

## Arbitrage and the Expectations Hypothesis

FRANCIS A. LONGSTAFF\*

### ABSTRACT

This paper shows that all traditional forms of the expectations hypothesis can be consistent with the absence of arbitrage if markets are incomplete. A key implication is that the validity of the expectations hypothesis is purely an empirical issue; the expectations hypothesis cannot be ruled out on a priori theoretical grounds.

ONE OF THE OLDEST AND BEST-KNOWN MODELS in finance is the expectations hypothesis of the term structure. Introduced by Fisher (1896) more than a century ago, the expectations hypothesis has become a standard framework for explaining how yields of different maturities are related and it provides an important empirical benchmark.

From a theoretical perspective, however, the expectations hypothesis is not generally viewed as a viable model of the term structure. In particular, Cox, Ingersoll and Ross (1981) show that the traditional expectations hypothesis is actually a set of mutually exclusive propositions and offer a proof that most forms of the expectations hypothesis admit arbitrage. Recently, McCulloch (1993) and Fisher and Gilles (1998) present counterexamples to the Cox et al. proof, showing that the traditional forms of the expectations hypothesis can be consistent with the absence of arbitrage. They acknowledge, however, that their counterexamples are somewhat contrived and view them as economically implausible. These results leave the impression that the expectations hypothesis is only consistent with the absence of arbitrage in pathological cases of little practical interest.

In this paper, we argue that the expectations hypothesis is much more theoretically defensible than commonly believed. An important point often overlooked is that the Cox et al. (1981) results are developed in an economic setting where bonds are redundant securities. In the more realistic case where bond prices display security-specific variation (and there is mounting evidence of this), traditional forms of the expectations hypothesis can hold quite generally without arbitrage. Only weak conditions guaranteeing the existence of an equivalent martingale measure need be satisfied for the various forms of the expectations hypothesis to hold without arbitrage. A key

\* The Anderson School at the University of California at Los Angeles. I am grateful for the comments of Pedro Santa-Clara, Olivier Ledoit, Eduardo Schwartz, two anonymous referees, and the editor, René Stulz. All errors are my responsibility.

implication is that the expectations hypothesis cannot be ruled out on a priori theoretical grounds; the validity of the expectations hypothesis is purely an empirical issue.

The paper is organized as follows. Section I generalizes the Cox et al. (1981) framework to allow markets where bonds are not redundant securities and discusses the restrictions imposed on expected returns. Section II reviews recent empirical evidence of security-specific variation in fixed income prices. Section III makes concluding remarks.

### I. Restrictions on Expected Returns

In a complete markets setting, Cox et al. (1981) show that expected returns of discount bonds must satisfy specific restrictions to avoid arbitrage. They also identify restrictions imposed by the different forms of the expectations hypothesis and argue that most are inconsistent with the absence of arbitrage. In this section, we extend their framework to allow markets to be incomplete.

We assume that there are  $M < \infty$  distinct riskless discount bonds traded in the financial markets, where all coupon bonds can be represented as portfolios of these discount bonds. The market price  $P(t, T_i)$  of the  $i$ th discount bond follows the dynamics

$$\frac{dP(t, T_i)}{P(t, T_i)} = \alpha(t, T_i) dt + \delta'(t, T_i) dZ(t), \quad (1)$$

where  $T_i$  is the maturity date of the bond,  $Z(t)$  is a standard  $N$ -dimensional Brownian motion, and  $\alpha(t, T_i)$  and  $\delta(t, T_i)$  are maturity-specific functions that may depend on the current and past values of  $Z(t)$ . In this setting, the  $N$  individual Brownian motions in  $Z(t)$  play the role of state variables; an individual Brownian motion  $Z_i(t)$  may be a common factor affecting the returns on all discount bonds, or may be a security-specific factor affecting only the returns of a particular discount bond. To keep notation simple, we assume that the covariance matrix of bond returns is of full rank—for example,  $\min(N, M)$ .

