How Does the Bond Market Perceive Macroeconomic Risks under Zero Lower Bound?*

Yuji Sakurai†

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Abstract

I present a joint model of yield curves and macroeconomic variables with an explicit effective zero lower bound by employing the concept of shadow interest rates. Bond yields are derived by assuming no arbitrage opportunities. However, they are not affine due to the zero lower bound. I thus develop a new approximate bond pricing formula that is correct up to a second order. In order to conduct a counterfactual analysis of monetary policy, I employ a standard New Keynesian macroeconomic model and estimate the model parameters for the US and Japan. I find that the effective zero lower bound is small and positive for both countries. The long-run real interest rate is equal to 1.0% in the US during the 1991-2015 period and it is -2.8% in Japan during the 2004-2015 period. Finally, I conduct a counterfactual analysis of raising the target inflation level. In both US and Japan, a higher inflation target steepens the yield curve when the current policy interest rate is not constrained by the zero lower bound. On the other hand, a higher inflation target increases long-term yields while keeping short-term yields unchanged when the current policy interest rate is constrained by the zero lower bound.

Keywords: Zero interest rate policy; Non-linear term structure modeling; No arbitrage; Negative interest rates;

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†Corresponding author. UCLA Anderson School of Management;
E-mail address: yuji.sakurai.2015@anderson.ucla.edu
1 Introduction

Central banks in developed countries have been confronting the zero lower bound and inflation that is below their target over several years. A central bank can end the zero interest rate policy any time it wants. Yet, such a sudden interest rate hike could be contractionary to the economy. As an alternative, some prominent policy makers have proposed that a central bank should raise the inflation target level to generate a higher expected inflation and thus increase the future nominal interest rate. They argue that higher expected nominal interest rates ease the zero lower bound problem. This paper evaluates the merit of this approach.

A common approach in describing interactions between various macroeconomic variables and how they are impacted by policy shocks is to estimate Vector Autoregressions (VARs). However, when an economy is constrained by the zero lower bound, such a method is not applicable for two reasons. First, a policy interest rate is constant and therefore difficult to directly incorporate into VARs. Second, evaluating a policy that has never been implemented requires a counterfactual analysis, which needs a structural model.

With respect to the first issue, the term structure of interest rates contains the information about the future policy interest rate that helps us describe the relationship between interest rates and macroeconomic variables even under the zero lower bound. I extract a shadow policy interest rate that is equal to the nominal policy interest rate as long as that rate is above the zero lower bound. The concept of a shadow interest rate was developed by Black (1995). In this paper, I generalize the original definition of the shadow interest rate to allow for a non-zero lower bound and estimate it using data. The approach of a shadow interest rate is a parsimonious way to allow a short-term nominal rate to stay at zero for a certain period of time. The shadow interest rate can be easily included as a proxy of an actual policy interest rate in specifying the joint dynamics of interest rates and macroeconomic variables.

To resolve the second issue, I assume that macroeconomic variables are described by a standard New Keynesian model that is widely used among policy makers. I take the zero lower bound into account in the New-Keynesian model. This feature allows us to conduct a counterfactual analysis such as raising the inflation target and introducing a negative interest rate on central bank’s reserve. It also naturally captures a possible change in macroeconomic dynamics when an economy is constrained by the zero lower bound.

Specifically, I develop and estimate a joint model of yield curves and macroeconomic variables with an explicit modeling of the zero lower bound both in nominal bond pricing and macroeconomic dynamics. There are four state variables: real GDP, potential GDP, inflation and the shadow interest rates. These variables fol-


\[2\] For instance, Krugman (2014) argues “Escaping from this feedback loop appears to require more radical economic policies than are likely to be forthcoming. As a result, a relatively high inflation target in normal times can be regarded as a crucial form of insurance, a way of foreclosing the possibility of very bad outcomes.”
I employ a no-arbitrage term structure model of interest rates that is flexible enough to capture the cross-section of nominal bond yields before and during the zero interest rate policy. Since the model is non-affine to the state variables due to the zero lower bound, I develop a second order approximation for pricing bonds. The no-arbitrage restrictions allow us to transform the private sector’s expectations about macroeconomic variables into the bond market’s expectations contained in yield curves while keeping the joint model internally consistent. Once the no-arbitrage term structure model is estimated with the cross-section of nominal bond yields, one can decompose nominal bond yields into the path of expected nominal interest rates and term premium. The expected nominal interest rate is a key input in a standard New Keynesian model as I will explain below. For simplicity, I assume a linear market price of risk.\(^3\)

Four equations describe the standard New Keynesian model. The first one is the Investment-Saving (IS) equation that determines the relationship between expected nominal interest rates, expected inflation and real output. The IS equation is derived from an inter-temporal consumption Euler equation. The zero lower bound is explicitly modeled in the IS equation. The other three equations follow a standard New-Keynesian model. First, the shadow interest rate mean-reverts to a policy target level that is specified by a Taylor rule. Second, current inflation depends on lagged inflation, expected inflation and the output gap. Third, potential GDP is assumed to be exogenously driven and mean-reverting. These structural assumptions are widely used in macroeconomic analysis and made for conducting a counterfactual analysis. They also facilitate the interpretation and estimation of the parameters in the VAR (1) system.

I apply my model of yield curves and macroeconomic variables for the US and Japan. I jointly estimate parameters of a standard New Keynesian model and market price of risk parameters. Using market price of risk parameters, I compute nominal bond prices with the zero lower bound and fit model-implied nominal bond yields to actual yields with some measurement errors. The parameters of a standard New Keynesian model are estimated to capture the dynamics of shadow interest rates and other macroeconomic variables in the physical measure.

After estimating the model parameters, I conduct different types of counterfactual analyses of the interaction between interest rates and macroeconomic variables given a change in monetary policy. The estimated term structure model also provides a unified understanding of the relationship between bond yields and macroeconomic variables before and during the zero lower bound. I perform similar analysis as done in other empirical studies of term structure models.

Not surprisingly, extracted shadow interest rates for the US show that the ef-

\(^3\)In general, it is difficult to match nominal bond yields generated from a standard macroeconomic model to actual data because of undervaluation of term premium. To fix this issue, one needs more sophisticated utility functions such as Epstein-Zin preferences documented by Rudebusch and Swanson (2012). Yet, these more sophisticated utility functions are then difficult to incorporate into the term structure model with the zero lower bound. Thus, I employ a simple no-arbitrage term structure model with the linear market price of risk.
fective (zero) lower bound became binding in 2009. It is approaching the bound recently, again. In the case of Japan, the shadow interest rates have been negative over years and became more negative since 2014 as the long-term nominal yield is further lowered by the Bank of Japan’s QQE (Quantitative and Qualitative Easing). When the long-run real interest rate is computed as a mean-reverting level of the real shadow interest rate with zero output and inflation gaps, it is equal to 1.0% in the US during the 1991-2015 period and it is −2.8% in Japan during the 2004-2015 period.

I conduct a counterfactual analysis of raising the target inflation level with negative shadow interest rates for the initial time period. Recall that a negative current shadow interest rate means that the zero lower bound is currently binding. In both US and Japan, a higher inflation target increases long-term nominal yields while keeping short-term yields unchanged. For comparison, I also study the case when the zero interest rate policy is suddenly abandoned. In this case, the long-term yields do not increase as much as those in the case of higher inflation target. Plus, a sudden ending of the zero interest rate policy is contractionary while raising inflation target is expansionary. In this respect, raising inflation target is preferred as it is expansionary.

It is also possible to conduct other types of counterfactual analyses. For example, I evaluate an introduction of a negative lower bound. I consider the case when a current shadow interest rate is negative. I document that it steepens the nominal yield curve by allowing the short-term nominal yields to be negative and by increasing the long-term nominal yields.

As in the previous studies, I investigate how macroeconomic risks are priced in the yield curves by looking at decompositions of the yield curves and the factor loading of the shape of the yield curves. There are several notable findings.

First, I decompose the nominal bond yields into the expected interest rate and the term premium. For both US and Japan, the term premium is larger for longer maturities. This finding is consistent with previous studies. In the case of Japan, the nominal bond yields are almost entirely explained by the term premium. This is not surprising since the Bank of Japan has been employing the zero interest rate policy during the 2004-2015 period studied in this paper.4

Second, I decompose nominal bond yields into expected real interest rates, real interest rate risk premium, expected inflation, and inflation risk premium. I show that the nominal bond yield curve is upward sloping mostly due to real interest rate risk premium in US during the 1991-2015 period. In Japan, the nominal bond yields are mostly explained by real interest rate risk premium than other three components, especially for longer maturities. Expected inflation implied by the nominal bond yields is negative or close to zero for all maturities. This is natural because the historical average of realized inflation is almost zero as confirmed in Table 2.

Third, following Wright (2011), I investigate whether a larger cross-sectional dispersion in survey-based inflation forecasts explains a larger inflation risk premium. Wright (2011) conducts the same analysis for ten industrialized countries during the 1990-2010 period. He finds a statistically significant positive relationship be-

4Precisely speaking, the Bank of Japan hiked its policy interest rate to the positive level during the period between July of 2006 and December of 2008.
tween the cross-sectional dispersion and the inflation premium. Yet, he employs a conventional affine term structure model to obtain the inflation premium. I extend his analysis by modeling the zero lower bound explicitly. I find that the signs of the coefficients are consistent with his findings in almost all cases for both US and Japan although the results are not statistically significant in the case of US. Thus, the results weakly support that Wright (2011)’s finding is applicable to the zero lower bound period.

Fourth, the model also provides a unified understanding of the macroeconomic drivers of yield curves. It is well known that (1) variations in yield curves are explained by level, slope and curvature factors and (2) level and slope factors are often associated with inflation and real economic activity. Yet, once the zero lower bound is binding, there is no clear level factor. Without a macro-finance term structure model that incorporates a zero lower bound, it is difficult to have a consistent explanation of macroeconomic drivers of yield curves. The model in this paper offers a better understanding of macroeconomic effect on term structure both before and after the zero interest rate policy is employed. For example, in the case of the US, the results show that shadow interest rates function as a level factor during normal times but it works as a slope factor during the zero lower bound time period.

As an additional application, I study the excess sensitivity of long-distant real forward interest rates to changes in the short-term nominal rate in Japan. This has been studied as a puzzle in the US. I document that there is a similar or even more pronounced excess sensitivity in Japan. Utilizing a comprehensive dataset of fixed income investors, I investigate the duration adjustment effect proposed by Hansen and Stein (2015). Their hypothesis is as follows. When short-term interest rates are lowered, fixed income investors re-balance their bond portfolios toward longer-term bonds in an effort to keep returns from their portfolio unchanged. Such a re-balancing further lowers the longer-term bond yields. This feedback effect is a key mechanism in creating the excess sensitivity in the hypothesis.

One issue in their study is that they use a crude measure of portfolio duration held by commercial banks and primary dealers due to limited availability of data in the US. For commercial banks, as a proxy for duration, they use the average fraction of non-trading account securities with a current remaining maturity or next repricing date of one year or longer. For primary dealers, they use the data from the Federal Reserve Bank of New York that categorizes bonds into four buckets: shorter than 3 years, 3 to 6 years, 6 to 11 years and longer than 11 years.

The data set of Japanese fixed income investors used in this paper is more detailed. It allows us to see each institutional investor’s duration across several financial sectors such as banking, asset management and insurance. Equipped with this data set, I investigate the hypothesis of Hansen and Stein (2015). Specifically, I conduct two different types of regressions. In the first regression analysis, I study the impact

\footnote{Figures 11 and 12 show the results of the principal component analysis before and during the zero lower bound period. In Figure 12, there is no clear level factor.}

\footnote{The puzzle is that a short-term nominal interest rate impacts long-distant real forward rate although nominal changes are supposed not to have such a strong effect on long-distant real interest rates in a standard New Keynesian model.}
of each financial sector’s change in duration on nominal bond yields. In the second regression analysis, I investigate the reverse causality: whether a change in nominal bond yields induces investor’s adjustment of duration. Although I document some empirical support for the causality in the second regression analysis, I do not find statistically significant results in the first regression analysis. Overall, the results partly supports the hypothesis of Hansen and Stein (2015) but not entirely.

The rest of the paper is structured as follows. Section 2 provides a literature review. Section 3 describes a joint model of macroeconomic and term structures dynamics with an explicit zero lower bound. Section 4 extends the model by incorporating regime switching. Section 5 discusses the data sets used and the estimation methodology employed in this study. Section 6 analyzes implications of the model for term structure before and during the zero lower bound. Section 7 concludes.

2 Literature review

This paper contributes to four strands of the literature. The first two strands of the literature are closely related to the model I developed in this paper. The third one is associated with an extension of the model. The last one is related only to my additional empirical exercise. I briefly review relevant papers in each strand in this section. The list of the papers is not exhaustive but chosen based on relevance.

2.1 Shadow interest rate models

One theoretical contribution of this paper is modeling yield curves under zero lower bound. After the Federal Reserve lowered its policy rate to zero, there have been many studies on modeling the zero lower bound in the term structure of interest rates. As an early study, Longstaff (1992) examines the CIR term structure model and discusses a sticky boundary behavior of interest rates. Black (1995) interprets the nominal rate as an option on a hypothetical interest rate called a shadow interest rate. Gorovoi and Linetsky (2004) revisit Black (1995)’s work and obtain analytical expression for yield curves when a shadow interest rate follows Vasicek process or CIR process. Equipped with this analytical formula, Ueno, Baba and Sakurai (2006) calibrate the term structure model of shadow interest rates to the Japanese government bond markets and find that it fits better than the conventional Vasicek model. Oda and Ueda (2005) is an early study of macro-finance term structure models using a concept of a shadow interest rate. As in this paper, they assume that a shadow interest rate is specified by the Taylor rule and find that the Bank of Japan’s unconventional monetary policy functioned through the zero interest rate commitment. More recently, Kim and Singleton (2012) develop a two-factor model of shadow interest rates and estimate their model parameters using extended Kalman filtering. They show that the model outperforms conventional affine counterpart.

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7 Fixed income investors are quite segregated in terms of their bond maturities. For example, banks are active in short-term maturities while insurance companies are more active in long-term bonds. In my regression analysis, I take into account this heterogeneity.

8 Bomfim (2003) estimates the probability that the federal funds rate hits zero lower bound. Yet, he employs a conventional affine model for that.
and quadratic Gaussian models. Christensen and Rudebusch (2014) and Ichiue and Ueno (2013) estimate two-factor shadow interest rate models based on an approximation developed by Krippner (2013a,b). Bauer and Rudebusch (2013) employ a simulation-based formula for bond pricing. Wu and Xia (2014) approximate forward interest rates with zero lower bound, but they ignore a convexity term in discrete time.\(^9\) Lombardi and Feng (2014) estimate a dynamic factor model using a dataset up until the time when the Federal Reserve started the zero interest rate policy and extrapolate a shadow policy rate during the zero lower bound period. Yet, none of those studies develop multi-factor shadow interest rate models with an explicit modeling of the effect of the zero lower bound with macroeconomic dynamics.\(^10\)

My approximation formula is related to Krippner (2013a,b) in which he applies an option-based approximation for nominal bond prices. A technical contribution of this paper is that it develops a second-order approximation for bond pricing with the zero lower bound. The approximation is interpreted as the convexity adjustment based on the delta of the option arising from zero lower bound. I also provide a formal proof as well as an intuitive derivation.

\subsection*{2.2 New Keynesian macro-finance models}

The empirical part of this paper contributes to testing term structure models that explicitly incorporates structural macroeconomic dynamics. In these models, forward-looking agents optimize their behavior. The dynamics of real output is described by the Investment-Saving (IS) equation derived from an intertemporal consumption Euler equation. An evolution of inflation is determined by the Aggregate-Supply (AS) equation. To the best of my knowledge, Hördhal, Tristani and Vestin (2006) and Wu (2006) pioneer the literature by incorporating a New Keynesian macro framework into a no-arbitrage affine term structure model. Hördhal, Tristani and Vestin (2006) find that forecasting performance of their model is superior to affine counterparts without a structural macroeconomic dynamics. In a similar framework, Wu (2006) reports that the slope factor of the yield curve is driven by monetary policy shocks while the level factor is explained by technology shocks. Bakaert, Cho and Moreno (2010) further extend Hördhal, Tristani and Vestin (2006) model by allowing potential GDP to be time-varying. In this respect, the model estimated in this paper is very close to theirs. However, they assume constant risk premia. Thus, it is difficult to understand the effect of monetary policy on the expected path of nominal policy interest rates and the term premium separately. The other closely related papers are Bikbov and Chernov (2013), Dew-Becker (2014) and Kung (2015). Bikbov and Cher-
nov (2013) incorporate a log-linearized New Keynesian model with regime switching into a no-arbitrage affine term structure model. Dew-Becker (2014) and Kung (2015) also construct term structure models with New Keynesian macroeconomic dynamics and solve their forward-looking macroeconomic models using higher-order approximation. One difference from these three papers is an explicit consideration of zero lower bound that is important to analyze a recent behavior of interest rates. Campbell, Pflueger and Viceira (2015) study the impacts of monetary policy rules and macroeconomic shocks on nominal bond risks by employing New Keynesian macroeconomic term structure models. They conduct a counterfactual analysis and find that nominal bond risk increases after 1977 due to a more anti-inflationary stance. In this paper, I primarily focus on implications of the zero lower bound for term structure and macroeconomic dynamics. To do so, I employ a textbook-style New Keynesian model since solving a large-scale forward-looking macroeconomic model with the zero lower bound is computationally difficult.

2.3 Regime-switching term structure model of interest rates

The other contribution of this paper is associated with regime switching term structure modeling. Motivated by empirical evidence of a structural change in macroeconomic variables or a central bank’s policy stance, several papers employ Markov-regime switching term structure models. Ang, Bakaert and Wei (2008) present an affine term structure model with regime switching where a nominal short rate is driven by three factors including expected inflation. They show that unconditional real rate curve is flat and around 1.3% although it is strongly downward sloping in one regime. Li, Li and Yu (2013) study the effect of a central bank’s monetary policy stance on the term structure of interest rates using regime switching. They find that the Federal Reserve is more proactive in one regime than in other regime and the stance contributes to Great Moderation. Bikbov and Chernov (2013) formulate a term structure model in which inflation and real GDP are governed by a forward-looking macroeconomic model with regime switching. They allow both monetary policy stance and macroeconomic volatility to be regime-dependent. A new aspect of the model in this paper is an explicit consideration of zero lower bound under regime-switching term structure model.

