apply the theory on affine processes (see Duffie, 2008) to compute this quantity. Indeed, given that the state-vector \( (\zeta, \tilde{f}_B, \tilde{g}, \mu)^\top \) is affine-quadratic (see Cheng and Scaillet, 2007), the vector of state variables \( X \) defined by

\[ X = \left( \zeta \ \tilde{f}_B \ \tilde{g} \ \mu \ \tilde{g}^2 \ \tilde{g}\tilde{f}_B \ \tilde{f}_B^2 \right)^\top \]

This rather light assumption greatly simplifies the calculus. Still, if the coefficient of relative risk aversion was real, then the computations would be performed using Newton’s generalized binomial theorem.
is standard affine. The stock price is then obtained by numerically integrating over the single-dividend paying stocks. All computational details are provided in Appendix A.6.

Because we focus exclusively on the behavior of stock return volatility, the risk free rate, market price of risk, and portfolio allocations are not reported in the paper but are available in Appendix A.3. The time \( t \) stock return volatility satisfies

\[
|\sigma_t| = \left| \frac{\sigma(X_t) \top \partial S_t}{S_t} \right| = \left| \frac{\sigma(X_t) \top \int_0^{+\infty} \frac{\partial S_t}{\partial X_t} \; d\tau}{\int_0^{+\infty} S_t \; d\tau} \right|
\]

where \( \sigma(X_t) \) denotes the diffusion of the state vector \( X \).

3 Disagreement and Volatility

In this section we provide evidence that the disagreement among agents has a strong impact on the stock return volatility. We consider three important aspects about volatility. They are level, variation, and persistence. We show that the level of volatility is mostly driven by long-run risk, whereas variation and persistence are mostly driven by disagreement.

Considerable evidence shows that stock return volatility is persistent (e.g., Engle, 1982; Bollerslev, 1986; Nelson, 1991). Yet, there is no clear theoretical answer to the question: why is volatility persistent? Our aim is to show that when agents disagree on the length of the business cycle, in the sense that at least one of the two types of agents believes in long-run risk, then the disagreement persists and generates volatility clustering.

The calibration used from now on is provided in Table 1. These parameters are adapted from Brennan and Xia (2001) and Dumas et al. (2009) in order to insure finiteness of the price. We adopt, however, significantly lower values for the volatility of the fundamental and the dividend growth volatility. Since aggregate consumption is equal to the dividend in equilibrium, we choose to calibrate the output process to aggregate consumption data instead of aggregate dividend data. In that respect, we are relatively close to the consumption calibration provided in Brennan and Xia (2001). Our preference parameters, however, significantly differ from those exposed in Brennan and Xia (2001). Indeed, we choose a considerably smaller coefficient of relative risk aversion (3 instead of 15) and a positive subjective discount rate.

The mean-reversion speed chosen by Agent \( B \) is 0.1, corresponding to a business cycle half-life of approximately 7 years. Agent \( B \) consequently believes in long-run risk. On the other hand, Agent \( A \) believes the length of the business is shorter, as its perceived half-life is equal to 2.3 years.

Note that if one assumes that the true mean-reversion speed of the fundamental is 0.2, then the values of \( \lambda_A \) and \( \lambda_B \) correspond to the inference bias obtained under the assumption that the estimation period is 20 years (see Yu, 2012).
<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
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<tr>
<td>Relative Risk Aversion</td>
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<tr>
<td>Subjective Discount Rate</td>
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</tr>
<tr>
<td>Agent A’s Initial Share of Consumption</td>
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<td>0.5</td>
</tr>
<tr>
<td>Consumption Growth Volatility</td>
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<td>0.03</td>
</tr>
<tr>
<td>Mean-Reversion Speed of the Fundamental</td>
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<tr>
<td></td>
<td>$\lambda_B$</td>
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</tr>
<tr>
<td>Volatility of the Fundamental</td>
<td>$\sigma_f$</td>
<td>0.015</td>
</tr>
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</table>

Table 1: Calibration

### 3.1 Volatility Dynamics: Level, Variation, and Persistence

The stock return diffusion, $\sigma$, can be written

$$
\sigma_t = \sigma_\delta + \frac{S_f \gamma_B}{\sigma_{f,lr}} + \frac{S_\delta \left( \gamma_B - \gamma_A \right)}{\sigma_{f,lr}} \frac{S_\mu}{\sigma_{g,lr}} \hat{g}_t + \frac{S_\mu}{\sigma_{g,lr}} \hat{g}_t
$$

where $S_f$, $S_g$, and $S_\mu$ represent partial derivatives of stock price with respect to $\bar{f}_B$, $\hat{g}$, and $\mu \equiv \ln \eta$. Let aside the dependence of these partial derivatives on state variables, the fourth term in Equation (6) clearly shows that disagreement is the only state variable that directly drives volatility.

Furthermore, Equation (6) shows that the stock return diffusion $\sigma$ consists in the standard Lucas (1978) volatility $\sigma_\delta$ and three terms representing the *long-run* impact of changes in the estimated fundamental $\bar{f}_B$ ($\sigma_{f,lr}$), the *long-run* impact of changes in the disagreement $\hat{g}$ ($\sigma_{g,lr}$), and the *instantaneous* impact of changes in the disagreement $\hat{g}$ ($\sigma_{g,i}$).

It is worth mentioning that in a single agent model the last two terms in Equation (6) disappear and volatility depends only on $\sigma_\delta$ and $\sigma_{f,lr}$. We show below that stock return volatility does not move significantly in such a single agent framework. Hence in our heterogeneous agent model, changes in volatility and its persistence are expected to be mostly driven by the last two terms in Equation (6).

The presence of the partial derivatives $S_f$, $S_g$, and $S_\mu$ complicates the analysis, for these derivatives depend on the state variables themselves and thus are time-varying. In order to gain more intuition, we proceed in two steps. First, we simulate the last three terms in Equation (6). This will help us understand which of these components drive the level of volatility and which ones generate fluctuations in volatility. Second, we decompose these—rather opaque—components by means of Malliavin calculus, in Section 3.2 below. This will help us to interpret them and to understand clearly the train of thought by which disagreement
One simulated path (100 years) of the stock return diffusion and its components. Simulations are performed at weekly frequency, but lines are plotted at quarterly frequency to avoid graph cluttering. The diffusion components $\sigma_{f,lr}$, $\sigma_{g,lr}$, and $\sigma_{g,i}$ are defined in Equation (6). The calibration is provided in Table 1.

The fourth term in Equation (6) is clearly driven by disagreement, but also by fluctuations in the partial derivative of the stock price with respect to sentiment, $S_{\mu}$. To clarify that disagreement is the main driver, we plot in the left panel of Figure 2 the fourth diffusion component and disagreement. The correlation coefficient between the two lines in this particular example yields a value of 0.95. In the right panel of Figure 2 we plot the distribution of the correlation between the diffusion term $\sigma_{g,i}$ and the disagreement $\hat{q}$ for 1,000 simulations and we find that the correlation coefficient stays mainly between 0.8 and 1. This means that the partial derivative of price with respect to sentiment does not significantly drive the diffusion term $\sigma_{g,i}$ and therefore have a weak influence on the
dynamics of stock return volatility. In other words, changes in the fourth diffusion component are almost solely driven by changes in disagreement.

To confirm that the dynamics illustrated on Figure 1 are not particular to one simulation, we plot in Figures 3 and 4 the distributions of the averages and variances of $\sigma_{f,lr}$, $\sigma_{g,lr}$, and $\sigma_{g,i}$. Averages and variances are computed over the length of each simulation which is chosen to be 100 years at weekly frequency.

Figure 3 shows that the diffusion components $\sigma_{g,lr}$ and $\sigma_{g,i}$ do not have a significant impact on the level of volatility. The level of volatility is primarily determined by the $\tilde{f}_B$-term defined by $\sigma_{f,lr}$. It is worth mentioning that the $\tilde{f}_B$-term is negative because the precautionary savings effect dominates the substitution effect in our model. Indeed, a positive shock in the fundamental increases future consumption. Because agents want to smooth consumption over time, they increase their current consumption and so reduce their current investment. This tendency to disinvest outweighs the substitution effect (which pushes investors to invest more) and implies a drop in prices as long as agents are sufficiently risk averse ($\alpha > 1$). Hence the stock return diffusion component determined by changes in the fundamental, $\sigma_{f,lr}$, is negative.

