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Arbitrage in Stock Index Futures*

The textbook description of arbitrage suggests that it is a straightforward matter of taking offsetting positions in different securities and realizing the arbitrage profit. Such descriptions, however, typically ignore the transaction costs that give rise to the arbitrage opportunity in the first place. Taking proper account of these transactions costs may considerably complicate the problem, particularly when, as is usually the case, the arbitrage potential is restricted.[1] This article is concerned with optimal arbitrage strategies with transaction costs when the arbitrage potential is restricted by position limits. The particular case we shall analyze is the Standard and Poor's (S&P) 500 Stock Index Futures contract. Optimal arbitrage strategies for a trader who does not incur transaction costs but who is subject to a position limit have been analyzed in a recent article by Brennan and Schwartz (1988).[2]

Recent evidence on stock index and other financial futures has shown that these contracts do not always trade at the prices predicted by a simple arbitrage relation with the spot price. For example, Figlewski (1985) reports that the annualized standard deviation of daily returns on a portfolio corresponding to the New York Stock Exchange (NYSE) Index, hedged by a short position in the nearest NYSE futures contract, was 19.72% for the period January 1981–March 1982; the corresponding figure for the S&P 500 portfolio for the same period was

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1. Stoll and Whaley (1987) point out that the ability of a brokerage firm, e.g., to take arbitrage positions "can be constrained by net capital requirements and the manner in which the 'hair-cut' provisions are applied to positions in futures and stocks as well as the manner in which margin requirements are applied to arbitrage positions."

2. In Brennan and Schwartz (1988) a simplified transaction-cost structure was also briefly considered.

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16.46%. Using more recent data, Figlewski (1984*b*) documents a decline in basis variability for the S&P 500 contract since its inception: nevertheless, he finds that the standard deviation of the basis for the nearby contract was still 60 basis points for the period May–September 1983; this corresponds to a standard deviation of 6.23% for annualized arbitrage returns, suggesting that "there were many attractive (arbitrage) opportunities" (Figlewski 1984*b*, p. 667). Casual empiricism suggests that this variability in the basis has persisted.[3]

Explanations proposed for the variability in the basis include the market-to-market requirement for futures contracts,[4] the differential tax treatment of spot and futures,[5] and the existence of a tax-timing option in a spot position but not in a futures position,[6] as well as the difficulties of arbitrage between a large portfolio of 500 stocks and a futures contract.[7] However, Figlewski (1984*b*) suggests that "noise" is the primary explanation for mispricing and shows that once the market matured, approximately 70% of the arbitrage opportunity was eliminated by the close of the following day.

Whatever its cause, variability in the basis is limited by the actions of arbitrageurs, and, therefore, in order to understand the behavior of the basis, it is necessary to analyze the problem of arbitrageurs. In this article, we take a step in this direction by analyzing the optimal strategy of an arbitrageur, taking as given the stochastic evolution of the basis.

The classical analysis of futures arbitrage assumes that the arbitrage position is held until expiration. This is legitimate if there are no position limits, either institutional or self imposed, and if the cost of closing out an existing arbitrage position is equal to the cost of initiating a new arbitrage position; under these conditions each arbitrage opportunity may be analyzed independently. However, if the costs of closing out an existing position are *less than* that of initiating a new position, then the early close-out option has value. For this reason, it may pay to open an arbitrage position even though the simple arbitrage profit is less than the cost incurred in opening and closing the position at maturity.[8] This

3. Other studies of basis risk in stock index futures include Cornell and French (1983), Modest and Sundaresan (1983), and Figlewski (1984*a*, 1984*b*); Arditti et al. (1986) show that stock index futures arbitrage would have outperformed high-ranking mutual funds over several recent periods.

4. See Cox, Ingersoll, and Ross (1981). French (1983) provides some empirical evidence on the differences between forward and futures prices.

5. See Cornell (1985).

6. See Cornell and French (1983).

7. See Figlewski (1984*a*).

8. The equilibrium implications of this have been noticed by Arditti et al. (1986, p. 63), who write: "The underpricing is often close to, but less than, the round-trip transaction costs. Consequently for the strategy to show a profit, the trade must be unwound when the futures contract is overpriced; overpricing is a frequent occurrence."

makes the strategy followed in closing-out an existing arbitrage position important.

