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Portfolio selection with stable distributed returns*

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Abstract. This paper analyzes and discusses the stable distributional approach in portfolio choice theory. We consider different hypotheses of portfolio selection with stable distributed returns and, more generally, with heavy-tailed distributed returns. In particular, we examine empirical differences among the optimal allocations obtained with the Gaussian and the stable non-Gaussian distributional assumption for the financial returns. Finally, we compare performances among stable multivariate models.

1 Introduction

The purpose of this paper is to describe and compare stable portfolio selection models. We first consider portfolio choice models coherent with the asymptotic behavior of the return data and consistent with the maximization of the expected utility. Secondly, we examine empirical optimal allocation differences among Gaussian and heavy tailed models. Finally, we analyze and compare the performance among some of the proposed portfolio choice models.

It is well-known that asset returns are not normally distributed, but many of the concepts in theoretical and empirical finance developed over the past decades rest upon the assumption that asset returns follow a normal distribution. The fundamental work of Mandelbrot (1963a–b, 1967a–b) and Fama (1963, 1965a–b) has sparked considerable interest in studying the empirical distribution of financial assets. The excess kurtosis found in Mandelbrot's and Fama's investigations led them to reject the normal assumption and to pro-

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pose the stable Paretian distribution as a statistical model for asset returns. The Fama and Mandelbrot's conjecture was supported by numerous empirical investigations in the subsequent years, (see Mittnik, Rachev and Paolella (1997) and Rachev and Mittnik (2000)).

In this work, we first study models which consider the asymptotic distributional behavior of data. The behavior, generally stationary over time of returns, the Central Limit Theorem and Central Pre-limit Theorem (see Klebanov, Rachev, Szekely (2000) and Klebanov, Rachev, Safarian (2001)) for normalized sums of i.i.d. random variables (see Zolatorev (1986)) theoretically justify the stable Paretian approach proposed by Mandelbrot and Fama. The practical and theoretical appeal of the stable non-Gaussian approach is given by its attractive properties that are almost the same as the normal one. A relevant desirable property of a stable distributional assumption is that stable distributions have domain of attraction. Therefore, any distribution in the domain of attraction of a specified stable distribution will have properties close to those of the stable distribution. Another attractive aspect of the stable Paretian assumption is the stability property, i.e. stable distributions are stable with respect to summation of i.i.d. random stable variables. Hence, the stability governs the main properties of the underlying distribution (detailed accounts for theoretical aspects of stable distributed random variables can be found in Samorodnitsky and Taqqu (1994) and Janicki and Weron (1994)). Here, we find an equivalent parameterization of the stable laws (in terms of some moments) that characterizes the stable laws used in portfolio choice theory. Using this parametrization and stochastic dominance properties of stable laws we can characterize the efficient frontiers of non-satiable and risk averse investors. Moreover, we recall three admissible fund separation models where the asset returns are in the domain of attraction of stable laws (see Ortobelli, Rachev and Schwartz (2000), Ortobelli, Huber, Rachev and Schwartz (2001), Ortobelli and Rachev (2001)). We first consider the portfolio allocation among n α-stable sub-Gaussian distributed risky assets (with $1 < \alpha < 2$) and the riskless one. The joint stable sub-Gaussian family is an elliptical family. Hence, as argued by Owen and Rabinovitch (1984), in this case, we can use a mean-dispersion analysis. The resulting efficient frontier is formally the same as Markowitz-Tobin's mean-variance analysis, but, instead of considering the variance as a risk parameter, we have to consider the scale parameter of the stable distributions. All the stable parameters can be estimated (see Rachev and Mittnik (2000) and the references therein). In order to consider the possible asymmetry of asset returns, we describe a three-fund separation model for returns in the domain of attraction of a stable law. In case of asymmetry, the model results from a new stable version of the Simaan's model, see Simaan (1993). In case of symmetry of returns, we obtain a version of a model recently studied by Götzenberger, Rachev and Schwartz (1999), that can also be viewed as a particular version of the two-fund separation of Fama's (1965b) model. In this case too, it is possible to estimate all parameters with a maximum likelihood method (see Rachev and Mittnik (2000) and the references therein). Finally, the last model proposed deals with the case of optimal allocation among stable distributed portfolios with different indexes of stability. To overcome the difficulties of the most general case of the stable law, we introduce a $k+1$ fund separation model. Then, we show how to express the model's multi-parameter admissible frontier.

Secondly, we analyze empirical optimal allocation differences among

Gaussian and stable non-Gaussian models. The investment allocation problem consists of the maximization of the mean minus a measure of portfolio risk. We propose a mean risk analysis that facilitates the interpretation of the results. In the allocation problem, we consider as the risk measure the expected value of a power absolute deviation. When the power is equal to two, we obtain the classic quadratic utility functional. We first examine the optimal allocation between a riskless return and a risky stable distributed return, then we compare the allocation obtained with the Gaussian and the stable non-Gaussian distributional assumption for the risky return. We choose the 6% annual rate as riskless return. As a possible risky asset, we consider the stock indexes S&P500, DAX30 and CAC40. The models' parameters are estimated in Khindanova, Rachev and Schwartz (1999). This first comparison shows that there are significant differences in the allocation when the data fit the stable non-Gaussian or the normal distributions. Secondly, we analyze the optimal allocation among a riskless return and 23 risky stable distributed returns, then we compare the allocation obtained with the Gaussian and the stable sub-Gaussian distributional assumption for the risky returns. The model parameters are estimated using the methodology based on the moment method. We show that there are significant differences in the allocation when the data fit the stable sub-Gaussian or the normal distributions. By comparing the joint normal distribution with the joint stable sub-Guassian law one, it has occurred that the results performed under the examined optimal allocation problems are substantially different. In particular, the stable market portfolio is generally less risky than the Gaussian market portfolio. This intuitive result is confirmed by the comparison of the optimal allocations when different distributional hypotheses are assumed. Therefore, the investors who fit the data with the stable distributions are generally more risk preserving than the investors who fit the data with the normal laws because they consider the component of risk due to the heavy tails.

Finally, we propose a performance comparison among the mean-variance approach and some stable sub-Gaussian models considering the same data set of previous empirical analysis. For this purpose we analyze two allocation problems for investors with different risk aversion coefficients. We determine the efficient frontiers given by the minimization of the dispersion measures for different levels of expected value. Each investor, characterized by his/her utility function, will prefer the mean-dispersion model which maximizes his/her expected utility on the efficient frontier. The portfolios obtained with this methodology represent the optimal investors' choices in the different approaches.

Section 2 analyzes and introduces the asymptotic distributional assumption. In Section 3 we compare the stable non-Gaussian approach with the Gaussian one. Section 4 proposes a performance comparison among stable sub-Gaussian and mean-variance models. In the last section, we briefly summarize the results.

2 Portfolio choice models in the domain of attraction of stable laws

In this section we study the portfolio choice problem analyzing the asymptotic behavior of data. In particular, we consider portfolio choice problem among $n+1$ assets: n of those assets are risky with returns (continuously compounded) $r = [r_1, \ldots, r_n]'$, and the $(n+1)th$ asset has risk-free return z_0.

The recent crashes observed in the stock market showed that the stock returns are more volatile than those predicted by the models with finite variance of the asset returns. In the empirical financial literature, it is well documented that the asset returns have a distribution whose tail is heavier than that of the distributions with finite variance, i.e.,

$$P(|r_i| > x) \sim x^{-\alpha_i} L_i(x) \quad \text{as } x \to \infty, \tag{1}$$

where $0 < \alpha < 2$ and $L_i(x)$ is a slowly varying function at infinity, i.e.,

$$\lim_{x\to\infty} \frac{L_i(cx)}{L_i(x)} \to 1 \quad \text{for all } c > 0,$$

see Rachev and Mittnik (2000) and the references therein. In particular, in the data observed until now $1 < \alpha < 2$. The constrain $1 < \alpha < 2$ and the relation (1) imply that returns r_i admit finite mean and infinite variance. The tail condition in (1) also implies that the vector of returns $r = [r_1, \dots, r_n]'$ is in the domain of attraction of $(\alpha_1, \dots, \alpha_n)$-stable law. That is, given T i.i.d (independent and identically distributed) observations on r, namely

$$r^{(t)} = [r_1^{(t)}, \dots, r_n^{(t)}]' \quad t = 1, 2, \dots, T,$$

then, there exist normalizing constants

$$a^{(T)} = (a_1^{(T)}, \dots, a_n^{(T)}) \in R_+^n \quad \text{and} \quad b^{(T)} = (b_1^{(T)}, \dots, b_n^{(T)}) \in R^n,$$

such that

$$\left(\sum_{i=1}^{T} \frac{r_1^{(i)}}{a_1^{(T)}} + b_1^{(T)}, \dots, \sum_{i=1}^{T} \frac{r_1^{(i)}}{a_n^{(T)}} + b_n^{(T)} \right) \xrightarrow{d} S(\alpha_1, \dots, \alpha_n) \quad \text{as } T \to \infty, \tag{2}$$

where $S(\alpha_1, \dots, \alpha_n)$ is $(\alpha_1, \dots, \alpha_n)$-stable random vector. This convergence result is a consequence of the stationary behavior of returns and of the Central Limit Theorem for normalized sums of i.i.d. random variables which determines the domain of attraction of each stable law (see Zolatorev (1986)). Therefore, any distribution in the domain of attraction of a specified stable distribution will have properties close to those of the stable distribution. The constants $a_j^{(T)}$ in (2) have the form

$$a_j^{(T)} = T^{1/\alpha_j} L_j(T),$$

where $L_j(T)$ are slowly varying functions as $T \to \infty$.

Each component of $S(\alpha_1, \dots, \alpha_n) = (s_1, \dots, s_n)$ has a Pareto-Lévy stable distribution, i.e., its characteristic function is given by

$$\Phi_j(t) = \begin{cases} \exp[-\sigma_j^{\alpha_j} |t|^{\alpha_j} (1 - i\beta_j sgn(t) \tan(\frac{\pi\alpha_j}{2})) + i\mu_j t] & \text{if } \alpha_j \neq 1 \\ \exp[-\sigma_j |t| (1 + i\beta_j \frac{2}{\pi} sgn(t) \log|t|) + i\mu_j t] & \text{if } \alpha_j = 1 \end{cases}, \tag{3}$$

where $\alpha_j \in (0, 2)$ is the so-called stable (tail) index of s_j, $\sigma_j > 0$ is the scale (or dispersion) parameter, $\beta_j \in [-1, 1]$ is a skewness parameter and μ_j is a location parameter. When $\alpha_j > 1$ the location parameter μ_j is the mean. However, there is a considerable debate in literature concerning the applicability of α-stable distributions as they appear in Lévy's central limit theorems. A serious drawback of Lévy's approach is that in practice one can never know whether the underlying distribution is heavy tailed, or just has a long but truncated tail. Limit theorems for stable laws are not robust with respect to truncation of the tail or with respect to any change from light to heavy tail, or conversely. Based on finite samples, one can never justify the specification of a particular tail behavior. Hence, one cannot justify the applicability of classical limit theorems in probability theory. Therefore, instead of relying on limit theorems, we can use the so-called pre-limit theorem which provides an approximation for distribution functions in case the number of observation T is "large" but not too "large" (see Klebanov, Rachev, Szekely (2000) and Klebanov, Rachev, Safarian (2001)). In particular the "pre-limiting" approach helps to overcome the drawback of Lévy-type central limit theorems. As a matter of fact, we can assume that returns are bounded "far away", say daily returns cannot be outside the interval $[-0.5, 0.5]$. Thus, considering the empirical observation on asset returns, we can assume that the asset returns r_i are truncated α_i-stable distributed with support, $[-0.5, 0.5]$. Even if the returns will be attracted by the CLT to the Gaussian law, pre-limit theorems show that for any reasonable T the truncated stable laws will be attracted to the stable laws. Therefore, it is plausible assuming that the vector of returns $r = [r_1, \dots, r_n]'$ is in the domain of attraction of a n-dimensional $(\alpha_1, \dots, \alpha_n)$-stable law.

Recall that when unlimited short selling is allowed, every portfolio of returns is a linear combination of the constant riskless return z_0, and the risky returns r_i, (i.e. $(x_0 z_0 + \sum_{i=1}^{n} x_i r_i)$ where $(x_0, x) \in R^{n+1}$, $x \in R^n$). Therefore, the distribution functions of all admissible portfolios (of returns) belong to a translation and scalar invariant family[1] determined by a finite number of parameters. Most distributional approaches in portfolio selection theory assume that the distribution functions of portfolios belong to a translation and scalar invariant family, denoted $\sigma\tau_k(\bar{a})$, with the following characteristics:

A) Every distribution F_X belonging to $\sigma\tau_k(\bar{a})$ is identified by k parameters $(m_X, \sigma_X, a_{1,X}, \dots, a_{k-2,X}) \in A \subseteq R^k$ where m_X is the mean of X, σ_X is the positive scale parameter of X. We assume that the class $\sigma\tau_k(\bar{a})$ is *weakly determined* by its parameterization. That is the equality

$$(m_X, \sigma_X, a_{1,X}, \dots, a_{k-2,X}) = (m_Y, \sigma_Y, a_{1,Y}, \dots, a_{k-2,Y}),$$

implies that $F_X \overset{d}{=} F_Y$ but the converse is not necessarily true.