It is worth noting that the smaller the mean-reversion speed $\lambda_B$ is, the more negative the $\sigma_{f,lr}$ component is, and consequently the larger stock return volatility becomes. The reason is that a small mean-reversion speed implies a significant amount of long-run risk and therefore the stock price is very sensitive to movements in the fundamental, as in Bansal and Yaron...
The average over 1,000 simulations of each of the last three diffusion components in Equation (6) is computed over a 100 years horizon, at weekly frequency. The calibration is provided in Table 1.

This argument will be made clear in Section 3.2, where we decompose $\sigma_{f,lr}$ by means of Malliavin calculus.

To summarize the message of Figure 3, the average level of volatility depends strongly on future movements in the estimated fundamental and weakly on instantaneous and future changes in disagreement. A different message emerges when we try to understand which components drive the variability of stock return diffusion. This is shown in Figure 4, which depicts the variances of the diffusion components and confirms the conclusions drawn from the example depicted in Figure 1. Variations incurred by the stock return diffusion are almost exclusively generated by variations in the third and fourth diffusion terms, $\sigma_{g,lr}$ and $\sigma_{g,i}$, which are both driven by disagreement. Indeed, variations in $\sigma_{f,lr}$ are relatively small.

The message to be learned here is that the level of the volatility is mainly driven by the persistence of the consumption growth, whereas fluctuations in volatility are driven by differences of beliefs regarding the persistence of the consumption growth. In Section 3.2 we make this point explicit by decomposing the diffusion components with Malliavin calculus. Before turning to that, there is a third aspect of stock return volatility worth analyzing, namely persistence.

While we have just provided evidence that the disagreement is the main driver of fluctuations in stock return volatility, we haven’t yet provided any intuition on the time-varying properties of volatility. More to the point, are these fluctuations in volatility persistent? We show here that stock return volatility clusters in our model because of the following mechanism. As explained in Section 2.1, disagreement $\hat{g}$ mean-reverts to the stochastic long-term mean $\hat{f}_B$ (see Equation 2). Because one of the agents (in this case Agent $B$) believes the fundamental is persistent, Agent $B$’s estimation of the fundamental, $\hat{f}_B$, is persistent and so becomes the disagreement. Given that the disagreement enters the diffusion of state-price density through the sentiment variable $\eta$ (see Equation 1) and then enters volatility through the

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.png}
\caption{Distribution of the Average of the Diffusion Components}
\end{figure}

Figure 3: Distribution of the Average of the Diffusion Components

The average over 1,000 simulations of each of the last three diffusion components in Equation (6) is computed over a 100 years horizon, at weekly frequency. The calibration is provided in Table 1.
last component in Equation (6), stock return volatility clusters. This mechanism, new to our knowledge, shows how persistence in the fundamental (a component of the drift) can transmute into the diffusion of stock return and generate volatility clustering effects.

We emphasize that volatility clustering arises only because agents disagree. Without disagreement, as explained above, there are no significant fluctuations in volatility and thus no volatility clustering. In what follows, we provide evidence that persistent disagreement indeed implies GARCH-type stock return volatility dynamics. To this end, we simulate 1,000 paths of stock returns over a 100 years horizon at weekly frequency. For each simulated path we compute the demeaned returns, \( \epsilon \), by extracting the residuals of the AR(1) regression

\[
    r_{t,t+1} = \alpha_0 + \alpha_1 r_{t-1,t} + \epsilon_{t+1},
\]

where \( r_{t,t+1} \) stands for the stock return between time \( t \) and \( t + 1 \). The demeaned returns \( \epsilon \) is then fitted to a GARCH(1,1) process defined by

\[
    \epsilon_t = \sigma_t z_t, \text{ where } z_t \sim N(0, 1)

    \sigma_{t+1}^2 = \beta_0 + \beta_1 \epsilon_t^2 + \beta_2 \sigma_t^2.
\]

Figure 5 illustrates the distribution of the ARCH parameter \( \beta_1 \) and the GARCH parameter \( \beta_2 \). Their associated t-stats range between 6 and 11 for the ARCH parameter and between 150 and 350 for the GARCH parameter. First, the large t-stats suggest that the estimated parameters are significant. Second, the values of \( \beta_1, \beta_2 \), and in particular their sum show that stock return volatility clusters. That is, the model implied volatility clusters because its main driver—the disagreement among agents—is persistent.

Note that if the disagreement were not persistent, then stock return volatility would
still primarily be driven by the disagreement but wouldn’t be persistent. Therefore, it is crucial that one of the two types agents believes in long-run risk in order to obtain volatility clustering effects. Given the size and the significance of the long-run risk literature, we are convinced that our main assumption, namely that a part of the population believes in long-run risk, is realistic and relatively weak. Moreover, it is also well-known that the long-run risk literature has risen some doubts in certain researchers mind, strengthening our assumption that another part of the population doesn’t believe in that type of risk (see van Binsbergen, Brandt, and Koijen, 2012; van Binsbergen, Hueskes, Koijen, and Vrugt, 2013; Beeler and Campbell, 2012). To summarize, volatility neither clusters in a world where each agent believes in long-run risk nor in a world where neither agent believes in it. Stock market volatility clusters, however, in a world where one of the agents believes in long-run risk and where both types of agents trade with each other.

As a final note, we mention that disagreement about signal informativeness, as in Dumas et al. (2009), is not easily persistent, even though both agents would be long-term believers. The reason is that disagreement mean-reverts around zero with a parameter equal to $\lambda + \gamma/\sigma^2_\delta$ (see Lemma 2 in Dumas et al. 2009). Because of the second term, this parameter is usually larger than 0.3, which is not enough to generate persistent dynamics.

### 3.2 Further Insights From Malliavin Calculus

The decomposition of the stock return diffusion in Equation (6) comprises partial derivatives of the stock price with respect to the state variables. Technically, these partial derivatives are straightforward to understand: they measure the local change of the stock price to a small
change in the state variable. Yet, the economic intuition for why the stock price moves when a state variable moves is not clearly understood. This is where Malliavin derivatives can help.\textsuperscript{7}

Loosely speaking, the Malliavin derivative is an “impulse-response.” It measures the change in a function of Brownian motions implied by a shock to the Brownians. In the context of the present model, consider the state variables $\hat{f}_B$ and $\hat{g}$. These state variables are path-dependent functions of the Brownian $W^B_\delta$. Imagine now that a unit $dW^B_\delta$ shock occurs at time $t$. What are the responses at time $u \geq t$ of $\hat{f}_B$ and $\hat{g}$ to this unit shock? Figure 6 depicts these impulse-responses.

A unit $dW^B_\delta$ shock drives $\hat{f}_B$ up at time $t$ and then slowly decays (left panel of Figure 6). How fast it decays depends obviously on the persistence of $\hat{f}_B$, which is dictated by the parameter $\lambda_B$. If $\lambda_B$ is low, i.e., if $\hat{f}_B$ is persistent, then the response to the unit $dW^B_\delta$ shock decays slowly, still having a positive impact after 30 years. The blue solid line shows this. For higher values of $\lambda_B$, the response decays much faster, as shown by the dashed red line. A similar analysis can be done for $\hat{g}$, whose impulse responses are depicted in the right panel of Figure 6.

Malliavin derivatives thus clearly show the effect of long-run risk. If a process is persistent, then a shock today will have a long-lived impact. Let’s see now how these Malliavin derivatives can deepen the analysis of volatility. Denoting by $\mathcal{D}_u X_u$ the response at time $u$ of a process

\textsuperscript{7}Details on Malliavin calculus can be found in Malliavin and Thalmaier (2005). See Detemple, Garcia, and Rindisbacher (2003), Detemple, Garcia, and Rindisbacher (2005), Berrada (2006), and Dumas et al. (2009) for applications of Malliavin calculus in finance.
to a unit $dW_B^t$ shock having occurred at time $t \leq u$, and following the derivations in Dumas et al. (2009), the stock return diffusion can be written (details on the derivations are provided in Appendix A.8)

$$
\sigma_t = \sigma_0 + \frac{1 - \alpha}{S_t} \mathbb{E}_t \left[ \int_t^\infty \frac{\xi_s}{\xi_t} \delta_s \int_s^t \mathcal{D}_t \tilde{f}_B dW_B duds \right] \\
- \frac{1}{S_t} \mathbb{E}_t \left[ \int_t^\infty \frac{\xi_s}{\xi_t} \delta_s \left( \int_t^s \mathcal{D}_t \tilde{g}_u d\tilde{W}_B^u + \frac{1}{\sigma_s^2} \int_t^s \tilde{g}_u \mathcal{D}_t \tilde{g}_u du \right) ds \right] + \frac{1}{\sigma_s} \mathbb{E}_t \left[ \int_t^\infty \frac{\xi_s}{\xi_t} \delta_s \left( \omega (\eta_t) - \omega (\eta_s) \right) ds \right] \\
\equiv \sigma_0 + \sigma_{fr} + \sigma_{g,lr} + \sigma_{g,i}.
$$

We uncover the same components as in Equation (6), except that now they do not involve partial derivatives of the stock price with respect to state variables, but they involve Malliavin derivatives. For instance, the second line of Equation (7) clearly shows how the accumulated impulse responses of $\tilde{f}_B$ to $dW_B^t$ shocks enter in the component $\sigma_{fr}$. It is obvious now that if $\tilde{f}_B$ is persistent, then the time integral in the expectation becomes significant, and thus $\sigma_{fr}$ is large. This is the long-run risk effect from Bansal and Yaron (2004). A similar reasoning applies for the $\sigma_{g,lr}$ component, where the sensitivity of price to shocks is magnified by long-run movements in disagreement.