2.4 Excess sensitivity of long-term real interest rates

Finally, this paper contributes to empirical studies of excess sensitivity of long-term forward interest rates to short-term nominal interest rates. Gürkaynak, Sack and Swanson (2005) is the first study to document it for the US case during 1990 and 2002. They find that it is driven by a private sector’s change in long-term expected inflation. They also show that the similar pattern is not observable in the case of UK treasury bond markets. Nakamura and Steinsson (2013) conduct a similar study and show that it is mostly driven by a change in forward real rates. Hanson and Stein (2015) propose a hypothesis that some investors are yield-oriented and those investor’s re-balancing of their fixed-income portfolio impacts on long-term real interest rates. I study the excess sensitivity of long-term forward interest rates
in the case of Japan with a comprehensive data set of fixed income investors.

3 Model

I develop a joint model of yield curves and macroeconomic variables with the zero lower bound in this section. First, I explain the building blocks of a textbook-style New-Keynesian macroeconomic model. I show that the New-Keynesian model has a VAR (1) representation. For expository simplicity, I first explain the model without the zero lower bound and then discuss the case with an explicit zero lower bound.

Second, I explain a no-arbitrage term structure model of interest rates with an explicit modeling of the zero lower bound. The dynamics of state variables in physical measure are modeled as a VAR(1). The coefficient matrices of VAR (1) is determined by the New Keynesian model. Given the physical measure process, I need to specify the market price of risk for pricing nominal bonds. I employ a linear market price of risk. I then briefly discuss the approximation to price the bonds with the zero lower bound.

3.1 A structural New-Keynesian macroeconomic dynamics

I rely on a simple New-Keynesian macroeconomic model. Variants of this model are widely used for macroeconomic analysis. The model has micro foundations in the sense that it can be derived by assuming forward-looking households and profit-maximizing firms. Thus, it is suitable for conducting a counterfactual analysis of monetary policy.

The main reason to adopt a structural model is to conduct a counterfactual analysis. Yet, there are some other advantages. First, a stylized structural macroeconomic model helps us interpret estimated parameters. Second, it reduces the number of model parameters to be estimated. I admit that the model may be too simple to describe a complicated behavior of an economy. However, this is the first step towards a joint modeling of yield curves and macro economy. The model can be easily extended if one is interested in some other aspects.

In what follows, I explain each building block of the model in detail. There are four key equations: an inflation equation, policy rule equation, and two output equations for real and potential output. I show how the structural model is represented as a reduced VAR form.

First, let us describe how inflation evolves over time. As in textbook-style New Keynesian models, current inflation is determined by three components: an expected inflation \( E_t[\pi_{t+1}] \), lagged inflation \( \pi_{t-1} \) as well as the output gap \( \Delta y_t = y_t - y^n_t \).

\[
\pi_t = \mu_p E_t[\pi_{t+1}|I_t] + (1 - \mu_p)\pi_{t-1} + \kappa(y_t - y^n_t) + \epsilon_{AS,t}, \tag{3.1}
\]

There are 2 \times 4 \times 4 = 32 parameters in the VAR coefficient \( F \) and the matrix \( H \). The vector \( G \) has 4 parameters. In total there are 36 parameters. When we allow the time to exit from the zero lower bound affects VAR coefficients, \( F, G, H \), in macroeconomic dynamics, these VAR coefficients become time-varying so that the number of model parameters increase dramatically and makes their estimation difficult.
where $\epsilon_{AS,t}$ is sampled from a normal distribution $N(0, \sigma_{AS})$. This equation is often called the Aggregate-Supply (AS) equation or as the New-Keynesian Phillips curve. The information set $I_t$ is defined as $I_t = \{y_t, \pi_t, x_t, y^n_t\}$.

This inflation equation (3.1) is derived as a first-order condition of the monopolistically competitive firms’ optimal price setting. Output gap $y_t - y^n_t$ in the third term measures real marginal costs for the firms. The dependence of current inflation on the lagged inflation is motivated by empirical studies of the dynamics of inflation. It captures the degree of backward-looking behavior of the firms or nominal price indexation.\(^{12}\)

Second, I assume that a central bank determines its policy target by following a Taylor rule with current real output $y_t$, potential output $y^n_t$ and inflation $\pi_t$.

$$x^\text{target}_t = i^*_t + \gamma_y(y_t - y^n_t) + \gamma_\pi(\pi_t - \bar{\pi}), \quad (3.2)$$

where $i^*_t$ is a policy-neutral nominal interest rate. The sum of the second and the third terms reflects the central bank’s adjustment of its target interest rate. I assume that the policy-neutral nominal interest rate $i^*_t$ is decomposed into

$$i^*_t = r^* + \pi_t, \quad (3.3)$$

where $r^*$ is constant. I interpret $r^*$ as the long-run real interest rate or an equilibrium real interest rate. I set a target inflation rate $\bar{\pi} = 0.02$ in my empirical analysis as there are two constant terms, $r^*$ and $\gamma_\pi \bar{\pi}$. 2% inflation target is realistic in both US and Japan.\(^{13}\)

A shadow interest rate follows AR(1) process and mean-reverts to the target shadow interest rate $x^\text{target}_t$.

$$x_t = \mu_x x^\text{target}_t + (1 - \mu_x) x_{t-1} + \epsilon_{x,t}, \quad (3.4)$$

where $\epsilon_{x,t}$ is sampled from a normal distribution $N(0, \sigma_x)$. The speed of mean reversion $\mu_x$ is smaller than one and captures the fact that a central bank gradually changes its policy interest rate to a desired target interest rate. Empirical studies on US Treasury bonds often document $\mu_x$ to be very close to zero and thus a policy interest rate is close to a random walk.\(^{14}\)

Third, potential GDP $y^n_t$ follows AR(1) process.

$$y^n_t = \mu_y y^n + (1 - \mu_y) y^n_{t-1} + \epsilon_{y^n,t}, \quad (3.5)$$

\(^{12}\)I also estimate the case when expected inflation $E_t[\pi_{t+1}]$ is replaced with a survey-based inflation expectation $\pi^e_t$ that is defined as in the following relationship $s_t = \mu_s \pi^e_t + (1 - \mu_s) s_{t-1}$ where $\mu_s$ captures stickiness of survey-based inflation forecasts. This assumption is employed by Baele et al. (2015). The results of decomposition of nominal bond yields are not quantitatively different. To conduct a counterfactual analysis of raising inflation target, I choose endogenous expected inflation.

\(^{13}\)The Bank of Japan announced that they will strictly target 2% inflation in April of 2013. Before that, the target inflation was not explicitly stated but Masaaki Shirakawa, the former governor of the Bank of Japan, stated that the inflation consistent with the price stability is in a positive range of 2 percent or lower. See Shirakawa (2012).

where $\epsilon_{y^n,t}$ is sampled from a normal distribution $N(0, \sigma_{y^n})$. Notice that this specification allows both (1) a very persistent process as $\mu_y$ goes to zero and (2) a very fast mean-reverting process as $\mu_y$ goes to 1. The former case is relevant to US while the latter is relevant to Japan. The equation (3.5) is also employed by Bakaert, Cho and Moreno (2010).

Finally, I close the model by introducing one more equation for real output and a real interest rate. The equation is called Investment-Saving (IS) equation. I consider two different specifications. The first one is

$$[\text{IS-ZLB}] \quad y_t = \alpha_{IS} + \mu_y^+ E_t[y_{t+1}|I_t] + (1 - \mu_y^-) y_{t-1} - \phi(i_t - E_t[\pi_{t+1}|I_t]) + \epsilon_{IS,t}, \tag{3.6}$$

where $\epsilon_{IS,t}$ is sampled from a normal distribution $N(0, \sigma_{IS})$. Notice that a current real output depends on both the expected real output and the lagged real output. This type of the IS equation is derived as a first-order condition of a utility-maximizing representative agent with external habit formation. The parameter $\mu_y$ measures the degree of the agent’s external habit. I call (3.6) IS-ZLB. The second one is given by

$$[\text{IS-SR}] \quad y_t = \alpha_{IS} + \mu_y^+ E_t[y_{t+1}|I_t] + (1 - \mu_y^-) y_{t-1} - \phi(x_t - E_t[\pi_{t+1}]) + \epsilon_{IS,t}, \tag{3.7}$$

I call the specification above IS-SR. The only difference between IS-ZLB and IS-SR is that a nominal policy rate $i_t$ is replaced with a shadow policy rate $x_t$ in IS-SR.

The reason why I consider these two specifications is as follows. Recall that a nominal interest rate (nominal policy rate) is a non-linear function of the shadow policy interest rate.

$$i_t = \max(x_t, \tilde{i}) \quad \tag{3.8}$$

where the max operator arises from the effective (zero) lower bound on nominal interest rates. Substituting (3.8) for (3.6), one obtains a nonlinear forward-backward-looking equation.

$$y_t = \alpha_{IS} + \mu_y^+ E_t[y_{t+1}|I_t] + (1 - \mu_y^-) y_{t-1} - \phi(\max(x_t, \tilde{i}) - E_t[\pi_{t+1}|I_t]) + \epsilon_{IS,t}. \tag{3.9}$$

A non-linearity arising from the zero lower bound makes it difficult to apply a conventional method to solve forward-backward equations. In this respect, the IS-SR case is more tractable than the IS-ZLB case. I first explain how a VAR (1) form is obtained from the structural New Keynesian macroeconomic dynamics in the IS-SR case for expository simplicity. I will discuss IS-ZLB case in the next subsection. I focus on IS-ZLB case in my empirical analysis. Notice that the zero lower bound in nominal bond pricing is not abstracted in the IS-SR case even if the max operator in (3.8) is dropped.

There can be another reason to consider the IS-SR case in addition to its tractability. Suppose that a shadow interest rate $x_t$ is negative at time $t$. In the IS-ZLB case, (3.6) is reduced to

$$y_t = \alpha_{IS} + \mu_y^+ E_t[y_{t+1}|I_t] + (1 - \mu_y^-) y_{t-1} - \phi(i_t - E_t[\pi_{t+1}]) + \epsilon_{IS,t}. \tag{3.9}$$
There is no term involving a current shadow interest rate \( x_t \) in (3.9). Thus, a current shadow interest rate does not have any direct impact on current real output although it may have an indirect impact on future real output \( E_t[y_{t+1}] \) by changing the future path of the nominal policy rate. In IS-SR case, a current shadow interest rate has a direct impact on current real output. Recall that a more negative shadow interest rate leads to lower long-term nominal bond yields. One can interpret the IS-SR case as a parsimonious way to capture a possible relationship between current long-term interest rates and current real output when the zero lower bound is binding.

Combining these four equations of macroeconomic dynamics above, an entire system is described as follows. For notational simplicity, I write \( E_t[y_{t+1}] \) and \( E_t[\pi_{t+1}|I_t] \) as \( E_t[y_{t+1}] \) and \( E_t[\pi_{t+1}] \), respectively.

\[
y_t = \alpha IS + \mu_y^t E_t[y_{t+1}] + (1 - \mu_y^t)y_{t-1} - \phi(x_t - E_t[\pi_{t+1}]) + \epsilon_{IS,t}, \tag{3.10}
\]

\[
\pi_t = \mu_\pi E_t[\pi_{t+1}] + (1 - \mu_\pi)\pi_{t-1} + \kappa(y_t - y^n_t) + \epsilon_{AS,t}, \tag{3.11}
\]

\[
x_t = \mu_x (r^* + \pi_t + \gamma_x(y_t - y^n_t) + \gamma_x(\pi_t - \pi)) + (1 - \mu_x)x_{t-1} + \epsilon_{x,t}. \tag{3.12}
\]

\[
y^n_t = \mu_y^n \tilde{y}^n + (1 - \mu_y^n)y^n_{t-1} + \epsilon_{y^n,t}. \tag{3.13}
\]

The system of these four equations is almost the same as the one employed in Gürkaynak, Sack and Swanson (2005) and Hördal and Vestin (2006) except that (1) they abstract the zero lower bound and (2) they don’t have the equation (3.13) and model output gap \( \Delta y_t = y_t - y^n_t \) directly.\(^{15}\) In this paper, I separately treat these two equations to make it clear that the zero lower bound arises in the IS equation for real output, not for output gap in the IS-ZLB case. The other closely related papers such as Bekaert, Cho and Moreno (2010) and Campbell, Pflueger and Viceira (2015) also abstract the zero lower bound. Precisely speaking, Bekaert, Cho and Moreno (2010) have one additional equation to associate the long-run expected inflation with the inflation target. Similarly, Campbell, Pflueger and Viceira (2015) assume that the inflation target follows a random walk and describe the macroeconomic dynamics by the four equations. To keep my macroeconomic model as simple as possible, I employ the four equations described above.

The IS equation (3.10) and the AS equation (3.11) are rewritten as

\[
-\mu_y^t E_t[y_{t+1}] - \phi E_t[\pi_{t+1}] = -y_t - \phi x_t + (1 - \mu_y^t)y_{t-1} + \alpha IS + \epsilon_{IS,t}, \tag{3.14}
\]

\[
-\mu_\pi E_t[\pi_{t+1}] = -\pi_t + (1 - \mu_\pi)\pi_{t-1} + \kappa(y_t - y^n_t) + \epsilon_{AS,t}. \tag{3.15}
\]

The policy rule equation (3.12) and the potential output equation (3.13) are also rearranged as

\[
0 = \mu_x \gamma_x y_t + \mu_x (1 + \gamma_x)\pi_t - x_t - \mu_x \gamma y^n_t + (1 - \mu_\pi)x_{t-1} + \mu_x r^* - \mu_x \gamma \pi + \epsilon_{x,t}, \tag{3.16}
\]

\[
0 = -y^n_t + \mu_y^n \tilde{y}^n + (1 - \mu_y^n)y^n_{t-1} + \epsilon_{y^n,t}. \tag{3.17}
\]

The four equations, (3.14), (3.15), (3.16) and (3.17) are represented as a VAR (1) system:

\[
AE_t[X_{t+1}] = BX_t + CX_{t-1} + D + \epsilon_t, \tag{3.18}
\]

\(^{15}\)A textbook-style New Keynesian model is often represented by the first three equations: the IS equation for output gap, the AS equation and a policy rule equation.
where \( X = (y_t, \pi_t, x_t, y_n^t)^T \) and \( \epsilon_t = (\epsilon_{IS,t}, \epsilon_{AS,t}, \epsilon_{x,t}, \epsilon_{y,n,t})^T \). Specifically, coefficient matrices \( A \) and \( B \) are given by

\[
A = \begin{pmatrix}
-\mu_y^+ & -\phi & 0 & 0 \\
0 & -\mu_x & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
B = \begin{pmatrix}
-1 & 0 & -\phi & 0 \\
\kappa & -1 & 0 & -\kappa \\
\mu_x\gamma_y & \mu_x(1 + \gamma_x) & -1 & -\mu_x\gamma_y \\
0 & 0 & 0 & -1
\end{pmatrix}
\]

(3.19)

Matrix \( C \) is specified as

\[
C = \begin{pmatrix}
1 - \mu_y^- & 0 & 0 & 0 \\
0 & 1 - \mu_x & 0 & 0 \\
0 & 0 & 1 - \mu_x & 0 \\
0 & 0 & 0 & 1 - \mu_{yn}
\end{pmatrix}
\]

(3.20)

Vector \( D \) is given by

\[
D = (\alpha_{IS}, 0, \mu_x(r^* - \gamma_x\pi), \mu_y\tilde{y}^n)^T.
\]

(3.21)

In the literature of a structural New Keynesian macroeconomics, the techniques to solve the VAR system are widely known. So I briefly review how to solve (3.18). Let us assume the solution of (3.18) is represented as

\[
X_{t+1} = FX_t + G + H\epsilon_t,
\]

(3.22)

Substituting (3.22) into (3.18) yields

\[
X_t = (AF - B)^{-1}(CX_{t-1} + D - AG + \epsilon_t).
\]

(3.23)

Applying the method of undetermined coefficients, one obtains

\[
F = (AF - B)^{-1}C,
G = (AF - B)^{-1}(D - AG),
H = (AF - B)^{-1}.
\]

(3.24) \hspace{1cm} (3.25) \hspace{1cm} (3.26)

(3.24) and (3.25) are simplified as

\[
AF^2 - BF - C = 0,
G = (AF - B + A)^{-1}D.
\]

(3.27) \hspace{1cm} (3.28)

It is straightforward to solve the quadratic equation (3.27) for the matrix \( F \). Once \( F \) is obtained, substituting \( F \) for (3.28) and (3.26) yields \( G \) and \( H \). One technical issue is the existence of multiple solutions when the quadratic equation (3.27) is numerically solved. I select a solution that is non-explosive solution.\(^{16}\)

\(^{16}\)When \( \phi \) is small compared to \( \mu_y^+ \) and \( \mu_y^- \), one can approximately solve the IS equation separately from other equations. Then, one gets \( F_{11} \approx (1 \pm \sqrt{1 - 2(1 - \mu_y^-)\mu_y^+})/(2\mu_y^+) \). One can use this as an initial guess for \( F_{11} \). It is also worth mentioning that \( F_{44} = 1 - \mu_{yn} \) and \( G_4 = \mu_{yn}\tilde{n} \).
Once (3.22) is obtained, one can forecast the state variables $X = (y_t, \pi_t, x_t, y^n_t)^T$. The forecasts of $k$-period-ahead state variables $E_t[X_{t+k}]$ are given by

$$E_t[X_{t+k}] = F^kX_t + (I + F + F^2 + \cdots + F^{k-1})G,$$

(3.29)

where $I$ is a 4 x 4 unit matrix. The forecasts of the time average of state variables over $n$ periods are given by

$$E_t[\bar{X}_{t+n}] = \sum_{k=1}^{n}E_t[X_{t+k}]/n = \bar{F}_nX_t + \bar{G}_n,$$

(3.30)

where $\bar{F}_n = \sum_{k=1}^{n}F^k/n$ and $\bar{G}_n = \sum_{k=1}^{n}\sum_{j=1}^{k}F^{j-1}G/n$. The second element of $E_t[\bar{X}_{t+k}]$ is the forecast of the average inflation over $n$ periods at time $t$, denoted as $E_t[\bar{\pi}_{t+k}]$. In estimating the model, I use survey-based forecasts of the average inflation. I denote the survey-based forecasts of the average inflation over $n$ periods at time $t$ with $s^n_t$. I assume that $s^n_t$ is determined by

$$s^n_t = E_t[\bar{\pi}_{t+k}] + \alpha^n_s + \epsilon^n_t,$$

(3.31)

where $\alpha^n_s$ captures a bias of the survey-based forecast of inflation. $\epsilon^n_t$ is sampled from $N(0, \sigma^n)$. Notice that $s^n_t$ is a linear function of all state variables $X_t$ because $E_t[\bar{\pi}_{t+k}]$ is a linear function of $X_t$. Therefore, one can rewrite (3.31) as

$$s^n_t = \bar{F}_{n,2}X_t + \bar{G}_{n,2} + \alpha^n_s + \epsilon^n_t,$$

(3.32)

where $\bar{F}_{n,2}$ is the second row of $\bar{F}_n$ and $\bar{G}_{n,2}$ is the second element of $\bar{G}_n$.