B) For every admissible real t, the distribution function $F_X \in \sigma\tau_k(\bar{a})$ has the same parameters, except the mean, as $F_{X+t} \in \sigma\tau_k(\bar{a})$ (the translated of F_X).

[1] Recall that a parametric family $\Im$ of distribution functions is *translation invariant* if whenever the distribution $F_X(x) = P(X \leq x)$ belongs to $\Im$, then for every $t \in R$, $F_{X+t} \in \Im$ as well. Similarly, we say that a family $\Im$ is *scalar invariant* if whenever the distribution F_X belongs to $\Im$, then for every $\alpha > 0$, $F_{\alpha X}$ belongs to $\Im$ as well.

C) For every admissible positive α, the distribution function $F_X \in \sigma\tau_k(\bar{a})$ has the same parameters of the distribution $F_{\alpha X} \in \sigma\tau_k(\bar{a})$ except for the mean that is αm_X and the scale parameter that is $\alpha\sigma_X$ (where m_X and σ_X are respectively the mean and the scale parameter of the random variable X).

When the distribution functions of portfolios belong to a $\sigma\tau_k(\bar{a})$ class, we can identify the following stochastic dominance relations[2] among portfolios (see Ortobelli (2001) and Ortobelli, Huber, Rachev, Schwartz (2001)).

Theorem 1. *Suppose the distribution functions of all random portfolios belong to the same class $\sigma\tau_k(\bar{a})$. Let $w'r$ and $y'r$ be a couple of random portfolios unbounded from below respectively determined by the parameters*

$$(m_{w'r}, \sigma_{w'r}, a_{1,p}, \ldots, a_{k-2,p}) \quad \text{and} \quad (m_{y'r}, \sigma_{y'r}, a_{1,p}, \ldots, a_{k-2,p}).$$

Then, the following properties are equivalent

1. *$E(w'r) \geq E(y'r)$, $\sigma_{w'r} \leq \sigma_{y'r}$ with at least one inequality strict.*
2. *$w'r$ SSD $y'r$ and $y'r \stackrel{d}{=} w'r - (E(w'r) - E(y'r)) + \varepsilon$ and $E(\varepsilon/w'r) = 0$.*

The tail behavior of returns implies that the vector of returns $r = [r_1, \ldots, r_n]'$ is in the domain of attraction of a n-dimensional $(\alpha_1, \ldots, \alpha_n)$-stable law. In order to express a multi-parameter choice in portfolio selection theory coherent with the empirical evidence and consistent with the expected utility maximization, we need the *asymptotic distributional assumption* consisting in:

1. (*Heavy tailedness assumption*) Portfolios $x'r$ are random variables belonging to L^p with $1 < p \leq 2$ and the return vector $r = [r_1, \ldots, r_n]'$ is in the domain of attraction of $(\alpha_1, \ldots, \alpha_n)$-stable law. The assumption $1 < \alpha_i \leq 2$ is supported by increasing empirical results as shown by Mandelbrot (1963a–b, 1967a–b), Fama (1963, 1965a–b), Mittnik, Rachev and Paolella (1997), Rachev and Mittnik (2000).
2. (*Consistency with the expected utility maximization*) The distributions of the portfolio returns $x'r$ belong to the same $\sigma\tau_k(\bar{a})$ class of distribution functions.

Under these assumptions, as for Theorem 1, we obtain an admissible frontier for non-satiable and non-satiable risk averse investors[3].

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[2] Recall that the portfolio $x'Z$ first order stochastically dominates (FSD) $y'Z$ if and only if for every increasing utility functions u, $E(u(x'Z)) \geq E(u(y'Z))$ and the inequality is strict for some u. Equivalently $x'Z$ FSD $y'Z$ if and only if $P(x'Z \leq t) \leq P(y'Z \leq t)$ for every real t and strictly for some t. Analogously, we say that $x'Z$ second order stochastically dominates (SSD) $y'Z$, if and only if for every increasing, concave utility functions u, $E(u(x'Z)) \geq E(u(y'Z))$ and the inequality is strict for some u. Equivalently, $x'Z$ SSD $y'Z$, if and only if $\int_{-\infty}^{t} F_{x'Z}(v)\,dv \leq \int_{-\infty}^{t} F_{y'Z}(v)\,dv$ for every real t and strictly for some t (see, among others, Fishburn (1964), Hanoch and Levy (1969), Levy (1992)).

We also say that $x'Z$ Rothschild Stiglitz stochastically dominates (R-S) $y'Z$ if and only if for every concave utility functions u, $E(u(x'Z)) \geq E(u(y'Z))$ and the inequality is strict for some u. Equivalently $x'Z$ R-S $y'Z$ if and only if $E(x'Z) = E(y'Z)$ and $x'Z$ SSD $y'Z$ (see Rothschild and Stiglitz (1970)). However, there exist many other stochastic orders used in Economics and Finance, see, among others, Shaked and Shanthikumar (1994).

[3] Recall that non-satiable agents are investors with increasing utility function. Instead, risk averse decision makers are investors with concave utility functions.

A simpler way to express the asymptotic behavior of data consists in considering every portfolio in the domain of attraction of a Pareto-Lévy α stable distribution with $\alpha > 1$. Given that, we implicitly assume that all optimal choices are identified by four parameters of the underlined stable law. Therefore, every portfolio $x'r$ can be well approximated by a stable distribution, i.e. we can assume:

$$x'r + (1 - x'e)z_0 \stackrel{d}{=} S_{\alpha(x)}(\sigma(x), \beta(x), \mu(x)), \tag{4}$$

where z_0 is the riskless return, $\alpha(x) \in \left(\min_{1<i<n} \alpha_i, 2\right)$ is the index of stability, $\alpha_j > 1$ is the index of stability of the jth asset return, $\sigma(x)$ is the scale parameter, $\mu(x) = x'E(r) + (1 - x'e)z_0$ is the mean and $\beta(x)$ is the skewness parameter. Properties of $\sigma\tau_4(\bar{a})$ class are verified with this parameterization, so according to Theorem 2 every risk averse investor will choose a portfolio weight, solution of the following constrained problem

$$\begin{aligned} &\min_x \sigma(x) \text{ subject to} \\ &x'E(r) + (1 - x'e)z_0 = m \\ &\beta(x) = \beta^* \\ &\alpha(x) = \alpha^* \end{aligned}, \tag{5}$$

for some m, β^*, α^*. In this case, we are not able to find a closed form of the efficient frontier because we do not know a priori the joint distribution of the asset returns. In order to overcome this problem, we could consider another admissible parameterization of the stable distribution for problem (4). For example, we can prove that the mean $\mu(x) = x'E(r) + (1 - x'e)z_0$, the scale parameter $s(x) = E(|x'r - x'E(r)|)$ and the fundamental ratios $\rho_1(x) = \frac{E(|x'r - x'E(r)|^{q_1})}{(s(x))^{q_1}}$ and $\rho_2(x) = \frac{E((x'r - x'E(r))^{\langle q_2 \rangle})}{(s(x))^{q_2}}$ where $q_1, q_2 \in \left(1, \lim_{1<i<n} \alpha_i\right)$; represent a parameterization which verifies the properties of $\sigma\tau_4(\bar{a})$ class[4]. In fact, first observe that $\rho_1(x)$ and $\rho_2(x)$ do not depend on portfolio mean $\mu(x)$ and scale parameter $\sigma(x)$ because

$$|x'r - x'E(r)|^{q_1} \stackrel{d}{=} \sigma(x)^{q_1} |S_{\alpha(x)}(1, \beta(x), 0)|^{q_1},$$

and also

$$[x'r - x'E(r)]^{\langle q_2 \rangle} \stackrel{d}{=} \sigma(x)^{q_2} [S_{\alpha(x)}(1, \beta(x), 0)]^{\langle q_2 \rangle}.$$

Thus, as a consequence of Property 1.2.17 in Samorodnisky and Taqqu (1994)

$$\begin{aligned} \rho_1(x) &= \frac{E(|x'r - x'E(r)|^{q_1})}{(s(x))^{q_1}} \\ &= K \frac{\Gamma\left(1 - \frac{q_1}{\alpha(x)}\right) \cos\left(\arctan\left(\beta(x) \tan\left(\frac{\pi\alpha(x)}{2}\right)\right) \frac{q_1}{\alpha(x)}\right)}{\left(\Gamma\left(1 - \frac{1}{\alpha(x)}\right) \cos\left(\arctan\left(\beta(x) \tan\left(\frac{\pi\alpha(x)}{2}\right)\right) \frac{1}{\alpha(x)}\right)\right)^{q_1}}, \end{aligned}$$

[4] The symbology $x^{\langle t \rangle}$ stands for $sgn(x)|x|^t$.

where K is a constant that depends only on q_1. Hence, for every $q_1 \in \left(1, \min_{1<i<n} \alpha_i\right)$ and for every fixed β, $\rho_1(x)$ is a decreasing function of $\alpha(x)$ on the existence interval. Moreover, $\rho_1(x)$ is an even function of $\beta(x)$ and it decreases in $|\beta(x)|$ for fixed $\alpha(x) \in \left(\min_{1<i<n} \alpha_i, 2\right)$. Instead, $\rho_2(x)$ is an increasing odd function of $\beta(x)$ for every $q_2 \in \left(1, \min_{1<i<n} \alpha_i\right)$ and for every fixed $\alpha(x) \in \left(\min_{1<i<n} \alpha_i, 2\right)$. These relations imply that $\rho_1(x)$ and $\rho_2(x)$ uniquely determinate $\alpha(x)$ and $\beta(x)$. Then, under the assumption (4), every risk averse investor will choose a portfolio weight, solution of the following constrained problem

$$\begin{aligned}
&\min_x E(|x'r - x'E(r)|) \text{ subject to} \\
&x'E(r) + (1 - x'e)z_0 = m \\
&\frac{E(|x'r - x'E(r)|^{q_1})}{(s(x))^{q_1}} = \rho_1 \\
&\frac{E((x'r - x'E(r))^{\langle q_2 \rangle})}{(s(x))^{q_2}} = \rho_2
\end{aligned} \qquad (6)$$

for some m, ρ_1, ρ_2. Differently from problem (5), problem (6) does not require the knowledge of the joint distribution of asset returns but it is still computationally too complex. Generally, in order to identify the efficient frontier and reduce the number of parameters, we assume that $\alpha_j = \alpha$ for all $j = 1, \ldots, n$. Observe that stable distributions are stable with respect to summation of i.i.d. random stable variables and the vector of returns $r = [r_1, \ldots, r_n]'$ is α-stable distributed with $\alpha > 1$ if and only if all linear combinations are stable (see Samordinsky and Taqqu (1994) Theorems 2.1.2 and 2.1.5). In this case the joint characteristic function of returns is given by

$$\Phi_r(t) = \exp\left(-\int_{S_n} |t's|^{\alpha}\left(1 - isgn(t's)\tan\left(\frac{\pi\alpha}{2}\right)\right)\gamma(ds) + it'\mu\right),$$

where α is the index of stability, $\gamma(ds)$ is the spectral measure concentrated on $S_n = \{s \in R^n / \|s\| = 1\}$.

Thus, when the vector of returns is α stable distributed (with $\alpha > 1$), every portfolio $x'r + (1 - x'e)z_0$ (except the riskless return i.e. $x = 0$) is distributed as

$$x'r + (1 - x'e)z_0 \stackrel{d}{=} S_{\alpha}(\sigma(x), \beta(x), \mu(x)),$$

where

$$\mu(x) = x'E(r) + (1 - x'e)z_0;$$

$$\sigma(x) = \left(\int_{S_n} |x's|^{\alpha}\gamma(ds)\right)^{1/\alpha} \quad \text{and} \quad \beta(x) = \frac{\int_{S_n} |x's|^{\alpha} sgn(x's)\gamma(ds)}{(\sigma(x))^{\alpha}}$$

are respectively the mean, the scale parameter and the skewness parameter of the portfolio $x'r + (1 - x'e)z_0$. Under this distributional assumption, every risk averse investor will choose a portfolio weight, solution of the following constrained problem

$$\begin{aligned}&\min_{x} \sigma(x) \text{ subject to}\\ &x'E(r) + (1 - x'e)z_0 = m\\ &\beta(x) = \beta^*\end{aligned} \tag{7}$$

for some m and β^*. In order to determine estimates of the scale parameter and of the skewness parameter, we can consider the tail estimator for the index of stability α and the estimator for the spectral measure $\gamma(ds)$ proposed by Rachev and Xin (1993) and Cheng and Rachev (1995). However, even if the estimates of the scale parameter and the skewness parameter are computationally feasible, they require numerical calculations. Thus, model (7) does not present an easy applicability from an empirical point of view. Similarly to problem (6), we can fix $q < \alpha$ and propose a different representation based on the moments type constrains. Therefore, instead of model (7), we obtain the following constrained problem

$$\begin{aligned}&\min_{x} E(|x'r - x'E(r)|) \text{ subject to}\\ &x'E(r) + (1 - x'e)z_0 = m\\ &\frac{E((x'r - x'E(r))^{\langle q_2 \rangle})}{(E(|x'r - x'E(r)|))^{q_2}} = \rho_2\end{aligned} \tag{8}$$

for some m and ρ_2. Optimization problems (8) and (6) can be used in a more general setting than optimization problems (5), (7). In fact, a priori other classes of distribution functions (not only stable distributions) for returns uniquely determined by the parameters $m(x)$, $s(x)$, $\rho_1(x)$ and $\rho_2(x)$ could exist. Next, in order to overcome the intrinsic difficulties of the problems (5), (6), (7) and (8), we analyze different fund separation models that consider the asymptotic distributional assumption.