The fourth component of the stock return diffusion, $\sigma_{g,i}$, shows how disagreement instantaneously affects volatility. The key is in the expectation term. Investors know that disagreement today implies future changes in consumption shares in the future (because in the future disagreement tends to revert to its mean). These accumulated changes in future consumption share change the valuation of the stock today, and consequently impacts the sensitivity of the stock price to shocks. This sensitivity thus depends on the value of disagreement today.

Malliavin derivative help us also understand the result from Figure 4, i.e., the fact that variations in volatility are mostly driven by the $\sigma_{g,i}$ term and not by $\sigma_{fr}$. As Figure 6 shows, $\mathcal{D}_t \tilde{f}_B$ is a deterministic function of time and thus does not vary a lot (its instantaneous variance is zero). On the contrary, $\sigma_{g,i}$ has a positive instantaneous variance since it directly depends on disagreement $\tilde{g}$. It is therefore clear which terms drive the level of the volatility of stock returns and which drive its fluctuations.

To summarize, Malliavin derivatives allow us to look further inside the components of the volatility. They make clear the point that, in an economy where agents disagree about the speed of mean reversion, the level of volatility in stock returns is mostly driven by long-run
risk whereas fluctuations in volatility are mostly driven by disagreement between agents. Furthermore, if disagreement itself is persistent, then volatility becomes persistent.

### 3.3 Model Disagreement Amplifying Volatility

Now, we use a numerical example to show how model disagreement leads to excess volatility, even when \( \hat{g} \) is currently zero. This latter condition allows us to compare an economy populated by a representative agent to that populated by two agents with model disagreement, when the two settings are observationally equivalent in terms of their average expected growth rate and average uncertainty.

Suppose that in the first economy, the representative agent uses a mean-reversion parameter \( \lambda_{\text{rep}} \in [0.1, 0.3] \). Different levels of \( \lambda_{\text{rep}} \) result in different levels of uncertainty, denoted hereafter \( \gamma_{\text{rep}} \). As such, if the agent believes the growth rate of the economy to be persistent, uncertainty is higher—due to the long-run risk effect—and thus volatility is higher. This is reflected by the blue solid line in Figure 7. If the representative agent believes \( \lambda_{\text{rep}} \) to be 0.3, then uncertainty takes the value \( \gamma_A \). As \( \lambda_{\text{rep}} \) decreases, uncertainty rises up to \( \gamma_B \), which is attained for \( \lambda_{\text{rep}} = 0.1 \). To keep it simple, we assume that the filtered growth rate of the representative agent is \( \hat{f}_{\text{rep}} = \bar{f} \). Figure 7 thus confirms the direct, positive, effect of uncertainty on volatility in a representative agent economy.

Now, compare this to a second economy where there is model disagreement and assume
that there are equal consumption weights for the agents (i.e., $\omega_A = \omega_B = 1/2$) and equal expected growth rates (i.e., $\bar{f}_{\text{rep}} = \bar{f}_A = \bar{f}_B = \bar{f}$). To make the comparison meaningful, we keep the underlying uncertainty equal to the previous case, so that with $\lambda_B = 0.1$ and $\lambda_A \geq 0.1$, uncertainty in the representative agent economy equals the weighted average uncertainty in the heterogeneous agent economy; that is, $\lambda_A$ solves

$$\gamma_{\text{rep}} \equiv \omega_A \gamma_A + \omega_B \gamma_B.$$  

(8)

We then compute the volatility that arises for all values of $\lambda_A$ which solve (8) and with $\lambda_B = 0.1$ fixed.

The red dashed line in Figure 7 shows that model disagreement amplifies volatility with respect to an observationally equivalent representative agent economy. This is meaningful because the two economies are observationally equivalent. Indeed, (i) uncertainty is the same and equals $\gamma_{\text{rep}}$, (ii) the average views of agents on the growth rate are the same and equal $\bar{f}$, and (iii) disagreement is the same and equals $\tilde{g} = 0$. The only difference between these economies is the existence of model disagreement, which presumably is not observable by the econometrician. But there is excess volatility that arises because the agents know that they will certainly disagree in the future, even if they agree today.\(^8\)

Given this, it appears that model disagreement not only induces persistent fluctuations in volatility as we have shown previously, but also amplifies it to higher levels than what a representative agent economy observationally equivalent in all respects would predict.

4 Disagreement and Trading Volume

Now, we consider how model disagreement affects trading volume in the economy. Trading volume represents the absolute value of the change in investors’ position in the risky asset. Measuring trading volume is straightforward in discrete time. In continuous time, however, diffusion processes have infinite variation. We therefore follow Xiong and Yan (2010) and proxy trading volume with the volatility of investors’ position changes.\(^9\) For this matter, picking type $A$ or type $B$ investors gives the same absolute value and thus we focus here on type $B$ investors.

\(^8\)Separate calculations show that volatility is further amplified with respect to the dashed red line when $\tilde{g}_t < 0$ (i.e., when the long-term agent $B$ is pessimistic and remains almost unchanged with respect to the dashed red line when $\tilde{g}_t > 0$ (i.e., when the long-term agent $B$ is optimistic).

\(^9\)Trading actually occurs in discrete time and it is thus reasonable to measure changes in position across small intervals (but finite). On average these changes increase with the volatility of investors’ position changes.
The wealth of type $B$ agent at time $t$ equals

$$W_t = \int_t^\infty E_t \left( \frac{\xi_T}{\xi_t} (1 - \omega_T) \delta_T \right)$$

$$= \int_t^\infty e^{-\rho(T-t)} \delta_t^\alpha (1 - \omega_t)^\alpha \delta_t^\alpha \sum_{j=0}^{\alpha-1} \left( \frac{\alpha - 1}{j} \right) \left( \frac{1}{\eta_t} \right)^j \left( \frac{\omega_t}{1 - \omega_t} \right)^j E_t \left( \delta_{T-t}^{1 - \alpha} \eta_T^{\frac{j}{T}} \right)$$

where $1 - \omega$ represents the consumption share of agent $B$. A proof of this statement is provided in Appendix A.5.

Our task is to obtain the number of assets held by agent $B$. Calling this number $\pi_B$, the martingale representation theorem indicates that:

$$\pi_{B,t} \sigma_t = \frac{\partial W_t}{\partial x_t} \sigma_t$$

(9)

where $x_t = (\zeta, \hat{f}_B, \hat{g}, \mu)$ is the state vector of the economy. Equation (9) states that fluctuations in the price of the risky asset, scaled by the number of assets held by investor $B$, are perfectly matched to fluctuations in investors $B$’s wealth. In other words, investor’s position in the risky asset is set in such a way to replicate wealth fluctuations. Naming the term on the right hand side $\sigma_{W,t}$, the position in the risky asset is

$$\pi_{B,t} = \frac{\sigma_{W,t}}{\sigma_t}$$

We are interested in measuring fluctuations in this position. These fluctuations can be gauged either by simulations, or by simply computing the absolute value of the position’s diffusion:

$$\sigma(\pi_{B,t}) = \left| \frac{\partial \pi_{B,t}}{\partial \zeta_t} \sigma_\delta + \frac{\partial \pi_{B,t}}{\partial f_B} \frac{\gamma_B}{\sigma_\delta} + \frac{\partial \pi_{B,t}}{\partial \hat{g}} \frac{\gamma_B - \gamma_A}{\sigma_\delta} - \frac{\partial \pi_{B,t}}{\partial \mu_t} \frac{1}{\sigma_\delta} \frac{\delta_t}{g_t} \right|$$

(10)

Inspecting (10), the last term shows how disagreement directly moves trading volume. Of course, it does enter indirectly as well through the partial derivatives, as do the other state variables.