### 3.2 Zero lower bound in macroeconomic dynamics

In the previous subsection, I replace a nominal policy rate with a shadow (nominal) policy rate and focused on IS-SR case (3.7). In the following subsection, I discuss IS-ZLB case (3.6) where the zero lower bound is explicitly considered in a structural New Keynesian macroeconomic model.

The system of four equations with zero lower bound is given by

$$y_t = \alpha_{IS} + \mu^n_yE_t[y_{t+1}] + (1 - \mu^n_y)y_{t-1} - \phi(i_t - E_t[\bar{\pi}_{t+1}]) + \epsilon_{IS,t},$$

(3.33)

$$\pi_t = \mu_\pi E_t[\bar{\pi}_{t+1}] + (1 - \mu_\pi)\bar{\pi}_{t-1} + \kappa(y_t - y^n_t) + \epsilon_{AS,t},$$

(3.34)

$$x_t = \mu_x (s^n + \pi_t + \gamma_y(y_t - y^n_t) + \gamma_\pi(\bar{\pi}_t - \bar{\pi})) + (1 - \mu_x)x_{t-1} + \epsilon_{x,t},$$

(3.35)

$$y^n_t = \mu_{y^n}y^n + (1 - \mu_{y^n})y^n_{t-1} + \epsilon_{y^n,t},$$

(3.36)

where $i_t = \max(x_t, \bar{i})$. Notice that a shadow interest rate $x_t$ cannot have any impact on current real output $y_t$ if the zero lower bound is binding ($x_t < 0$) in (3.33). Yet, a shadow interest rate could have an impact on current real output by changing the expected real output.\(^{17}\) This effect depends on how long a central bank keeps its

\(^{17}\)To see this, consider a simplified IS equation below:

$$y_t = E_t[y_{t+1}] - \phi(i_t - E_t[\bar{\pi}_{t+1}]),$$

$$= -\phi E_t \left[ \sum_{k=1}^{\infty} (i_{t+k} - \pi_{t+k+1}) \right] - \phi(i_t - E_t[\bar{\pi}_{t+1}]),$$

14
policy rate at zero percent. As a result, coefficients $F$ and $G$ are time-dependent when the shadow interest rate is negative.

The four equations above can be expressed in the form of matrix equation.

$$
A^*E_t[X_{t+1}] = B^*X_t + C^*X_{t-1} + D + \epsilon_t, \quad \text{if } \ x_t \leq \tilde{i}, \quad (3.37)
$$

$$
AE_t[X_{t+1}] = BX_t + CX_{t-1} + D + \epsilon_t, \quad \text{if } \ x_t > \tilde{i}, \quad (3.38)
$$

$A, B, C$ and $D$ are already defined in the previous subsection. It is easy to confirm the coefficient matrix $A^* = A$ that is defined by (3.19) and a diagonal matrix $C^* = C$ in (3.20) because the zero lower bound does not appear in these matrices. The coefficient matrix $B^*$ and the vector $D^*$ are different from their counterparts, $B$ and $D$.

$$
B^* = 
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
\kappa & -1 & 0 & -\kappa \\
\mu_x \gamma_y & \mu_x (1 + \gamma_x) & -1 & -\mu_x \gamma_y \\
0 & 0 & 0 & -1
\end{pmatrix}
$$

(3.39)

Vector $D^*$ is given by

$$
D^* = (\alpha_{IS} - \phi \tilde{i}, 0, \mu_x (r^* - \gamma_x \tilde{\pi}), \mu_y \tilde{y})^T.
$$

(3.40)

To solve a set of the two forward-backward looking matrix equations, (3.37) and (3.38), I employ the method developed by Guerrieri and Iacoviello (2015) with a little modification. Loosely speaking, I extrapolate a solution obtained in the region without the zero lower bound to the region with the zero lower bound by taking into account the ending time of the zero interest rate policy.

The key assumption of their method is that a nominal policy interest rate reverts to a positive level and the economy will not be constrained by the zero lower bound after a certain time period $\tau$. This assumption allows us to have a boundary condition and dramatically reduces a complexity of (3.33)-(3.36). My modification of their method is that (1) I maintain shocks $\epsilon_t$ in (3.37) while they drop them. (2) I restrict the case when $\tau$ is the first exit time from the zero lower bound while they allow regimes to switch from one to another before $t < \tau$.

I denote the first expected exit time of zero interest rate policy with $\tau$ that is mathematically defined as

$$
\tau = \min(s; E_t[x_s] > \tilde{i}, E_t[x_{s-1}] \leq \tilde{i}).
$$

(3.41)

When a shadow interest rate $x_t < 0$ at time $t = 1$, the solution at the time period $t = 1$ takes the following form.

$$
X_{t+1} = F^*X_t + H^*\epsilon_t, \quad \text{if } \ 1 \leq t < \tau \quad (3.42)
$$

$$
X_{t+1} = FX_t + G + H\epsilon_t, \quad \text{if } \ t \geq \tau \quad (3.43)
$$

From the first term in the equation above, one can see a theoretical possibility that a real output can be increased if the future nominal interest rate is lowered even when a current policy interest rate $i_t$ is at zero percent in the second term.
where $F$, $G$ and $H$ are VAR coefficients without the zero lower bound and numerically computed by solving (3.18). $F_t^*$, $G_t^*$ and $H_t^*$ are solved recursively from $t = 1$ to $t = T$, given $F$, $G$ and $H$. My additional assumption is that the expected shadow interest rate hits to zero and gets above the zero lower bound only when $t = T$. Notice that the expected exit time of zero interest rate policy $\tau$ becomes more distant from now when a current shadow interest rate $x_t$ is more negative. Thus, the expected exit time $\tau$ must be consistent with the current shadow interest rate $x_t$ when solved. The numerical solution is postponed to the Appendix A.\textsuperscript{18}

### 3.3 No-arbitrage term structure model

When the zero lower bound is binding, a policy interest rate is kept constant and thus a shadow policy interest rate $x_t$ is not directly observable. I use the information contained in the cross section of nominal bond yields to extract a shadow interest rate $x_t$. In doing so, I construct a term structure model of interest rates with the zero lower bound. Since the model is non-linear and thus a closed-form solution is not available, I develop an approximation that is correct up to second order.

I incorporate the structural macroeconomic model as parameter restrictions on the dynamics of state variables into a no-arbitrage term structure model.\textsuperscript{19} Recall that I have shown that forward-looking equations in a New Keynesian macroeconomic model is reduced to a VAR (1). That VAR (1) describes the dynamics of the state variables in physical measure when modeling term structure.

As in empirical studies of no-arbitrage term structure models, the model enables us to decompose nominal bond yields into the expected real rates, the real interest rate risk premium, the expected inflation, and the inflation risk premium. Furthermore, one can investigate how factor loadings of yield curves change before and after the zero interest rate policy without treating these two periods separately.

Following Black (1995), I employ the concept of a shadow interest rate to model the zero lower bound on nominal interest rates. A shadow interest rate $x_t$ can take a negative value and a positive part of the shadow interest rate is set equal to an observed nominal policy rate $i_t$. The mathematical definition is

$$i_t = \max(x_t, \tilde{i}),$$  \hspace{1cm} (3.44)

In (3.44), I slightly generalize the original definition of shadow interest rates by introducing an additional parameter $\tilde{i}$. I call $\tilde{i}$ effective lower bound. The effective lower bound $\tilde{i}$ can be understood as interest rate on reserve (IOR). It arises from the fact that banks cannot keep cash physically and they are required to hold reserves at a central bank. I estimate $\tilde{i}$ for actual data.

\textsuperscript{18}A detailed comparison between analytically solvable case (with only IS equation and policy rule equation) and the approximate formula is available upon request.

\textsuperscript{19}There are $2 \times 4 \times 4 = 32$ parameters in the VAR coefficient $F$ and the matrix $H$. The vector $G$ has 4 parameters. In total there are 36 parameters. When we allow the time to exit from the zero lower bound affects VAR coefficients, $F$, $G$, $H$, in macroeconomic dynamics, these VAR coefficients become time-varying so that the number of model parameters increase dramatically and makes their estimation difficult.
Next, let us introduce a vector of stochastic factors $X_t = (y_t, \pi_t, x_t, y^n_t)^T$ that drive a shadow interest rate at the next time step, $x_{t+1}$. There are three other macroeconomic variables: real output $y_t$, inflation $\pi_t$, potential output $y^n_t$.

I assume that the vector of these stochastic factors $X_t$ follows a VAR(1).

$$X_t = FX_{t-1} + G + H\epsilon_t,$$

where $F$ and $H$ are $4 \times 4$ matrix and $G$ is $4 \times 1$ vector. $\epsilon_t$ is a $4 \times 1$ vector of fundamental shocks. $\epsilon_{i,t}$ is sampled from a normal distribution $N(0, \sigma_i)$. These innovations $\epsilon_{i,t}$ are not correlated with each other. Correlations between them are captured by $H$.

If we look at the dynamics of shadow interest rate $x_t$, it is given by

$$x_t = F_{31}y_{t-1} + F_{32}\pi_{t-1} + F_{33}x_{t-1} + F_{34}y^n_{t-1} + G_3 + H_3\epsilon_t,$$

where $F_{3j}$ is the $ij$ element of the matrix $F$. $H_3$ are a 3rd-row of each matrix. $G_3$ is the 3rd element of the vector $G$. The equation (3.46) tells us that the $H_3$ mixes different fundamental shocks on a shadow interest rate $x_t$ and thus the same $H_3\epsilon_t$ has different impacts on the expected path of the shadow interest rates $x_t$.

One potential concern for (3.45) is that a shadow interest rate may not be able to have any impact on real output when it is negative and a nominal rate is equal to zero. In other words, $F_{13}$ may depend on the level of $x_t$. To avoid this issue, I have explicitly modeled the dependence of macroeconomic dynamics on a shadow interest rate using the IS-ZLB specification in Section 3.2.

As it is often used in the literature, I assume that a market price of risk $\lambda_t$ is a linear function the state variables $X_t$ in order to model the dynamics of the vector $X_t$ under the risk neutral measure.

$$\lambda_t = \lambda^0 + \lambda^1X_t,$$

where $\lambda^0 = [\lambda^0_y, \lambda^0_{\pi}, \lambda^0_x, \lambda^0_{y^n}]^T$ and $\lambda^1$ is $4 \times 4$ upper triangle matrix except modification that allows $\lambda^1_{2,1}$ to be non-zero.

Nominal bond price $P^n_t$ with maturity $\tau$ at time $t$ is recursively computed. Suppose that we are pricing bonds under the time frequency $\Delta t$ such that $\tau = n\Delta t$. The pricing formula for a nominal bond is given by

$$P^n_t(X) = E^Q_t[ e^{-\sum_{k=0}^{n}(t_k+k\Delta)} | X_t = X],$$

where the expectations are computed under the risk neutral measure. Henceforth, $E^Q_t$ is the expectations under the risk-neutral measure. $X$ is an initial value of macroeconomic variables including a shadow interest rate $(X_t = X)$.

The dynamics of the four stochastic factors under the risk-neutral measure are given by

$$X_t = F^Q X_{t-1} + G^Q + H\epsilon_t,$$

Non-zero $\lambda^1_{2,1}$ allows us to have different dependence of inflation on previous real output between the risk neutral measure and the physical measure.
where $F^Q$ and $G^Q$ are given by
\[
F^Q = F - H\mu^1, \quad (3.50) \\
G^Q = G - H\mu^0. \quad (3.51)
\]

Similarly, real bond price $D^n_t$ with maturity $\tau$ at time $t$ is also recursively computed. The formula for real bond pricing is given by
\[
D^n_t(X) = E^Q_t \left[ e^{-\sum_{k=0}^{n} r_{t+k\Delta t} |X_t = X|} \right], \quad (3.52)
\]
where $r_t$ is a real interest rate $r_t = i_t - \pi_t = \max(x_t, \bar{i}) - \pi_t$. Nominal bond yields $i^n_t$ with maturity $\tau$ at time $t$ are defined as
\[
i^n_t(X) = -\log(P^n_t(X))/\tau. \quad (3.53)
\]

Similarly, real bond yields $r^n_t$ with maturity $\tau$ at time $t$ are defined as
\[
r^n_t(X) = -\log(D^n_t(X))/\tau. \quad (3.54)
\]

In estimating the model for actual nominal bond yields, I assume that there is an observation noise for bond yields. The observation noise $w_t$ is sampled from the normal distribution $N(0, \sigma_R)$ with the standard deviation $\sigma_R$.
\[
i^n_{t,\text{data}}(X) = i^n_{t,\text{model}}(X) + w_t. \quad (3.55)
\]

I assume that the observation noise $w_t$ is independent of shocks to other macroeconomic variables $\epsilon_t$. I also assume that $w_t$ for the maturity $\tau$ is independent from $w_t$ for other maturity $\tau' (\neq \tau)$.

When estimating a joint model of yield curves and macroeconomic variables, bond yields are computed many times given different parameters. Thus, it is important to obtain either a closed-form solution or approximate solution for computational feasibility. One cannot use a well-known affine bond pricing formula here even though stochastic factors $X_t$ follow VAR(1) system since stochastic factors $X_t$ are non-linearly related to a short-term shadow interest rate $x_t$.

To overcome this issue, I develop an approximate formula for nominal bond prices when there exists an effective (zero) lower bound on nominal interest rates. Recall the definition of the shadow interest rate and notice that it can be represented as
\[
i_t = \max(x_t, \bar{i}) = x_t + \max(\bar{i} - x_t, 0). \quad (3.56)
\]
The second term is analogous to a put option with the strike $\bar{i}$ on the shadow interest rate $x_t$. The nominal bond yield is
\[
i^n_t(X) = i^n_{t,\text{affine}}(X) + P^n_t(X, \bar{i}). \quad (3.57)
\]

The value of the put option $P^n_t$ depends on (1) the volatility of the underlying shadow interest rate and (2) to what extent the shadow interest rate is negative. For example, the option value arising from the zero lower bound is ignorable if the shadow interest rate is strongly positive and its volatility is low. Loosely speaking,
the approximation takes the volatility effect into account up to a second order. The second effect is captured as delta of the option.\textsuperscript{21} I provide an intuitive derivation in Appendix B. A formal proof is postponed to the Appendix C.\textsuperscript{22}

4 Extension: Regime-switching macroeconomic dynamics

Recently, there is a debate that an equilibrium real interest rate has been declining during the last decade. For example, Summers (2014) argues that an equilibrium real interest rate in the US has recently declined by citing the estimates based on Laubach and Williams (2003) model and lists several factors explaining such a downward trend of the real interest rates.\textsuperscript{23} King and Low (2014) document that weighted real interest rates across G7 countries have been gradually declining since 1985. Motivated by such a debate, I incorporate regime switching feature to a target shadow interest rate. Specifically, I assume that a target shadow interest rate depends on two regimes, \( s_t = u, d \).

\[ x_{t}^{\text{target}} = r^*(s_t) + \pi_t + \gamma_y(y_t - y^n_t) + \gamma_n(\pi_t - \bar{\pi}_t), \quad (4.1) \]

It is tough to allow regime switching feature in the case of IS-ZLB. Thus, I focus on the case of IS-SR when employing (4.1). Given this modification, the four equations are rewritten as the following forward-backward looking equation.

\[ AE_t[X_{t+1}] = BX_t + CX_{t-1} + D(s_t) + \epsilon_t, \quad (4.2) \]

where the coefficient matrices \( A, B, C \) are same as those in the case of IS-SR. The only vector \( D \) becomes regime-dependent.

\[ D(s_t) = \begin{pmatrix} \alpha_{IS} & 0 \\ \mu_r & \gamma^* - \gamma^\pi \bar{\pi} \\ \mu_y & \gamma^n \end{pmatrix}^T. \quad (4.3) \]

There are several techniques to solve regime-dependent forward-backward equation.\textsuperscript{24} In this special case, one can apply a similar method used to solve non-regime-switching forward and backward looking equation. Let us guess the solution takes the following form.

\[ X_{t+1} = FX_t + G(s_t) + H\epsilon_t \quad (4.4) \]

\( F, G, H \) solve

\[ AF^2 - BF - C = 0, \quad G(s_t) = (AF - B + A)^{-1}D(s_t), \quad H = (AF - B)^{-1}. \quad (4.5) \]

\textsuperscript{21}The term, “delta” is widely used in option pricing. It refers to as the derivative of the option value with respect to the underlying asset. Delta captures moneyness of the put option. In other words, delta naturally takes into account to what extent the shadow interest rate is negative.

\textsuperscript{22}A detailed comparison of bond prices based on the approximate formula with Monte Carlo simulation is available upon request.

\textsuperscript{23}See also Teulings and Baldwin (2014) for reference.

\textsuperscript{24}Davig and Leeper (2007), Farmer, Waggoner and Zha (2009, 2011) and Cho (2009) define and discuss the concept of solutions based on minimum state variable. Svensson and Williams (2005) also proposed a simple algorithm to solve this type of equations.
It is easy to see that $F$ does not depend on regime $s_t$. Thus, one can obtain $F$ first and then compute $G$ and $H$ with already obtained $F$.

In pricing nominal bonds, we need to specify a function of market price of risk. For simplicity, I assume that factor-dependent market price of risk is independent of regimes.

$$
\mu_t = \mu^0(s_t) + \lambda^1 X_t. \tag{4.6}
$$

Thus, only constant market price of risk depends on regime $s_t$. The evolution of the four stochastic factors under the risk-neutral measure is given by

$$
X_{t+1} = F^Q X_t + G^Q(s_t) + H \epsilon_t, \tag{4.7}
$$

where $F^Q$ and $G^Q(s_t)$ are given by

$$
F^Q = F - H \mu^1, \tag{4.8}
$$

$$
G^Q(s_t) = G - H \mu^0(s_t). \tag{4.9}
$$

The approximate formula for nominal bond yields under regime switching is explained in the Appendix C. The approach to obtain the formula is same as the one without regime switching.

## 5 Data and estimation methodology

### 5.1 Data

I study government bond yields and macroeconomic variables in the US and Japan. For the case of US, I study the period from October of 1991 (4th Quarter) to January of 2015 (1st quarter). I obtain the US Treasury bond yields from the website of the Federal Reserve Board. The data is constructed based on Gürkaynak, Sack and Wright (2007). GDP growth and CPI inflation are downloaded from FRED at the website of the Federal Reserve Bank of St. Louis. I also use 10-year CPI forecast from the Survey of Professional Forecasters. The starting date is determined due to availability of 10-year CPI forecast. I use nominal bond yields data constructed by Gürkaynak, Sack and Wright (2006). Figure 1 depicts evolution of nominal bond yields used in estimating the model for US. It is clear that the zero lower bound has been binding in the US since December of 2008.