Moreover, the existence of a limit on the maximum position that can be taken makes the option to open an arbitrage position a scarce resource. This affects the early close-out strategy and gives rise to the issue of timing the opening of an arbitrage position; under position limits, the opening of a long arbitrage position today forecloses the possibility of opening a similar position tomorrow on possibly more advantageous terms.

In Section I, we describe the structure of transaction costs. In Section II, we develop basic arbitrage restrictions that must be satisfied by the value of claims that give the right to open and to close out an arbitrage position before maturity. Section III shows how these claims and the optimal strategies for exercising them may be determined under the assumption that the deviation from the theoretical arbitrage relation follows a "Brownian Bridge" process. Section IV describes the data, and Section V presents the empirical results. Section VI concludes the article.

I. Structure of Transaction Costs

Throughout this article, we shall neglect the distinction between futures and forward prices and shall treat all futures contracts as though they were forward contracts.[9] To derive the theoretical arbitrage relation between spot and futures prices, consider a futures contract of maturity τ, and let $F(\tau)$ be the futures price, $P(\tau)$ be the price of a τ-period unit discount bond, and S be the current spot price of the underlying portfolio. Define $G = F(\tau) \cdot P(\tau) + \mathrm{PV}_\tau\,(\mathrm{div})$, where $\mathrm{PV}_\tau\,(\mathrm{div})$ is the present value of the dividends payable on the underlying portfolio up to the maturity of the contract: these dividends are assumed to be riskless. Let ϵ denote the arbitrage profit in the absence of transaction costs to be obtained by taking a long position in the underlying portfolio, hedging it with a short position in the futures contract, and holding the position until maturity of the futures contract: we shall refer to this as a *simple* long arbitrage position; it is simple because it is to be held until maturity. Then

$$\epsilon = G - S. \tag{1}$$

Thus, consider the strategy of buying one unit of the underlying portfolio and borrowing an amount G, which is equal to the present value of the futures price plus the present value of the dividends payable on the underlying portfolio. This strategy yields an immediate cash inflow of ϵ and no further net cash flows since the dividends

9. See Cox, Ingersoll, and Ross (1981).

received on the portfolio will pay off part of the loan, the balance being paid by delivering the underlying portfolio against the futures contract. Therefore, ϵ is the value of the arbitrage profit to be reaped from this simple long arbitrage position.

Note that, if ϵ is negative, an arbitrage profit of $-\epsilon$ can be obtained by reversing the above strategy to obtain what we shall call a simple short arbitrage position.

Since stock index arbitrage involves transactions in both the stock and futures markets, account must be taken of commissions and bid-ask spreads in the two markets. To open an arbitrage position involves a futures commission, a stock commission, and the market impact associated with the stock transaction, due to the bid-ask spread. If the arbitrage position is held to expiration, the only additional cost is the commission to close out the futures position and the stock commission associated with the reversal of the stock position. No market-impact costs are incurred since the stock can be sold at the market-closing price, which is the same as the terminal futures price. However, if the arbitrage position is closed out early, there is an additional cost consisting of the market-impact cost on the stock position.

Therefore, the costs can be conveniently classified as those associated with the simple arbitrage and the incremental costs associated with early close out. The former, which we denote by $C1$, consist of two futures commissions, two stock commissions, and one market-impact cost. The latter, which we denote by $C2$, consists of one market-impact cost.[10]

II. Basic Arbitrage Relations

In the absence of transaction costs or position limits, competitive traders would take an unlimited long or short simple arbitrage position whenever ϵ was nonzero, so that arbitrage would ensure that ϵ was always identically zero. In this section, we consider some arbitrage relations that must be satisfied by the values of claims that give the right to unwind an existing arbitrage position before maturity (the early close-out option) or to initiate a new arbitrage position (the arbitrage option) for a trader who is subject to transaction costs, as described in the previous section, and possibly to position limits also.