We conclude this section by considering how model disagreement affects trading volume. Similar to Section 3.3, we assume that $\hat{g} = 0$ so we can compare an economies with two agents that have model disagreement with those that have a representative agent. As before, all of the economies considered are otherwise observationally equivalent in terms of their average expected growth rate and average uncertainty. Table 2 describes the five distinct economies we analyze, which are different with respect to the set of parameters $(\lambda_A, \lambda_B)$ considered. Severity of model disagreement is measured by the distance between the parameters $\lambda_A$ and
\( \lambda_B \). As such, two of the economies feature moderate model disagreement (economies 2 and 4), one economy features more severe model disagreement (economy 3), and the last two economies have a representative agent (economies 1 and 5).

The last column of Table 2 shows that the level of trading volume increases with the severity of model disagreement. Clearly, trading volume is zero in the representative agent cases (economies 1 and 5). In between, agents take on speculative positions against each other, which increases trading volume. These results also show that investors change their positions even though disagreement today is zero, i.e., \( \hat{g} = 0 \). They do so because they know that their underlying models are different and thus they anticipate future fluctuations in disagreement.

\section*{5 Survival}

In our model we make the assumption that the fundamental is unobservable. It is consequently reasonable to assume that both investors have different beliefs regarding the dynamics of an unobservable process. Furthermore, we are convinced that it would be arbitrary and non-realistic to assume that one of the two agents has the correct beliefs i.e. the right model in mind. This raises two questions: what is the true data-generating process and how long do agents survive given this true data-generating process? This section is devoted to a discussion of these two questions.

In order to investigate how long each agent survives, we have to assume a realistic data-generating process in the sense that it has to be in line with agents beliefs. Indeed, although both agents might realistically have wrong beliefs, both agents cannot be too far from the truth because otherwise one would assume that both are fools. Therefore, we assume that the true data-generating process is

\[
\frac{d\delta_t}{\delta_t} = f_t dt + \sigma dW^\delta_t \tag{11}
\]

\[
df_t = \lambda(\bar{f} - f_t)dt + \sigma f dW^f_t, \tag{12}
\]

where \( W^\delta \) and \( W^f \) are two independent Brownian motions under the true probability measure \( \mathbb{P} \). The true mean-reversion speed \( \lambda \) is assumed to be the average of Agents A and B estimated
mean-reversion speeds.

Table 3 provides the values of the true and perceived mean-reversion speeds as well as their corresponding half-lives. Both agents misperceive the true length of the business cycle, one overestimates it and the other underestimates it. Given the high probability that both agents don’t use the same dataset and estimation method to infer the mean-reversion speed of the fundamental, the probability that agents come up with different estimates is high too. Moreover, agents have at most 100 years of data at hand to estimate the mean-reversion speed of a process having a half-life of roughly 3.5 years. This lack of data motivates the fact that both agents certainly infer inaccurate estimates.

<table>
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<th>Belief</th>
<th>$\lambda$</th>
<th>Half-Life</th>
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<tbody>
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<td>2.31</td>
</tr>
<tr>
<td>True value</td>
<td>0.2</td>
<td>3.46</td>
</tr>
<tr>
<td>Agent $B$</td>
<td>0.1</td>
<td>6.93</td>
</tr>
</tbody>
</table>

Table 3: Mean-Reversion Speed and Length of the Business Cycle

To investigate the speed at which Agent $A$ or Agent $B$ disappears from the economy, we compute the $P$-expectation of the consumption share of Agent $A$. To this end we first simulate the dividend process using the true data-generating process provided in (11) and (12). Then we perform both Agents learning exercise to infer the perceived fundamental $\hat{f}_B$, the disagreement $\hat{g}$, and the sentiment $\eta$. Finally, we compute the expectation of the consumption share of Agent $A$, $\omega(\eta_T)$, for $T$ ranging from 0 to 1000 years.

Figure 8 depicts the $P$-expectation (i.e., under the true probability measure) of the consumption share of agent $A$ over 1000 years. Surprisingly, it shows that the consumption share of Agent $A$ decreases on average, even though both agents incur a mean-reversion speed estimation error of 0.1 and Agent $A$’s perceived half-life is closer to the truth than Agent $B$’s perceived half-life. This analysis suggests that investors who believe in long-run risk are expected to gain a slight share of consumption. Both shares of consumption remain, however, very close to each other and both agents survive for way more than 1000 years. Therefore, the type of disagreement we consider affects asset prices for an extremely long period of time.

To summarize, we have constructed a model where volatility clusters as long as the consumption share of both agents remains close to 50%. In this last section we have provided evidence that this situation is likely to hold for an extremely long time, if not forever.

6 Conclusion

The premise of this paper is that individuals often interpret information using different economic models. We consider a theoretical framework in which two agents pick out different
models of the economy. Specifically, in our setup agents disagree on the length of the business cycle. We analyze the asset pricing implications of such disagreement.

We first show that the disagreement regarding the actual state of the business cycle persists. This persistent disagreement affects the volatility of the risk-adjusted discount factor and consequently implies stock return volatility clustering. We cleanly decompose the dynamics of volatility by means of Malliavin calculus and show that disagreement is indeed the main driver of volatility fluctuations, while the absolute level of volatility is driven primarily by long-run risk. We thus provide a theoretical foundation of volatility clustering which is based on different but realistic perceptions of the macro-economy. Importantly, we show that the mere existence of model disagreement is enough to amplify volatility in the market, beyond the level obtained by simply raising uncertainty in a representative agent model. Moreover, volatility and trading volume are now strongly related through disagreement.

Several questions are the subject of our ongoing research. First, in this paper we assume that investors do not change their perceptions of the length of the business cycle. It is important to understand how our results would change if agents were to perform the full learning exercise (that is, if they estimate along the way the mean-reversion parameter $\lambda$). In such a case, how quickly would agents get to the true reversion coefficient? Our expectation is that investors’ estimators of the mean-reversion parameter should end up close to the true one only after a very long time, making our assumption of dogmatic beliefs is relatively weak. The reason is that a lot of data are needed to get an accurate estimator of the mean-reversion speed of a relatively persistent process. Moreover, even if now agents learn about this parameter, it is still assumed that they know and agree on the functional of the economy from the infinite set of all possible characterizations of the world. If agents where to start with large sets of plausible models, then we believe it is virtually impossible for them to end up in agreement.

Figure 8: Expected Share of Consumption of Agent $A$

Expectation under the true probability measure $\mathbb{P}$ of the consumption share of Agent $A$, $\omega(\eta_T)$. 10’000 simulations over 1000 years are performed. The calibration is provided in Table 1.
about the model governing the economy.

Finally, learning (Bayesian) uncertainty is by construction constant in our setup. This is because we do not consider here any additional news (newspapers, quarterly reports, economic data, and so on). In a setting with additional news and in which investors’ attention to those news is fluctuating, Bayesian uncertainty fluctuates. Then our conjecture is that spikes in attention should exacerbate the disagreement among agents, amplifying significantly the effects on volatility described in this paper. Therefore, it is important to study the synergistic relationships between attention, uncertainty, and disagreement and their impact on asset prices.
A Appendix

A.1 Filtering Problem

- Agent A’s learning problem

Following the notations of Liptser and Shiryaev (2001), the observable process is

\[
\frac{d\delta_t}{\delta_t} = (A_0 + A_1 f_{At})dt + B_1 dW^f_{At} + B_2 dW^\delta_{At}
\]

\[
= (0 + 1 \cdot f_{At})dt + 0 \cdot dW^f_{At} + \sigma_d dW^\delta_{At}.
\]

The unobservable process \( f_{At} \) satisfies

\[
df_{At} = (a_0 + a_1 f_{At})dt + b_1 dW^f_{At} + b_2 dW^\delta_{At}
\]

\[
= (\lambda_A f + (-\lambda_A) f_{At})dt + \sigma_f dW^f_{At} + 0 \cdot dW^\delta_{At}.
\]

Thus,

\[
\begin{align*}
&bob = b_1 b'_1 + b_2 b'_2 = \sigma_f^2 \\
&BoB = B_1 B'_1 + B_2 B'_2 = \sigma_\delta^2 \\
&boB = b_1 B'_1 + b_2 B'_2 = 0.
\end{align*}
\]