In the complete markets case where  $N < M$ , Cox et al. (1981) show that an instantaneously riskless portfolio consisting of  $N + 1$  different discount bonds can be formed.<sup>1</sup> Since the expected return on this portfolio must equal the riskless rate  $r(t)$  to avoid arbitrage, this implies the following restriction on the expected return of the  $i$ th discount bond:

$$\alpha(t, T_i) = r(t) + \lambda'(t) \delta(t, T_i), \quad (2)$$

<sup>1</sup> As an alternative to using the terms *complete* and *incomplete* to describe the market, we could also refer to the case where  $N < M$  as *overspanned*, the case where  $N = M$  as *just spanned*, and the case where  $N > M$  as *underspanned*. Clearly, bonds are only redundant securities when the market is overspanned.

where  $\lambda(t)$  is a vector of maturity-independent market prices of risk. Note that both  $r(t)$  and  $\lambda(t)$  may be stochastic. Cox et al. then show that the various forms of the expectations hypothesis imply

$$\alpha(t, T_i) = r(t) + a\delta'(t, T_i)\delta(t, T_i), \quad (3)$$

where  $a = 1$  for the return-to-maturity expectations hypothesis,  $a = \frac{1}{2}$  for the yield-to-maturity expectations hypothesis, and  $a = 0$  for the local expectations hypothesis. Cox et al. argue that equations (2) and (3) can be jointly satisfied only when  $a = 0$ ; the other forms of the expectations hypothesis imply arbitrage. In contrast, McCulloch (1993) and Fisher and Gilles (1998) provide counterexamples showing that equations (2) and (3) can both be satisfied when  $a > 0$ .

Leaving aside the issue of whether equations (2) and (3) can be jointly satisfied when  $N < M$ , it is important to observe that the above analysis is applicable only when bonds are redundant securities. In the more general case where discount bond prices are affected by common economic factors as well as security-specific liquidity factors, discount bonds are no longer redundant securities. In this case, the following proposition shows that the linear restriction in equation (2) need not hold.

**PROPOSITION:** *If  $N \geq M$ , then equation (2) is not a necessary condition for the absence of arbitrage.*

*Proof:* Let  $\delta'(t)$  be the  $M \times N$  matrix formed by stacking the individual  $\delta'(t, T_i)$  vectors, and let  $\mathbf{0}$  be a vector of zeros. By assumption, the rank of  $\delta(t)$  is  $M$ . This implies that the homogeneous equation  $\delta(t)X = \mathbf{0}$  has only the trivial solution for the  $M$ -vector  $X$  and that the null space of  $\delta(t)$  is  $\mathbf{0}$ . Since the weights of a positive investment portfolio must sum to one and since the individual  $N$  Brownian motions are independent, the set of instantaneously riskless portfolios of the discount bonds is empty. Since this set is empty, the set of riskless arbitrage portfolios is a fortiori empty, even if equation (2) does not hold. Q.E.D.

The intuition behind this proposition is very simple. When discount bonds are not redundant securities, riskless arbitrage portfolios cannot be formed even if equation (2) does not hold. Thus, equation (2) is not a necessary condition for the absence of arbitrage.<sup>2</sup>

Clearly, if the restriction in equation (2) need not be satisfied, then the question of whether the different forms of the expectations hypothesis are consistent with equation (2) does not arise. This means that when  $N \geq M$ , the Cox et al. (1981) critique is not applicable, and there is no a priori theo-

<sup>2</sup> Although equation (2) need not be satisfied in general, it is clearly possible to find sets of additional assumptions under which this restriction might be satisfied. For example, given the similarity of equation (2) to the APT model of Ross (1976), additional structure could possibly be imposed allowing the APT to be applied to discount bonds in this framework.

retical ground for ruling out any of the various forms of the expectations hypothesis, provided certain weak conditions on  $\alpha(t, T_i)$  and  $\delta(t, T_i)$  are satisfied. These well-known conditions ensure the existence of a martingale measure and are described in Duffie (1996a).<sup>3</sup> The bottom line is that if  $N \geq M$ , the validity of a particular form of the expectations hypothesis can only be determined on the basis of empirical evidence.