A simple long arbitrage position as defined in Section I involves a long position in the underlying portfolio and a short position in the futures contract, held to maturity; ϵ is the riskless profit obtained by establishing such a position. Similarly, we define a short arbitrage posi-

10. A useful discussion of the costs of stock index arbitrage or "program trading" can be found in Stoll and Whaley (1987). For simplicity, we ignore the timing of the second stock commission.

tion as a short position in the underlying portfolio offset by a long position in the futures contract, held to maturity; $-\epsilon$ is the riskless profit from establishing a short position.

A long (short) arbitrage position can be closed-out prior to maturity by taking an offsetting short (long) arbitrage position. In the absence of transaction costs, this yields an additional arbitrage profit of $-\epsilon$ (ϵ).

Let $V(\epsilon, \tau)$ ($V^*(\epsilon, \tau)$) be the value of the right to close a long (short) arbitrage position prior to maturity when the simple arbitrage profit before transaction costs is ϵ and the time to maturity of the futures contract is τ. Similarly, let $W(\epsilon, \tau)$ be the value of the right to initiate an arbitrage position.[11]

A. No Position Limits

In the absence of position limits, closing out a long (short) position yields a net benefit after transaction costs of $-\epsilon - C2$ ($\epsilon - C2$) so that the values of the early close-out options satisfy

$$V(\epsilon, \tau) \geq \max[-\epsilon - C2, 0], \quad (2)$$

$$V^*(\epsilon, \tau) \geq \max[\epsilon - C2, 0]. \quad (3)$$

Initiating an arbitrage position yields not only the immediate arbitrage profit but also the right to close the position prior to maturity. Therefore, the value of the arbitrage option satisfies

$$W(\epsilon, \tau) \geq \max[\epsilon + V(\epsilon, \tau) - C1, -\epsilon + V^*(\epsilon, \tau) - C1, 0]. \quad (4)$$

B. Position Limits

We assume without loss of generality that the arbitrageur is restricted to a single net long or short arbitrage position at any moment in time, perhaps because of capital requirements or self-imposed exposure limits.

With a position limit, closing an outstanding arbitrage position not only yields an immediate arbitrage profit but also gives the arbitrageur the right to initiate a new arbitrage position in the future. Therefore, the values of the early close-out options satisfy

$$V(\epsilon, \tau) \geq \max[W(\epsilon, \tau) - \epsilon - C2, 0], \quad (5)$$

$$V^*(\epsilon, \tau) \geq \max[W(\epsilon, \tau) + \epsilon - C2, 0]. \quad (6)$$

The value of the arbitrage option will still satisfy condition (4).

Of course, at expiration, $\epsilon = 0$, and all three options have zero value, so that

$$V(0, 0) = V^*(0, 0) = W(0, 0) = 0. \quad (7)$$

11. In principle, the value of these options, $V(\cdot)$, etc., may depend on additional state variables. The assumptions we shall make below are sufficient to ensure that the values depend only on ϵ and τ.

III. The Valuation Model

In order to value the arbitrage and early close-out options and determine the optimal strategies for exercising them, it is necessary to make some assumptions about the stochastic evolution of ϵ, the profit from initiating a simple arbitrage position. In this article, we assume that the simple arbitrage profit associated with a given futures contract follows a continuous-time Brownian Bridge process,

$$d\epsilon(\tau) = -\frac{\mu\epsilon}{\tau}dt + \gamma dz, \tag{8}$$

where τ denotes the time to maturity of the futures contract and dz is the increment to a Gauss-Wiener process. This Brownian Bridge process has the property that the arbitrage profit tends to return to zero and is zero at maturity with probability one; μ determines the speed of mean reversion, and γ is the instantaneous standard deviation of the process.[12]

We assume that the values of the options are determined by discounting their expected payoffs at the risk-free interest rate. This assumption is consistent either with risk neutrality or with a representative individual model in which innovations in ϵ are orthogonal to innovations in aggregate consumption. Then it is straightforward to show that, for $\tau > 0$, the values of all three options satisfy a partial differential equation of the form

$$\tfrac{1}{2}\gamma^2 V_{\epsilon\epsilon} - \frac{\mu\epsilon}{\tau}V_\epsilon - V_\tau - rV = 0, \tag{9}$$

where r is the riskless interest rate that is assumed to be constant.