For the case of Japan, I focus on the period from July of 2004 to December of 2014 because fixed income investors’ inflation forecasts are available only from July of 2004. Survey-based inflation forecasts are from Quick. For computing the average of CPI forecasts, I adjust a consumption tax hike in 2013 April. I obtain the Japanese government bond yields from the website of Ministry of Finance. GDP growth and CPI inflation are downloaded from Cabinet Office and Statistics Bureau. Figure 2 shows historical data of nominal bond yields used in estimating the model for Japan. One can see that even 10-year bond yield is very low and even

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I use demeaned real GDP and potential GDP as a proxy for real output $y_t$ and potential output $y^*_n$, respectively. I employ CPI inflation as a proxy for inflation $\pi_t$.

Table 1 shows the average nominal yields of the US Treasury bond yields and Japanese government bonds, respectively. US Treasury bond yields are constrained by the zero lower bound after the Federal Reserve employed the zero interest rate policy. To formally test this argument, I estimate the effective lower bound in the next section. Japanese government bond yields have been very low during the entire sample period. After QQE (Quantitative Qualitative Easing), they are further lowered.

Table 2 shows summary statistics of survey-based CPI inflation forecasts. In the case of US, the historical average of realized CPI inflation was around 2.37% during 1991-2015 period. The average of 10-year-average CPI inflation forecast is 0.4% lower than that. The difference seems not large. This is in sharp contrast to Japan. The historical average of realized CPI inflation was almost zero during 2004-2015 period in Japan. The historical average of 10-year-average CPI inflation forecast is 1.08%.

### 5.2 Estimation methodology

As described in the previous sections, my joint model of yield curves and macroeconomic variables is described by (3.22), (3.32) and (3.55). To clarify what I am going to estimate, I reproduce those equations below:

$$i^n_{t,\text{data}} = i^n_{t,\text{model}}(X_t, \Theta, \lambda_0, \lambda_1) + w^n_t,$$

$$X_t = F(\Theta)X_{t-1} + G(\Theta) + H(\Theta)\epsilon_t,$$

$$s^n_t = \tilde{F}_{n,2}(\Theta)X_t + \tilde{G}_{n,2}(\Theta) + \alpha^n_s + \epsilon^n_s,$$

where $X_t = (y_t, \pi_t, x_t, y^*_n)^T$. $s^n_t$ is $n$-quarter-average inflation forecast at time $t$. I set $n = 40$ and use 10-year CPI forecast as a proxy of the long-term inflation forecast. Measurement error $w^n_t$ is sampled from a normal distribution $N(0, \sigma_M)$. $\lambda_0$ and $\lambda_1$ are parameters of market price risk as discussed in the previous section. $\Theta$ is a vector of the New Keynesian model parameters.

$$\Theta = \{\alpha_{IS}, \mu^+_y, \mu^-_y, \phi, \mu_\pi, \kappa, \mu_x, r^*, \gamma_y, \gamma_\pi, \bar{\pi}, \overline{y^n}, \mu_{yn}, \overline{i}\}.$$  

Notice that volatilities of four macroeconomic shocks $\sigma = \{\sigma_{IS}, \sigma_{AS}, \sigma_x, \sigma_{yn}\}$ are standard deviations of a vector of these shocks $\epsilon_t$. When IS-ZLB is employed, (5.2) should be replaced with (3.42) and (3.43). I estimate the model under the specification of IS-ZLB unless otherwise mentioned. Recall that the shadow interest rate $x_t$ is latent when the zero lower bound is binding.

I estimate the model by two methods. In the first method, I extract a shadow interest rate $x_t$ at time $t$ by fitting the model-implied nominal yield curve $i^n_{t,\text{model}}$ to cross-sectional bond yields $i^n_{t,\text{data}}$. This approach is widely used in empirical studies of affine term structure models. Unlike affine term structure models, one cannot...
analytically solve the state variables as a function of observable bond yields.\footnote{In $M$-factor affine term structure models, a vector of $M$ number of yields, $i_t^M$ is given by}

\[
\hat{x}_t = \text{argmin} \sum_{n \in N} \left( i_{t}^{n, \text{data}} - i_{t}^{n, \text{model}}(X_t, \Theta, \lambda_0, \lambda_1) \right)^2, \tag{5.5}
\]

Thus, I numerically solve the state variable $x_t$.

where a vector of maturities $N = \{1, 2, 3, 4, 5, 7, 10\}$ Rather than assuming that nominal bond yields with specific maturities can be observed without noise, I use all cross-sectional bond yields to extract the shadow interest rate.\footnote{I have also estimated the model by assuming no observable noise for nominal bond yield for one specific maturity. Yet, it makes solving a nonlinear equation unstable due to non-linearity of the model, especially when the yield is approaching zero.} Then, I employ maximum likelihood estimation for the VARs (1) ((5.2) and (5.3)) where the four state variables are given by $\hat{X}_t = (y_t, \pi_t, \hat{x}_t, y_t^n)^T$.

To reduce the time for estimation, I estimate model parameters $\bar{\gamma}_y, \mu_{y^n}, \sigma_{y^n}$ in the equation of potential output (potential GDP), (3.13) (or equivalently (3.36)) separately because the potential output is completely exogenous. I also fix Taylor rule coefficients $\gamma_y = 0.5$ and $\gamma_\pi = 0.5$ for both US and Japan, as in the original Taylor rule. This assumption is not necessary but helps us obtain an estimate of the long-run real interest rate $r^*$ in the shadow policy rule equation (3.12) (or (3.35)).

When estimating shadow interest rate models with regime switching, I simply apply a standard methodology to estimate regime switching model for VAR system developed by Hamilton (1989).

In the second method, I treat shadow interest rates as a latent factor and thus employ unscented Kalman Filtering. Since bond yields in my term structure model are not linear to latent variables, I cannot use conventional Kalman filtering to compute likelihood. Unscented Kalman Filtering (UKF) developed by Juller and Uhlmann (2004) is applicable to non-linear models. It is possible to estimate the model parameters by extended Kalman filtering based by linearizing the model. Yet, Christoffersen et al. (2012) report that unscented Kalman filter performs better than extended Kalman Filtering.

For shadow interest rate models with regime switching, I develop a regime-switching unscented Kalman filtering as an additional robustness check. I integrate Kim(1994)’s method for regime-switching dynamic linear model with unscented Kalman filtering.\footnote{Kim (1994) developed algorithm to estimate state space models with Markov regime switching. The detail is Appendix D.} The second approach is just for a robustness check. All of results in the next section are based on the first approach.
6 Results

6.1 Estimated parameters and pricing errors

Table 3 shows absolute pricing errors of the model. The average errors across all maturities are 33bps for US and 8bps for Japan. These numbers are reasonably low, considering that the model has some restrictions on the dynamics of the state variables $X_t$.

Tables 4 and 5 present estimates of structural New Keynesian model parameters as well as market price of risk parameters for the two countries. We observe the following: First, the effective lower bound $\tilde{i}$ is slightly positive for both cases. For Japan, $\tilde{i}$ is 13bps while it is 10bps for US. This indicates that specifying exactly zero percent as the lower bound in modeling might give misleading results in modeling.\(^{30}\)

Second, the long-run real rate is negative for Japan with $r^* = -2.8\%$. The Bank of Japan is currently targeting for a 2\% inflation. If the Japanese inflation $\pi_t$ converges to the official target inflation of 2.0\%, the long-run (policy-neutral) nominal interest rate $i_t^*(= r^* + \pi_t)$ is equal to -0.8\%. Yet, it is still lower than the effective (zero) lower bound $\tilde{i}(= 0.1\%)$. It indicates that the Bank of Japan might need to have a higher inflation ($> 2.9\%$) in order to exit from the zero lower bound. In the case of the US, the long-run real rate is equal to 1.0\%. Thus, policy-neutral nominal interest rate should be positive if the inflation is higher than -1.0\%.

Third, the bias of inflation survey $\alpha_*$ is 0.7\% for the US while it is around 1.1\% for Japan. Survey-based inflation forecasts are biased upward, especially for Japan. This confirms what I have discussed in the previous section for Table 2.

Figure 3 shows the evolution of the shadow interest rate extracted from the yield curves. The shadow interest rate hit the zero lower bound in early 2009 and became very negative in 2012. It is approaching the bound recently, again. In the case of Japan, the shadow interest rates have been negative over years and became more negative since 2014 as the long-term nominal yield is further lowered by Quantitative and Qualitative Easing (QQE).

6.2 Counterfactual analysis

Equipped with the estimated New Keynesian macroeconomic model, I conduct a counterfactual analysis of monetary policy. Figures 4 and 5 show the nominal bond yield curve given a higher inflation target level for the US. In Figure 4, the zero lower bound is not binding at the current time period ($x_t = 1.0\%$). Thus, a higher inflation target lowers a policy interest rate in the short run, but it increases long-run expected inflation. Thus, long-term nominal bond yields increase.

In Figure 5, the zero lower bound is binding ($x_t = -1.0\%$). As a result, a higher inflation target cannot further lower short-term nominal yield. Figure 5 also shows the yield curve when the zero interest rate policy is suddenly abandoned. The short-term yields increase more than those in the case of a higher inflation target, but the long-term yields do not. Furthermore, although not reported here, real

\(^{30}\)The Bank of Japan has been keeping an interest rate on reserve (IRO) to 10bps since 2008 October.
output increases when a higher inflation target is adopted. Real output decreases when the zero interest rate policy suddenly ends. These results indicate that raising the inflation target is better than suddenly ending the zero interest rate policy in stimulating the macro economy and generating higher long-run nominal interest rates.

Figure 6 documents that the increase in the nominal yields caused by higher inflation target is mostly explained by the increase in the expected nominal interest rate, not in the term premium. That is not surprising since higher inflation target generates higher expected inflation and thus increases the expected nominal interest rates.

Figures 7 and 8 show the nominal bond yield curves given a higher inflation target level for Japan. The result is similar to the US case although long-term yields are more impacted in the case of Japan.

Figures 9 and 10 show the nominal bond yield curves given an introduction of a negative lower bound. The negative lower bound steepens the nominal yield curve by allowing short-term nominal yields to be negative by increasing the long-term nominal yields.

6.3 Decompositions of yields

Table 6 shows the decomposition of nominal bond yields into expected interest rate and term premium for both US and Japan. The numbers are the average values during entire sample period. For both countries, the term premium is larger for longer maturities. This finding is consistent with previous studies. In the case of Japan, the nominal bond yields are almost entirely explained by the term premium. This is not surprising since the Bank of Japan has been employing the zero interest rate policy during the sample period, which excludes the period between July of 2006 and December of 2008.

Table 7 shows decomposition of nominal bond yields into the expected real rates, the real interest rate risk premium, the expected inflation, and the inflation risk premium for the US and Japan.

In the US, one can see that upward-sloping nominal bond yield curve is arising from the real interest rate risk premium. Expected real interest rates and expected inflation are almost constant across maturities.

In Japan, the nominal bond yields are mostly explained by real interest rate risk premium for longer maturity. Expected inflation implied by the nominal bond yields is negative or close to zero for all maturities. This is natural because the historical average of realized inflation is almost zero as confirmed in Table 2.

6.4 Dispersion of inflation forecasts on inflation risk premium

Following Wright (2011), I conduct regression analysis to investigate whether the inflation risk premium can be explained by disagreement about future inflation as summarized in Table 8. Wright (2011) finds that a larger cross-sectional dispersion of inflation forecasts predicts a larger inflation risk premium. Yet, his term struc-
ture model abstracts the zero lower bound. I extend his analysis by taking this into account. I find that the signs of coefficients are positive except for the ten year maturity in both US and Japan. However, none of these coefficients are statistically significant in the US case. By contrast, some of the coefficients are statistically significant in Japan.\textsuperscript{31} Overall, I confirm the positive relationship between the inflation risk premium and dispersion of inflation forecasts.

### 6.5 Variance decompositions

Table 9 presents variance decompositions of nominal bond yields into expected interest rate variance and term premium variance. Notice that the sum of variances of these two components is not equal to the variance of nominal yields since there is a covariance term. Also notice that the table shows standard deviations, not variance itself. In the case of Japan, the variance of nominal bond yields arises from the variance of the term premium, not the variance of expected nominal interest rates. This is realistic because the policy interest rate in Japan has been kept zero percent during almost the entire sample period.

In the US case, the variance of nominal yields is explained mostly by expected interest rate variance for the short maturity, but the term premium variance is more dominant for longer maturity.

Table 10 presents variance decompositions of nominal bond yields into the variances of expected real interest rate, real rate premium, expected inflation, and inflation risk premium. Again, notice that the sum of variances of these two components is not equal to the variance of nominal yields since there are covariance terms. Consistent with the finding in Table 9, the contributions of the variances of the real interest rate risk premium and inflation risk premium are larger for longer maturity for both US and Japan.

### 6.6 Factor loadings of yield curve

Figures 11 and 12 show factor loadings of the first three factors of nominal bond yields in a principal component analysis (PCA). In Figure 11, I conducted PCA for the nominal bond yields during from October of 1991 to October of 2008. In Figure 12, I use the data during January of 2009 to January of 2015 when the Federal Reserve employed the zero interest rate policy. In Figure 11, it is clear that PCA1, PCA2, PCA3 correspond to level, slope, curvature factors, respectively. In Figure 12, such a clear mapping is difficult. Both PCA 1 and PCA 2 behave like slope factors. This means that the principle component analysis cannot provide us

\[ \Delta ip_{t+1}(\tau) = \alpha ip + \beta STD + STD_t \cdot 1_{\pi_t < 0} + \beta STD - STD_t \cdot 1_{\pi_t > 0} \]

where \( ip_t(\tau) \) is the inflation risk premium of \( \tau \)-year maturity at time \( t \). \( \omega_t^{ip} \) is distributed from \( N(0, \sigma^{ip}) \). \( STD_t \) and is cross-sectional standard deviation of CPI forecasts. Yet, the results are not statistically significant.

\textsuperscript{31}There is a possibility that deflation risk premium (negative inflation risk premium) positively depends on cross-selectional inflation dispersion. Thus, I conduct the following asymmetric regression.

Yet, the results are not statistically significant.
a unified understanding of the macroeconomic drivers of yield curves.

Figures 13 and 14 enable us to understand economic drivers of nominal bond yields with and without the zero interest rate policy. These two figures show factor loadings to four different macroeconomic variables. In Figure 13, the initial shadow interest rate is positive \( x = 1.0\% \) but it is negative \( x = -1.0\% \) in Figure 14. Figure 13 shows that the shadow interest rate works as a level factor. The other three variables function as slope factors. In Figure 14, there is no clear level factor. All four macroeconomic factors function as slope factor but with different magnitudes.

Figures 15 and 16 show factor loadings of those four variables in the case of Japan. The results are almost same as the US case. One difference is that inflation functions as a curvature factor in Japan rather than a slope factor.

6.7 Results of regime-switching long-run real interest rates

Table 11 shows estimates of the parameters of an extended joint model with Markov regime switching. I report only regime-dependent parameters, the long-run real interest rates \( r^* \) in this table. The long-run real rate \( r^* \) is equal to 3.07% in the high regime and \( r^* \) is equal to -1.88% in the low regime. This estimate is in line with the estimated range of the long-run real interest rates reported in Laubach and Williams (2015). Recall that the target shadow interest rate \( x^\text{target} \) is still positive if the sum of inflation, inflation gap, and output gap is large enough.

Figure 17 shows that low-rate regime has been dominant since 2008. This indicates that there was an irreversible shift in the long-run real interest rate for the US. This finding can be associated with the recent discussion by Summers (2014) as mentioned in Section 4.

6.8 Additional applications: Impact of yield-oriented investors on bond yields

Hanson and Stein (2015) argue that investors’ adjustment of their fixed-income portfolio can explain excess sensitivities of long-term interest rates to short-term rates. Motivated by their study, I conduct a simple statistical analysis to see whether duration adjustments by fixed income investors are impacting duration and, reversely, that duration changes incentive to the fixed income investors to adjust their durations (average maturity). One big difference in my paper from Hanson and Stein (2015) is that the data set in this paper allows me to see each fixed income investor’s duration (average maturity of their bonds).

A comprehensive survey of Japanese fixed income investors provided by QUICK Corp allows us to see the average duration (maturity) of each fixed income investor. Utilizing that data, I study how medium and long-term bond yields are lowered by fixed income investors when they extend their average maturity given lower bond yields of maturities to which they are exposed. For brevity, I call this impact the duration adjustment effect.

The duration adjustment effect is related to a portfolio re-balancing effect often used to justify the real economic effect of large-scale asset purchases. A portfolio
re-balancing effect is that investors increase the position in risky assets when the returns from those assets currently held by them are lowered.

To quantify the duration adjustment, I consider the two way effects between financial sector’s duration and bond yields in VAR framework.

First, I formulate the impact of duration on bond yields.

\[
\Delta y_{t, \text{actual}}(\tau) = \Delta y_{t-1, \text{model}}(\tau) + \beta^{D-y} \Delta D_t + \epsilon_{D, t}; \quad (6.1)
\]

where \(D_t\) is a vector of durations for financial sectors. In the following analysis, I define \(D_t = (D_{t, \text{bank}}, D_{t, \text{mutual funds}} , D_{t, \text{insurance}})^T\) where \(D_{t, \text{bank}}, D_{t, \text{mutual funds}},\) and \(D_{t, \text{insurance}}\) are the average duration of banking sector, mutual funds sector and insurance sector, respectively. \(y_{t, \text{actual}}(\tau)\) and \(y_{t, \text{model}}(\tau)\) are a \(N \times 1\) vector of actual and model-implied \(\tau\)-year bond yields. \(N\) is the number of bond yields used for the estimation. \(\Delta\) is a time lag operator. \(\beta^{D-y}\) is a \(N \times 3\) matrix. The distribution of error term \(\epsilon_{D, t}\) follows a multi-variate normal distribution with zero mean and covariance matrix \(\Sigma^{D-y}\).

Next, I model the impact of bond yields on duration.

\[
\Delta D_{t+1} = \alpha^{y-D} \Delta y_{t, \text{actual}}(\tau) + \epsilon_{D, t}^{y-D}, \quad (6.2)
\]

where \(\alpha^{y-D}\) is a \(3 \times N\) vector. The distribution of error term \(\epsilon_{D, t}^{y-D}\) follows a multi-variate normal distribution with zero mean and covariance matrix \(\Sigma^{y-D}\).

Figure 18 shows the model-implied sensitivity of real forward rates to 2-year yield change given a shadow rate policy shock under the physical measure as well as actual sensitivities.\(^{32}\) The actual one is computed using yield changes before and after monetary policy meetings during 2006 April to 2008 December in which a policy rate is above zero percent. It tells us that model-implied sensitivity is not so persistent compared to the actual one. This is in line with previous studies in the US.

Table 12 is the summary statistics of this data. It shows the average durations across three different financial sectors. Notice that bank’s duration ranges from 3 to 5 years while asset management sector’s duration ranges from 5 to 9 years. Insurance sector’s duration is distributed from 5 to 10 years.