The estimated process defined by \( \hat{f}_{At} = \mathbb{E}^\mathcal{A}(f_{At}|\mathcal{O}_t) \) has dynamics

\[
d\hat{f}_{At} = (a_0 + a_1 \hat{f}_{At})dt + (boB + \gamma_{At} A'_1)(BoB)^{-1}(\frac{d\delta_t}{\delta_t} - (A_0 + A_1 \hat{f}_{At})dt),
\]

where the posterior variance \( \gamma_{At} \) solves the ODE

\[
\dot{\gamma}_{At} = a_1 \gamma_{At} + \gamma_{At} a'_1 + bob - (boB + \gamma_{At} A'_1)(BoB)^{-1}(boB + \gamma_{At} A'_1)'.
\]

Assuming that we are at the steady-state yields

\[
a_1 \gamma_{At} + \gamma_{At} a'_1 + bob - (boB + \gamma_{At} A'_1)(BoB)^{-1}(boB + \gamma_{At} A'_1)' = 0.
\]

Consequently,

\[
d\hat{f}_{At} = \lambda_A (\bar{f} - \hat{f}_{At})dt + \frac{\gamma_A}{\sigma_\delta} d\hat{W}_{At^}\delta
\]

where

\[
\gamma_A = \sqrt{\sigma_\delta^2(\sigma_\delta^2 \lambda_A^2 + \sigma_f^2) - \lambda_A \sigma_\delta^2}
\]

\[
d\hat{W}_{At^}\delta = \frac{1}{\sigma_\delta} \left( \frac{d\delta_t}{\delta_t} - \hat{f}_{At} dt \right).
\]

- Agent B’s learning problem

The estimated process is defined by \( \hat{f}_{Bt} = \mathbb{E}^\mathcal{B}(f_{Bt}|\mathcal{O}_t) \). Doing the same computations as before yields

\[
d\hat{f}_{Bt} = \lambda_B (\bar{f} - \hat{f}_{Bt})dt + \frac{\gamma_B}{\sigma_\delta} d\hat{W}_{Bt^}\delta,
\]

where

\[
\gamma_B = \sqrt{\sigma_\delta^2(\sigma_\delta^2 \lambda_B^2 + \sigma_f^2) - \lambda_B \sigma_\delta^2}
\]

\[
d\hat{W}_{Bt^}\delta = \frac{1}{\sigma_\delta} \left( \frac{d\delta_t}{\delta_t} - \hat{f}_{Bt} dt \right).
\]
A.2 Dynamics of the Disagreement \( \hat{g} \)

The dynamics of \( \hat{f}_A \) under the measure \( \mathbb{P}^B \) are written

\[
d\hat{f}_A = \lambda_A(\hat{f} - \hat{f}_A)dt + \frac{\gamma_A}{\sigma_\delta}(\hat{f}_B - \hat{f}_A)dt + \frac{\gamma_A}{\sigma_\delta}d\hat{W}_B, \\
n = \lambda_A\hat{f}dt + \lambda_A\hat{g}_t dt - \lambda_A\hat{f}_B dt + \frac{\gamma_A}{\sigma_\delta}g_t dt + \frac{\gamma_A}{\sigma_\delta}d\hat{W}_B.
\]

because by Girsanov’s Theorem

\[
d\hat{W}_A = d\hat{W}_B + \frac{1}{\sigma_\delta}g_t dt.
\]

Consequently, the dynamics of \( \hat{g} \) satisfy

\[
d\hat{g} = d\hat{f}_B - d\hat{f}_A = \left(\lambda_A - \lambda_B\right)(\hat{f}_B - \hat{f}) + \frac{\gamma_A}{\sigma_\delta}g_t dt + \frac{\gamma_B - \gamma_A}{\sigma_\delta}d\hat{W}_B.
\]

A.3 Optimal Consumption and State-Price Density

We assume that both agents have the same CRRA utility function defined by

\[
U(t,c) = e^{-\rho t} \frac{c^{1-\alpha}}{1-\alpha}.
\]

The market is complete in equilibrium since there is one stock and a single source or risk. Consequently, we can solve the problem using the martingale approach of Karatzas et al. (1987) and Cox and Huang (1989).

The optimization problem of Agent \( B \) is written

\[
\max_{c_{Bt}} \mathbb{E}\left(\int_0^\infty e^{-\rho t} \frac{c_{Bt}^{1-\alpha}}{1-\alpha} dt\right) \\
\text{s.t. } \mathbb{E}\left(\int_0^\infty \xi_t c_{Bt} dt\right) \leq x_{B0},
\]

where \( \xi \) denotes the state-price density perceived by Agent \( B \) and \( x_{B0} \) his initial wealth. The problem of Agent \( A \) under the measure \( \mathbb{P}^B \) is written

\[
\max_{c_{At}} \mathbb{E}\left(\int_0^\infty \eta_t e^{-\rho t} \frac{c_{At}^{1-\alpha}}{1-\alpha} dt\right) \\
\text{s.t. } \mathbb{E}\left(\int_0^\infty \xi_t c_{At} dt\right) \leq x_{A0}.
\]

The first order conditions are

\[
c_{Bt} = \left(\kappa_B e^{\rho t} \xi_t\right)^{-\frac{1}{\alpha}}, \\
c_{At} = \left(\kappa_A e^{\rho t} \xi_t\right)^{-\frac{1}{\alpha}},
\]

where \( \kappa_A \) and \( \kappa_B \) are the Lagrange multipliers associated to the budget constraints of Agents \( A \) and \( B \), respectively. The good market clearing condition is

\[
\left(\frac{\kappa_A}{\eta_t} e^{\rho t} \xi_t\right)^{-\frac{1}{\alpha}} + \left(\kappa_B e^{\rho t} \xi_t\right)^{-\frac{1}{\alpha}} = \delta_t.
\]
Therefore, the state-price density satisfies

\[ \xi_t = e^{-\rho t} \hat{\delta}_t^{-\alpha} \left( \left( \frac{\eta_t}{\kappa_A} \right)^{1/\alpha} + \left( \frac{1}{\kappa_B} \right)^{1/\alpha} \right)^{\alpha}. \]

Substituting this expression in the optimal consumption policies yields

\[ c_A t = \omega(\eta t) \hat{\delta}_t, \]

\[ c_B t = (1 - \omega(\eta t)) \hat{\delta}_t, \]

where

\[ \omega(\eta t) = \left( \frac{\eta t}{\kappa_A} \right)^{1/\alpha} + \left( \frac{1}{\kappa_B} \right)^{1/\alpha} \]

is the share of consumption of Agent A.

We turn now to the computation of the risk-free rate, market price of risk, stock return volatility, and optimal portfolio choice.

Applying Itô’s lemma to the state-price density leads to the following characterization of the risk-free rate \( r \) and the market price of risk \( \theta \)

\[ r_t = \rho + \alpha \hat{f}_{Bt} - \alpha \omega(\eta t) \hat{g}_t + \frac{1}{2} \left( \frac{\alpha - 1}{\alpha \delta^2} \omega(\eta t)(1 - \omega(\eta t)) \hat{g}_t^2 - \alpha(\alpha + 1) \sigma^2 \right) \]

\[ \theta_t = \alpha \sigma \delta + \omega(\eta t) \frac{\hat{g}_t}{\sigma \delta}. \]

The number of stocks \( \nu_{Bt} \) held by Agent B

\[ \nu_{Bt} = \frac{1}{\sigma_t \hat{S}_t} \left( \frac{\partial V_{Bt}}{\partial X_t} \right)^T \sigma(X_t) \]

where \( V_B \) represents Agent B’s wealth.