It follows from the symmetry of the stochastic process that

$$V(\epsilon, \tau) = V^*(-\epsilon, \tau). \tag{10}$$

In the absence of position limits, the value of the early close-out option satisfies equation (9) subject to the boundary conditions (2) and (7). Under an optimal exercise strategy, the Merton-Samuelson high-contact condition implies

$$V(\epsilon_c(\tau), \tau) = -\epsilon_c(\tau) - C2, \tag{11}$$

and

$$V_\epsilon(\epsilon_c(\tau), \tau) = -1, \tag{12}$$

where $\epsilon_c(\tau)$ is the maximum value of ϵ at which it is optimal to close out a long arbitrage position. It follows from the symmetry of the problem

12. MacKinlay and Ramaswamy (1988) present some preliminary evidence that the actions of arbitrageurs cause the stochastic process for ϵ to be path dependent, rather than Markov, as we have assumed.

that $\epsilon_c(\tau) = -\epsilon_c^*(\tau)$, where $\epsilon_c^*(\tau)$ is the minimum value of ϵ at which it is optimal to close out a short arbitrage position.

Without position limits, it will pay to initiate arbitrage positions whenever the right-hand side of expression (4) is positive. The assumed exogeneity of the stochastic process for ϵ implies that the arbitrage profits would be infinite. For this reason we shall concentrate below on the more realistic case in which the arbitrageur is subject to position limits.

Under position limits, the value of the early close-out option satisfies equation (9) subject to the boundary conditions (5) and (7), while the value of the arbitrage option satisfies equation (9) subject to the boundary conditions (4) and (7). Under optimal exercise policies for the close-out options, their values will satisfy the high-contact condition

$$V(\epsilon_c(\tau), \tau) = W(\epsilon_c(\tau), \tau) - \epsilon_c(\tau) - C2, \tag{13}$$

$$V_\epsilon(\epsilon_c(\tau), \tau) = W_\epsilon(\epsilon_c(\tau), \tau) - 1, \tag{14}$$

where, as before, ϵ_c is the optimal value of ϵ at which to close a long arbitrage position, and $\epsilon_c^* = -\epsilon_c$.

Under the optimal exercise policy for the arbitrage option, its value satisfies the corresponding high-contact condition:

$$W(\epsilon_o(\tau), \tau) = \epsilon_o(\tau) + V(\epsilon_o(\tau), \tau) - C1, \tag{15}$$

$$W_\epsilon(\epsilon_o(\tau), \tau) = 1 + V_\epsilon(\epsilon_o(\tau), \tau), \tag{16}$$

$$W(-\epsilon_o(\tau), \tau) = -\epsilon_o(\tau) + V(-\epsilon_o(\tau), \tau) - C1, \tag{17}$$

$$W_\epsilon(-\epsilon_o(\tau), \tau) = -1 + V_\epsilon(-\epsilon_o(\tau), \tau), \tag{18}$$

where ϵ_o $(-\epsilon_o)$ is the optimal value of ϵ at which to open a long (short) arbitrage position.

Figure 1 illustrates the valuation schedules for the arbitrage and early close-out options. The vertical dashed lines correspond to the values of ϵ at which it is optimal to initiate and close out an arbitrage position. It is generally the case that $\epsilon_c < \epsilon_c$ because the early close-out cost is less than the cost of opening a new position. When $\epsilon > \epsilon_c$, it is optimal to close out the short position, which yields a direct profit of $\epsilon - C2$ after transaction costs and the option, W, to initiate a new position. For $\epsilon > \epsilon_o$, it is also optimal to exercise this option so that $W - V = \epsilon - C1$ because initiating a long position yields a direct profit $\epsilon - C1$ and the close-out option V. Combining these results, we have for $\epsilon > \epsilon_c$, $V^* - V = 2\epsilon - C1 - C2$.