Table 13 shows the interaction between duration and nominal bond yields. The top and the bottom of the table shows estimated coefficients of regressions, (6.1) and (6.2), respectively. In these regressions, I take into account the range of each financial sector’s duration. For example, I did not regress residual component of 10-year nominal bond yields on the banking sector’s duration since the banking sector’s duration ranges from 3 to 5 years. It is difficult to imagine that this could impact on 10-year nominal yield directly.

The bottom of the table indicates that the average durations have little effect on residual components of nominal yields. It shows that the average maturities of bonds held by the banking sector and mutual fund sector become longer if nominal bond yields are lowered. However, the coefficients of the insurance sector’s duration are not negative.\(^{33}\) Given these two results, it is difficult to argue that there exists

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\(^{32}\)When the sensitivities are computed under the risk-neutral measure, they are almost flat and around one since shadow interest rates function as a level factor.

\(^{33}\)The insurance sector in Japan has been making their duration longer over years even though the nominal yields were rising during 2005-2006 period. That might lead to this result.
a strong effect of fixed-income investors’ bond portfolio adjustment on bond yields.

7 Conclusion

In this paper, I present a joint model of term structure and macroeconomic variables with an explicit zero lower bound that employs shadow interest rates as policy interest rates. Bond yields satisfy no-arbitrage condition, but not affine due the zero lower bound. A well-known closed form solution for bond prices under an affine term structure model is not available. Thus, I develop a new approximate bond pricing formula that is correct up to a second order of shadow interest rate volatility. The formula is intuitive and compatible with other features such as regime switching. I assume that the market price of risk is a linear function of the state variables.

To conduct a counterfactual analysis of monetary policy, I assume that macroeconomic dynamics is described by a standard New Keynesian model with the zero lower bound. The New Keynesian model has a VAR (1) representation with time-varying coefficient matrices that depend on the level of the shadow interest rate. I include the shadow policy interest rate as a state variable in the VAR (1). The other macroeconomic variables are real output, inflation, and potential output.

The yield curve contains the information about the expected (nominal) interest rate that is a key input in the IS equation of the New Keynesian model. Thus, I extract a shadow policy interest rate by using yield curves.

I apply the model for the US and Japanese economy. I jointly estimate New Keynesian model parameters and the linear market price of risk parameters using yield curve, survey-based inflation forecasts and macroeconomic variables.

I find that the long-run real interest rate is equal to 1.0% in the US during the 1991-2015 period and it is -2.8% in Japan during the 2004-2015 period. I also document that the effective (lower) bound on nominal interest rates is slightly positive for both US and Japan.

I conduct different types of counterfactual analyses. As the main feature of my model, I conduct a counterfactual analysis of raising the target inflation level. In both US and Japan, a higher inflation target steepens the yield curve when the zero lower bound is not binding. On the other hand, a higher inflation target increases long-term yields while keeping short-term yields unchanged under the zero lower bound. For comparison, I also conduct a counterfactual analysis of suddenly ending the zero interest rate policy. In this case, the long-term nominal yields do not increase as much as those in the case of a higher inflation target. When I look at the effects of these policies on real output, raising inflation target is expansionary while ending the zero interest rate policy is contractionary. In this respect, raising inflation target is more appropriate.

One methodological contribution of this paper is the use of information contained in the yield curves for estimating structural macroeconomic models with a zero lower bound. The structural macroeconomic model employed in this paper is so stylized that many aspects of a real economy are not captured. I assume a textbook-style New Keynesian macroeconomic dynamics in this paper to keep the key aspects of my joint model clear. More empirical investigations about macro-finance models
with a zero lower bound should be conducted to facilitate a robust monetary policy.
References


A Solving a log-linearized New Keynesian model with the zero lower bound

The numerical procedure is as follows. During the time period when the zero lower bound is binding, VAR coefficients $F^*_t$, $G^*_t$ and $H^*_t$ are computed given $F$, $G$ and $H$. First, guess the expected exit time $\tau$. From (3.43), the evolution of $X_t$ at time $t = \tau$ is determined by

$$X_{\tau+1} = FX_\tau + G + H\epsilon_\tau.$$ (A.1)

Substituting (A.1) for (3.37) at time $t = \tau$, one obtains

$$A^*(FX_\tau + G) = B^*X_\tau + C^*X_{\tau-1} + D^* + \epsilon_t,$$

$$\Leftrightarrow X_\tau = (A^*F - B^*)^{-1}(C^*X_{\tau-1} + D^* - A^*G + \epsilon_t).$$ (A.2)

Comparing (A.2) with (3.42) at time $t = \tau - 1$, we have

$$F^*_{\tau-1} = (A^*F - B^*)^{-1}C^*,$$

$$G^*_{\tau-1} = (A^*F - B^*)^{-1}(D^* - A^*G),$$ (A.3)

$$H^*_{\tau-1} = (A^*F - B^*)^{-1}.$$ (A.4)

Given $F_{\tau-1}$, $G_{\tau-1}$, $H_{\tau-1}$, one can recursively compute the previous coefficients $F_t$, $G_t$, $H_t$ ($1 \leq t \leq \tau - 1$).

$$F^*_t = (A^*F^*_{t+1} - B^*)^{-1}C^*,$$

$$G_t = (A^*F^*_{t+1} - B^*)^{-1}(D^* - A^*G_{t+1}),$$ (A.5)

$$H^*_t = (A^*F^*_{t+1} - B^*)^{-1}. (A.6)$$

Equipped with $F_t$, $G_t$, one can compute the expected exit time $\tau'$. If $\tau' < \tau$, lower $\tau$. Otherwise raise $\tau$. Repeating this procedure, one should obtain $\tau$ that is consistent with $x_t$ at time $t = 1$.

The algorithm is summarized is outlined below.

- Solve VAR coefficient $F$, $G$ and $H$ without the zero lower bound.

  **Step 1** Guess the ending time of the zero interest rate policy $\tau$.

  **Step 2** Compute VAR coefficient $F^*_t$, $G^*_t$, and $H^*$ during the time period of zero interest rate policy (from $t = \tau - 1, \cdots, 1$) using (A.5) and (A.6).

  **Step 3** Generate the expected path of shadow interest rates $E_1[x_t]$ for $t = 1, \cdots, \tau - 1$ using (3.42) and compute $\tau'$ as a first hitting time of the shadow interest rate to the effective (zero) lower bound $\tilde{i}$. $E_1[x_t]$ is obtained as one element of the vector $E_1[X_t]$.

  **Step 4** compare whether $\tau'$ is equal to $\tau$ or not. If not, update $\tau$.

- Repeat the Step 1 to Step 4 until convergence.

In practice, I set max of the hitting time $\tau = 40$ (10 years) since I can only extract the information about expected interest rate up to 10 years.
Approximate bond pricing formula with delta-based convexity: An intuitive derivation

In this section, I give an intuitive derivation of the approximate bond pricing formula with zero lower bound. A formal proof is in the Appendix C.

In a continuous-time framework, the approximate formula is given by

\[ i_t^n(X) = \frac{1}{T} \int_t^{t+\tau} V(X, t, s) ds - \frac{1}{T} \int_t^{t+\tau} \mathcal{E}_t^Q \left[ \frac{\sigma^2}{2} \left( \int_s^{t+\tau} \Delta(X_s, s, u) du \right)^2 \right] ds. \] (B.1)

In a discrete-time framework, the approximate formula is represented as

\[ i_t^n(X) = \frac{1}{n} \sum_{k=0}^{n} V(X, t, t+kdT) - \frac{1}{n} \sum_{k=0}^{n} \mathcal{E}_t^Q \left[ \frac{\sigma^2}{2} \sum_{j=k}^{n} \Delta(X_{t+jdT}, t+jdT, T)^2 \right]. \] (B.2)

where I define a call option on shadow interest rates with \( \bar{i} \) strike \( V(X, t, T) = \mathcal{E}_t^Q [\max(x_T, \bar{i})|X_t = X] \), \( \Delta(X, t, T) = \frac{\partial V(X, t, T)}{\partial x} \),

where \( x \) is the value of a shadow interest rate at time \( t(x_t = x) \).

In the following, I briefly explain the intuition behind the approximate formula. First, consider the the identity

\[ \log \mathcal{E}_0 [e^{y_t}] = \mathcal{E}_0 [y_t] + \log \mathcal{E}_0 \left[ e^{y_t - \mathcal{E}_0[y_t]} \right], \] (B.4)

where \( y_t \) is some stochastic variable. One can approximate the second term in the equation (B.4).

\[ \log \mathcal{E}_0 [e^{y_t}] = \mathcal{E}_0 [y_t] + \log \left[ 1 + \frac{1}{2} \mathcal{E}_0 \left( (y_t - \mathcal{E}_0[y_t])^2 \right) + \cdots \right] \]

\[ \approx \mathcal{E}_0 [y_t] + \frac{1}{2} \text{Var}_0[y_t] \] (B.5)

Consider a special case when \( y_t \) is given by

\[ y_t = -\int_0^\tau \max(x_s, \bar{i}) ds, \] (B.6)

where \( x_t \) is a shadow interest rate at time \( t \). Substituting (B.6) for (B.4), we obtain

\[ \log \mathcal{E}_0 \left[ e^{-\int_0^\tau \max(x_s, \bar{i}) ds} \right] \approx -\int_0^\tau \mathcal{E}_0 [\max(x_s, \bar{i})] ds + \frac{1}{2} \text{Var}_0 \left[ \int_0^\tau \max(x_s, \bar{i}) ds \right] \] (B.7)

The first term in the right hand side of equation corresponds to \( V(X, t, T) \). The second term is a bit complicated and difficult to approximate. A rigorous proof of
the approximate formula is provided in the Appendix C. So suppose that one can rewrite (B.7) as

$$\log E_0 \left[ e^{-\int_0^\tau \max(x_s, \bar{i}) ds} \right] \approx -\int_0^\tau E_0 \left[ \max(x_s, \bar{i}) \right] ds + \frac{\Delta(x_0, 0, \tau)^2}{2} \text{Var}_0[x_\tau],$$

(B.8)

where $\Delta(t)$ is defined as

$$\Delta(x_t, t, T) = \frac{\partial}{\partial x_t} E_t \left[ \int_t^T \max(x_s, \bar{i}) ds \right].$$

(B.9)

Finally, one obtains

$$\bar{i}_0^\tau \approx \frac{1}{\tau} \int_0^\tau E_0 \left[ \max(x_s, \bar{i}) \right] ds - \frac{\Delta(x, 0, \tau)^2}{2\tau} \text{Var}_0[x_\tau].$$

(B.10)

The second term in (B.10) is now easily associated with the second term in (B.1).

Also notice that (B.10) is represented as

$$\bar{i}_0^\tau \approx \frac{1}{\tau} \int_0^\tau E_0 [x_s] ds + \frac{1}{\tau} \int_0^\tau E_0 \left[ \max(\bar{i} - x_s, 0) \right] ds - \frac{\Delta(x, 0, \tau)^2}{2\tau} \text{Var}_0[x_\tau].$$

(B.11)

In (B.11), the first term can be associated with nominal bond yield without the zero lower bound. The second and the third term are analogous to a put option arising from the zero lower bound. As discussed in the main text, the value of this put option naturally reflects (1) the volatility of the underlying shadow interest rate and (2) to what extent the shadow interest rate is negative.
C Approximate bond pricing formula with delta-based convexity: a formal proof

In this section, I provide a formal proof of the approximate bond pricing formula with the zero lower bound.

C.1 Approximate formula for nominal bond yields (single-factor case)

In this appendix, I drive an approximate pricing formula for nominal bonds under zero lower bound using an asymptotic expansion. I first obtain the approximate formula in a continuous time and then discretize it. Consider that a shadow interest rate follows a stochastic differential equation.

\[ dx_t = \mu(x_t)dt + \sigma(x_t)dW_t, \]  

(C.1)

where \( W_t \) is a Brownian motion. Let us denote \( x_t = x \). The Feynman-Kac formula provides a link between a stochastic process and its partial differential equation tells us that the price of a nominal discount bond \( P(x, t, T) \) with expiry \( T \) at time \( t \) is a solution of the following partial differential equation.

\[ \frac{\partial P}{\partial t} + \mu(x)\frac{\partial P}{\partial x} + \frac{\sigma(x)^2}{2} \frac{\partial^2 P}{\partial x^2} = x^+P, \]

(C.2)

where \( x^+ \) is defined as \( \max(x, \tilde{x}) \). Notice that this definition is slightly different from the conventional notation in which \( x^+ = \max(x_t, 0) \). Suppose that the bond price is represented as \( P(x, t, T) = e^{-f(x, \tau)} \) where \( \tau = T - t \). Substituting this for (C.2), one obtains

\[ \frac{\partial f}{\partial \tau} = \mu(x)\frac{\partial f}{\partial x} + \frac{\sigma(x)^2}{2} \frac{\partial^2 f}{\partial x^2} + x^+ - \frac{\sigma(x)^2}{2} \left( \frac{\partial f}{\partial x} \right)^2. \]

(C.3)

Suppose that the last term in the right-hand side of (C.3) is replaced with \( \epsilon \frac{\sigma(x)^2}{2} \left( \frac{\partial f}{\partial x} \right)^2 \) where \( 0 < \epsilon \leq 1 \). Also, consider that \( f(x, \tau) \) has an asymptotic expansion.

\[ f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \cdots \]

(C.4)

Substituting (C.4) for (C.3), one obtains a system of PDEs.

\[ \frac{\partial f_0}{\partial \tau} = \mu(x)\frac{\partial f_0}{\partial x} + \frac{\sigma(x)^2}{2} \frac{\partial^2 f_0}{\partial x^2} + x^+, \]

(C.5)

\[ \frac{\partial f_1}{\partial \tau} = \mu(x)\frac{\partial f_1}{\partial x} + \frac{\sigma(x)^2}{2} \frac{\partial^2 f_1}{\partial x^2} - \frac{\sigma(x)^2}{2} \left( \frac{\partial f_0}{\partial x} \right)^2, \]

(C.6)

\[ \frac{\partial f_2}{\partial \tau} = \mu(x)\frac{\partial f_2}{\partial x} + \frac{\sigma(x)^2}{2} \frac{\partial^2 f_2}{\partial x^2} - \sigma(x)^2 \left( \frac{\partial f_0}{\partial x} \right) \left( \frac{\partial f_1}{\partial x} \right), \]

(C.7)
The first term $f_0$ in the asymptotic expansion is given by

$$f_0(x, t, T) = \int_t^T \mathbb{E}_t[x^+_{s}] ds = \int_t^T \int_{-\infty}^{+\infty} x^+_s \phi(x_s, s|x_t, t) ds,$$

(C.8)

where $\phi(x_s, s|x_t, t)$ is a transition density of a shadow interest rate $x_s$ at time $s$ from $x_t = x$ at time $t$. This transition density $\phi(x_s, s|x_t, t)$ becomes a normal distribution, when $x_t$ follows Vasicek process (AR(1) process).

Substituting (C.8) for (C.6) leads to

$$f_1(x, t, T) = \int_t^T \mathbb{E}_t \left[ -\frac{\sigma(x_s)^2}{2} \left( \frac{\partial f_0(x_s, s, T)}{\partial x_s} \right)^2 \right] ds.$$

(C.9)

The equation above clearly shows that a convexity effect depends on the volatility of a shadow interest rate and moneyness of a call option arising from the zero lower bound.

An asymptotic expansion for a nominal bond price $P(x, t, T)$ is

$$f(x, \tau) = -\log P(x, t, T) = \int_t^T \mathbb{E}_t[x^+_{s}] ds + \epsilon \int_t^T \mathbb{E}_t \left[ -\frac{\sigma(x_s)^2}{2} \left( \frac{\partial f_0(x_s, s, T)}{\partial x_s} \right)^2 \right] ds + O(\epsilon^2),$$

(C.10)

Let us denote a nominal bond yield with maturity $\tau = n\delta t$ at time $t$ with $y^n_t(x)$. As in the main text, I define a call option $V(x, t, T)$ arising from the zero lower bound and its first-order derivative $\Delta(x, t, T)$ as

$$V(x, t, s) = \mathbb{E}_t[\max(x_s, 0)], \quad \Delta(x, t, s) = \frac{\partial V(x, t, s)}{\partial x}.$$  \hspace{1cm} (C.11)

It is easy to confirm that

$$\int_t^T V(x, t, s) ds = f_0(x, t, T) \quad \int_s^T \Delta(x_s, s, u) du = \frac{\partial f_0(x_s, s, T)}{\partial x_s}.$$ \hspace{1cm} (C.12)

One can obtain an approximation for a nominal bond yield by the order of $\epsilon$ is computed as

$$i^n_t(x) = -\log P(x, t, T)/\tau = \frac{1}{\tau} \int_t^{t+\tau} V(x, t, s) ds - \frac{1}{\tau} \int_t^{t+\tau} \mathbb{E}_t \left[ \frac{\sigma(x_s)^2}{2} \left( \int_s^{t+\tau} \Delta(x_s, s, u) du \right)^2 \right] ds.$$

(C.13)

In a discrete-time framework, the pricing formula above is written as

$$i^n_t(x) \approx \frac{1}{n} \sum_{k=0}^n V(x, t, t + k\delta t) - \frac{1}{n} \sum_{k=0}^n \mathbb{E}_t \left[ \frac{\sigma(x_{t+j\delta t})^2}{2} \sum_{j=0}^n \Delta(x_{t+j\delta t}, t + j\delta t, T)^2 \right]$$

(C.14)

Notice that in both (C.13) and (C.14), it suffices to have a transition density $\phi(x_s, s|x_t, t)$ of a shadow interest rate.
C.2 Example

Consider that a shadow interest rate follows a random walk.

\[ dx_t = \sigma_x dW^x_t, \quad (C.15) \]

Suppose that there exists non-negative zero lower bound \( \bar{i} = 0 \). Applying the approximate formula developed in the previous subsection, one obtains:

\[ V(x, t, s) = x \left( 1 - N \left( \frac{-x}{\sigma_x \sqrt{s - t}} \right) \right) + \frac{\sigma_x \sqrt{s - t}}{\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma_x^2(s-t)}}, \quad (C.16) \]

\[ \Delta(x, t, s) = 1 - N \left( \frac{-x}{\sigma_x \sqrt{s - t}} \right). \quad (C.17) \]

If \( \frac{x}{\sigma_x} \) is large, \( V(x, t, s) \approx x \) and \( \Delta(x, t, s) \approx 1 \). Substituting these for (C.13), one obtains:

\[ i^\tau_i(x) \approx \frac{1}{\tau} \int_t^{t+\tau} x ds - \frac{1}{\tau} \int_t^{t+\tau} \frac{\sigma_x^2}{2} \left( \int_s^{t+\tau} 1 du \right)^2 ds. \]

\[ = x - \frac{1}{\tau} \int_t^{t+\tau} \frac{\sigma_x^2}{2} (t + \tau - s)^2 ds = x - \frac{\sigma_x^2}{6\tau}. \quad (C.18) \]

This is consistent with a theoretical bond price under a random walk without the zero lower bound. Thus, when a shadow interest rate follows a random walk, the approximate formula is exact in the limit of \( \frac{x}{\sigma_x} \gg \bar{i} \).