2.1 *The sub-Gaussian stable model*

Assume the vector of returns $r = [r_1, \ldots, r_n]'$ is sub-Gaussian α-stable distributed with $1 < \alpha < 2$. Then, the characteristic function of r has the following form

$$\Phi_r(t) = E(\exp(it'r)) = \exp(-(t'Qt)^{\alpha/2} + it'\mu), \tag{9}$$

where $Q = \left[\frac{R_{i,j}}{2}\right]$ is a positive definite $(n \times n)$-matrix, $\mu = E(r)$ is the mean vector, and $\gamma(ds)$ is the spectral measure with support concentrated on $S_n = \{s \in R^n / \|s\| = 1\}$. The term $R_{i,j}$ is defined by

$$\frac{R_{i,j}}{2} = [\widetilde{r}_i, \widetilde{r}_j]_\alpha \|\widetilde{r}_j\|_\alpha^{2-\alpha}, \tag{10}$$

where $\widetilde{r}_j = r_j - \mu_j$ are the centralized return, the covariation $[\widetilde{r}_i, \widetilde{r}_j]_\alpha$ between two jointly symmetric stable random variables $\widetilde{r}_i$ and $\widetilde{r}_j$ is given by

$$[\widetilde{r}_i, \widetilde{r}_j]_\alpha = \int_{S_2} s_i |s_j|^{\alpha-1} sgn(s_j)\gamma(ds),$$

in particular, $\|\widetilde{r}_j\|_\alpha = \left(\int_{S_2} |s_j|^\alpha \gamma(ds)\right)^{1/\alpha} = \left([\widetilde{r}_j, \widetilde{r}_j]_\alpha\right)^{1/\alpha}$. Here the spectral measure $\gamma(ds)$ has support on the unit circle S_2.

This model can be considered as a special case of Owen-Rabinovitch's elliptical model (see Owen and Rabinovitch (1984)). However, no estimate procedure of the model parameters is given in the elliptical models with infinite variance. In our approach we use (9) and (10) to provide a statistical estimator of the stable efficient frontier. To estimate the efficient frontier for returns given by (9), we need to consider an estimator for the mean vector μ and an estimator for the dispersion matrix Q. The estimator of μ is given by the vector $\hat{\mu}$ of sample averages. Using lemma 2.7.16 in Samorodnitsky, Taqqu (1994) we can write for every p such that $1 < p < \alpha$

$$\frac{[\widetilde{r}_i, \widetilde{r}_j]_\alpha}{\|\widetilde{r}_j\|_\alpha^\alpha} = \frac{E(\widetilde{r}_i \widetilde{r}_j^{\langle p-1 \rangle})}{E(|\widetilde{r}_j|^p)}, \tag{11}$$

where the scale parameter σ_j can be written $\|\widetilde{r}_j\|_\alpha = \sigma_j$. It can be approximated by the moment method suggested by Samorodnitsky, Taqqu (1994) Property 1.2.17 in the case $\beta = 0$

$$\sigma_j^p = \|\widetilde{r}_j\|_\alpha^p = \frac{E(|r_j - \mu_j|^p) p \int_0^{+\infty} u^{-p-1} \sin^2 u \, du}{2^{p-1} \Gamma\left(1 - \frac{p}{\alpha}\right)}, \tag{12}$$

It follows

$$\frac{R_{i,j}}{2} = \sigma_j^2 \frac{E(\widetilde{r}_i \widetilde{r}_j^{\langle p-1 \rangle})}{E(|\widetilde{r}_j|^p)}.$$

The above suggests the following estimator $\hat{Q} = \left[\frac{\hat{R}_{i,j}}{2}\right]$ for the entries of the unknown covariation matrix Q

$$\frac{\hat{R}_{i,j}}{2} = \hat{\sigma}_j^2 \frac{\sum_{k=1}^N \widetilde{r}_i^{(k)} (\widetilde{r}_j^{(k)})^{\langle p-1 \rangle}}{\sum_{k=1}^N |\widetilde{r}_j^{(k)}|^p}, \tag{13}$$

where the σ_j^2 is estimated as follows

$$\hat{\sigma}_j^2 = \frac{\hat{R}_{j,j}}{2} = \left(\frac{\frac{1}{N} \sum_{k=1}^N |\widetilde{r}_j^{(k)}|^p p \int_0^{+\infty} u^{-p-1} \sin^2 u \, du}{2^{p-1} \Gamma\left(1 - \frac{p}{\alpha}\right)} \right)^{2/p}. \tag{14}$$

The rate of convergence of the empirical matrix $\hat{Q} = \left[\frac{\hat{R}_{i,j}}{2}\right]$ to the unknown matrix Q (to be estimated), will be faster, if p is as large as possible, see Rachev (1991).

Now, let us recall that our portfolio satisfies the relation

$$x'r \stackrel{d}{=} S_\alpha(\sigma_{x'r}, \beta_{x'r}, m_{x'r})$$

and furthermore, $W = z_0$ when $x = 0$, otherwise $W = x'r + (1 - x'e)z_0 \stackrel{d}{=} S_\alpha(\sigma_{x'r}, \beta_{x'r}, m_W)$, where α is the index of stability, $\sigma_{x'r} = \sqrt{x'Qx}$ is the scale (dispersion) parameter, $\beta_{x'r} = 0$ is the skewness parameter and $m_W = x'E(r) + (1 - x'e)z_0$ is the mean of W. In particular, every sub-Gaussian α-stable family is a particular $\sigma\tau_2(m, \sigma)$ class.

In view of what stated before, when the returns $r = [r_1, \dots, r_n]'$ are jointly sub-Gaussian α-stable distributed, every risk averse investor will choose an optimal portfolio among all portfolio solutions of the following optimization problem:

$$\begin{aligned} &\min_x x'Qx \text{ subject to} \\ &x'\mu + (1 - x'e)z_0 = m_W \end{aligned} ; \tag{15}$$

for some given mean m_W where $W = x'r + (1 - x'e)z_0$. Thus, every optimal portfolio that maximizes a given concave utility function u,

$$\max_x E(u(x'r + (1 - x'e)z_0)),$$

belongs to the mean-dispersion frontier

$$\sigma = \begin{cases} \frac{m - z_0}{\sqrt{(\mu - ez_0)'Q^{-1}(\mu - ez_0)}} & \text{if } m \geq z_0 \\ \frac{z_0 - m}{\sqrt{(\mu - ez_0)'Q^{-1}(\mu - ez_0)}} & \text{if } m < z_0 \end{cases} ; \tag{16}$$

where $\mu = E(r)$; $m = x'\mu + (1 - x'e)z_0$; $e = [1, \dots, 1]'$; and $\sigma^2 = x'Qx$. Besides, the optimal portfolio weights x satisfy the following relation:

$$x = Q^{-1}(\mu - z_0 e)\frac{m - z_0}{(\mu - ez_0)'Q^{-1}(\mu - ez_0)}. \tag{17}$$

Note that (16) and (17) have the same forms as the mean-variance frontier. However, even if Q is a symmetric matrix (it is definite positive), the estimator proposed in the sub-Gaussian cases (see formulas (13) and (14)) generally is not symmetric. Therefore, in some extreme cases we could obtain the inconsistent situation of stable distributions associated to portfolios $x'r$ whose square scale parameter estimator is lower than zero[5]. This is the first reason

[5] Observe that for every $x \in R^n$, we get $x'\hat{Q}x > 0$ if and only if $\frac{(\hat{Q}+(\hat{Q})')}{2}$ is a definite positive matrix. Thus, we can verify that $\frac{(\hat{Q}+(\hat{Q})')}{2}$ is definite positive in order to avoid stable portfolios $x'r$ with negative scale parameter estimators. Moreover, we observe that the symmetric matrix $\frac{(\hat{Q}+(\hat{Q})')}{2}$ is an alternative estimator of dispersion matrix Q whose statistical properties have to be proved. Therefore, further studies on this and other alternative estimators will be object of future research.

for considering and studying the convergence properties of the estimator (see Rachev (1991)) and the suitability of the model. Moreover, (17) exhibits the two fund separation property for both the stable and the normal case, but the matrix Q and the parameter σ have different meaning. In the normal case, Q is the variance-covariance matrix and σ is the standard deviation, while in the stable case Q is a dispersion matrix and σ is the scale (dispersion) parameter, $\sigma = \sqrt{x'Qx}$. According to the two-fund separation property of the sub-Gaussian α-stable approach, we can assume that the market portfolio is equal to the risky tangent portfolio under the equilibrium conditions (as in the classical mean-variance Capital Asset Pricing Model (CAPM)). Therefore, every optimal portfolio can be seen as the linear combination between the market portfolio

$$\bar{x}'r = \frac{r'Q^{-1}(\mu - z_0 e)}{e'Q^{-1}\mu - e'Q^{-1}ez_0}, \tag{18}$$

and the riskless asset return z_0. Following the same arguments as in Sharpe, Lintner, Mossin's mean-variance equilibrium model, the return of asset i is given by:

$$E(r_i) = z_0 + \beta_{i,m}(E(\bar{x}'r) - z_0), \tag{19}$$

where $\beta_{i,m} = \frac{\bar{x}'Qe^i}{\bar{x}'Q\bar{x}}$, with e^i the vector with 1 in the *i-th* component and zero in all the other components. As a consequence of Ross' necessary and sufficient conditions of two-fund separation (see Ross (1978a)), the above model admits the form

$$r_i = \mu_i + b_i Y + \varepsilon_i, \quad i = 1, \dots, n,$$

where $\mu_i = E(r_i)$, $E(\varepsilon/Y) = 0$, $\varepsilon = [\varepsilon_1, \dots, \varepsilon_n]'$, $b = [b_1, \dots, b_n]'$ and the vector $bY + \varepsilon$ is sub-Gaussian α-stable distributed with zero mean.

Hence, our sub-Gaussian α-stable version of CAPM is not much different from Gamrowski-Rachev's (1999) version of the two-fund separation α-stable model. As a matter of fact, Gamrowski and Rachev (1999) propose a generalization of Fama's α-stable model (1965b) assuming $r_i = \mu_i + b_i Y + \varepsilon_i$, for every $i = 1, \dots, n$, where ε_i and Y are α-stable distributed and $E(\varepsilon/Y) = 0$. In view of their assumptions,

$$E(r_i) = z_0 + \tilde{\beta}_{i,m}(E(\bar{x}'r) - z_0),$$

where $\tilde{\beta}_{i,m} = \frac{1}{\alpha\|\bar{x}'\tilde{r}\|_\alpha^\alpha} \frac{\partial\|\bar{x}'\tilde{r}\|_\alpha^\alpha}{\partial\bar{x}_i} = \frac{[\tilde{r}_i, \bar{x}'\tilde{r}]_\alpha}{\|\bar{x}'\tilde{r}\|_\alpha^\alpha}$. Furthermore, the coefficient $\frac{[\tilde{r}_i, \bar{x}'\tilde{r}]_\alpha}{\|\bar{x}'\tilde{r}\|_\alpha^\alpha}$ can be estimated using the above formula (11).

Now, we see that in the above sub-Gaussian symmetric α-stable model $\bar{x}'Q\bar{x} = \|\bar{x}'\tilde{r}\|_\alpha^2$ and $\bar{x}'Qe^i = \frac{1}{2}\frac{\partial\|\bar{x}'\tilde{r}\|_\alpha^2}{\partial\bar{x}_i}$. Thus, we get the equivalence between the coefficient $\beta_{i,m}$ of model (19) and $\tilde{\beta}_{i,m}$ of Gamrowski-Rachev's model i.e.:

$$\beta_{i,m} = \frac{\bar{x}'Qe^i}{\bar{x}'Q\bar{x}} = \frac{1}{\sigma_{\bar{x}'r}}\frac{\partial\sigma_{\bar{x}'r}}{\partial\bar{x}_i} = \frac{[\tilde{r}_i, \bar{x}'\tilde{r}]_\alpha}{\|\bar{x}'\tilde{r}\|_\alpha^\alpha} = \tilde{\beta}_{i,m},$$

where $\sigma_{\bar{x}'r}$ is the scale parameter of market portfolio.