The V and V^* schedules are monotone in ϵ because they relate to the option to close out a long position in one case and a short position in the other. The W schedule, in contrast, is symmetric because it relates to the option to initiate either a long or a short position, and this option has the same value whether exercised at ϵ or $-\epsilon$.

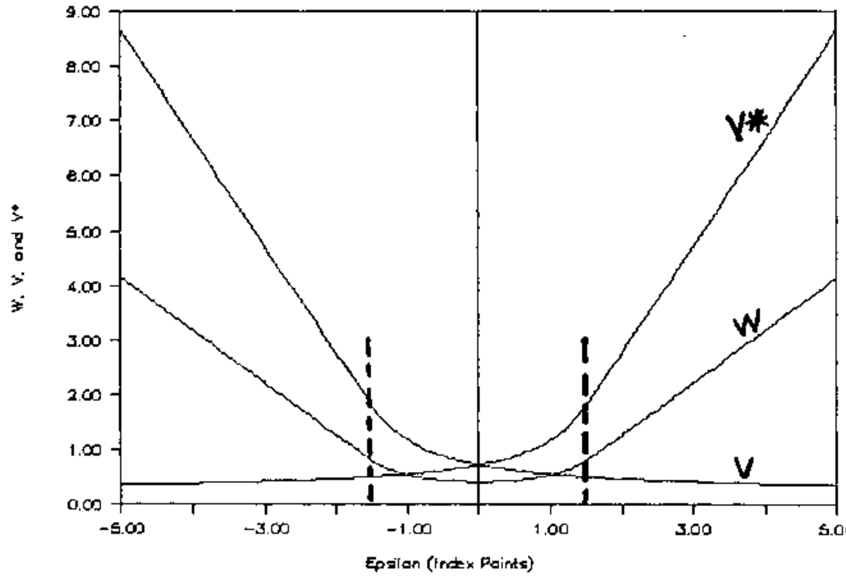

FIG. 1.—Values of the arbitrage (W) and early close-out (V and V^*) options; $\mu = 2.28$, $\gamma = 0.30$, $r = 0.07$, $C1 = 1.20$, $C2 = 0.50$.

IV. Data

The value of the simple arbitrage opportunity is defined by

$$\epsilon(t) = F(T, t) \cdot e^{-r(T-t)} + \mathrm{PV}_t\,(\mathrm{div}) - S(t), \tag{19}$$

where $F(T, t)$ is the futures price at time t for a contract maturing at time T, r is the riskless interest rate, PV_t (div) is the present value of the daily dividends on the S&P 500 index portfolio up to the maturity of the contract, and $S(t)$ is the value of the index at time t.

In this article, the measure of the value of the simple arbitrage opportunity is a "mispricing" series calculated by Craig MacKinlay and Krishna Ramaswamy (personal communication, 1988) and kindly supplied to us. This series is derived from transactions data on the S&P 500 spot and nearby futures prices every 15 minutes for the period June 17, 1983, to June 18, 1987;[13] it employs the daily Center for Research in Security Prices (CRSP) value-weighted dividend yield and the certificate of deposit (CD) rate as the riskless interest rate.

This measure of the value of the simple arbitrage opportunity is imperfect for two reasons. First, it assumes that the dividends are known with certainty. The error introduced by this assumption is likely to be small since Kipnis and Tsang (1984) found that dividends can

13. It excludes the 3:15 P.M. futures observations for which there is no synchronous spot observation.

vary by as much as 10% while moving the implied futures price .20 index points, only or about 0.1%. Second, the value of the S&P 500 index itself is based on the last trade prices, which may cause the index to lag in changes in equilibrium prices. However, Collins (1986) has found that the value of the index almost always lies within the bounds established by bid-and-ask prices of the underlying stocks.