### A.4 Stock Price

Since we assume, as in Dumas et al. (2009), that the relative risk aversion parameter is an integer, the state-price density at time \( T \) can be rewritten

\[ \xi_T = e^{-\rho T} \hat{\delta}_T^{-\alpha} \left( \left( \frac{1}{\kappa_B} \right)^{1/\alpha} + \left( \frac{\eta_T}{\kappa_A} \right)^{1/\alpha} \right)^{\alpha} \]

\[ = e^{-\rho T} \hat{\delta}_T^{-\alpha} \frac{1}{\kappa_B} \sum_{j=0}^{\alpha} \frac{\alpha}{j} \left( \frac{\eta_T \kappa_B}{\kappa_A} \right)^{\frac{\alpha}{j}} \]

\[ = e^{-\rho T} \hat{\delta}_T^{-\alpha} \frac{1}{\kappa_B} \sum_{j=0}^{\alpha} \frac{\alpha}{j} \left( \frac{1}{\eta_t} \right)^{\frac{\alpha}{j}} \left( \frac{\eta_T \kappa_B}{\kappa_A} \right)^{\frac{\alpha}{j}} \eta_T \]

\[ = e^{-\rho T} \hat{\delta}_T^{-\alpha} \frac{1}{\kappa_B} \sum_{j=0}^{\alpha} \frac{\alpha}{j} \left( \frac{1}{\eta_t} \right)^{\frac{\alpha}{j}} \left( \frac{\omega(\eta_t)}{1 - \omega(\eta_t)} \right)^{\frac{\alpha}{j}} \eta_T \]

(13)
where the last equality comes from the fact that

\[
\omega(\eta_t) = \left(\frac{\eta_t}{\kappa_A}\right)^{1/\alpha} + \left(\frac{\eta_t}{\kappa_B}\right)^{1/\alpha}
\]

\[
1 - \omega(\eta_t) = \left(\frac{1}{\kappa_B}\right)^{1/\alpha} + \left(\frac{\eta_t}{\kappa_A}\right)^{1/\alpha}
\]

(14)

and consequently

\[
\left(\frac{\eta_t \kappa_B}{\kappa_A}\right)^{1/\alpha} = \frac{\omega(\eta_t)}{1 - \omega(\eta_t)}.
\]

Rewriting Equation (14) yields

\[
\left(\frac{1}{\kappa_B}\right)^{1/\alpha} + \left(\frac{\eta_t}{\kappa_A}\right)^{1/\alpha} = \left(\frac{1}{1 - \omega(\eta_t)}\right)^\alpha \frac{1}{\kappa_B}.
\]

(15)

Thus the single-dividend paying stock price satisfies

\[
S_t^T = E_t \left( \frac{\xi_T}{\delta_T} \right)
\]

\[
= \left(\frac{1}{\kappa_B}\right)^{1/\alpha} + \left(\frac{\eta_t}{\kappa_A}\right)^{1/\alpha} = \left(\frac{1}{1 - \omega(\eta_t)}\right)^\alpha \frac{1}{\kappa_B}.
\]

Finally the stock price is given by

\[
S_t = \int_t^\infty S_t^u \, du.
\]

A.5 Wealth of Agent B

[TO BE COMPLETED]

A.6 State Vector and Transform Analysis

Following Cheng and Scaillet (2007), the augmented affine state-vector is

\[
dX_t = \begin{pmatrix}
d\zeta_t \\
df_{Bt} \\
dg_t \\
d\mu_t \\
dg^2_t \\
dg_t df_{Bt} \\
df^2_{Bt}
\end{pmatrix}.
\]
In other words,

\[
\begin{align*}
    dX_t &= \mu(X_t)dt + \sigma(X_t)d\tilde{W}_B, \\
    \mu(x) &= K_0 + K_1x \\
    (\sigma(x)\sigma(x)^T)_{ij} &= H_{0ij} + H_{1ij} \cdot x,
\end{align*}
\]

where \(K_0, K_1,\) and the matrices \(H\) are provided in Section A.7. From Duffie (2010) we know that

\[
\mathbb{E}_t(\delta_u \eta_u) = \mathbb{E}_t\left[e^{\epsilon \eta_u + \chi \mu_u}\right] = e^{\bar{\eta}(\tau) + \bar{\eta}(\tau)X_t},
\]

where \(\tau = u - t\) and \(\epsilon\) and \(\chi\) are arbitrary constants. \(\bar{\eta}\) and \(\bar{\eta}\) solve the following system of 8 Ricatti ODEs

\[
\begin{align*}
    \bar{\eta}(s) &= \frac{1}{2} \bar{\eta}(s)H_1\bar{\eta}(s) \\
    \bar{\eta}(s) &= \frac{1}{2} \bar{\eta}(s)H_0\bar{\eta}(s)
\end{align*}
\]

with boundary conditions \(\bar{\eta}_1(0) = \epsilon, \bar{\eta}_2(0) = 0, \bar{\eta}_3(0) = 0, \bar{\eta}_4(0) = \chi, \bar{\eta}_5(0) = 0, \bar{\eta}_6(0) = 0, \bar{\eta}_7(0) = 0,\) and \(\alpha(0) = 0.\) This system cannot be directly solved in closed form. However, we know that \(\bar{\eta}_1(\tau) = \epsilon\) and \(\bar{\eta}_4(\tau) = \chi.\) Thus, the system can be rewritten in a matrix Riccati form as follows\(^{10}\)

\[
Z' = J + B^\top Z + ZB + ZQZ,
\]

where

\[
Z = \begin{pmatrix}
    \Gamma & \beta_3/2 & \beta_2/2 \\
    \beta_3/2 & \beta_5 & \beta_6/2 \\
    \beta_2/2 & \beta_6/2 & \beta_7
\end{pmatrix}
\]

and \(\Gamma\) is a function of \(\tau.\) The matrices \(J, B,\) and \(Q\) satisfy

\[
\begin{align*}
    J &= \begin{pmatrix}
        0 & -\frac{\chi}{2} & \frac{\chi}{2} \\
        -\frac{\chi}{2} & \frac{(\chi - 1)\chi}{2\sigma^2} & 0 \\
        \frac{\chi}{2} & 0 & 0
    \end{pmatrix} \\
    B &= \begin{pmatrix}
        -\gamma_A \epsilon + \gamma_B \epsilon - (\lambda_A - \lambda_B)\bar{f} & 0 & -\lambda_A \sigma^2 + \gamma_A x + \gamma_B x & 0 & \lambda_A - \lambda_B \\
        \gamma_B \epsilon + \bar{f}\lambda_B & 0 & 0 & -\lambda_B & 0
    \end{pmatrix} \\
    Q &= \begin{pmatrix}
        0 & 0 & 0 \\
        0 & 2\gamma_A \gamma_B + \gamma_A^2 & 2\gamma_B \gamma_A \\
        0 & 2\gamma_B \gamma_A & \gamma_A^2 + \gamma_B^2
    \end{pmatrix}. \quad (\sigma^2, \sigma^2)
\end{align*}
\]

Note that we chose to set \(J_{11}\) and \(J_{23}\) to zero since these can be any real numbers. Using Radon’s lemma, we get

\[
Z(\tau) = Y^{-1}(\tau)X(\tau) \quad \text{where} \quad X\text{ and } Y \text{ satisfy}
\]

\[
\begin{align*}
    X' &= BX + JY, \quad X(0) = [0]_{3 \times 3} \\
    Y' &= -QX - B^\top Y, \quad Y(0) = I_{3 \times 3}.
\end{align*}
\]

The solution of this system is

\[
(X(\tau) \quad Y(\tau)) = (X(0) \quad Y(0)) M(\tau), \quad \text{where} \quad M(\tau) \text{ is the matrix exponential}
\]

\[
M(\tau) = \exp\left(\begin{pmatrix}
    B & -Q \\
    J & -B^\top
\end{pmatrix} \tau\right).
\]

\(^{10}\)See Andrei and Cujean (2010) for detailed explanations related to this methodology.
The state variables have the following dynamics under $(\mathbb{P}^B, \mathcal{G}_t)$

\[
\begin{align*}
d\zeta_t &\equiv d\ln \delta_t = \left( \tilde{f}_{B_t} - \frac{1}{2} \sigma^2_t \right) dt + \sigma_d d\tilde{W}^\delta_{B_t} \\
d\tilde{f}_{B_t} &= \lambda_B \left( \tilde{f} - \tilde{f}_{B_t} \right) dt + \frac{\gamma_B}{\sigma^2} d\tilde{W}^\delta_{B_t} \\
d\tilde{g}_t &= \left( \lambda_A - \lambda_B (\tilde{f}_{B_t} - \tilde{f}) \right) dt + \frac{\gamma_A}{\sigma^2} d\tilde{W}^\delta_{B_t} \\
d\mu_t &\equiv d\ln \eta_t = -\frac{1}{2} \eta^2 dt - \frac{1}{\sigma^2} \tilde{g}_t d\tilde{W}^\delta_{B_t}.
\end{align*}
\]