C.3 Approximate formula for nominal bond yields (multi-factor case)

Next, suppose that a shadow interest is driven by multiple stochastic factors.

\[ dx_t = \mu_x(x_t, Z_t) dt + \sigma_x(x_t, Z_t) dW^x_t, \quad (C.19) \]

where the vector \( Z_t = [z^1_t, z^2_t, \ldots, z^N_t] \) and a stochastic process for each element of stochastic factor \( z^i_t \) is given by

\[ dz^i_t = \mu_i(x_t, Z_t) dt + \sigma_i(x_t, Z_t) dW^i_t, \quad (C.20) \]

\( W^x_t \) and \( W^i_t \) are Brownian motions with correlation \( \rho_i \). The correlations between \( W^i_t \) and \( W^j_t \) are \( \rho_{ij} \). I denote \( x_t = x \) and \( z^i_t = z^i \). The Feynman-Kac formula shows the price of a nominal discount bond \( P(x, Z, t, T) \) with expiry \( T \) at time \( t \) as a solution of the following partial differential equation.

\[ x^+ P = \frac{\partial P}{\partial t} + \mu_x \frac{\partial P}{\partial x} + \frac{\sigma_x^2}{2} \frac{\partial^2 P}{\partial x^2} + \sum_{i=1}^{N} \left( \mu_i \frac{\partial P}{\partial z_i} + \frac{\sigma_i^2}{2} \frac{\partial^2 P}{\partial z_i^2} + \rho_{i} \sigma_x \sigma_i \frac{\partial^2 P}{\partial z_i \partial x} \right) + \sum_{i=1, j>i}^{N} \rho_{ij} \sigma_i \sigma_j \frac{\partial^2 P}{\partial z_i \partial z_j}, \quad (C.21) \]
where I drop the subscripts of $\sigma_i(x_t, Z_t)$ for notation simplicity.

Suppose that the real bond price is represented as $P(x, Z, t, T) = e^{-f(x, Z, \tau)}$ where $\tau = T - t$. Substituting this for (C.21), one obtains

$$\frac{\partial f}{\partial \tau} = x^+ + \mu_x \frac{\partial f}{\partial x} + \frac{\sigma_x^2}{2} \frac{\partial^2 f}{\partial x^2} + \sum_{i=1}^{N} \left( \mu_i \frac{\partial f}{\partial z_i} + \frac{\sigma_i^2}{2} \frac{\partial^2 f}{\partial z_i^2} + \rho_i \sigma_x \sigma_i \frac{\partial f}{\partial z_i} \frac{\partial f}{\partial x} \right)$$

$$+ \sum_{i=1, j>i}^{N} \rho_{ij} \sigma_i \sigma_j \left( \frac{\partial f}{\partial z_i} \frac{\partial f}{\partial z_j} \right) - \sum_{i=1}^{N} \left( \frac{\sigma_x^2}{2} \left( \frac{\partial f}{\partial x} \right)^2 + \rho_i \sigma_x \sigma_i \left( \frac{\partial f}{\partial z_i} \right) \left( \frac{\partial f}{\partial x} \right) \right)$$

$$- \sum_{i=1, j>i}^{N} \rho_{ij} \sigma_i \sigma_j \left( \frac{\partial f}{\partial z_i} \frac{\partial f}{\partial z_j} \right) \right) \right)$$

(C.22)

Suppose that the following nonlinear terms in the right-hand side of (C.22) are multiplied by $\epsilon$ where $0 < \epsilon \leq 1$.

$$- \frac{\sigma_x^2}{2} \left( \frac{\partial f}{\partial x} \right)^2 - \sum_{i=1}^{N} \left( \frac{\sigma_i^2}{2} \left( \frac{\partial f}{\partial z_i} \right)^2 + \rho_i \sigma_x \sigma_i \left( \frac{\partial f}{\partial z_i} \right) \left( \frac{\partial f}{\partial x} \right) \right) \right) - \sum_{i=1, j>i}^{N} \rho_{ij} \sigma_i \sigma_j \left( \frac{\partial f}{\partial z_i} \frac{\partial f}{\partial z_j} \right) \right) \right)$$

(C.23)

Also, consider that $f(x, Z, \tau)$ has an asymptotic expansion.

$$f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \cdots$$

(C.24)

Repeating the same analysis as we did for nominal bonds, the first term $f_0$ is given by

$$f_0(x, \pi, t, T) = \int_t^T E_t[x_s^+] ds$$

(C.25)

The second term $f_1$ is given by

$$f_1(x, Z, t, T) = \int_t^T E_t \left[ \frac{\sigma_x^2}{2} \left( \frac{\partial f}{\partial x} \right)^2 + \sum_{i=1}^{N} \left( \frac{\sigma_i^2}{2} \left( \frac{\partial f}{\partial z_i} \right)^2 + \rho_i \sigma_x \sigma_i \left( \frac{\partial f}{\partial z_i} \right) \left( \frac{\partial f}{\partial x} \right) \right) \right] ds.$$}

(C.26)
bound option $V$ with respect to $y$, $\pi$, $y_n$ are small compared to $x$ in empirical applications since a (shadow) interest rate is usually persistent. Thus, the largest sensitivity is the derivative with respect to the shadow interest rate $x_t$. Thus, we can effectively reduce the number of terms in (C.26).

When a regime switching feature is introduced, expectations should be calculated not only over shadow interest rates $x$ but also across regimes $s_t$.

C.4 Approximate formula for real bond yields

For real bond yields, we can apply a similar method. Let us consider that a stochastic process for inflation and a shadow interest rate are given by

$$dx_t = \mu_x(x_t)dt + \sigma_x(x_t)dW_t^x,$$  \hfill (C.27)

$$d\pi_t = \mu_{\pi}(\pi_t)dt + \sigma_{\pi}(\pi_t)dW_t^\pi.$$  \hfill (C.28)

where $W_t^x$ and $W_t^\pi$ are Brownian motions with correlation $\rho$. Let us denote $x_t = x$ and $\pi_t = \pi$. The Feynman-Kac formula gives us the price of a real discount bond $D(x, t, T)$ with expiry $T$ at time $t$ as a solution of the following partial differential equation.

$$\frac{\partial D}{\partial t} + \mu_x \frac{\partial D}{\partial x} + \mu_{\pi} \frac{\partial D}{\partial \pi} + \frac{\sigma_x^2 \frac{\partial^2 D}{\partial x^2}}{2} + \frac{\sigma_{\pi}^2 \frac{\partial^2 D}{\partial \pi^2}}{2} + \rho \sigma_x \sigma_{\pi} \frac{\partial^2 D}{\partial \pi \partial x} = (x^+ - \pi)D,$$  \hfill (C.29)

Suppose that the real bond price is represented as $D(x, t, T) = e^{f(x, \pi, \tau)}$ where $\tau = T - t$. Substituting this for (C.2), one obtains

$$f_t = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \cdots$$  \hfill (C.30)

Suppose that the last three terms in the right-hand side of (C.30) are multiplied by $\epsilon$ where $0 < \epsilon \leq 1$. Also, consider that $f(x, \tau)$ has an asymptotic expansion.

$$f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \cdots$$  \hfill (C.31)

Repeating the same analysis as we did for nominal bonds, the first term $f_0$ is given by

$$f_0(x, \pi, t, T) = \int_t^T E_t[x_s^+]ds - \int_t^T E_t[\pi_s]ds.$$  \hfill (C.32)

The second term $f_1$ is given by

$$f_1(x, t, T) = \int_t^T E_t \left[ -\frac{\sigma_x^2}{2} \left( \frac{\partial f_0(x_s, \pi_s, s, T)}{\partial x_s} \right)^2 - \frac{\sigma_{\pi}^2}{2} \left( \frac{\partial f_0(x_s, \pi_s, s, T)}{\partial \pi_s} \right)^2 \right. \right. \left. \left. -\rho \sigma_x \sigma_{\pi} \frac{\partial f_0(x_s, \pi_s, s, T)}{\partial x_s} \frac{\partial f_0(x_s, \pi_s, s, T)}{\partial \pi_s} \right] ds.$$  \hfill (C.33)
Discretizing (C.33), one obtains an approximate real bond pricing formula. Let us denote a real bond yield with maturity $\tau = n\delta t$ at time $t$ with $r^n_t(x)$. Then we have

$$r^n_t(x) \approx \frac{1}{n} \sum_{k=0}^{n} \left( V^x_t(x, t, t + k\delta t) + V^\pi_t(\pi, t, t + k\delta t) \right)$$

$$- \frac{1}{n} \sum_{k=0}^{n} E_t \left[ \frac{\sigma^2_{x_t + j\delta t}}{2} \sum_{j=k}^{n} \Delta^x(x_t + j\delta t, t + j\delta t, T)^2 + \frac{\sigma^2_{\pi_{t + j\delta t}}}{2} \sum_{j=k}^{n} \Delta^\pi(\pi_{t + j\delta t}, t + j\delta t, T)^2 \right].$$

$$- \frac{1}{n} \sum_{k=0}^{n} \sum_{j=k+1}^{n} E_t \left[ \rho \sigma_{x_{t + j\delta t}} \sigma_{\pi_{t + j\delta t}} \Delta^x(x_{t + j\delta t}, t + j\delta t, T) \Delta^\pi(\pi_{t + j\delta t}, t + j\delta t, T) \right], \quad (C.34)$$

where $V^x, V^\pi, \Delta^x$ and $\Delta^\pi$ are defined as

$$V^x(x, t, s) = E_t[\max(x_s, 0)], \quad \Delta^x(x, t, s) = \frac{\partial V(x, t, s)}{\partial x}, \quad (C.35)$$

$$V^\pi(\pi, t, s) = E_t[\pi_s], \quad \Delta^\pi(\pi, t, s) = \frac{\partial V^\pi(\pi, t, s)}{\partial \pi}. \quad (C.36)$$
D Regime-switching unscented Kalman filtering

I explain the algorithm of regime-switching unscented Kalman filtering that combines unscented Kalman filtering developed by Julier and Uhlmann (2004) and Kalman filtering for regime-switching state space model proposed by Kim (1994). In unscented Kalman filtering, I match the first and second moments of the distribution of observable variables using several sequences of latent variables. In fact, Unscented Kalman Filtering can be understood as the special case of Quasi-Monte Carlo filtering where the distribution is approximated by the quasi-random sequences.

Let us denote the vector of yields with different maturities at time $t$ with $y_t = (y(t, \tau_1), y(t, \tau_2), \cdots, y(t, \tau_N))$. Here, $\tau_j$ denotes $j$-th maturity. I also define $h(x_t)$ as the function of the yield vector $y_t$ with respect to the vector of latent variables. $x_t$ denotes the state variables.

Similar to linear Kalman filtering, unscented Kalman filtering consists of two operations: prediction and filtering. First, I describe the prediction algorithm. Second, I explain filtering algorithm. Both of these algorithms are done given the current and previous regimes. Thus, in the final part, I show how to take into account regime changes given transition matrix and per-period log likelihood.

D.1 Prediction

I introduce $\sigma$ points that are the collection of the vectors $\{\hat{x}_{t|t-1}(m)\}$ ($m = 1, \cdots, 2d+1$). Here, $d$ is the number of latent variables. In this paper, the model has $d = 4$ factors. Each $\sigma$ point is calculated as follows.

\begin{align}
\hat{x}_{t|t}(0) &= \hat{x}_{t|t-1}, \\
\hat{x}_{t|t}(m) &= \hat{x}_{t|t-1} + \left(\sqrt{(d+\lambda)}P_{t|t-1}^m\right)_m, \\
\hat{x}_{t|t}(m + d) &= \hat{x}_{t|t-1} - \left(\sqrt{(d+\lambda)}P_{t|t-1}^m\right)_m.
\end{align}

To step forward in time, I calculate the vector of latent variables $\hat{x}$ at time $t+1$ for each $\sigma$ point.

\begin{align}
\hat{x}_{t+1|t}(m) = f^j(\hat{x}_{t|t}(m)),
\end{align}

where the function $f^j(x_t)$ governs the time evolution of latent variables. In this paper, given linear market price of risks, the function $f(\cdot)$ is specified as

\begin{align}
x_{t+1|t}(m) = F^j x_{t|t}(m) + G^j.
\end{align}

where $F^j$ and $G^j$ are defined in (3.45).

I then take an average of the $2n$ vectors $\hat{x}_{t+1|t}^{i,j}$ as the predicted value.

\begin{align}
\hat{x}_{t+1|t} = \sum_{m=0}^{2d} W^{(m)} \hat{x}_{t+1|t}^{i,j}(m),
\end{align}
I also calculate error covariance matrix $P_{t+1|t}^{i,j}$:

$$P_{t+1|t}^{i,j} = \sum_{m=0}^{2d} W(m)[\hat{x}_{t+1|t}^{i,j}(m) - \hat{x}_{t+1|t}^{i,j}] [\hat{x}_{t+1|t}^{i,j}(m) - \hat{x}_{t+1|t}^{i,j}]' + Q^i, \quad (D.7)$$

where $Q^i$ is the covariance matrix of system error and it is given by $H \Sigma H'$ where the matrix $\Sigma = \text{diag}([\sigma_y^2, \sigma_\pi^2, \sigma_x^2, \sigma_{y_n}^2])$.

**D.2 Filtering**

Second, I explain the filtering algorithm. I compute $\sigma$ points as I have done in the prediction algorithm.

$$\hat{x}_{t|t-1}^{i,j}(0) = \hat{x}_{t|t-1}^{i,j}, \quad (D.8)$$

$$\hat{x}_{t|t-1}^{i,j}(m) = \hat{x}_{t|t-1}^{i,j} + \left(\sqrt{(d + \lambda)}P_{t|t-1}^{i,j}\right)_m, \quad (D.9)$$

$$\hat{x}_{t|t-1}^{i,j}(m + d) = \hat{x}_{t|t-1}^{i,j} - \left(\sqrt{(d + \lambda)}P_{t|t-1}^{i,j}\right)_m, \quad (D.10)$$

where $\lambda$ is a free parameter and need to be adjusted for each specific case.

I calculate the yield vector $y_t$ for each $\sigma$ point.

$$\hat{y}_{t|t-1}^{i,j}(m) = h(\hat{x}_{t|t-1}^{i,j}(m)), \quad (D.11)$$

where $h(\cdot)$ is a non-linear function. In this paper, $h(\cdot)$ is a function of bond yields with respect to stochastic factors $X_t = [y_t, \pi_t, x_t, y_{n}^t]$. The weighted prediction is given by

$$\hat{y}_{t|t-1}^{i,j} = \sum_{m=0}^{2d} W_h(m)\hat{y}_{t|t-1}^{i,j}(m), \quad (D.12)$$

where the weight $W(m)$ is defined as

$$W(0) = \lambda \quad (d + \lambda), \quad (D.13)$$

$$W(m) = \frac{\lambda}{2(d + \lambda)}, \quad (D.14)$$

$$W(m + d) = \frac{\lambda}{2(d + \lambda)}. \quad (D.15)$$

Conditional covariance matrices are given by

$$V_{t|t-1}^{i,j} = \sum_{m=0}^{2n} W_h(m)[\hat{y}_{t|t-1}^{i,j}(m) - \hat{y}_{t|t-1}^{i,j}][\hat{y}_{t|t-1}^{i,j}(m) - \hat{y}_{t|t-1}^{i,j}]' + R, \quad (D.16)$$

$$U_{t|t-1}^{i,j} = \sum_{m=0}^{2n} W_h(m)[\hat{x}_{t|t-1}^{i,j}(m) - \hat{x}_{t|t-1}^{i,j}][\hat{y}_{t|t-1}^{i,j}(m) - \hat{y}_{t|t-1}^{i,j}]', \quad (D.17)$$
where $R$ is the covariance matrix of measurement error with the standard deviation $\sigma_y$

Kalman gain is given by

$$K_{i;j}^t = U_{i;j}^t (V_{i;j}^t)^{-1}. \quad \text{(D.18)}$$

Filtered latent variables and covariance matrix are given by

$$\hat{x}_{i;j}^t = \hat{x}_{i;j}^{t-1} + K_{i;j}^t [y_t - \hat{y}_{i;j}^{t-1}], \quad \text{(D.19)}$$

$$P_{i;j}^t = P_{i;j}^{t-1} - U_{i;j}^t (V_{i;j}^t)^{-1} (U_{i;j}^t)^T. \quad \text{(D.20)}$$

Equipped with the time series of $y_{i;j}^{t-1}$ and $V_{i;j}^{t-1}$, I can calculate the log likelihood as I have done in linear Kalman filtering.