2.2 A three fund separation model in the domain of attraction of a stable law

Let us assume that the vector $r = [r_1, \ldots, r_n]'$ describes the following three-fund separating stable model of security returns:

$$r_i = \mu_i + b_i Y + \varepsilon_i, \quad i = 1, \ldots, n, \tag{20}$$

where the random vector $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)'$ is independent from Y and follows a joint sub-Gaussian α_1-stable distribution $(1 < \alpha_1 < 2)$, with zero mean and characteristic function

$$\Phi_\varepsilon(t) = \exp(-|t'Qt|^{\alpha_1/2}),$$

where Q is the definite positive dispersion matrix. On the other hand,

$$Y \stackrel{d}{=} S_{\alpha_2}(\sigma_Y, \beta_Y, 0)$$

is α_2-stable distributed random variable, independent from ε, with $1 < \alpha_2 < 2$ and zero mean. Under these assumptions, the portfolios are in the domain of attraction of an α stable law with $\alpha = \min(\alpha_1, \alpha_2)$ and belong to a $\sigma\tau_3(\bar{a})$ family. A testable case in which Y is α_2-stable symmetric distributed (i.e. $\beta_Y = 0$), was recently studied by Götzenberger, Rachev and Schwartz (1999). When $\beta_Y = 0$ and $\alpha_1 = \alpha_2$, our model can lead to the two-fund separation Fama's model. The characteristic function of the vector of returns $r = [r_1, r_2, \ldots, r_n]'$ is given by:

$$\Phi_r(t) = \Phi_\varepsilon(t)\Phi_Y(t'b)e^{it'\mu} = \exp\biggl(-|t'Qt|^{\alpha_1/2} + \\ -|t'b\sigma_Y|^{\alpha_2}\left(1 - i\beta_Y sgn(t'b)\tan\left(\frac{\pi\alpha_2}{2}\right)\right) + it'\mu\biggr), \tag{21}$$

where $b = [b_1, \ldots, b_n]'$ is the coefficient vector and $\mu = [\mu_1, \ldots, \mu_n]'$ is the mean vector.

Next we shall estimate the parameter in model (20), (21). First, the estimator of μ is given by the vector $\hat{\mu}$ of sample average. Then, we consider as factor Y a centralized index return (for example the market portfolio (18) given by the above sub-Gaussian model). Therefore, given the sequence of observations $Y^{(k)}$, we can estimate its stable parameters. Observe that the random vector ε admits a representation as a product of random variable V and Gaussian vector G:

$$\varepsilon = VG,$$

$V = \sqrt{A}$, where A is an $\frac{\alpha_1}{2}$-stable subordinator, that is

$$A \stackrel{d}{=} S_{\alpha_1/2}\left(\left(\cos\left(\frac{\pi\alpha_1}{4}\right)\right)^{2/\alpha_1}, 1, 0\right);$$

G is a $(n \times 1)$ Gaussian vector with null mean and variance covariance matrix

Q and it is independent from A. We can generate values A_k, $k = 1, \dots, N$ of A independent from G. We address to Paulauskas and Rachev's work (1999) the problem of generating such values A_k. Using the centralizing returns $\tilde{r}_j = r_j - \mu_j$ on Y we write the following OLS estimators[6] for $b = [b_1, \dots, b_n]'$ and Q:

$$\hat{b}_i = \frac{\sum_{k=1}^{N} \frac{Y^{(k)}\tilde{r}_i^{(k)}}{A_k}}{\sum_{k=1}^{N} \frac{(Y^{(k)})^2}{A_k}}; \quad i = 1, \dots, n,$$

and

$$\hat{Q} = \frac{1}{N}\sum_{k=1}^{N} \frac{(\tilde{r}^{(k)} - \hat{b}Y^{(k)})(\tilde{r}^{(k)} - \hat{b}Y^{(k)})'}{A_k}.$$

The selection of α_1 is a separate problem. A possible way to estimate α_1 is to consider the OLS estimator $\tilde{b}_i = \frac{\sum_{k=1}^{N} Y^{(k)}\tilde{r}_i^{(k)}}{\sum_{k=1}^{N} (Y^{(k)})^2}$ and then to evaluate the sample residuals $\tilde{\varepsilon}^{(k)} = \tilde{r}^{(k)} - \tilde{b}Y^{(k)}$. If these residuals are heavy tailed, one can take the tail exponent as an estimator for α_1. The asymptotic properties of the above estimator can be derived arguing similarly with Paulauskas and Rachev (1999) and Götzenberger, Rachev and Schwartz (1999).

In order to determine portfolios that are R-S non-dominated when unlimited short selling is allowed, we have to minimize the scale parameter $\sigma_W = \sqrt{x'Qx}$ for some fixed mean $m_W = x'\mu + (1 - x'e)z_0$ and $\tilde{b} = \frac{x'b}{\sqrt{x'Qx}}$. Alternatively, as shown by Ortobelli, Rachev and Schwartz (2000), we can obtain these portfolios from the solution of the following quadratic programming problem:

$$\begin{aligned} &\min_x x'Qx \text{ subject to} \\ &x'\mu + (1 - x'e)z_0 = m_W, \\ &x'b = b^* \end{aligned} \tag{22}$$

for some m_W and b^*. Thus, under our assumptions, every portfolio that maximizes the expected value of a given concave utility function u,

$$\max_x E(u(x'r))$$

belongs to the following frontier

$$(1 - \lambda_2 - \lambda_3)z_0 + \lambda_2 \frac{r'Q^{-1}(\mu - z_0 e)}{e'Q^{-1}(\mu - z_0 e)} + \lambda_3 \frac{r'Q^{-1}b}{e'Q^{-1}b} \tag{23}$$

spanned by the riskless return z_0, and the two risky portfolios

[6] For a discussion see Tokat, Rachev and Schwartz (2001).

$$u^{(1)} = \frac{r'Q^{-1}(\mu - z_0 e)}{e'Q^{-1}(\mu - z_0 e)} \quad \text{and} \quad u^{(2)} = \frac{r'Q^{-1}b}{e'Q^{-1}b}.$$

Observe in (21) that when $\alpha = \alpha_1 = \alpha_2 > 1$, every portfolio $x'r$ is an α-stable distribution and satisfies the relation

$$W = (1 - x'e)z_0 + x'r \stackrel{d}{=} S_\alpha(\sigma_{x'r}, \beta_{x'r}, (1 - x'e)z_0 + m_{x'r})$$

and $W = z_0$ when $x = 0$, where

$$\sigma^\alpha_{x'r} = (x'Qx)^{\alpha/2} + |x'b\sigma_Y|^\alpha, \quad \beta_{x'r} = \frac{|x'b\sigma_Y|^\alpha sgn(x'b)\beta_Y}{\sigma^\alpha_{x'r}}, \quad m_{x'r} = x'E(r).$$

Hence, this jointly α-stable model is a fund separation model whose solutions are given by the optimization problem (7) and these solutions satisfy the quadratic programming problem (22).

2.3 A $k+1$ fund separation model in the domain of attraction of a stable law

As empirical studies show in the stable case one of the most severe restrictions of performance measurement and asset pricing is the assumption of a common index of stability for all assets – individual securities and portfolio alike. It is well understood that asset returns are not normally distributed. We also know that the return distributions do not have the same index of stability. However, under the assumption that returns have different indexes of stability, it is not generally possible to find a closed form to the efficient frontier. Generalizing the above model instead, we get the following $k+1$ fund separation model, (for details on k fund separation models see Ross (1978a)):

$$r_i = \mu_i + b_{i,1}Y_1 + \cdots + b_{i,k-1}Y_{k-1} + \varepsilon_i, \quad i = 1, \ldots, n. \tag{24}$$

Here, $n \geq k \geq 2$, the vector $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)'$ is independent from $Y_1, \ldots, Y_{k-1}$ and follows a joint sub-Gaussian symmetric α_k-stable distribution with $1 < \alpha_k < 2$, zero mean and characteristic function $\Phi_\varepsilon(t) = \exp(-|t'Qt|^{\alpha_k/2})$, and the random variables $Y_j \stackrel{d}{=} S_{\alpha_j}(\sigma_{Y_j}, \beta_{Y_j}, 0)$, $j = 1, \ldots, k-1$ are mutually independent[7] α_j-stable distributed with $1 < \alpha_j < 2$ and zero mean. If we need to insure the separation obtained in situations where the above model degenerates into a p-fund separation model with $p < k+1$, we

[7] In order to estimate the parameters, we need to know the joint law of the vector $(Y_1, \ldots, Y_{k-1})$. Therefore, we assume independent random variables Y_j, $j = 1, \ldots, k-1$. Then the characteristic function of the vector of returns $r = [r_1, \ldots, r_n]'$ is given by

$$\Phi_r(t) = \Phi_\varepsilon(t) \prod_{j=1}^{k-1} \Phi_{Y_j}(t'b_{\cdot,j})e^{it'\mu}.$$

Under this additional assumption, we can approximate all parameters of any optimal portfolio using a similar procedure of the previous three fund separation model. However, if we assume a given joint $(\alpha_1, \ldots, \alpha_{k-1})$ stable law for the vector $(Y_1, \ldots, Y_{k-1})$, we can generally determine estimators of the parameters studying the characteristics of the multivariate stable law.

require the rank condition (see Ross (1978a)). However, under these assumptions, the portfolios belong to a $\sigma\tau_{k+1}(\bar{a})$ class. In order to determine portfolios that are R-S non-dominated, when unlimited short selling is allowed, we have to minimize the scale parameter $\sigma_W = \sqrt{x'Qx}$ for some fixed mean $m_W = x'\mu + (1 - x'e)z_0$ and $\tilde{b}_j = \frac{x'b_{\cdot,j}}{\sqrt{x'Qx}}$, $j = 1, \dots, k-1$. Alternatively, as shown by Ortobelli, Rachev and Schwartz (2000), we can obtain these portfolios from the solution of the following quadratic programming problem:

$$\min_x x'Qx \text{ subject to}$$

$$x'\mu + (1 - x'e)z_0 = m_W \tag{25}$$

$$x'b_{\cdot,j} = c_j \quad j = 1, \dots, k-1$$

By solving the optimization problem (25), we obtain that the riskless portfolio and other k risky portfolios span the efficient frontier for the risk averse investors given by

$$\left(1 - \sum_{j=1}^{k} \lambda_j\right) z_0 + \lambda_1 \frac{r'Q^{-1}(\mu - z_0 e)}{e'Q^{-1}(\mu - z_0 e)} + \sum_{j=1}^{k-1} \lambda_{j+1} \frac{r'Q^{-1}b_{\cdot,j}}{e'Q^{-1}b_{\cdot,j}}.$$

The above multivariate models are motivated by arbitrage considerations as in the Arbitrage Pricing Theory (APT) (see Ross (1976)). Without going into details, it should be noted that there are two versions of the APT for α-stable distributed returns, a so-called equilibrium (see Chen and Ingersoll (1983), Dybvig (1983), Grinblatt and Titman (1983)) and an asymptotic version (see Huberman (1982)). Connor (1984) and Milne (1988) introduced a general theory which encompassed the equilibrium APT as well as the mutual fund separation theory for returns belonging to any normed vector space (hence also symmetric α-stable distributed returns). While Gamrowski and Rachev (1999) provide the proof for the asymptotic version of α-stable distributed returns. Hence, it follows from Connor and Milne's theory that the above random law in the domain of attraction of a stable law of the return is coherent with the classic arbitrage pricing theory and the mean returns can be approximated by the linear pricing relation

$$\mu_i \sim z_0 + b_{i,1}\delta_1 + \dots + b_{i,k-1}\delta_{k-1},$$

where δ_p, for $p = 1, \dots, k-1$, are the risk premiums relative to the different factors. The above $k + 1$ fund separation model concludes the examples of models in the domain of attraction of stable laws. In the next section we compare the Gaussian multivariate approach with the sub-Gaussian stable one.

3 A first comparison between the Gaussian distributional assumption and the stable non-Gaussian one

In this section we examine and compare the stable non-Gaussian assumption with the normal distributional one. First we consider the problem of finding the optimal allocation λ in an investment consisting of two positions: a risky asset with stable distributed return and a riskless asset. We assume the investors wish to maximize the following utility functional:

$$U(W) = E(W) - cE(|W - E(W)|^q), \tag{26}$$

where c and q are positive real numbers, $W = \lambda z_0 + (1-\lambda)z$ is the return on the portfolio, z_0 is the risk-free asset return, and z is the risky asset return. We observe:

1. Problem (26) is equivalent to the following maximization of the utility functional

$$aE(W) - bE(|W - E(W)|^q), \tag{27}$$

   assuming $c = \frac{b}{a}$ in (26) for every $a, b > 0$. Thus, $E(|W - E(W)|^q)$ represents a particular risk measure of portfolio loss, which satisfies the main characteristics of the dispersion measures. Solving the optimal allocation problem (26), the investor implicitly maximizes the expected mean of the increment wealth aW as well as minimizes the individual risk $bE(|W - E(W)|^q)$.
2. Furthermore, when $q = 2$, the maximization of utility functional (26) motivates the mean variance approach in terms of preference relations.

Suppose X dominates Y in the sense of R–S. Since $E(X) = E(Y)$ and $f(x) = c|x - E(X)|^q$ is a concave utility function, for every $q \in [1, \alpha)$, it follows that:

$$U(X) = E(X) - cE(|X - E(X)|^q) \geq U(Y); \quad \forall q \in [1, \alpha).$$

The above inequality implies that every risk averse investor with utility functional (27) should choose a portfolio $W = \lambda z_0 + (1-\lambda)z$ that maximizes the utility functional (26) for some real λ and some $q \in [1, \alpha)$.