Note that the matrix exponential $M(\tau)$ has to be computed using a Jordan decomposition. Indeed, we have

\[M(\tau) = S \exp(J_0 \tau) S^{-1},\]

where $J_0$ and $S$ are respectively, the Jordan and the similarity matrix extracted from the Jordan decomposition. The betas are consequently given by

\[
\bar{\beta}_1(\tau) = \epsilon \\
\bar{\beta}_2(\tau) = \frac{n_{01} + \sum_{i=1}^s n_{i1} e^{\delta_j \tau}}{b_{01} + \sum_{i=1}^s b_{i1} e^{\delta_i \tau}} \\
\bar{\beta}_3(\tau) = \frac{n_{02} + \sum_{i=1}^s n_{i2} e^{\delta_j \tau}}{b_{02} + \sum_{i=1}^s b_{i2} e^{\delta_i \tau}} \\
\bar{\beta}_4(\tau) = \chi \\
\bar{\beta}_5(\tau) = \frac{n_{03} + \sum_{i=1}^s n_{i3} e^{\delta_j \tau}}{b_{03} + \sum_{i=1}^s b_{i3} e^{\delta_i \tau}} \\
\bar{\beta}_6(\tau) = \frac{n_{04} + \sum_{i=1}^s n_{i4} e^{\delta_j \tau}}{b_{04} + \sum_{i=1}^s b_{i4} e^{\delta_i \tau}} \\
\bar{\beta}_7(\tau) = \frac{n_{05} + \sum_{i=1}^s n_{i5} e^{\delta_j \tau}}{b_{05} + \sum_{i=1}^s b_{i5} e^{\delta_i \tau}}.
\]

The coefficients are available upon request. Notice that the function $\bar{\pi}(\tau)$ is obtained through a numerical integration. Thus, this function is not obtained in closed form. Since in our setup $\chi = \frac{1}{6}$ and $\epsilon = 1 - \alpha$, the stock price simplifies to

\[
S_t = \int_0^{+\infty} S_t \; d\tau \\
= \delta_t (1 - \omega(\eta_t)) \alpha \sum_{j=0}^\infty \left( \frac{\omega(\eta_t)}{1 - \omega(\eta_t)} \right) \times \\
\times \int_0^{+\infty} e^{-\rho \tau} e^{\bar{\pi}(\tau) + \bar{\pi}_{21}(\tau) \tilde{f}_{B_t} + \bar{\pi}_{31}(\tau) \tilde{g}_t + \bar{\pi}_{32}(\tau) \tilde{g}_t + \bar{\pi}_{41}(\tau) \tilde{f}_{B_t} + \bar{\pi}_{51}(\tau) \tilde{f}_{B_t}} \; d\tau.
\]

Even though the integral has to be performed numerically, this formula permits to simulate the price process in a very efficient way.

### A.7 State-Vector

The state variables have the following dynamics under $(\mathbb{P}^B, \mathcal{G}_t)$
The diffusion vector is given by

\[ d(\hat{g}_t) = \left(2(\lambda_A - \lambda_B) \left(\hat{g}_t \hat{f}_B - \hat{f}\right) - 2 \left(\lambda_A + \frac{\gamma A}{\sigma_d^2}\right) \hat{g}_t^2 + \left(\frac{\gamma B - \gamma A}{\sigma_d^2}\right)^2 \right) dt + 2 \frac{\gamma B - \gamma A}{\sigma_d} \hat{g}_t d\hat{W}_B^\delta \]

\[ d(\hat{g}_t \hat{f}_B) = \left(\frac{\gamma B(\gamma B - \gamma A)}{\sigma_d^2}\right) + \lambda_B \hat{g}_t + (\lambda_B - \lambda_A) \hat{f}_B - \left(\frac{\gamma A}{\sigma_d^2} + \lambda_A + \lambda_B\right) \hat{g}_t \hat{f}_B \]

\[ + (\lambda_A - \lambda_B) \hat{f}_B^2 dt + \gamma B \hat{g}_t + (\gamma B - \gamma A) \hat{f}_B d\hat{W}_B^\delta \]

\[ d(\hat{f}_B^2) = \left(2 \hat{f} \lambda_B \hat{f}_B - 2 \lambda_B \hat{f}_B^2 + \frac{\gamma B^2}{\sigma_d^2}\right) dt + 2 \frac{\gamma B \hat{f}_B}{\sigma_d} d\hat{W}_B^\delta. \]

Hence the vector \( K_0 \) and the matrix \( K_1 \) are

\[
K_0 = \begin{pmatrix}
-\frac{\sigma_d^2}{2} \\
\hat{f} \lambda_B \\
\lambda_B - \lambda_A \hat{f} \\
0 \\
\frac{(\gamma B - \gamma A)^2}{\sigma_d^2} \\
\frac{\gamma B(\gamma B - \gamma A)}{\sigma_d^2} \\
\frac{\gamma B^2}{\sigma_d^2}
\end{pmatrix}
\]

\[
K_1 = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -\lambda_B & 0 & 0 & 0 & 0 & 0 \\
0 & \lambda_A - \lambda_B & -\frac{2A}{\sigma_d^2} - \lambda_A & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\frac{1}{2\sigma_d^2} & 0 & 0 & 0 \\
0 & 0 & 2(\hat{f} \lambda_B - \hat{f} \lambda_A) & -2(\lambda_B - \lambda_A) & 0 & 0 & 0 \\
0 & (\lambda_B - \lambda_A) \hat{f} & \hat{f} \lambda_B & 0 & 0 & \hat{f} \lambda_B - \lambda_B & \lambda_B - \lambda_B \\
0 & 2 \hat{f} \lambda_B & 0 & -\frac{2A}{\sigma_d^2} - \lambda_A - \lambda_B & \lambda_A - \lambda_B & 0 & 0 \\
\end{pmatrix}
\]

The diffusion vector is given by

\[
\sigma(X_t) = \begin{pmatrix}
\sigma_d \\
\frac{\sigma_d}{\gamma B} \\
\frac{\gamma B - \gamma A}{\sigma_d^2} \\
\frac{\gamma_B^2}{\sigma_d^2} \\
\frac{\gamma_B^2}{\gamma_B(\gamma B - \gamma A)} \\
\frac{\gamma_B(\gamma B - \gamma A)}{\sigma_d} \\
\frac{2 \hat{f} \lambda_B}{\sigma_d}
\end{pmatrix}
\]

and the \( H \) matrices satisfy

\[
H_0 = \begin{pmatrix}
\sigma_d^2 & \gamma_B & \gamma_B - \gamma A & 0 & 0 & 0 \\
\gamma_B & \gamma_B^2 & \frac{\gamma_B(\gamma B - \gamma A)}{\sigma_d^2} & 0 & 0 & 0 \\
\gamma_B - \gamma A & \gamma_B(\gamma B - \gamma A) & \frac{(\gamma B - \gamma A)^2}{\sigma_d^2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
\[ H_{11} \text{ and } H_{14} = [0]_{7 \times 7} \]

\[
H_{12} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & \gamma_B - \gamma_A & 2\gamma_B \\
0 & 0 & 0 & 0 & 0 & 2\gamma_B & -\gamma_A \\
0 & 0 & 0 & 0 & 0 & \gamma_A^2 - 2\gamma_B \gamma_A + \gamma_B^2 & \gamma_A^2 - 2\gamma_B \gamma_A + \gamma_B^2 \\
0 & 0 & 0 & 0 & 0 & 2\gamma_B & -\gamma_A \\
0 & 0 & 0 & 0 & 0 & 2\gamma_B & -\gamma_A \\
0 & 0 & 0 & 0 & 0 & 2\gamma_B & -\gamma_A \\
0 & 0 & 0 & 0 & 0 & 2\gamma_B & -\gamma_A \\
\end{pmatrix}
\]

\[
H_{13} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2\gamma_B - 2\gamma_A \\
-1 & -\gamma_B & \gamma_A - \gamma_B & 2\gamma_B^2 - 2\gamma_A \gamma_B & \gamma_B^2 - 2\gamma_A \gamma_B \\
2\gamma_B - 2\gamma_A & \gamma_A - \gamma_B & 2\gamma_B^2 - 2\gamma_A \gamma_B & \gamma_B^2 - 2\gamma_A \gamma_B \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
H_{15} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
H_{16} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

\[
H_{17} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
A.8 Volatility Decomposition through Malliavin Calculus

In order to pin down the main driver of volatility, we decompose the latter by means of Malliavin derivatives. Do so, let us first define the bivariate Ornstein-Uhlenbeck process

\[
Y_t = \left( \frac{\tilde{f}_{Bt}}{\tilde{g}_t} \right) \\
\ \ dY_t \equiv (A - BY_t) \, dt + C \tilde{W}^\delta_{Bt}
\]

where the matrices \(A, B,\) and \(C\) satisfy

\[
A = \begin{pmatrix} \tilde{f}_{\lambda_B} \\ (\lambda_B - \lambda_A) \tilde{f} \end{pmatrix}, \\
B = \begin{pmatrix} \lambda_B \\ \lambda_B - \lambda_A \end{pmatrix} \begin{pmatrix} \sigma^2_A \\ 0 + \lambda_A \end{pmatrix}, \\
C = \begin{pmatrix} \frac{\gamma_B}{\sigma^2_A} \\ \frac{\gamma_B}{\sigma^2_A} - \lambda_A \end{pmatrix}.
\]