### D.3 How to eliminate lagged regime dependence

What is remaining is to compute lagged-regime-dependent latent factors, $x_{i;j}^t$, given $x_{i;j}^{t-1}$. In order to eliminate the dependence of lagged regime dependence, one can take expectations over regime transitions.

$$x_{i;j}^t = \sum_{i=1}^L \Pr[S_{t-1} = j, S_t = j|i] \hat{x}_{i;j}^{t-1} \quad \Pr[S_t = j|i] \text{ (D.21)}$$

$$P_{i;j}^t = \sum_{i=1}^L \Pr[S_{t-1} = j, S_t = j|i] \left( P_{i;j}^{t-1} + (x_{i;j}^t - \hat{x}_{i;j}^t)(x_{i;j}^t - \hat{x}_{i;j}^t)^T \right) \Pr[S_t = j|i] \text{ (D.22)}$$

The next problem is how to compute the probabilities of having two specific regimes in current and previous states. That can be done in the following way:

**Step 1** Predicting the probabilities of having two specific regimes in current and previous states from the previous one using a given transition matrix.

$$\Pr[S_{t-1} = j, S_t = j|i] = \Pr[S_t = j|i] \times \sum_{i=1}^L \Pr[S_{t-2} = i', S_{t-1} = i|\phi_{t-1}]. \quad \text{(D.23)}$$

**Step 2** Compute the joint density of observable variables $y_t$, the current regime $S_{t-1}$ and the previous regime $S_{t-1}$.

$$f(y_t, S_{t-1}, S_t = j|\phi_{t-1}) = f(y_t, S_{t-1} = i, S_t = j, \phi_{t-1}) \times \Pr[S_{t-1} = i, S_t = j|\phi_{t-1}], \quad \text{(D.24)}$$

where $f(y_t|S_{t-1}, S_t = j|\phi_{t-1})$ is a density function given current and previous regimes that is per-period log likelihood.
**Step 3** "Filtering" the probabilities of having two specific regimes in current and previous states from the previous one using per-period likelihood.

\[
Pr[S_{t-1} = i, S_t = j | \phi_t] = \frac{f(y_t, S_{t-1}, S_t = j | \phi_{t-1})}{f(y_t | \phi_{t-1})}, \tag{D.25}
\]

where \(f(y_t | \phi_{t-1})\) is given by

\[
f(y_t | \phi_{t-1}) = \sum_{j=1}^{L} \sum_{i=1}^{M} f(y_t, S_{t-1} = i, S_t = j | \phi_{t-1}). \tag{D.26}
\]

**Step 4** This part is not mandate but needed when one wants to see the probability of being in a specific regime.

\[
Pr(S_t = j | \phi_t) = \sum_{j=1}^{L} \sum_{i=1}^{M} Pr[S_{t-1} = i, S_t = j | \phi_{t-1}]. \tag{D.27}
\]
Table 1: **Summary statistics of nominal bond yields**
The table shows the average nominal bond yields for US Treasury bonds from October 1991 to January 2015 and Japanese government bonds from July 2004 to January 2015. “Before ZIRP” means the sub-sample period starts on October 1991 and ends on October 2008, which is one quarter before the Federal Reserve announced the zero interest rate policy (ZIRP). “After ZIRP” is the sub-sample period during January 2009 to January 2015. “FFR” is the Federal Funds Rates. “Before QQE” is the sub-sample period that starts on July 2004 and ends March 2013, which is one month before the Bank of Japan announced Quantitative Qualitative Easing (QQE). “After QQE” is the sub-sample period from April 2013 to January 2015. “ON” stands for overnight uncollateralized interest rate.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>FFR/ON</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>7</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>US</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Full Sample</td>
<td>3.06%</td>
<td>3.14%</td>
<td>3.41%</td>
<td>3.66%</td>
<td>3.89%</td>
<td>4.11%</td>
<td>4.48%</td>
<td>4.90%</td>
</tr>
<tr>
<td>Before ZIRP</td>
<td>4.13%</td>
<td>4.18%</td>
<td>4.44%</td>
<td>4.66%</td>
<td>4.85%</td>
<td>5.02%</td>
<td>5.31%</td>
<td>5.65%</td>
</tr>
<tr>
<td>After ZIRP</td>
<td>0.11%</td>
<td>0.28%</td>
<td>0.54%</td>
<td>0.89%</td>
<td>1.26%</td>
<td>1.60%</td>
<td>2.20%</td>
<td>2.85%</td>
</tr>
<tr>
<td>Japan</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Full Sample</td>
<td>0.16%</td>
<td>0.23%</td>
<td>0.31%</td>
<td>0.40%</td>
<td>0.52%</td>
<td>0.62%</td>
<td>0.82%</td>
<td>1.22%</td>
</tr>
<tr>
<td>Before QQE</td>
<td>0.18%</td>
<td>0.27%</td>
<td>0.36%</td>
<td>0.47%</td>
<td>0.60%</td>
<td>0.72%</td>
<td>0.96%</td>
<td>1.35%</td>
</tr>
<tr>
<td>After QQE</td>
<td>0.06%</td>
<td>0.06%</td>
<td>0.07%</td>
<td>0.09%</td>
<td>0.14%</td>
<td>0.18%</td>
<td>0.34%</td>
<td>0.61%</td>
</tr>
</tbody>
</table>
Table 2: Summary statistics of CPI inflation forecasts

The table shows summary statistics of survey-based CPI forecasts for the US and Japan. For the US case, the data is from Survey of Professional Forecasters. For the Japanese case, the data is from Quick Monthly Market Survey of Bond. The average CPI inflation forecasts are computed after adjusting a consumption tax hike in April 2013. “Avg” is the average forecast number of CPI for each horizon. “Std” denotes the dispersion of opinions that is computed as the cross-sectional standard deviation of CPI forecasts. “Before ZIRP” means the sub-sample period from October 1991 to October 2008, which is one quarter before the Federal Reserve announced the zero interest rate policy (ZIRP). “After ZIRP” is the sub-sample period from January 2009 to January 2015. “Before QQE” means that the sub-sample period that starts on July 2004 and ends on March 2013, which is one month before the Bank of Japan announced Quantitative Qualitative Easing (QQE). “After QQE” is the sub-sample period from April 2013 to January 2015. “Realized” means realized inflation during the corresponding time period.

<table>
<thead>
<tr>
<th></th>
<th>Realized</th>
<th>1y-Avg</th>
<th>2y-Avg</th>
<th>10y-Avg</th>
<th>1y-Std</th>
<th>2y-Std</th>
<th>10y-Std</th>
</tr>
</thead>
<tbody>
<tr>
<td>US</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Entire Sample</td>
<td>2.37%</td>
<td>2.14%</td>
<td>-</td>
<td>2.67%</td>
<td>0.97%</td>
<td>-</td>
<td>0.52%</td>
</tr>
<tr>
<td>Before ZIRP</td>
<td>2.68%</td>
<td>2.30%</td>
<td>-</td>
<td>2.79%</td>
<td>0.88%</td>
<td>-</td>
<td>0.51%</td>
</tr>
<tr>
<td>After ZIRP</td>
<td>1.52%</td>
<td>1.73%</td>
<td>-</td>
<td>2.33%</td>
<td>1.21%</td>
<td>-</td>
<td>0.56%</td>
</tr>
<tr>
<td>Japan</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Entire Sample</td>
<td>0.02%</td>
<td>0.15%</td>
<td>0.48%</td>
<td>1.08%</td>
<td>0.36%</td>
<td>0.40%</td>
<td>0.60%</td>
</tr>
<tr>
<td>Before QQE</td>
<td>-0.15%</td>
<td>0.07%</td>
<td>0.35%</td>
<td>1.03%</td>
<td>0.27%</td>
<td>0.35%</td>
<td>0.59%</td>
</tr>
<tr>
<td>After QQE</td>
<td>0.80%</td>
<td>0.49%</td>
<td>1.01%</td>
<td>1.30%</td>
<td>0.73%</td>
<td>0.64%</td>
<td>0.64%</td>
</tr>
</tbody>
</table>
Table 3: Absolute pricing error
The table shows summary statistics of the absolute pricing error for each maturity in both US and Japanese cases. Absolute errors are computed as the difference between model-implied bond yields and actual bond yields. “Before ZIRP” means the sub-sample period that starts on October 1991 and ends on October 2008, which is one quarter before the Federal Reserve announced zero interest rate policy (ZIRP). “After ZIRP” is the sub-sample period from January 2009 to January 2015. “Before QQE” means the sub-sample period that starts on July 2004 and ends March 2013, which is one month before the Bank of Japan announced Quantitative Qualitative Easing (QQE). “After QQE” is the sub-sample period from April 2013 to January 2015. “Avg” is the average of absolute pricing error across all maturities.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>7</th>
<th>10</th>
<th>Avg</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>US</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Full Sample</td>
<td>0.52%</td>
<td>0.27%</td>
<td>0.16%</td>
<td>0.17%</td>
<td>0.25%</td>
<td>0.41%</td>
<td>0.54%</td>
<td>0.33%</td>
</tr>
<tr>
<td>Before ZIRP</td>
<td>0.62%</td>
<td>0.24%</td>
<td>0.13%</td>
<td>0.19%</td>
<td>0.29%</td>
<td>0.44%</td>
<td>0.58%</td>
<td>0.36%</td>
</tr>
<tr>
<td>After ZIRP</td>
<td>0.24%</td>
<td>0.34%</td>
<td>0.24%</td>
<td>0.12%</td>
<td>0.13%</td>
<td>0.30%</td>
<td>0.45%</td>
<td>0.26%</td>
</tr>
<tr>
<td><strong>Japan</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Full Sample</td>
<td>0.14%</td>
<td>0.08%</td>
<td>0.06%</td>
<td>0.05%</td>
<td>0.06%</td>
<td>0.07%</td>
<td>0.12%</td>
<td>0.08%</td>
</tr>
<tr>
<td>Before QQE</td>
<td>0.24%</td>
<td>0.16%</td>
<td>0.11%</td>
<td>0.07%</td>
<td>0.04%</td>
<td>0.07%</td>
<td>0.13%</td>
<td>0.09%</td>
</tr>
<tr>
<td>After QQE</td>
<td>0.07%</td>
<td>0.06%</td>
<td>0.05%</td>
<td>0.05%</td>
<td>0.04%</td>
<td>0.06%</td>
<td>0.05%</td>
<td>0.06%</td>
</tr>
</tbody>
</table>
Table 4: Estimates of the New Keynesian model parameters

The following are the estimated parameters of a structural New Keynesian model for the US and Japan. Asymptotic standard errors are computed as the estimate of the Fisher information matrix. For each parameter, * denotes statistical significance at the 5% level. † means that the number is given by the assumptions explained in the main text.

<table>
<thead>
<tr>
<th>Description of parameters</th>
<th>Notation</th>
<th>US</th>
<th>Japan</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sensitivity of real output to real interest rate</td>
<td>$\phi$</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>Sensitivity of inflation to output gap</td>
<td>$\kappa$</td>
<td>0.01</td>
<td>0.02*</td>
</tr>
<tr>
<td>Mean-reverting level of potential output</td>
<td>$\bar{y}_n$</td>
<td>0.00†</td>
<td>0.00†</td>
</tr>
<tr>
<td>Sensitivity of policy rate to output gap</td>
<td>$\gamma_y$</td>
<td>0.50†</td>
<td>0.50†</td>
</tr>
<tr>
<td>Sensitivity of policy rate to inflation gap</td>
<td>$\gamma_\pi$</td>
<td>0.50†</td>
<td>0.50†</td>
</tr>
<tr>
<td>Constant term in IS equation</td>
<td>$\alpha_{IS}$</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>Equilibrium real interest rate</td>
<td>$i^*$</td>
<td>0.010*</td>
<td>-0.028*</td>
</tr>
<tr>
<td>Effective lower bound</td>
<td>$\bar{i}$</td>
<td>0.0010</td>
<td>0.0012</td>
</tr>
<tr>
<td>Bias of survey-based inflation forecast</td>
<td>$\alpha_s$</td>
<td>0.007</td>
<td>0.011</td>
</tr>
<tr>
<td>Dependence on lagged real output</td>
<td>$\mu_y$</td>
<td>0.53</td>
<td>0.50</td>
</tr>
<tr>
<td>Dependence on expected real output</td>
<td>$\mu_y^e$</td>
<td>0.52</td>
<td>0.48</td>
</tr>
<tr>
<td>Dependence on expected inflation</td>
<td>$\mu_\pi$</td>
<td>0.90</td>
<td>0.70*</td>
</tr>
<tr>
<td>Dependence on lagged shadow rate</td>
<td>$\mu_x$</td>
<td>0.070*</td>
<td>0.094</td>
</tr>
<tr>
<td>Dependence on lagged potential output</td>
<td>$\mu_{yn}$</td>
<td>0.001</td>
<td>0.541*</td>
</tr>
<tr>
<td>IS shock volatility</td>
<td>$\sigma_{IS}$</td>
<td>0.003</td>
<td>0.003</td>
</tr>
<tr>
<td>AS shock volatility</td>
<td>$\sigma_{AS}$</td>
<td>0.003*</td>
<td>0.003*</td>
</tr>
<tr>
<td>Shadow interest rate volatility</td>
<td>$\sigma_x$</td>
<td>0.01*</td>
<td>0.008</td>
</tr>
<tr>
<td>Potential output volatility</td>
<td>$\sigma_{yn}$</td>
<td>0.002*</td>
<td>0.003*</td>
</tr>
<tr>
<td>Inflation survey noise</td>
<td>$\sigma_s$</td>
<td>0.004*</td>
<td>0.002*</td>
</tr>
</tbody>
</table>
Table 5: Estimates of the market price of risk
This table provides the estimated parameters of the market price of risk for US and Japan. \( \lambda^0 \) is a constant term. \( \lambda^1 \) is the sensitivity with respect to state variables \( X_t \). For example, \( \lambda_{13}^1 \) is the third column of \( \lambda^1 \). Recall that \( \lambda_{41}^1, \lambda_{42}^1, \lambda_{43}^1 \) are equal to zero by assumption. Asymptotic standard errors are computed as the estimate of the Fisher information matrix. For each parameter, * and ** denote statistical significance at the 5% level and at the 1% level, respectively.

<table>
<thead>
<tr>
<th></th>
<th>( \lambda^0 )</th>
<th>( \lambda_{11}^1 )</th>
<th>( \lambda_{12}^1 )</th>
<th>( \lambda_{13}^1 )</th>
<th>( \lambda_{14}^1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>US</td>
<td>0.91</td>
<td>-59.78</td>
<td>-0.26**</td>
<td>-15.61**</td>
<td>-14.12**</td>
</tr>
<tr>
<td></td>
<td>6.68**</td>
<td>6.20**</td>
<td>-23.86</td>
<td>0.00**</td>
<td>-6.20**</td>
</tr>
<tr>
<td></td>
<td>-0.10</td>
<td>3.46</td>
<td>3.46**</td>
<td>-6.39**</td>
<td>-3.46**</td>
</tr>
<tr>
<td></td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>2.38**</td>
</tr>
<tr>
<td>Japan</td>
<td>-0.53</td>
<td>-28.9**</td>
<td>1.3</td>
<td>-13.5</td>
<td>0.3</td>
</tr>
<tr>
<td></td>
<td>-0.89**</td>
<td>45.5</td>
<td>-133.5</td>
<td>-23.5</td>
<td>-13.7</td>
</tr>
<tr>
<td></td>
<td>-0.57</td>
<td>5.8</td>
<td>17.4</td>
<td>-10.5</td>
<td>-5.8**</td>
</tr>
<tr>
<td></td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>-204.2</td>
</tr>
</tbody>
</table>
Table 6: **Decomposition of nominal bond yields into two components**

This table provides a decomposition of nominal bond yields into (1) the expected path of the nominal interest rate and (2) the term premium. The details of the computation are as follows: First, I compute nominal bond yields under both the physical measure and the risk-neutral measure. Second, expected interest rate path components are computed as nominal bond yields under the physical measure. Third, term premium components are computed as the difference between the nominal bond yields under the physical and the risk-neutral measure. Nominal yields are estimated model-implied yields. These nominal yields are equal to the sum of expectation component of the nominal interest rate and term premium. The numbers are the time average values for a specific maturity of each component during the sample period.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>7</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>US</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nominal yields</td>
<td>2.78%</td>
<td>3.39%</td>
<td>3.69%</td>
<td>3.90%</td>
<td>4.08%</td>
<td>4.40%</td>
<td>4.82%</td>
</tr>
<tr>
<td>Expectation</td>
<td>3.27%</td>
<td>3.28%</td>
<td>3.27%</td>
<td>3.26%</td>
<td>3.26%</td>
<td>3.25%</td>
<td>3.24%</td>
</tr>
<tr>
<td>Term premium</td>
<td>-0.49%</td>
<td>0.12%</td>
<td>0.42%</td>
<td>0.64%</td>
<td>0.83%</td>
<td>1.16%</td>
<td>1.58%</td>
</tr>
<tr>
<td>Japan</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nominal yields</td>
<td>0.26%</td>
<td>0.32%</td>
<td>0.38%</td>
<td>0.46%</td>
<td>0.55%</td>
<td>0.84%</td>
<td>1.31%</td>
</tr>
<tr>
<td>Expectation</td>
<td>0.13%</td>
<td>0.13%</td>
<td>0.13%</td>
<td>0.13%</td>
<td>0.13%</td>
<td>0.13%</td>
<td>0.13%</td>
</tr>
<tr>
<td>Term premium</td>
<td>0.13%</td>
<td>0.19%</td>
<td>0.25%</td>
<td>0.33%</td>
<td>0.42%</td>
<td>0.71%</td>
<td>1.19%</td>
</tr>
</tbody>
</table>
Table 7: Decomposition of nominal bond yields into four components

This table provides a decomposition of nominal bond yields into (1) the expected path of the real interest rate, (2) the real risk premium, (3) the expected inflation, and (4) the inflation risk premium. The details of the computation are as follows: First, I obtain nominal and real bond yields under both the physical measure and the risk-neutral measure. Second, expected real yields are computed as those yields under the physical measure. Third, expected inflation is calculated as a difference between nominal yields and real yields under the physical measure. Fourth, the real rate risk premium is computed as the differences between the real bond yields under the physical and the risk-neutral measure. Fifth, the inflation premium is computed as the differences between the nominal bond yields under the physical and the risk-neutral measure and further subtracting the real premium. Nominal yields are estimated model-implied yields. They are equal to the sum of the four components. The numbers are the time average values of each component for a specific maturity during the sample period.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>7</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>US</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nominal yields</td>
<td>2.78%</td>
<td>3.39%</td>
<td>3.69%</td>
<td>3.90%</td>
<td>4.08%</td>
<td>4.40%</td>
<td>4.82%</td>
</tr>
<tr>
<td>Expected real yields</td>
<td>1.17%</td>
<td>1.18%</td>
<td>1.18%</td>
<td>1.17%</td>
<td>1.17%</td>
<td>1.16%</td>
<td>1.15%</td>
</tr>
<tr>
<td>Real risk premium</td>
<td>-0.74%</td>
<td>-0.02%</td>
<td>0.37%</td>
<td>0.67%</td>
<td>0.94%</td>
<td>1.42%</td>
<td>2.05%</td>
</tr>
<tr>
<td>Expected inflation</td>
<td>2.08%</td>
<td>2.08%</td>
<td>2.08%</td>
<td>2.08%</td>
<td>2.09%</td>
<td>2.08%</td>
<td>2.08%</td>
</tr>
<tr>
<td>Inflation risk premium</td>
<td>0.26%</td>
<td>0.13%</td>
<td>0.03%</td>
<td>-0.05%</td>
<td>-0.13%</td>
<td>-0.28%</td>
<td>-0.50%</td>
</tr>
<tr>
<td>Japan</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nominal yields</td>
<td>0.26%</td>
<td>0.32%</td>
<td>0.38%</td>
<td>0.46%</td>
<td>0.55%</td>
<td>0.84%</td>
<td>1.31%</td>
</tr>
<tr>
<td>Expected real yields</td>
<td>0.45%</td>
<td>0.41%</td>
<td>0.36%</td>
<td>0.32%</td>
<td>0.29%</td>
<td>0.24%</td>
<td>0.20%</td>
</tr>
<tr>
<td>Real risk premium</td>
<td>-0.42%</td>
<td>-0.25%</td>
<td>-0.13%</td>
<td>-0.01%</td>
<td>0.12%</td>
<td>0.47%</td>
<td>1.02%</td>
</tr>
<tr>
<td>Expected inflation</td>
<td>-0.32%</td>
<td>-0.28%</td>
<td>-0.24%</td>
<td>-0.20%</td>
<td>-0.16%</td>
<td>-0.11%</td>
<td>-0.07%</td>
</tr>
<tr>
<td>Inflation risk premium</td>
<td>0.55%</td>
<td>0.44%</td>
<td>0.39%</td>
<td>0.34%</td>
<td>0.31%</td>
<td>0.24%</td>
<td>0.17%</td>
</tr>
</tbody>
</table>
Table 8: Regressions of the inflation risk premium on cross-sectional dispersion of inflation forecasts