We know that for $\lambda \neq 1$, all the portfolio returns $W = \lambda z_0 + (1-\lambda)z$ admits stable distribution

$$S_\alpha(|1-\lambda|\sigma_z, sign(1-\lambda)\beta_z, \lambda z_0 + (1-\lambda)m_z);$$

and $W = z_0$ when $\lambda = 1$. Now, in order to solve the asset allocation problem

$$\max_\lambda E(W) - cE(|W - E(W)|^q),$$

note that, for all $q \in [1, \alpha)$ and $1 < \alpha < 2$, we get

$$\begin{aligned} U(W) &= E(W) - cE(|W - E(W)|^q) \\ &= \lambda z_0 + (1-\lambda)m_z - c(H(\alpha, \beta_z, q))^q |1-\lambda|^q \sigma_z^q \end{aligned}$$

where

$$(H(\alpha,\beta_z,q))^q = \frac{2^{q-1}\Gamma\left(1-\frac{q}{\alpha}\right)}{q\int_0^\infty u^{-q-1}\sin^2 u\,du}\left(1+\beta_z^2\left(\tan^2\left(\frac{\alpha\pi}{2}\right)\right)\right)^{q/(2\alpha)}$$
$$\times\cos\left(\frac{q}{\alpha}\arctan\left(\beta_z\tan\left(\frac{\alpha\pi}{2}\right)\right)\right),$$

(see Samorodnitsky and Taqqu (1994), Hardin (1984)). The above relation holds only in the stable non-Gaussian case. When the vector r admits a joint normal distribution (i.e. $\alpha = 2$), then for all $q > 0$,

$$\begin{aligned} U(W) &= E(W) - cE(|W-E(W)|^q) \\ &= \lambda z_0 + (1-\lambda)m_z - c\frac{2^{q/2}\Gamma\left(\frac{q+1}{2}\right)}{\sqrt{\pi}}|1-\lambda|^q\sigma_z^q. \end{aligned}$$

Hence, the real optimal solution of the problem in the important case $q \in (1,\alpha)$, is given by

$$\bar{\lambda} = 1 - sgn(1-\bar{\lambda})\left(\frac{sgn(1-\bar{\lambda})(m_z - z_0)}{qc\sigma_z^q V(\alpha,\beta_z,q)}\right)^{1/(q-1)}, \tag{28}$$

where

$$V(\alpha,\beta_z,q) = \begin{cases} (H(\alpha,\beta_z,q))^q & \text{in the stable case } (1<\alpha<2) \\ \frac{2^{q/2}\Gamma\left(\frac{q-1}{2}\right)}{\sqrt{\pi}} & \text{in the normal case } (\alpha = 2) \end{cases}.$$

We would expect that the optimal allocation is different because the constant $V(\alpha,\beta_z,q)$ and the dispersion σ_z are different in the stable non-Gaussian and in the normal case.

Recall that the tail behavior of every stable non-Gaussian distribution $X \stackrel{d}{=} S_\alpha(\sigma,\beta,\mu)$, with $1<\alpha<2$, is given by

$$\lim_{\lambda\to+\infty}\lambda^\alpha P(\pm X > \lambda) = C_\alpha\frac{1\pm\beta}{2}\sigma^\alpha, \tag{29}$$

where $C_\alpha = \frac{1-\alpha}{\Gamma(2-\alpha)\cos\frac{\pi\alpha}{2}}$. Therefore, several indexes of stability imply deep differences in relation to the tail behavior. As a consequence of relation (29) it follows that every stable non-Gaussian distribution $X \stackrel{d}{=} S_\alpha(\sigma,\beta,\mu)$, with $1<\alpha<2$, admits

$$E(|X-E(X)|^q) < \infty \quad \text{for } q<\alpha \tag{30}$$

$$\text{and}\quad E(|X-E(X)|^q) = \infty \quad \text{for } q \geq \alpha. \tag{31}$$

Hence, the weight of the risk measure $E(|X-E(X)|^q)$ in optimization problem (26) is generally greater for the investors who use the stable laws for asset returns when q is quite close to the index of stability α.

In a recent work Ortobelli, Rachev and Schwartz (2000) compare the sta-

Table I. Maximum likelihood estimators of stable and normal daily index returns

Series	Normal		Stable			
	Mean	Standard Deviation	α	β	μ	σ
S&P 500	0.032	0.930	1.708	0.004	0.036	0.512
DAX30	0.026	1.002	1.823	−0.084	0.027	0.592
CAC40	0.028	1.198	1.784	−0.153	0.027	0.698

The above table by Khindanova, Rachev and Schwartz (1999) summarizes the estimated parameters of the normal and the stable distributions for daily index S\&P500 or DAX30 or CAC40. The DAX30 series includes 8630 observations from 1.04.65 to 1.30.98, the S\&P 500 series – 7327 observations from .01.70 to 1.30.98, the CAC40 series – 2756 observations from 7.10.87 to 1.30.98.

ble non-Gaussian assumption and the normal one by analyzing the optimal allocations between a riskless return and a benchmark index. Three different indexes have been taken into consideration: CAC40, DAX30 and S&P500 (see Table I reporting the same data used by Khindanova, Rachev and Schwartz (1999)). The riskless return is 6% p.a.

In table II, we listed the optimal allocation for the normal and the stable fit when no short sales are allowed. Recall that $\bar{\lambda}$ is the optimal proportion of funds invested in the risk free asset chosen and $q = 1.35$ so that q is strictly less than all indexes of stability in the data set. On the other hand, we want q to be large, far away from 1, because for $q = 1$, we obtain the trivial allocation (see Ortobelli, Rachev and Schwartz (2000)).

The analysis of table II shows that the optimal allocation in the normal and in the stable case is more sensitive to smaller risk aversion coefficient c. In particular, the stable optimal allocation can be up to **40%**, from the normal allocation (see Table II). These results also show that in the stable non-Gaussian case the riskless asset allocation is greater than the normal one (except for DAX30 when $q = 1.35$). This is indeed due to the fat tails of the stable distribution. Thus, when investors fit normal distributions for return assets, they miss an important component of portfolio risk. On the contrary, the investor who fits stable distributions for return assets, she/he implicitly tries to approximate the additional component of risk related to the heavy fat tailedness as return distributions. Let observe another consequence of the above relation, (see for example the DAX30 index). When q is more distant from the stability parameter, we have to expect that the greatest difference in the allocation is lower (about 10% in DAX30) and more influenced by the differences in the trivial allocation ($q = 1$). This can easily be confirmed in all the above indexes considering the lower $q = 1.35$, in the allocation problem. In this sense, the stability index plays a strategic role in the optimal portfolio selection and for this reason, it becomes very significant as an accurate estimate of this parameter. Conversely, the importance given to q is intuitively linked to the conditions of the market in which the investor operates. Hence, this empirical analysis shows that the component of risk due to heavy-tail distributions and the stability property can be extremely important in the choice of the optimal portfolio. We cannot be excessively surprised about these differences in the optimal allocations. As a matter of fact, also Mehra and Prescott's empirical analysis (1985) underlines that asset pricing puzzles

Table II. Optimal allocation for the optimization problem

$$\max_{\lambda} E(W) - cE(|W - E(W)|^{r})$$

when Gaussian and stable non-Gaussian distributional assumption are considered

SERIES	Coefficient "c" of the optimization problem	Normal optimal allocation $\bar{\lambda}$ when		α-stable optimal allocation $\bar{\lambda}$ when	
		$q = 1.35$	$q = 1.5$	$q = 1.35$	$q = 1.5$
S&P500	$c = 0.0276$	0.000	**0.006**	0.000	**0.415**
	$c = 0.03$	0.000	**0.159**	0.000	**0.505**
	$c = 0.032$	0.069	**0.261**	0.096	**0.565**
	$c = 0.033$	0.148	**0.305**	0.172	**0.591**
	$c = 0.034$	0.217	**0.345**	0.240	**0.615**
	$c = 0.036$	0.335	**0.416**	0.356	**0.656**
	$c = 0.038$	0.430	**0.476**	0.447	**0.691**
	$c = 0.04$	0.508	**0.527**	0.522	**0.721**
	$c = 0.045$	0.649	**0.626**	0.659	**0.780**
	$c = 0.05$	0.740	0.697	0.748	0.822
	$c = 0.055$	0.802	0.750	0.808	0.853
	$c = 0.065$	0.877	0.821	0.881	0.895
	$c = 0.1$	0.964	0.924	0.965	0.955
DAX30	$c = 0.021$	0.000	0.096	0.000	0.193
	$c = 0.022$	0.000	0.176	0.000	0.265
	$c = 0.023$	0.012	0.247	0.000	0.327
	$c = 0.024$	0.126	0.308	0.021	0.382
	$c = 0.025$	0.222	0.362	0.129	0.431
	$c = 0.027$	0.375	0.453	0.301	0.512
	$c = 0.0285$	0.465	0.509	0.401	0.562
	$c = 0.03$	0.538	0.557	0.482	0.605
	$c = 0.033$	0.648	0.634	0.606	0.673
	$c = 0.035$	0.702	0.675	0.667	0.709
	$c = 0.04$	0.797	0.751	0.772	0.778
	$c = 0.05$	0.893	0.841	0.880	0.858
	$c = 0.1$	0.985	0.960	0.983	0.964
CAC40	$c = 0.017$	0.000	**0.063**	0.000	**0.401**
	$c = 0.018$	0.000	**0.164**	0.029	**0.466**
	$c = 0.019$	*0.000*	**0.250**	*0.168*	**0.520**
	$c = 0.02$	*0.085*	**0.323**	*0.281*	**0.567**
	$c = 0.0205$	*0.148*	**0.356**	*0.330*	**0.588**
	$c = 0.0215$	*0.256*	**0.414**	*0.416*	**0.625**
	$c = 0.023$	0.387	**0.488**	0.518	**0.673**
	$c = 0.024$	0.457	**0.530**	0.573	**0.699**
	$c = 0.025$	0.517	**0.567**	0.620	**0.723**
	$c = 0.028$	0.650	0.655	0.725	0.779
	$c = 0.033$	0.781	0.751	0.828	0.841
	$c = 0.04$	0.874	0.831	0.901	0.892
	$c = 0.1$	0.991	0.973	0.993	0.983

This table computes the optimal allocation $\bar{\lambda}$ in the riskless return 6% annual rate (daily $z_0 = 0.000166$) for different risk aversion coefficient c of the optimization problem $\max_{\lambda} E(W) - cE(|W - E(W)|^{r})$ where $W = \lambda z_0 + (1 - \lambda)z$ and z is either the index S&P 500 or DAX30 or CAC40. We analyze the normal and the stable cases when no shortsales are allowed and $q = 1.35$ or $q = 1.5$. In the table we marked the differences between stable and Gaussian allocation greater than 15%.

can be justified thinking of people much more risk averse. Clearly, we do not believe that the equity premium puzzle can be explained only considering the stable distribution instead of the Gaussian one. However, we believe that the distributional differences between the data and the classic model used in finance can help to understand asset pricing puzzles. This conjecture is partly confirmed by assuming the stable distributions in place of the Gaussian one (see for example Kocherlakota's test on CCAPM with heavy-tailed pricing errors (1997)).

Next, we consider a comparison between Gaussian and sub-Gaussian multivariate optimal allocation. This comparison is formally and theoretically different from the previous one because the benchmark index is given by the market portfolio which generally will change, if the distributional assumptions change too. Thus, as a consequence of Roll (1977, 1978, 1979a–b), Dybvig and Ross' (1985a–b) analysis, we observe that:

a) an investor, who fits the return distributions with a joint α_1-stable sub-Gaussian distribution, will consider as inefficient the choice of another investor who fits the return distributions with a joint α_2-stable sub-Gaussian distribution with $\alpha_1 \neq \alpha_2$; and
b) the stable CAPM is still subject of some of the criticism already addressed to the classical one.

Nevertheless, it seems that the stable case explains better the empirical data. This is the main reason why here we interpret and analyze the different behavior between the investor who fits the data with joint stable sub-Gaussian distribution and the investor who fits the data with the joint normal distribution.

3.1 *A comparison between Gaussian and sub-Gaussian multivariate optimal allocation*

First, we consider the optimal allocation among 24 assets: 23 of those assets are risky assets with returns $r = [r_1, r_2, \ldots, r_{23}]'$ and the 24*th* is riskfree with annual rate 6%. Second, we draw our attention on 13 risky asset returns with non negative mean. We analyze the portfolio choice problems when short sales are allowed and when short sales are not allowed. In view of this comparison, we discuss and study the differences in portfolio choice problems without examining them so as to choose one of the two assumptions (Gaussian or sub-Gaussian).

In our comparison we use daily data taken from 23 international risky indexes valued in USD and quoted from January 1995 to January 1998. In the analysis proposed we first consider the maximum likelihood estimation of the stable parameters and of the Gaussian ones for every risky asset. Thus, Table III assembles the approximating parameters obtained from using the program STABLE[8].

In order to compare the different stable sub-Gaussian joint distributions and the joint normal distributions for the asset returns, we assume that the

[8] See Nolan (1997) and the web site www.ca.american.edu/~jpnolan.