V. Empirical Results

A. The Stochastic Process for ϵ

It can be shown by standard methods that, under the Brownian Bridge assumption (eq. [8]), the behavior of ϵ, the value of the simple arbitrage opportunity, can be represented by

$$\epsilon(t) = \gamma(T - t)^{\mu} B\left[\frac{(T - t)^{1-2\mu}}{2\mu - 1}\right], \tag{20}$$

where $\{B(s);\ s \geq 0\}$ is a standard Brownian motion.[14]

This implies that the log-likelihood function can be written as

$$\ln L(\mu, \gamma) = -\frac{T}{2} \ln 2\pi - \sum_{t=1}^{T} \ln \sigma_t - \frac{1}{2} \sum_{t=1}^{T} \frac{u_t^2}{\sigma_t^2}, \tag{21}$$

where

$$u_t \equiv \epsilon(t + 1) - \left(1 - \frac{1}{T - t}\right)^{\mu} \epsilon(t),$$

and

$$\sigma_t^2 \equiv \frac{\gamma^2(T - t - 1)}{2\mu - 1}\left[1 - \left(1 - \frac{1}{T - t}\right)^{2\mu - 1}\right].$$

The parameters μ and γ of the stochastic process for ϵ were estimated by maximizing the log-likelihood function (21), and the results are reported in table 1.[15] There is considerable variation in the parameter estimates from contract to contract. However, in all cases, the estimate of the mean reversion parameter is highly significant. The overall estimate of 2.28 implies that the expected half-life of ϵ is 26.2% of the remaining time to expiration.

B. Value of the Arbitrage Opportunity

To gain some insight into the value of the arbitrage opportunity when there are transaction costs and position limits, partial differential equa-

14. See, e.g., Karlin and Taylor (1981), ch. 15.

15. In estimating the stochastic process, data for the last 5 trading days of the contract were omitted since it was found that the parameter estimates were very sensitive to these data.

TABLE 1 Estimation of μ and γ from Maximization of Log-likelihood Function (21)

$$d\epsilon(t) = \left(\frac{-\mu\epsilon(t)}{\tau}\right)dt + \gamma dz$$

Contract	No. of Observations	μ	γ
September 1983	1,391	3.51 (.48)	.25
December 1983	1,391	.82 (.20)	.19
March 1984	1,367	1.13 (.21)	.21
June 1984	1,319	5.03 (.55)	.21
September 1984	1,511	2.04 (.32)	.25
December 1984	1,415	.55 (.16)	.22
March 1985	1,247	.98 (.20)	.22
June 1985	1,511	2.60 (.36)	.18
September 1985	1,391	1.22 (.23)	.17
December 1985	1,533	1.79 (.26)	.21
March 1986	1,491	3.59 (.39)	.32
June 1986	1,517	3.12 (.39)	.35
September 1986	1,517	1.41 (.21)	.36
December 1986	1,543	3.91 (.43)	.36
March 1987	1,491	14.37 (1.30)	.52
June 1987	1,515	10.27 (.92)	.51
All	23,169	2.28 (.08)	.30

NOTE.—See text for explanation. Values in parentheses are standard errors.

tion (9) was solved numerically subject to the boundary conditions (15)–(18) and (7). The values of μ and γ were those obtained using all the data, $\mu = 2.28$, and $\gamma = 0.30$.

The transaction-cost assumption was derived from the study of Stoll and Whaley (1987), which is summarized in table 2. For our purposes, these costs must be expressed in the same units as ϵ, that is, in index points. Assuming an S&P 500 level of 200, we obtain $C1 = (62{,}500/50{,}000) \approx 1.2$ index points, and $C2 = (25{,}000/50{,}000) \approx 0.5$ index points, where 50,000 is the index multiple of a round lot.

Table 3 gives us the values of the arbitrage and early close-out op-

TABLE 2 **Estimated Transaction Costs for 100 Standard and Poor's 500 Futures Contracts (Each Contract Being 500 Times the Value of the Index)**

	Cost in Dollars
Round-lot transaction cost:	
Futures commission	1,250
Stock commission	17,500
Market impact	25,000
Cost of opening and sunk close-out cost:	
2 futures commission	2,500
2 stock commissions	35,000
1 market impact	25,000
Total	62,500
Cost of *early* close out:	
1 market impact	25,000

SOURCE.—Stoll and Whaley (1987).