This table provides estimates of coefficients in the regressions of quarterly changes in the inflation risk premium with \( n \)-year maturity on previous quarter changes in the cross-sectional dispersion (\( ny-Std \)). For each parameter, * denotes statistical significance at the 5% level based on the t-statistic.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>7</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>US</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1y-Std</td>
<td>0.086</td>
<td>0.083</td>
<td>0.083</td>
<td>0.084</td>
<td>0.087</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>10y-Std</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-0.145</td>
</tr>
<tr>
<td>Japan</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1y-Std</td>
<td>2.964*</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td></td>
</tr>
<tr>
<td>2y-Std</td>
<td>-</td>
<td>2.684*</td>
<td>2.597*</td>
<td>2.515</td>
<td>2.441</td>
<td>2.317</td>
<td>-</td>
</tr>
<tr>
<td>10y-Std</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-0.7768</td>
</tr>
</tbody>
</table>

Table 9: Variance decomposition of nominal bond yields into two components

This table provides a variance decomposition of nominal bond yields \( y_{tn} \) into (1) the expected path of the nominal interest rate \( y_{t}^{ph} \) and (2) the term premium \( p_{t}(\tau) \). The following equation is used:

\[
\text{var}(y_{tn}(\tau)) = \text{var}(y_{t}(\tau)^{ph}) + \text{var}(p_{t}(\tau)) + 2\text{cov}(y_{t}^{ph}(\tau), p_{t}(\tau)).
\]

Notice that the numbers in this table are standard deviations.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>7</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>US</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nominal yield</td>
<td>0.47%</td>
<td>0.48%</td>
<td>0.48%</td>
<td>0.48%</td>
<td>0.47%</td>
<td>0.46%</td>
<td>0.44%</td>
</tr>
<tr>
<td>Expected yield</td>
<td>0.38%</td>
<td>0.31%</td>
<td>0.26%</td>
<td>0.21%</td>
<td>0.18%</td>
<td>0.13%</td>
<td>0.09%</td>
</tr>
<tr>
<td>Term premium</td>
<td>0.15%</td>
<td>0.18%</td>
<td>0.24%</td>
<td>0.29%</td>
<td>0.32%</td>
<td>0.35%</td>
<td>0.36%</td>
</tr>
<tr>
<td>Japan</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nominal yield</td>
<td>0.03%</td>
<td>0.07%</td>
<td>0.09%</td>
<td>0.10%</td>
<td>0.11%</td>
<td>0.13%</td>
<td>0.15%</td>
</tr>
<tr>
<td>Expected yield</td>
<td>0.02%</td>
<td>0.02%</td>
<td>0.02%</td>
<td>0.02%</td>
<td>0.02%</td>
<td>0.02%</td>
<td>0.02%</td>
</tr>
<tr>
<td>Term premium</td>
<td>0.00%</td>
<td>0.06%</td>
<td>0.08%</td>
<td>0.10%</td>
<td>0.11%</td>
<td>0.12%</td>
<td>0.15%</td>
</tr>
</tbody>
</table>
Table 10: **Variance decomposition of nominal bond yields into four components**

This table provides a variance decomposition of nominal bond yields $y^{rn}_t$ into (1) the expected path of the real interest rate $r_t$, (2) the real interest rate premium $rp_t$, (3) the expected inflation $\pi_t$ and (4) the inflation risk premium $ip_t$. The following equation is used:

$$\text{var}(y^{rn}_t(\tau)) = \text{var}(r_t(\tau)^{ph}) + \text{var}(rp_t(\tau)) + \text{var}(\pi_t(\tau)) + \text{var}(ip_t(\tau)) + \text{cov}.$$  

“cov” denotes the remaining covariance terms. Notice that the numbers in this table are standard deviations.

<table>
<thead>
<tr>
<th>Maturity</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>7</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>US</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nominal yield</td>
<td>0.47%</td>
<td>0.48%</td>
<td>0.48%</td>
<td>0.48%</td>
<td>0.47%</td>
<td>0.46%</td>
<td>0.44%</td>
</tr>
<tr>
<td>Expected real yield</td>
<td>0.44%</td>
<td>0.37%</td>
<td>0.30%</td>
<td>0.25%</td>
<td>0.21%</td>
<td>0.15%</td>
<td>0.11%</td>
</tr>
<tr>
<td>Real risk premium</td>
<td>0.89%</td>
<td>0.88%</td>
<td>0.88%</td>
<td>0.88%</td>
<td>0.88%</td>
<td>0.86%</td>
<td>0.82%</td>
</tr>
<tr>
<td>Expected inflation</td>
<td>0.07%</td>
<td>0.06%</td>
<td>0.05%</td>
<td>0.05%</td>
<td>0.04%</td>
<td>0.03%</td>
<td>0.02%</td>
</tr>
<tr>
<td>Inflation risk premium</td>
<td>0.87%</td>
<td>0.85%</td>
<td>0.83%</td>
<td>0.81%</td>
<td>0.80%</td>
<td>0.77%</td>
<td>0.73%</td>
</tr>
<tr>
<td><strong>Japan</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Nominal yield</td>
<td>0.10%</td>
<td>0.09%</td>
<td>0.09%</td>
<td>0.09%</td>
<td>0.10%</td>
<td>0.11%</td>
<td>0.14%</td>
</tr>
<tr>
<td>Expected real yield</td>
<td>0.11%</td>
<td>0.08%</td>
<td>0.06%</td>
<td>0.04%</td>
<td>0.03%</td>
<td>0.02%</td>
<td>0.02%</td>
</tr>
<tr>
<td>Real risk premium</td>
<td>0.39%</td>
<td>0.36%</td>
<td>0.35%</td>
<td>0.34%</td>
<td>0.33%</td>
<td>0.31%</td>
<td>0.29%</td>
</tr>
<tr>
<td>Expected inflation</td>
<td>0.11%</td>
<td>0.08%</td>
<td>0.06%</td>
<td>0.05%</td>
<td>0.04%</td>
<td>0.03%</td>
<td>0.02%</td>
</tr>
<tr>
<td>Inflation risk premium</td>
<td>0.41%</td>
<td>0.29%</td>
<td>0.37%</td>
<td>0.37%</td>
<td>0.36%</td>
<td>0.34%</td>
<td>0.32%</td>
</tr>
</tbody>
</table>

Table 11: **Estimates of regime-dependent long-run real interest rates**

The following are the estimates of regime-dependent long-run real interest rates for a structural New Keynesian term structure for the US case. I denote the low regime with $s_t = d$ and the high regime $s_t = u$. Asymptotic standard errors are computed as the estimate of the Fisher information matrix. For each parameter, * and ** denote statistical significance at the 5% and 1% level, respectively.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Long-run real interest rate (low)</td>
<td>$r^*(d)$</td>
</tr>
<tr>
<td>Long-run real interest rate (high)</td>
<td>$r^*(s)$</td>
</tr>
<tr>
<td>Transition probability (low regime to low regime)</td>
<td>$Pr(s_t = d</td>
</tr>
<tr>
<td>Transition probability (high regime to high regime)</td>
<td>$Pr(s_t = u</td>
</tr>
</tbody>
</table>
Table 12: **Summary statistics of the average durations across different financial sectors**

The table shows summary statistics of the average durations for the following three different financial sectors: banking, mutual funds, and insurance. In banking, there are commercial banks as well as regional banks, but no investment banks. Mutual funds include asset management companies. Insurance sector includes both life insurance firms and non-life insurance firms. For each category, I compute the average duration across the same-type financial firms in each month. The original data source is Quick Monthly Market Survey of Bond.

<table>
<thead>
<tr>
<th></th>
<th>Duration$_{bank}^t$</th>
<th>Duration$_{mutualfund}^t$</th>
<th>Duration$_{insurance}^t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>number of sample (average)</td>
<td>37.9</td>
<td>42.3</td>
<td>18.3</td>
</tr>
<tr>
<td>mean (year)</td>
<td>3.43</td>
<td>6.40</td>
<td>7.86</td>
</tr>
<tr>
<td>min (year)</td>
<td>2.99</td>
<td>5.01</td>
<td>5.47</td>
</tr>
<tr>
<td>max (year)</td>
<td>4.10</td>
<td>8.12</td>
<td>10.87</td>
</tr>
<tr>
<td>trend in monthly change</td>
<td>0.01</td>
<td>0.02</td>
<td>0.03</td>
</tr>
<tr>
<td>volatility of monthly change</td>
<td>0.14</td>
<td>0.20</td>
<td>0.42</td>
</tr>
</tbody>
</table>

Table 13: **Interaction between nominal bond yields on the average durations of different financial sectors**

This table shows the results of two different types of regressions. The top part of the table (A) provides estimates of coefficients in the regressions of quarterly changes in residual of nominal bond yields for each maturity on previous quarter changes in the average duration of different financial sectors. The residual bond yields are computed by subtracting model-implied bond yields from actual nominal bond yields for each maturity. The bottom part (B) provides estimates of coefficients in the regressions of changes in the average duration of different financial sectors on changes in nominal bond yields for each maturity in the previous quarter. For each parameter, * and ** denote statistical significance at the 5% and 1% level, respectively.

<table>
<thead>
<tr>
<th></th>
<th>$\Delta y_{t-1}(3)^{resi}$</th>
<th>$\Delta y_{t-1}(4)^{resi}$</th>
<th>$\Delta y_{t-1}(5)^{resi}$</th>
<th>$\Delta y_{t-1}(7)^{resi}$</th>
<th>$\Delta y_{t-1}(10)^{resi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta Duration_{bank}^{t-1}$</td>
<td>0.00</td>
<td>0.07</td>
<td>0.07</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\Delta Duration_{mutualfund}^{t-1}$</td>
<td>-</td>
<td>-</td>
<td>-0.01</td>
<td>0.03</td>
<td>0.00</td>
</tr>
<tr>
<td>$\Delta Duration_{insurance}^{t-1}$</td>
<td>-</td>
<td>-</td>
<td>0.06</td>
<td>-0.01</td>
<td>-0.01</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$\Delta y_{t-1}(3)$</th>
<th>$\Delta y_{t-1}(4)$</th>
<th>$\Delta y_{t-1}(5)$</th>
<th>$\Delta y_{t-1}(7)$</th>
<th>$\Delta y_{t-1}(10)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta Duration_{bank}^{t}$</td>
<td>-0.36*</td>
<td>-0.33**</td>
<td>-0.33**</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$\Delta Duration_{mutualfund}^{t}$</td>
<td>-</td>
<td>-</td>
<td>-0.12</td>
<td>-0.12*</td>
<td>-0.09</td>
</tr>
<tr>
<td>$\Delta Duration_{insurance}^{t}$</td>
<td>-</td>
<td>-</td>
<td>0.38</td>
<td>0.29</td>
<td>0.23</td>
</tr>
</tbody>
</table>
Figure 1: Evolution of US Treasury bond yields
This figure provides time series plot of US Treasury bond yields during the period from October 1991 to January 2015. The data is obtained from the Federal Reserve Board. The details of computation of the nominal bond yields is found in Gürkaynak, Sack and Wright (2006).

Figure 2: Evolution of Japanese government bond yields
This figure provides time series plot of Japanese government bond yields during the period from July 2004 to January 2015. The data is obtained from the Ministry of Finance in Japan.
Figure 3: **Evolution of the US shadow interest rates**
This figure provides time series plot of the shadow interest rate in US during the period from October 1991 to January 2015. Estimated parameters are in Table 4.

Figure 4: **Raising the inflation target in US during the normal period**
This figure provides a counterfactual analysis of raising the inflation target for US when the zero lower bound is not binding at the initial time period. Estimated parameters are in Table 4. The shadow interest rate $x_t$ is equal to +1% ($> 0$). I call this setting normal period. Inflation $\pi_t = 2\%$ at the initial time period. Real and potential output are $y_t = y^*_t = 0\%$. 
Figure 5: Raising the inflation target in US during the ZLB period
This figure provides counterfactual analyses of (A) raising the inflation target and (B) suddenly ending the zero interest rate policy (ZIRP) for US when the zero lower bound is binding at the initial time period. Estimated parameters are in Table 4. In benchmark case, I set $x_t = -1\% (< 0)$. I call this setting ZLB (zero lower bound) period. The benchmark case is colored black. For (A), I set the inflation target to 3% (red line). For (B), I set the shadow interest rate $x_t = 0\%$ (blue line).

Figure 6: Decomposition of nominal bond yield changes given increased inflation target during the ZLB period
This figure shows the changes in expected (nominal) interest rates and term premium if the inflation target is hypothetically risen for US when the zero lower bound is binding at the initial time period. Estimated parameters are in Table 4. Shadow interest rate $x_t$ is equal to -1%. I call this setting ZLB (zero lower bound) period. Inflation $\pi_t = 2\%$ at the initial time period. Real and potential output are $y_t = y^n_t = 0\%$. The target is changed from 2% to 3%.
Figure 7: **Raising the inflation target in Japan during the normal period**
This figure provides a counterfactual analysis of raising inflation target for Japan when the zero lower bound is not binding at the initial time period. Estimated parameters are in Table 4. The shadow interest rate $x_t$ is equal to $+1\% (> 0)$. I call this setting normal period. Inflation $\pi_t = 0\%$ at the current time period. Real and potential output are $y_t = y^n_t = 0\%$.

![Diagram showing the nominal yield curve for different inflation targets.](image)

**Inflation target: 2%**
**Inflation target: 3%**

Figure 8: **Raising the inflation target in Japan during the ZLB period**
This figure provides counterfactual analyses of (A) raising inflation target and (B) suddenly ending the zero interest rate policy (ZIRP) for Japan when the zero lower bound is binding at the initial time period. Estimated parameters are in Table 4. In benchmark case, I set $x_t = -1\% (< 0)$. I call this setting ZLB (zero lower bound) period. The benchmark case is colored black. For (A), I set the inflation target to 3\% (red line). For (B), I set the shadow interest rate $x_t = 0\%$ (blue line).

![Diagram showing the nominal yield curve for different scenarios.](image)

**Benchmark**
**Inflation target: 3%**
**Suddenly ending ZIRP**
Figure 9: **Introducing the negative lower bound of nominal interest rates in US**
This figure provides a counterfactual analysis of introducing the negative lower bound for US when the shadow interest rate is negative. The effective lower bound $\bar{i}$ is set to -0.5%. Estimates of parameters are in Table 4. Shadow interest rate $x_t$ is equal to -1.0%. Inflation $\pi_t$ is 2.0%. Real and potential output are $y_t = y^p_t = 0\%$.

![Nominal Yield Curve in US](image)

Figure 10: **Introducing the negative lower bound of nominal interest rates in Japan**
This figure provides a counterfactual analysis of introducing the negative lower bound for Japan when the shadow interest rate is negative. The effective lower bound $\bar{i}$ is set to -0.5%. Estimates of model parameters are in Table 4. Shadow interest rate $x_t$ is equal to -1.0%. Inflation $\pi_t$ is 0%. Real and potential output are $y_t = y^p_t = 0\%$.

![Nominal Yield Curve in Japan](image)
Figure 11: **Principal component analysis of nominal bond yields during the normal period**
This figure provides factor loadings of nominal bond yields to the first three factors of nominal bond yields in principal component analysis: I conduct principal component analysis for nominal bond yields from October 1991 to October 2008 (normal period) for US and plot factor loadings. During this time period, the policy rate is positive.

![Graph showing factor loadings for PCA1, PCA2, and PCA3 for nominal bond yields during the normal period.](image)

Figure 12: **Principal component analysis of nominal bond yields during the ZLB period**
This figure provides factor loadings of nominal bond yields to the first three factors of nominal bond yields in principal component analysis: I conduct principal component analysis for nominal bond yields from October 2008 to January 2015 (ZLB period) for US and plot factor loadings. During this time period, the Federal Reserve employed the zero interest rate policy.

![Graph showing factor loadings for PCA1, PCA2, and PCA3 for nominal bond yields during the ZLB period.](image)
Figure 13: **Factor loadings in US during the normal period**
This figure provides factor loadings of nominal bond yields to four factors in the case of the US. The shadow interest rate is positive $x_t = 1\% (> 0)$. I call this setting normal period. Inflation $\pi_t = 2\%$. Real and potential output are $y_t = y_t^n = 0\%$. I compute the change in the nominal bond yields given the change in each variable. The numbers shown are calculated relative to the 10-year change.

![Factor loadings in US during the normal period](image)

Figure 14: **Factor loadings in US during the ZLB period**
This figure provides factor loadings of nominal bond yields to four factors in the case of the US. The shadow interest rate is negative $x_t = -1\% (< 0)$. I call this setting ZLB (zero lower bound) period. Inflation $\pi_t = 2\%$. Real and potential output are $y_t = y_t^n = 0\%$. I compute the change in the nominal bond yields given the change in each variable. The numbers shown are calculated relative to the 10-year change.

![Factor loadings in US during the ZLB period](image)
Figure 15: **Factor loadings in Japan during normal period**
This figure provides factor loadings of nominal bond yields to four factors in the case of Japan. The shadow interest rate is positive $x_t = 1\%$. Inflation $\pi_t = 0\%$. Real and potential output are $y_t = y_t^* = 0\%$. I compute the change in the nominal bond yields given the change in each variable. The number shown are relative to the 10-year change.

Figure 16: **Factor loadings in Japan during zero lower bound period**
This figure provides factor loadings of nominal bond yields to four factors in the case of Japan. The shadow interest rate is negative $x_t = -1\%$. Inflation $\pi_t = 0\%$. Real and potential output are $y_t = y_t^* = 0\%$. I compute the change in the nominal bond yields given the change in each variable. The number shown are relative to the 10-year change.
Figure 17: Evolution of the probability of being at each regime
This figure shows time series plot of probability of being at each regime: The high regime ($s_t = u$) is defined as the regime where the long-run real interest rate is equal to $r^*(u)(r^*(u) > r^*(d))$. The low regime ($s_t = d$) is defined as the regime where the long-run real interest rate takes $r^*(d)(r^*(d) < r^*(u))$.

Figure 18: Sensitivity of real forward interest rates to policy interest rate
This figure shows actual and model-implied sensitivity of real forward interest rates to the policy interest rate when the zero lower bound is not binding. In computing the model-implied sensitivity, I calculate the sensitivity of real forward interest rates to the change in the shadow interest rate $x_t$. I then it by the 2-year sensitivity for standardization. The sensitivities are computed under the physical measure.