Table III. Maximum likelihood estimators of stable and normal asset return parameters considering daily data from 1/3/95 to 1/30/98

Assets	Normal		Stable			
	Mean	Standard Deviation	Index of stability α	Stable skewness β	Stable mean μ	Stable scale σ
DAX 30	0.0011	0.0137	1.6541	−0.3085	0.001	0.0076
DAX 100	0.0011	0.0129	1.6311	−0.287	0.001	0.007
Performance CAC 40	0.0008	0.0138	1.8107	−0.4292	0.0006	0.0087
FTSE all share	0.0006	0.0088	1.7453	−0.114	0.0005	0.0053
FTSE 100	0.0007	0.0099	1.8066	−0.0429	0.0006	0.0062
FTSE actuaries 350	0.0006	0.0091	1.7599	−0.1052	0.0006	0.0056
Reuters Commodities	−0.0004	0.0073	1.7721	0.0128	−0.0004	0.0045
Nikkei 225 simple average	0.0001	0.016	1.6728	−0.0916	−0.0002	0.009
Nikkei 300 weighted stock average	0.0002	0.0139	1.7244	0.0293	0.0001	0.008
Nikkei 300 simple stock average	0.0004	0.013	1.7167	−0.0036	0.0003	0.0074
Nikkei 500	0.0002	0.013	1.719	−0.0944	0	0.0075
Nikkei 225 stock average	0.0001	0.0159	1.6912	−0.0355	−0.0001	0.009
Nikkei 300	0.0001	0.014	1.7072	0.0309	0.0001	0.008
Brent Crude Physical	0	0.0185	1.7398	−0.2379	−0.0004	0.0112
Brent current month	0	0.0186	1.7432	−0.2026	0	0.0112
Corn No 2 Yellow cents	0.0002	0.0152	1.6833	−0.1907	0	0.0083
Coffee Brazilian	0.0002	0.027	1.5763	−0.0587	0	0.0143
Dow Jones Futures 1	−0.0001	0.0055	1.8058	−0.4484	−0.0001	0.0034
Dow Jones Commodities	−0.0001	0.0078	1.6847	−0.1141	−0.0001	0.0038
Dow Jones Industrials	0.0009	0.0086	1.7355	−0.2471	0.0009	0.0049
Fuel Oil No 2	−0.0001	0.0201	1.735	−0.1685	0	0.0116
Goldman Sachs Commodity	0	0.0092	1.8033	−0.259	−0.0002	0.0058
S&P 500	0.0009	0.0083	1.6976	−0.0677	0.001	0.0046

vector r is sub-Gaussian α-stable distributed, with $\alpha = \alpha_k$, $k = 1, 2, 3$, where $\alpha_1 = 1.5763$ represents the minimum of the index of stability of the given returns, $\alpha_2 = 1.7223$ represents the average of the indexes of stability and $\alpha_3 = 1.8107$ represents the maximum of the indexes of stability (see Table III)[9]. Moreover, when in the following tables we consider the index of stability $\alpha = 2$, we implicitly assume that the returns are jointly normal distributed. Thus, every portfolio of risky assets is stable distributed in the following way:

$$x'r \overset{d}{=} S_{\alpha_k}(\sigma_{x'r}, \beta_{x'r}, m_{x'r}),$$

where α_k is one of the considered index of stability $k = 1, 2, 3$, $\sigma_{x'r} = (x'Q_k x)^{1/2}$ is the respective scale parameter, $Q_k = \left[\frac{R_{ij}}{2}\right]_k$ is the dispersion matrix, with $k = 1, 2, 3$, $\beta_{x'r} = 0$ is the skewness parameter, and $m_{x'r}$ represents the mean of $x'r$. Observe that the matrix Q_k is estimated with the method defined in the previous section and thus it depends on the index of stability α_k for $k = 1, 2, 3$. As observed previously, the rate of convergence of the empirical matrix $\hat{Q}_k$ to,

[9] In order to value the effects of heavy – tailedness on the portfolio selection problems, we first consider different indexes of stability. Secondly, in the next section we value the performance of different stable paretian approaches.

the unknown matrix Q_k will be faster, if p is as large as possible. In our estimations we use $p_1 = 1.5$ (relative to $\alpha_1 = 1.5763$), $p_2 = 1.7$ (relative to α_2) and $p_3 = 1.8$ (relative to α_3).

We analyze the differences in optimal allocations with reference to problem (26) when the investor chooses:

1. joint normal distribution,
   or,
2. joint α_k stable sub-Gaussian distribution ($k = 1, 2, 3$), where $\alpha_1 = 1.5763$; $\alpha_2 = 1.7223$; $\alpha_3 = 1.8107$.

as a model for the asset returns in his/her portfolio. Under these distinctive assumptions, the investors with utility functional (26) have different information about the distributional behavior of data.

First, considering that unlimited short selling is allowed, we examine optimal allocation among the riskless return and 23 index-daily returns: DAX 30, DAX 100 Perfomance, CAC 40, FTSE all share, FTSE 100, FTSE actuaries 350, Reuters Commodities, Nikkei 225 Simple average, Nikkei 300 weighted stock average, Nikkei 300 simple stock average, Nikkei 500, Nikkei 225 stock average, Nikkei 300, Brent Crude Physical, Brent current month, Corn No2 Yellow cents, Coffee Brazilian, Dow Jones Futures1, Dow Jones Commodities, Dow Jones Industrials, Fuel Oil No2, Goldman Sachs Commodity, S&P 500. We use the riskless return 6% p.a..

Second, when short sales are allowed and when short sales are not allowed, we consider optimal allocation among the riskless return and 13 risky returns: DAX 30, DAX 100 Perfomance, CAC 40, FTSE all share, FTSE 100, FTSE actuaries 350, Nikkei 300 weighted stock average, Nikkei 300 simple stock average, Nikkei 500, Corn No2 Yellow cents, Coffee Brazilian, Dow Jones Industrials, S&P 500.

Using the estimated daily index parameters, we can compute the dispersion matrixes and the approximating "market" portfolios. The dispersion matrix Q is given by either the variance-covariance matrix (in the normal case) or the matrix Q_k (in the stable cases) which depends on the index of stability α_k for $k = 1, 2, 3$ ($\alpha_1 = 1.5763$, $\alpha_2 = 1.7223$ and $\alpha_3 = 1.8107$). Therefore, as shown by Tables IV, V and VI, the market portfolio weights

$$\bar{x} = \frac{Q^{-1}(\mu - z_0 e)}{e'Q^{-1}\mu - e'Q^{-1}ez_0}$$

change under the different distributional assumptions. In particular, Table IV presents the market portfolio weights when we consider all 23 asset returns and short sales are allowed. Table V gives the market portfolio weights when only 13 returns are examined and short sales are allowed. Finally, when no short sales are allowed[10], Table VI determines the market portfolio weights of 13 returns and we find that optimal allocation is reduced only among the two risky assets DAX 30, S&P 500 and the riskless one. As argued by Roll (1977, 1978), Dybvig and Ross (1985a), different market portfolios imply a com-

[10] Under this constrain, we value the market portfolio weights as the risky portfolio compositions which maximize *the extended Sharpe ratio* $\frac{E(x'r)-z_0}{\,}$.

Table IV. Stable sub-Gaussian and Gaussian market portfolio weights considering 23 assets when short sales are allowed

ASSETS	WEIGHTS FOR $\alpha = 1.5763$	WEIGHTS FOR $\alpha = 1.7223$	WEIGHTS FOR $\alpha = 1.8107$	GAUSSIAN WEIGHTS $\alpha = 2$
DAX 30	−0.2398	−0.3927	−0.4575	−0.5741
DAX 100	−0.6603	−0.4375	−0.3333	−0.1333
Performance CAC 40	0.5609	0.5306	0.5106	0.4646
FTSE all share	−9.5133	−11.39	−12.2419	−13.6148
FTSE 100	−0.7286	−1.4042	−1.7348	−2.327
FTSE actuaries 350	8.8303	11.4297	12.6436	14.684
Reuters Commodities	2.2652	2.1787	2.1277	2.0159
Nikkei 225 simple average	2.0343	1.6394	1.4615	1.1694
Nikkei 300 weighted stock average	−0.0568	0.032	0.0782	0.1612
Nikkei 300 simple stock average	−2.8172	−2.341	−2.1415	−1.804
Nikkei 500	0.7426	0.5248	0.4409	0.3054
Nikkei 225 stock average	−1.5557	−1.1391	−0.9654	−0.6754
Nikkei 300	1.6791	1.3019	1.1353	0.8512
Brent Crude Physical	−0.1122	−0.0996	−0.0935	−0.082
Brent current month	−0.0685	−0.0507	−0.0415	−0.0249
Corn No 2 Yellow cents	−0.2852	−0.2319	−0.2106	−0.175
Coffee Brazilian	−0.1689	−0.1498	−0.142	−0.129
Dow Jones Futures 1	1.2231	1.2673	1.2793	1.2837
Dow Jones Commodities	0.8252	0.6554	0.5814	0.4554
Dow Jones Industrials	0.6048	0.6601	0.6845	0.734
Fuel Oil No 2	0.1573	0.15	0.1423	0.1241
Goldman Sachs Commodity	0.2395	0.1851	0.1674	0.1421
S&P 500	−1.9559	−1.9185	−1.8963	−1.8518

Table V. Stable sub-Gaussian and Gaussian market portfolio weights considering 13 assets when short sales are allowed

ASSETS	WEIGHTS FOR $\alpha = 1.5763$	WEIGHTS FOR $\alpha = 1.7223$	WEIGHTS FOR $\alpha = 1.8107$	GAUSSIAN WEIGHTS $\alpha = 2$
DAX 30	0.4832	0.5669	0.6185	0.7358
DAX 100	0.1016	0.0338	−0.0119	−0.1193
Performance CAC 40	−0.2815	−0.2901	−0.2925	−0.2948
FTSE all share	2.633	2.9652	3.0618	3.1266
FTSE 100	3.4361	3.4945	3.4938	3.4623
FTSE actuaries 350	−6.2115	−6.5916	−6.6849	−6.7201
Nikkei 300 weighted stock average	−1.2237	−1.1463	−1.1119	−1.0546
Nikkei 300 simple stock average	1.8869	1.7899	1.7388	1.643
Nikkei 500	−0.6984	−0.6824	−0.6669	−0.6296
Corn No 2 Yellow cents	−0.0125	−0.0143	−0.0152	−0.0172
Coffee Brazilian	−0.0311	−0.0378	−0.0401	−0.0431
Dow Jones Industrials	−0.4718	−0.5972	−0.6535	−0.7644
S&P 500	1.3898	1.5093	1.5642	1.6756

pletely different security market line analysis. Thus, the approach which takes into account more assets (23 instead of 13) presents more opportunities of earning because it considers additional information. Therefore, it dominates the other approaches. Besides, if the returns are jointly α_k stable sub-Gaussian

Table VI. Stable sub-Gaussian and Gaussian market portfolio weights considering 13 assets when no short sales are allowed

ASSETS	Weights for $\alpha = 1.5763$	Weights for $\alpha = 1.7223$	Weights for $\alpha = 1.8107$	Gaussian weights $\alpha = 2$
DAX 30	0.276	0.2742	0.2737	0.2734
S&P 500	0.724	0.7258	0.7263	0.7266
Other assets	0	0	0	0

Table VII. Stable sub-Gaussian and Gaussian market portfolio parameters considering 23 assets when short sales are allowed

Index of stability	Mean	Dispersion parameter
1.5763	−0.0044	0.0054
1.7223	−0.0041	0.0037
1.8107	−0.0039	0.0089
2	−0.0036	0.0208

distributed (for some determined $k = 1, 2, 3$), then the Gaussian approach is inefficient. Since, in general, efficient and inefficient portfolios can plot above and below the "real" security market line.