Solving this 2-dimensional stochastic differential equation yields

\[
\tilde{f}_{Bt} = e^{(t-s)\lambda_B} \tilde{f}_{Bt} + \left( 1 - e^{(t-s)\lambda_B} \right) \tilde{f} + \int_t^s \frac{e^{(u-s)\lambda_B} \gamma_B}{\sigma^2_\delta} \tilde{W}^\delta_{Bu} \, d\tilde{W}^\delta_{Bu}
\]

and

\[
\tilde{g}_s = e^{-s \left( \frac{\lambda_A}{\sigma^2_\delta} + \lambda_A + \lambda_B \right)} \left( \frac{e^{\frac{\gamma_B}{\sigma^2_\delta} + s \lambda_A + \lambda_B} - e^{\frac{\gamma_B}{\sigma^2_\delta} + (\lambda_A + s \lambda_B)}}{(\lambda_B - \lambda_A) \sigma^2_\delta + \gamma_A} \right) (\lambda_A - \lambda_B) \tilde{f}_{Bt} \sigma^2_\delta + e^{-s (\lambda_A \sigma^2_\delta + \gamma_A)} \\
= \frac{\sigma^2_\delta}{\lambda_A \sigma^2_\delta + \gamma_A} \left( \left( 1 + e^{-s \left( \frac{\lambda_A}{\sigma^2_\delta} + \gamma_A \right)} \right) \tilde{f}_{\lambda_A} \\
+ e^{-s \left( \frac{\lambda_A}{\sigma^2_\delta} + \gamma_A \right)} \gamma_A \lambda_B + e^{(t-s)\lambda_B} (\lambda_B - \lambda_A) \left( \lambda_A \sigma^2_\delta + \gamma_A \right) \\
+ \lambda_A \left( (\lambda_A - \lambda_B) \sigma^2_\delta + \gamma_A \right) \right) \right)
\]

\[
= \frac{1}{\sigma^2_\delta} \int_t^s \left( \gamma_B \left( \frac{e^{(u-s)\lambda_B} \lambda_A - \lambda_B)}{(\lambda_B - \lambda_A) \sigma^2_\delta + \gamma_A} + e^{-s \left( \frac{\lambda_A}{\sigma^2_\delta} + \gamma_A \right)} \gamma_A \right) \\
+ e^{-s \left( \frac{\lambda_A}{\sigma^2_\delta} + \gamma_A \right)} \gamma_A \right) \, d\tilde{W}^\delta_{Bu}.
\]

Moreover, the dividend process satisfies

\[
\delta_s = \delta_t e^{s \int_t^s \tilde{f}_{Bu} \, du - \frac{1}{2} \sigma^2 \int_t^s \tilde{W}^\delta_{Bu} - \tilde{W}^\delta_{Bu}},
\]

the state-price density

\[
\xi_s = e^{-s \delta_s - \alpha} \left( \left( \frac{1}{\kappa_B} \right)^{1/\alpha} + \left( \frac{\eta_s}{\kappa_A} \right)^{1/\alpha} \right) ^\alpha,
\]

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and the sentiment variable
\[ \eta_t = \eta_t e^{-\frac{1}{2} \int_t^s \gamma_B \, du - \frac{1}{2} \int_t^s \sigma_u^2 \, dW_u}. \]  

Let us denote by \( \mathcal{D}_t \) the Malliavin derivative taken at time \( t \). Using Equation (17), the Malliavin derivative of the dividend process satisfies
\[ \mathcal{D}_t \delta_s = \delta_s \left( \sigma_x - \frac{\gamma_B}{\sigma_x} \left( e^{(t-s)\lambda_B} - 1 \right) \right), \]

because
\[ \mathcal{D}_t f_{Bu} = e^{(t-s)\lambda_B} \gamma_B. \]  

Equation (18) shows that
\[ \mathcal{D}_t \xi_s = \xi_s \left( -\alpha \frac{\mathcal{D}_t \delta_s}{\delta_s} + \omega(\eta_s) \frac{\mathcal{D}_t \eta_s}{\eta_s} \right). \]

Thanks to Equation (19), the Malliavin derivative of the sentiment \( \eta \) satisfies
\[ \mathcal{D}_t \eta_s = \eta_s \left( -\frac{1}{\sigma_s^2} \frac{\gamma_B}{\sigma_s} \left( e^{(t-s)\lambda_B} - 1 \right) \right), \]

Equation (16) implies that the Malliavin derivative of the disagreement \( \hat{g} \) is
\[ \mathcal{D}_t \hat{g}_s = \frac{1}{\sigma_s^2} \left( \frac{\gamma_B}{\sigma_s} \left( e^{(t-s)\lambda_B} - 1 \right) \right) \left( e^{(t-s)\lambda_B} \frac{\gamma_B}{\sigma_s^2} - e^{(t-s)\lambda_B} + e^{(t-s)(\lambda_A + \gamma_A) / \sigma_s^2} \gamma_A \right) \left( \frac{\lambda_A - \lambda_B}{\sigma_s^2} + \gamma_A \right). \]

Let us now derive the volatility decomposition by first considering the following martingale
\[ M_t \equiv S_t \xi_t + \int_0^t \xi_u \delta_u \, du = \int_0^{+\infty} E_t (\xi_u \delta_u) \, du. \]

Applying Itô’s lemma on both sides yields
\[ dM_t = \xi_t \sigma_t (\sigma_t - \theta_t) \, d\hat{W}_t, \]

where \( \theta \) is the market price of risk and \( \phi \) the integrand pertaining to the martingale representation theorem. Consequently, the stock return volatility satisfies
\[ \sigma_t = \theta_t + \frac{\phi_t}{\xi_t S_t} = \theta_t + \frac{\phi_t}{\int_t^{+\infty} E_t (\xi_u \delta_u) \, du}. \]

The Clark-Ocone Theorem states that the integrand \( \phi \) satisfies
\[ \phi_t = \int_t^{+\infty} E_t (\delta_u \mathcal{D}_t \xi_u) \, du + \int_t^{+\infty} E_t (\xi_u \mathcal{D}_t \delta_u) \, du. \]
Following closely the derivations in Dumas et al. (2009), the stock return volatility can be written

\[
\sigma_t = \sigma_\delta + \frac{1 - \alpha}{S_t} \mathbb{E}_t \left( \int_t^\infty \frac{\xi_s}{\xi_t} \delta_s \int_t^s D_t f_Bu \, ds \right) \\
+ \frac{1}{S_t} \mathbb{E}_t \left( \int_t^\infty \frac{\xi_s}{\xi_t} \delta_s (\omega(\eta_s) - \omega(\eta_t)) \frac{D_t \eta_t}{\eta_t} \, ds \right) \\
- \frac{1}{S_t} \mathbb{E}_t \left( \int_t^\infty \frac{\xi_s}{\xi_t} \delta_s \omega(\eta_s) \left( \frac{1}{\sigma_\delta} \int_t^s \hat{g}_u \tilde{W}_{Bu} + \frac{1}{\sigma_\delta^2} \int_t^s \hat{g}_u D_t \tilde{g}_u du \right) \, ds \right),
\]

(23)

because

\[
\phi_t = \mathbb{E}_t \left( \int_t^\infty \xi_s \frac{D_t \xi_t}{\xi_t} ds \right) + \mathbb{E}_t \left( \int_t^\infty \xi_s \delta_s \frac{D_t \delta_t}{\delta_t} ds \right) \\
+ (1 - \alpha) \mathbb{E}_t \left( \int_t^\infty \xi_s \delta_s \int_t^s D_t f_Bu \, ds \right) \\
+ \mathbb{E}_t \left( \int_t^\infty \xi_s \delta_s (\omega(\eta_s) - \omega(\eta_t)) \frac{D_t \eta_t}{\eta_t} \, ds \right) \\
- \mathbb{E}_t \left( \int_t^\infty \xi_s \delta_s \omega(\eta_s) \left( \frac{1}{\sigma_\delta} \int_t^s \hat{g}_u \tilde{W}_{Bu} + \frac{1}{\sigma_\delta^2} \int_t^s \hat{g}_u D_t \tilde{g}_u du \right) \, ds \right).
\]

Finally, substituting Equations (20), (21), and (22) in Equation (23) provides a decomposition of the stock return volatility.
References