## II. Are Fixed Income Markets Complete?

The issue of whether fixed income markets are complete is itself a fundamental empirical question. On one hand, factor analysis of the term structure generally suggests that three or four factors are able to capture most of the variation in bond returns.<sup>4</sup> It is important to observe, however, that the restriction on expected returns in equation (2) need not hold unless all of the variation in bond prices is completely captured by  $N < M$  factors. Recall that Campbell (1986) shows that differences between the alternative versions of the expectations hypothesis are very small; even a small amount of security-specific variation in bond prices may eliminate any possibility of arbitrage. Furthermore, security-specific variation in fixed income markets may be episodic or event related such as that experienced recently by a number of hedge funds; infrequent liquidity-related variation may be difficult to detect using standard factor analysis techniques.

On the other hand, a number of recent papers provide support for the hypothesis that fixed income markets are incomplete by documenting security-specific or liquidity-related anomalies in the pricing of fixed income securities. Examples of these papers include the following:

- Daves and Ehrhardt (1993) document large differences in the prices of principal and coupon STRIPS with identical maturity dates. Since the cash flows from these securities are identical in all states of the world, these pricing differences are attributed to security-specific liquidity features. Note that the prohibitive costs of short selling STRIPS to maturity rules out the possibility of exploiting these pricing differences to generate arbitrage profits.
- Amihud and Mendelson (1991) and Kamara (1994) find significant differences between the prices of highly liquid Treasury bills and older and less-liquid Treasury bonds with no remaining intermediate coupon payments and identical maturity dates. Again, since the cash flows from these securities are identical, the pricing differences can only be attributed to differences in liquidity.

<sup>3</sup> The existence of a martingale measure is a sufficient condition for the absence of arbitrage. Note that when markets are incomplete, there may be more than one equivalent martingale measure. For a discussion of equivalent martingale measures in incomplete markets, see He and Pearson (1991).

<sup>4</sup> Recent examples of this literature include Litterman and Scheinkman (1991) and Knez, Litterman, and Scheinkman (1994).

- Duffee (1996) reports evidence of significant idiosyncratic variation in the prices of Treasury bills. He finds that Treasury bills with less than two months to maturity are much more volatile than other Treasury bills, and that Treasury bills as a sector behave differently from other Treasury securities.
- Grinblatt and Longstaff (2000) find that Treasury notes and bonds trade at different prices from the corresponding portfolios of STRIPS formed by stripping these securities. The valuation differences between the portfolio of STRIPS and the fully constituted security are related to the liquidity of the Treasury note or bond.
- Longstaff (1992) examines the pricing of callable Treasury bonds relative to portfolios of noncallable Treasury bonds and STRIPS with otherwise identical cash flows. He finds that the callable bonds periodically trade at a premium to the less-liquid replicating portfolio, implying negative call option values.
- Boudoukh and Whitelaw (1991) study the pricing of the benchmark Japanese Government Bond relative to other Japanese Government bonds. They find that the highly liquid benchmark bond frequently trades at a large premium to nearly identical nonbenchmark bonds.
- Cornell and Shapiro (1990), Cornell (1993), and Jordan and Jordan (1997) examine the pricing of a set of Treasury bonds that became special in the repo market and document large price movements in these securities directly attributable to their special status. Discussions with government bond dealers indicate that more than a dozen Treasury issues may be special at any point in time and that any Treasury bond could potentially become special, not just on-the-run bonds. Special repo rates are also studied by Duffie (1996b).
- Elton and Green (1998) study the pricing of Treasury bonds and find that highly liquid on-the-run Treasury notes and bonds often trade at a premium to other less-liquid Treasury securities.