The analysis of Tables IV, V and VI points out that the composition of the market portfolio is strictly linked to the index of stability. In fact, we see that the allocation of the market portfolio in each asset component is generally monotone with respect to the stability index. The fat tails of smaller stability indexes underline the risk of the loss component of every portfolio. In particular, under the diverse distributional assumption, we distinguish the different perception of risk in the market portfolio components. This issue can be easily analyzed in the market portfolio weights with reference to the 13 returns when no short sales are allowed. In fact, Table III shows that the index of stability of S&P500 is greater than the index of stability of DAX 30, even if there is not a consistent difference between the means of the two assets. Precisely it is of the order 10^{-6} not reported in Table III for reasons of space. We also observe that in Table VI the component of the S&P500 in the market portfolio increases with the index of stability α_k of the sub-Gaussian approach. Intuitively, the Gaussian market portfolio (for $\alpha = 2$) will be riskier than the $\alpha_3 = 1.8107$ stable sub-Gaussian market portfolio, because the Gaussian market portfolio has greater component than the asset with fatter tail. Similarly, we can consider the α_3 stable sub-Gaussian market portfolio as riskier than the $\alpha_2 = 1.7223$ stable sub-Gaussian market portfolio in its turn riskier than the $\alpha_1 = 1.5763$ stable sub-Gaussian one. Therefore, intuition suggests that the stable sub-Gaussian approaches with lower indexes of stability generally are more risk preserving than those with greater indexes of stability. This is due to the fact that they take more into consideration the component of risk because of the fat tails. This analysis is partially confirmed when short sales are allowed with either 13 or 23 assets. In fact, in Tables VII, VIII, and IX we listed the parameters of the market portfolios for the normal and the

Table VIII. Stable sub-Gaussian and Gaussian market portfolio parameters considering 13 assets when short sales are allowed

Index of stability	Mean	Dispersion parameter
1.5763	0.001476	0.0025
1.7223	0.0014451	0.0019
1.8107	0.0014338	0.0017
2	0.0018381	0.0123

Table IX. Stable sub-Gaussian and Gaussian market portfolio parameters considering 13 assets when no short sales are allowed

Index of stability	Mean $\times 10^{-4}$	Dispersion parameter
1.5763	9.9992	0.0017
1.7223	9.9955	0.0012
1.8107	9.9945	0.0011
2	9.9939	0.0079

stable fit when we consider 23 assets (Table VII) and 13 assets (Table VIII, which considers when unlimited short selling is allowed and Table IX, which considers when no short sales are allowed). Therefore, Tables VII and VIII show that the market portfolios of sub-Gaussian approaches have a lower mean than the market portfolios of Gaussian approaches. According to the classic mean-risk interpretation, an optimal portfolio that has a greater mean, it has also a greater risk[11]. Thus, intuitively the sub-Gaussian approaches with lower indexes of stability are more risk preserving than the approaches with greater indexes of stability. This intuition is partially confirmed by the examination of the optimal allocation problem proposed.

In Tables X, XI we listed the optimal allocation $\bar{\lambda}$ for the normal and the stable fit. Recall that $\bar{\lambda}$ is the optimal proportion of funds invested in the risk free asset which maximizes $E(W) - cE(|W - E(W)|^q)$, where $W = \lambda z_0 + (1 - \lambda)\bar{x}'r$. We have chosen $q = 1.45$ in Table X and $q = 1.55$ in Table XI, so that q is strictly less than all indexes of stability α_k, $k = 1, 2, 3$ in the data set, where $\alpha_1 = 1.5763$; $\alpha_2 = 1.7223$; $\alpha_3 = 1.8107$. On the other hand, we want to evaluate and compare the different effects of q distant or closer to the stability parameters α_k. Both tables show the greater diversity among the optimal allocations considering small risk aversion coefficients c. Instead, the very risk averse investors assume a less risky position with every distributional hy-

[11] This fact appears clear enough when we consider and compare the dispersion measures $\sqrt{\bar{x}_k' Q_j \bar{x}_k}$ in every mean-risk plane for every market portfolio weights $\bar{x}_k = \frac{Q_k^{-1}(\mu - z_0 e)}{e' Q_k^{-1}\mu - e' Q_k^{-1} e z_0}$, for every k and j. Observe that $\tilde{\sigma}_{j,k} = \sqrt{\bar{x}_k' Q_j \bar{x}_k}$ is the dispersion measure of market portfolio $\bar{x}_k' r$ considering the α_j stable paretian approach. Therefore, for every fixed mean-risk plane (i.e. for every fixed α_j stable distributional approach) we can compare the market portfolio risk positions considering their risk position $\tilde{\sigma}_{j,k}$ (varying k). According to a mean-risk interpretation, we could observe that market portfolio with greater mean admits also a greater dispersion measure $\tilde{\sigma}_{j,k}$ in any mean-risk plane.

Table X. Optimal allocation for the optimization problem

$$\max_{\lambda} E(W) - cE(|W - E(W)|^{1.45})$$

when different distributional assumption are considered

	Coefficient "c" of the optimization problem	Optimal allocation $\bar{\lambda}$ when $\alpha = 1.5763$	Optimal allocation $\bar{\lambda}$ when $\alpha = 1.7223$	Optimal allocation $\bar{\lambda}$ when $\alpha = 1.8107$	Optimal allocation $\bar{\lambda}$ when $\alpha = 2$
Allocation $\bar{\lambda}$ in the riskless asset considering the market portfolio on 23 assets when unlimited short sales are allowed	$c = 1.3$	1.82156	12.5865	2.05907	1.37792
	$c = 1.5$	1.59777	9.43040	1.77058	1.27497
	$c = 1.8$	1.39863	6.62200	1.51388	1.18337
	$c = 3$	1.12811	2.80672	1.16514	1.05893
	$c = 4.2$	1.06065	1.85539	1.07819	1.02790
	$c = 4.8$	1.04508	1.63576	1.05811	1.02074
	$c = 5$	1.04117	1.58062	1.05307	1.01894
	$c = 6$	1.02745	1.38720	1.03539	1.01263
	$c = 7$	1.01949	1.27489	1.02153	1.00897
	$c = 10$	1.00882	1.12443	1.01137	1.00406
	$c = 15$	1.00358	1.05054	1.00462	1.00165
	$c = 21$	1.00170	1.02393	1.00219	1.00078
Allocation $\bar{\lambda}$ in the riskless asset considering the market portfolio on 13 assets when unlimited short sales are allowed	$c = 1.3$	0.38814	−5.8152	−15.449	0.66239
	$c = 1.5$	0.55481	−3.9588	−10.968	0.75435
	$c = 1.8$	0.70311	−2.3069	−6.9813	0.83619
	$c = 3$	0.90459	−0.0627	−1.5649	0.94736
	$c = 4.2$	0.95483	0.49686	−0.2143	0.97508
	$c = 4.8$	0.96643	0.62605	0.09743	0.98147
	$c = 5$	0.96934	0.65848	0.17571	0.98308
	$c = 6$	0.97955	0.77225	0.45030	0.98872
	$c = 7$	0.98548	0.83831	0.60974	0.99199
	$c = 10$	0.99343	0.92681	0.82335	0.99637
	$c = 15$	0.99733	0.97027	0.92825	0.99853
	$c = 21$	0.99874	0.98593	0.96603	0.99930
Allocation $\bar{\lambda}$ in the riskless asset considering the market portfolio on 13 assets when short sales are not allowed	$c = 1.3$	0.22342	0	0	0.70108
	$c = 1.5$	0.43496	0	0	0.78250
	$c = 1.8$	0.62319	0	0	0.85496
	$c = 3$	0.87891	0	0	0.95338
	$c = 4.2$	0.94267	0.14648	0	0.97793
	$c = 4.8$	0.95739	0.36563	0	0.98360
	$c = 5$	0.96108	0.42065	0	0.98502
	$c = 6$	0.97405	0.61365	0.12054	0.99001
	$c = 7$	0.98157	0.72571	0.37563	0.99291
	$c = 10$	0.99166	0.87584	0.71737	0.99679
	$c = 15$	0.99661	0.94957	0.88521	0.99870
	$c = 21$	0.99840	0.97612	0.94565	0.99938

This table computes the optimal allocation $\bar{\lambda}$ in the riskless return 6% annual rate (daily $z_0 = 0.000166$) for different risk aversion coefficient c of the optimization problem $\max_{\lambda} E(W) - cE(|W - E(W)|^{1.45})$ where $W = \lambda z_0 + (1 - \lambda)\bar{x}'r$ and $\bar{x}'r$ is either the Gaussian Market portfolio (for $\alpha = 2$) or the sub-Gaussian market portfolio (for $\alpha = 1.5763$, or $\alpha = 1.7223$ or $\alpha = 1.8107$).

pothesis and the allocations in the riskless asset do not change very much. When we consider 23 assets, the market portfolio is an inefficient portfolio in all the approaches considered. In this case, investors have a long position in the riskless asset and a short position in the tangent portfolio. Conversely

Table XI. Optimal allocation for the optimization problem

$$\max_{\lambda} E(W) - cE(|W - E(W)|^{1.55})$$

when different distributional assumption are considered

	Coefficient "c" of the optimization problem	Optimal allocation $\bar{\lambda}$ when $\alpha = 1.5763$	Optimal allocation $\bar{\lambda}$ when $\alpha = 1.7223$	Optimal allocation $\bar{\lambda}$ when $\alpha = 1.8107$	Optimal allocation $\bar{\lambda}$ when $\alpha = 2$
Allocation $\bar{\lambda}$ in the riskless asset considering the market portfolio on 23 assets when unlimited short sales are allowed	$c = 1.5$	1.10838	8.76731	2.20010	1.59704
	$c = 1.7$	1.08632	7.18641	1.95584	1.47552
	$c = 2.1$	1.05878	5.21292	1.65092	1.32383
	$c = 3$	1.03073	3.20264	1.34032	1.16931
	$c = 4$	1.01822	2.30552	1.20171	1.10035
	$c = 5$	1.01214	1.87013	1.13444	1.06688
	$c = 7$	1.00659	1.47195	1.07292	1.03628
	$c = 10$	1.00344	1.24675	1.03812	1.01897
	$c = 13$	1.00214	1.15314	1.02366	1.00897
	$c = 17$	1.00131	1.09403	1.01453	1.00723
	$c = 21$	1.00089	1.06403	1.00989	1.00492
	$c = 25$	1.00065	1.04664	1.00721	1.00358
Allocation $\bar{\lambda}$ in the riskless asset considering the market portfolio on 13 assets when unlimited short sales are allowed	$c = 1.5$	0.90204	−4.6798	−14.296	0.40102
	$c = 1.7$	0.92198	−3.5337	−11.182	0.52293
	$c = 2.1$	0.94687	−2.0807	−7.2962	0.67512
	$c = 3$	0.97222	−0.6107	−3.3375	0.83014
	$c = 4$	0.98353	0.04536	−1.5708	0.89933
	$c = 5$	0.98903	0.36373	−0.7135	0.93290
	$c = 7$	0.99405	0.65489	0.07063	0.96361
	$c = 10$	0.99689	0.81957	0.51410	0.98097
	$c = 13$	0.99807	0.88802	0.69844	0.98819
	$c = 17$	0.99881	0.93124	0.81484	0.99275
	$c = 21$	0.99919	0.95318	0.87391	0.99506
	$c = 25$	0.99941	0.96590	0.90816	0.99640
Allocation $\bar{\lambda}$ in the riskless asset considering the market portfolio on 13 assets when short sales are not allowed	$c = 1.5$	0.87230	0	0	0.41234
	$c = 1.7$	0.89829	0	0	0.53195
	$c = 2.1$	0.93074	0	0	0.68126
	$c = 3$	0.96379	0	0	0.83335
	$c = 4$	0.97854	0	0	0.90123
	$c = 5$	0.98569	0	0	0.93417
	$c = 7$	0.99224	0.42186	0	0.96429
	$c = 10$	0.99594	0.69773	0.22749	0.98133
	$c = 13$	0.99748	0.81241	0.52056	0.98841
	$c = 17$	0.99845	0.88482	0.70562	0.99289
	$c = 21$	0.99895	0.92156	0.79952	0.00516
	$c = 25$	0.99923	0.94287	0.85399	0.99647

This table computes the optimal allocation $\bar{\lambda}$ in the riskless return 6% annual rate (daily $z_0 = 0.000166$) for different risk aversion coefficient c of the optimization problem $\max_{\lambda} E(W) - cE(|W - E(W)|^{1.55})$ where $W = \lambda z_0 + (1 - \lambda)\bar{x}'r$ and $\bar{x}'r$ is either the Gaussian Market portfolio (for $\alpha = 2$) or the sub-Gaussian market portfolio (for $\alpha = 1.5763$, or $\alpha = 1.7223$ or $\alpha = 1.8107$).

when we consider only 13 assets in the market, the tangent portfolio is an efficient portfolio. Thus, the investors have a long position in the market portfolio and a short or long position in the riskless asset.

As we see from these tables, when $q = 1.45$ the investors who fit the data with the Gaussian approach generally assume a less risky position than the investors who fit the data with the sub-Gaussian approach. This is due to the fact that the Gaussian market portfolio is intuitively riskier than the sub-Gaussian ones. Instead, when $q = 1.55$ in optimization problem (26), the investors who fit the data with $\alpha_1 = 1.5763$ stable sub-Gaussian approach assume a less risky position than the investors who fit the data with the Gaussian approach. In this case, the "stable investor" not only has a very risk preserving behavior because the stable market portfolio is less risky than the Gaussian one but also prefers not allocating too much wealth in the risky asset. In this sense, the stability index plays a strategic role in the stable optimal portfolio selection.

4 Performance comparison among stable sub-Gaussian and mean-variance models

In this section we examine and compare the performances of Gaussian and sub-Gaussian approaches. As a matter of fact, in the previous sections we have underlined and discussed the theoretical and empirical differences among portfolio choice models. Now, we evaluate their real performances.

First, assuming that limited short sales are allowed, we examine the optimal allocation among the riskless return and 23 index-daily returns (the same of the previous section). In this analysis we approximate optimal solutions to the utility functional:

$$\max_{y} E(y'r + (1 - y'e)z_0) - cE(|y'r - E(y'r)|^{1.5}). \tag{32}$$

where c is an indicator of the aversion to the risk.