TABLE 3 **Values of Open (W) and Early Close-Out (V for a Long and V^* for a Short Position) Options**

$\mu = 2.28$; $\gamma = 0.30$; $\tau = 90$ days;

$r = 0.07$; $C1 = 1.20$; $C2 = 0.50$

ϵ	−1.50	−1.00	−.50	.00	.50	1.00	1.50
V	.51	.56	.63	.73	.89	1.19	1.82
V^*	1.82	1.19	.89	.73	.63	.56	.51
W	.82	.55	.45	.42	.45	.55	.82

tions in index points for different values of ϵ when the contract has 90 days to expiration. These values may be compared with the value of the index in the neighborhood of 200. This would imply that a holder of the S&P portfolio with the assumed transaction costs could increase her rate of return by approximately 0.25% per quarter if she were able to sell her stock and replace it with a long futures position, on the one hand, or purchase an equal amount of stock on margin and hedge it with a short futures position, on the other hand.

In solving the differential equation for the value of the options, we also obtain the optimal strategy for initiating and closing out arbitrage positions. In the next subsection we simulate the result of following the optimal strategy on the sample contracts.

C. *Simulated Arbitrage Strategy*

The Appendix shows plots of the evolution of ϵ, the simple arbitrage profit, against calendar time for each of the 16 contracts maturing from September 1983 to June 1987. Each graph contains 1,300–1,600 observations. Note that observations on the same day form vertical lines and gaps between vertical lines represent nontrading days. Superimposed on each graph are time series of estimated critical ϵ values above or below which it is optimal either to initiate or to terminate an arbitrage position. These were obtained using the stochastic-process parameter estimates derived from the whole sample of 16 contracts. The inner pair of critical values of ϵ determine a band outside which it is optimal to close out a preexisting arbitrage position. The outer pair determine a band outside which it is optimal to initiate a new arbitrage position.

It is of interest to note that for some parameter values it may be optimal to open a new arbitrage position even when the simple arbitrage profit is less than the cost of executing the simple arbitrage. The reason for this is that a simple arbitrage position carries with it an option to close out early and thereby make an additional arbitrage profit. In this case, as the contract approaches expiration and the probability of a profitable early closeout decreases, the minimum ϵ required to open a new arbitrage position increases. However, for the parameter values used here, the critical ϵ decreases monotonically as expiration approaches.

Maximum arbitrage profits are realized when ϵ passes from one critical bound to another. This occurs relatively infrequently for the 1983–85 contracts. However, the 1986 and 1987 contracts are considerably more profitable. Table 4 reports the trades that would have been made under the optimal strategy for each contract and the profits that would have been earned. Each row of the table corresponds to a trade. The first column is calendar time, calculated by the maximum number of days in the contract minus the time to maturity. The second column gives the value of the simple arbitrage opportunity. The third column describes the nature of the arbitrage position to be taken. Note that a switch from a long position to a short position or vice versa yields two simple arbitrage profits. "Out" denotes the closing of the preexisting position without opening a new position. Columns 4 and 5 ("In" and "Out") denote the critical values of ϵ at which a short position should be taken and a long position should be closed out. The critical values for opening a long position and closing a short position are just the negatives of these. Profit 1 reported in column 6 is based on the assumption that transactions take place at the first price after the critical value of ϵ is passed. Profit 2 in column 7 is based on the more conservative assumption that the transactions take place at the critical value of ϵ. Column 8 reports the costs incurred in each transaction.

The average value for all 16 contracts of profit 1 after transactions costs is 1.00. The corresponding figure for profit 2 is 0.54. These average profit figures may be compared with the theoretical value of the arbitrage opportunity of 0.42, which is the value of the right to open an arbitrage position on a 90-day contract when ϵ equals zero, as reported

TABLE 4 **Arbitrage Transactions by Calendar Time for All 16 Contracts (in Index Points)**