These examples provide evidence of price variation in fixed income securities due to security-specific features such as liquidity. Though not conclusive, these examples provide a serious challenge to the view that fixed income markets are complete.

### III. Conclusion

If fixed income markets are incomplete, then traditional forms of the expectations hypothesis can hold without arbitrage. This means that these forms of the expectations hypothesis cannot be ruled out on a priori theoretical grounds; the validity of the expectations hypothesis is ultimately an empirical issue.

### REFERENCES

- Amihud, Yakov, and Haim Mendelson, 1991, Liquidity, maturity, and the yields on U. S. Treasury securities, *Journal of Finance* 46, 1411–1425.

- Boudoukh, Jacob, and Robert F. Whitelaw, 1991, The benchmark effect in the Japanese government bond market, *Journal of Fixed Income* September, 52–59.
- Campbell, John, 1986, A defense of traditional hypotheses about the term structure of interest rates, *Journal of Finance* 41, 183–193.
- Cornell, Bradford, 1993, Adverse selection, squeezes, and the bid-ask spread on Treasury securities, *Journal of Fixed Income* June, 39–47.
- Cornell, Bradford, and Alan C. Shapiro, 1990, The mispricing of U. S. Treasury bonds: A case study, *Review of Financial Studies* 2, 297–310.
- Cox, John, Jonathan Ingersoll, and Stephen Ross, 1981, A reexamination of traditional hypotheses about the term structure of interest rates, *Journal of Finance* 36, 321–346.
- Daves, Phillip, and Michael C. Ehrhardt, 1993, Liquidity, reconstitution, and the value of U. S. Treasury strips, *Journal of Finance* 48, 315–329.
- Duffee, Gregory, 1996, Idiosyncratic variation of Treasury bill yields, *Journal of Finance* 51, 527–551.
- Duffie, Darrell, 1996a, *Dynamic Asset Pricing Theory*, 2nd ed. (Princeton University Press, Princeton, N.J.).
- Duffie, Darrell, 1996b, Special repo rates, *Journal of Finance* 51, 493–526.
- Elton, Edwin, and T. Clifton Green, 1998, Tax and liquidity effects in pricing government bonds, *Journal of Finance* 53, 1533–1562.
- Fisher, Irving, 1896, Appreciation and interest, *Publications of the American Economic Association* 11, 21–29.
- Fisher, Mark, and Christian Gilles, 1998, Around and around: The expectations hypothesis, *Journal of Finance* 53, 365–383.
- Grinblatt, Mark, and Francis Longstaff, 2000, Financial innovation and the role of derivative securities: An empirical analysis of the U.S. Treasury's STRIPS program, *Journal of Finance*, forthcoming.
- He, Hua, and Neil Pearson, 1991, Consumption and portfolio policies with incomplete markets and short-sale constraints: The infinite dimensional case, *Journal of Economic Theory* 54, 259–304.
- Jordan, Bradford, and Susan Jordan, 1997, Special repo rates: An empirical analysis, *Journal of Finance* 52, 2051–2072.
- Kamara, Avi, 1994, Liquidity, taxes, and short-term Treasury yields, *Journal of Financial and Quantitative Analysis* 29, 403–417.
- Knez, Peter, Robert Litterman, and Jose Scheinkman, 1994, Explorations into factors explaining money market returns, *Journal of Finance* 49, 1861–82.
- Litterman, Robert, and Jose Scheinkman, 1991, Common factors affecting bond returns, *Journal of Fixed Income* 1, 54–61.
- Longstaff, Francis, 1992, Are negative option prices possible? The callable U.S. Treasury bond puzzle, *Journal of Business* 65, 571–592.
- McCulloch, J. Huston, 1993, A reexamination of traditional hypotheses about the term structure: A comment, *Journal of Finance* 48, 779–789.
- Ross, Stephen, 1976, The arbitrage theory of capital asset pricing, *Journal of Economic Theory* 13, 341–360.