Secondly, assuming that limited short sales are allowed, we examine the optimal allocation among the riskless return and 13 index-daily returns (the same of the previous section). Thus, we consider the negative exponential utility function

$$u(x) = -\exp(-\gamma x)$$

with risk aversion coefficient $\gamma > 0$. In this case, the absolute risk aversion function $\frac{-u''(x)}{u'(x)} = \gamma$ is constant. Hence, for every distributional model considered we are interested in finding optimal solutions to the functional

$$\max_{y} -E(\exp(-\gamma(y'r + (1 - y'e)z_0))). \tag{33}$$

Observe that in case of α stable distributed returns with $1 < \alpha < 2$, the expected utility of formula (33) is infinite. However, assuming that the returns are truncated far enough, formula (33) is formally justified by pre-limit theorems (see Klebanov, Rachev, Szekely (2000) and Klebanov, Rachev, Safarian (2001)), which provide the theoretical basis for modeling heavy tailed bounded random variables with stable distributions. On the other hand, it is obvious that the incomes are always bounded random variables. Typically, the investor works with a finite number of data so she/he can always approximate his/her expected utility. Therefore, we use diverse utility functions which differ in their absolute risk aversion functions and depend on a risk aversion coefficient. The presence of a parameter enables us to study the investor optimal portfolio selection for different degrees of risk aversion. Practically, we distinguish three separate steps in the decision process:

1. Choose the distributional model.
2. Calculate the optimal portfolios of the efficient frontier. (Also in this case we often have to choose among different estimators in order to evaluate the optimization problem parameters).
3. Express a preference among efficient portfolios. (In particular, we assume that the investor's distributional belief is not correlated to his/her expected utility. Therefore, the investor finds efficient frontiers assuming stable or Gaussians distributed returns, but his/her utility function can be any increasing concave utility function. This hypothesis is realistic enough because investors try to approximate their maximum expected utility among the efficient portfolios previously selected).

We assume the vector of risky return, $r = [r_1, \ldots, r_n]'$, is jointly Gaussian distributed or in the domain of attraction of an α-stable non Gaussian distribution with $\alpha = \alpha_k$, $k = 1, 2, 3$, where $\alpha_1 = 1.5763$; $\alpha_2 = 1.7223$; $\alpha_3 = 1.8107$. Thus, we compare the performance of Gaussian and sub-Gaussian approaches for each optimal allocation proposed. In view of these comparisons, we discuss and study the differences in maximum expected utility for each allocation problem ((32) and (33)) and for every portfolio choice model (Gaussian or sub-Gaussian) proposed.

Note that every model, Gaussian or sub-Gaussian (15), is based on a different risk perception. In order to compare the different models, we use the same algorithm proposed by Giacometti and Ortobelli (2001), Ortobelli, Huber, Höchstötter, Rachev (2001). Thus, first we consider the optimal portfolio compositions obtained solving the optimization problems (15) for different levels of the mean. In this case we have the analytical formulation of the efficient frontier given by the linear combination of the market portfolio and the riskless one. The efficient frontiers have been obtained for each model, discretizing the expected optimal portfolio return between the riskless return and the expected market portfolio return. Second, we select the portfolios on the efficient frontiers that maximize some parametric expected utility functions for different risk aversion coefficients.

Thus, we need to select portfolios belonging to the efficient frontiers such that:

$$x^* = \arg\left(\max_{x \in \text{efficient frontier}} E(u(x'r + (1 - x'e)z_0))\right),$$

where u is a given utility function. Finally, in tables XII and XIII we compare the maximum expected utility obtained with the stable or normal model for different risk aversion coefficients.

Therefore, considering N i.i.d. observations. $r^{(i)}$ $(i = 1, \ldots, N)$ of the vector $r = [r_1, r_2, \ldots, r_{23}]'$, the main steps of our comparison are the following:

Step 1 Fit the four efficient frontiers corresponding to the different distributional hypothesis: Gaussian and sub Gaussian. Therefore, for every k we

Table XII. Maximum expected utility

$$\max_{y} E((1 - y'e)z_0 + y'r) - cE(|y'r - E(y'r)|^{1.5})$$

when different distributional assumption are considered

Coefficient "c" of the optimization problem	Maximum expected utility in the Gaussian case	α-Stable		
		Maximum expected utility when $\alpha = 1.5763$	Maximum expected utility when $\alpha = 1.7223$	Maximum expected utility when $\alpha = 1.8107$
$c = 1.5$	0.26069	*0.26078*	0.26077	0.26075
$c = 2.25$	0.20223	*0.20227*	*0.20227*	0.20226
$c = 3$	0.18054	*0.18058*	*0.18058*	0.18057
$c = 3.75$	0.16956	*0.16961*	0.16960	0.16959
$c = 4.5$	0.16269	0.16272	*0.16274*	0.16273
$c = 5.25$	0.15751	*0.15778*	0.15771	0.15765
$c = 6$	0.15362	*0.15295*	0.15387	0.15382
$c = 6.75$	0.15067	*0.15106*	0.15097	0.15088
$c = 7.5$	0.14833	*0.14871*	0.14858	0.14851
$c = 8.25$	0.14651	*0.14680*	0.14670	0.14664
$c = 9$	0.14494	*0.14524*	0.14513	0.14507
$c = 9.75$	0.14362	*0.14294*	0.14383	0.14376
$c = 10.5$	0.14257	*0.14285*	0.14276	0.14270
$c = 11.25$	0.14159	*0.14189*	0.14179	0.14172
$c = 12$	0.14060	*0.14093*	0.14082	0.14075
$c = 12.75$	0.13962	*0.13996*	0.13985	0.13978
$c = 13.5$	0.13864	*0.13900*	0.13888	0.13880
$c = 14.25$	0.13765	*0.13804*	0.13790	0.13783
$c = 15$	0.13667	*0.13708*	0.13694	0.13685
$c = 15.75$	0.13569	*0.13611*	0.13597	0.13588

This table consider allocation among 23 risky assets and the riskless one (6% annual rate, daily $z_0 = 0.000166$). For different risk aversion coefficient c the maximum of utility functional $E(W) - cE(|W - E(W)|^{1.5})$ is approximated, where $W = y'r + (1 - y'e)z_0$ is either Gaussian distributed or α-stable sub-Gaussian distributed (with $\alpha = 1.5763$, or $\alpha = 1.7223$ or $\alpha = 1.8107$). In the table we marked the greatest expected utility among the different distributional approaches.

estimate 200 optimal portfolio weights varying the daily mean m in the following constrained problem:

$$\begin{aligned} &\min_{x} x'\hat{Q}_k x \\ &\text{subject to} \\ &x'\mu + (1 - x'e)z_0 = m; \\ &a_1 \leq (1 - x'e) \leq a_2; \quad \text{and } \forall i; \quad a_1 \leq x_i \leq a_2 \end{aligned} \quad , \tag{34}$$

where $\mu = E(r)$ is the mean of the return vector, z_0 is the riskless return and $\hat{Q}_k$ is one of the estimated dispersion matrixes: either the variance covariance matrix ($k = 4$) or α_k stable sub Gaussian dispersion matrix. Instead, we assume $a_1 = -0.1$ and $a_2 = 1$ when we consider the optimal al-

Table XIII. Maximum expected utility

$$\max_{y} -E(\exp(-b((1-y'e)z_0+y'r)))$$

when different distributional assumption are considered

Coefficient "c" of the optimization problem	Maximum expected utility in the Gaussian case	α-Stable		
		Maximum expected utility when $\alpha = 1.5763$	Maximum expected utility when $\alpha = 1.7223$	Maximum expected utility when $\alpha = 1.8107$
$b = 12.5$	−795.7797	*−795.7378*	−795.7395	−795.7488
$b = 14$	−795.5488	*−795.4924*	−795.4979	−795.5105
$b = 15.5$	−795.3471	*−795.2913*	−795.2963	−795.3089
$b = 17$	−795.1460	*−795.0910*	−795.0953	−795.1077
$b = 18.5$	−794.9454	*−794.8922*	−794.8949	−794.9071
$b = 20$	−794.7453	*−794.6935*	−794.6950	−794.7071
$b = 21.5$	−794.5457	*−794.4948*	−794.4959	−794.5077
$b = 23$	−794.3468	*−794.2963*	−794.2973	−794.3089
$b = 24.5$	−794.1483	*−794.0977*	−794.0987	−794.1105
$b = 26$	−793.9497	*−793.8992*	−793.9003	−793.9118
$b = 27.5$	−793.7513	*−793.7007*	−793.7019	−793.7134
$b = 29$	−793.5528	*−793.5024*	−793.5034	−793.5150
$b = 30.5$	−793.3546	*−793.3040*	−793.3052	−793.3167
$b = 32$	−793.1562	*−793.1058*	−793.1068	−793.1183
$b = 33.5$	−792.9580	*−792.9075*	−792.9086	−792.9201
$b = 35$	−792.7597	*−792.7092*	−792.7103	−792.7221
$b = 36.5$	−792.5616	*−792.5111*	−792.5124	−792.5238
$b = 38$	−792.3635	*−792.3130*	−792.3143	−792.3257
$b = 39.5$	−792.1653	*−792.1149*	−792.1160	−792.1279
$b = 41$	−791.9675	*−792.9172*	−791.9180	−791.9295

This table considers allocation among 13 risky assets and the riskless one (6% annual rate, daily $z_0 = 0.000166$). For different risk aversion coefficient b the maximum of utility functional $\max_y -E(\exp(-bW))$ is approximated, where $W = y'r + (1-y'e)z_0$ is either Gaussian distributed or α-stable sub-Gaussian distributed (with $\alpha = 1.5763$, or $\alpha = 1.7223$ or $\alpha = 1.8107$). In the table we marked the greatest expected utility among the different distributional approaches.

location problem (32), while when we consider the optimal allocation problem (33), $a_1 = -3$ and $a_2 = 3$.

Step 2 Choose a utility function u with a given coefficient of aversion to risk

Step 3 Calculate for every efficient frontier (34)

$$\max_{y} \sum_{i=1}^{N} u(y'r^{(i)} + (1-y'e)z_0)$$

subject to

y belongs to the efficient frontier

Step 4 Repeat steps 2 and 3 for every utility function and for every risk aversion coefficient.

Finally, we obtain two tables (XII, XIII) with the approximated maximum expected utility (at less of the multiplicative factor N). In fact, we implicitly assume the approximation:

$$\frac{1}{N}\sum_{i=1}^{N} u(y'r^{(i)} + (1 - y'e)z_0) \approx E(u(y'r^{(i)} + (1 - y'e)z_0)).$$

Moreover in order to obtain significative results, we calibrate the risk aversion coefficients such that the portfolios which maximize the expected utility are optimal portfolios in the segment of the efficient frontier considered.

As we can observe from tables XII, XIII it follows that the sub-Gaussian models present a superior performance with respect to the mean-variance model. Even if in these tables the stable sub-Gaussian approaches do not seem diverging significantly from the mean-variance approach, we could ascertain that optimal portfolio weights which maximize the expected utility in the different distributional frameworks are quite diverging. This issue implicitly supports that stable distributions fit real data better than Gaussian distributions. Moreover, this ex-ante comparison confirms that the stable risk measure, the scale parameter σ, capture the data distributional behavior (typically the component of risk due to heavy tails) better than the Gaussian model.

We also observe that the stable sub-Gaussian approach with the lowest index of stability, $\alpha_1 = 1.5763$, shows better performances than the other stable approaches. Thus, considering also the previous comparisons, we conclude that the decision makers with utility functions (32) and (33) are much more risk preserving than what the mean variance model can forecast.

5 Conclusions

In this paper we first describe and examine the portfolio choices consistent with the maximization of the expected utility and coherently with the asymptotic behavior of returns with heavy tailed distributions. As a matter of fact, when returns have a stationary behavior they are in the domain of attraction of a stable law. Therefore, we present some examples of models in the domain of attraction of stable laws. The first distributional model considered is the case of the sub-Gaussian stable distributed returns. It permits a mean risk analysis pretty similar to Markowitz-Tobin's mean variance one. In fact, this model admits the same analytical form for the efficient frontier but the parameters differ in the two models. Thus, the most important difference is given by the way of estimating the parameters. In order to present heavy tailed models that consider the asymmetry of returns, we study a three fund separation model where the portfolios are in the domain of attraction of an (α_1, α_2) stable law. Next, we analyze the case of $k + 1$ fund separation model with portfolios in the domain of attraction of an $(\alpha_1, \ldots, \alpha_k)$ stable law. In all models we analyze the efficient frontier for the risk averse investors.

In second analysis, the comparison made between the stable and the normal approach in terms of the allocation problems has indicated that the stable allocation is more risk preserving than the normal one. Precisely, the stable approach, differently from the normal one, considers the component of risk due to the fat tails. Therefore, we find that the tail behavior of stable and

Gaussian approaches could imply substantial differences in the asset allocation. Taken into account that the stable approach is more adherent to the reality of the market, then, as argued by Götzenberger, Rachev and Schwartz (1999), we can obtain models that improve the performance measurements with the stable distributional assumption.

Finally, we propose a performance comparison among the sub-Gaussian and the mean variance model. The comparison holds from an ex-ante analysis on the data. We compare the maximum expected utility of an investor on different efficient frontiers considering daily data. The analysis shows significative differences in the allocation between the mean-variance model and the sub-Gaussian approach. In particular, the sub-Gaussian approaches present better performances than the mean-variance one.

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