Calendar Time (in Days) by Contract (1)	ϵ (2)	Position (3)	Critical In (4)	Critical Out (5)	Profit 1 (6)	Profit 2 (7)	Transaction Costs (8)
September 1983:							
10.76	−1.74	Short	1.50	1.40	1.74	1.50	1.20
42.96	1.21	Out	1.40	1.20	1.21	1.20	.50
47.93	1.48	Long	1.40	1.15	1.48	1.40	1.20
Total					4.43	4.10	2.90
Net profit					1.53	1.20	
December 1983:							
31.92	1.50	Long	1.45	1.30	1.50	1.45	1.20
Net profit					.30	.25	
March 1984:							
4.76	1.74	Long	1.55	1.45	1.74	1.55	1.20
Net profit					.54	.35	
June 1984			No transactions				
September 1984:							
46.79	−1.43	Short	1.40	1.20	1.43	1.40	1.20
52.78	1.21	Out	1.40	1.15	1.21	1.15	.50
59.90	1.42	Long	1.35	1.05	1.42	1.35	1.20
Total					4.06	3.90	2.90
Net profit					1.16	1.00	
December 1984:							
.76	1.91	Long	1.55	1.45	1.91	1.55	1.20
Net profit					.71	.35	
March 1985:							
.76	3.53	Long	1.55	1.45	3.53	1.55	1.20
Net profit					2.33	.35	
June 1985:							
.76	2.26	Long	1.60	1.50	2.26	1.60	1.20
Net profit					1.06	.40	
September 1985:							
11.70	1.50	Long	1.60	1.50	1.50	1.60	1.20
67.78	−.87	Out	1.3	.85	.87	.85	.50
Total					2.37	2.45	1.70
Net profit					.67	.75	
December 1985:							
10.99	−1.59	Short	1.50	1.40	1.59	1.50	1.20
73.94	.87	Out	1.30	.80	.87	.80	.50
Total					2.46	2.30	1.70
Net profit					.76	.60	
March 1986:							
16.99	−1.56	Short	1.50	1.40	1.56	1.50	1.20
42.94	1.27	Out	1.40	1.15	1.27	1.15	.50
Total					2.83	2.65	1.70
Net profit					1.13	.95	

TABLE 4 (Continued)

Calendar Time (in Days) by Contract (1)	ϵ (2)	Position (3)	Critical In (4)	Critical Out (5)	Profit 1 (6)	Profit 2 (7)	Transaction Costs (8)
June 1986:							
1.99	1.58	Long	1.55	1.45	1.58	1.55	1.20
10.99	−1.41	Out	1.50	1.40	1.41	1.40	.50
49.94	−1.46	Short	1.40	1.10	1.46	1.40	1.20
81.79	.67	Out	1.25	.65	.67	.65	.50
Total					3.54	3.45	2.20
Net profit					1.34	1.25	
September 1986:							
.97	−1.89	Short	1.55	1.45	1.89	1.55	1.20
77.81	.70	Out	1.30	.70	.70	.70	.50
80.81	1.82	Short	1.25	.65	1.82	1.25	1.20
88.95	.54	Out	1.20	.50	.54	.50	.50
Total					3.06	2.45	2.20
Net profit					.86	.25	
December 1986:							
.74	−2.30	Short	1.55	1.45	2.30	1.55	1.20
60.90	.95	Out	1.35	.96	.95	.96	.50
71.00	1.48	Long	1.30	.85	1.48	1.30	1.20
80.80	−1.42	Short	1.25	.65	2.84	1.90	1.70
84.92	.84	Out	1.25	.60	.84	.60	.50
Total					5.16	3.80	3.40
Net profit					1.76	.40	
March 1987:							
7.99	−1.76	Short	1.55	1.45	1.76	1.55	1.20
35.76	1.79	Long	1.45	1.25	3.58	2.70	1.70
52.75	−1.14	Out	1.35	1.05	1.14	1.05	.50
86.75	−1.53	Short	1.20	.55	1.53	1.20	1.20
88.78	.76	Out	1.20	.50	.76	.50	.50
88.96	1.58	Long	1.20	.50	1.58	1.20	1.20
88.88	−1.04	Out	1.20	.50	1.04	.50	.50
Total					3.38	2.20	2.20
Net profit					1.18	.00	
June 1987:							
21.98	−1.59	Short	1.50	1.35	1.59	1.50	1.20
77.80	.72	Out	1.30	.70	.72	.70	.50
Total					2.31	2.20	1.70
Net profit					.61	.50	
Average for all 16 contracts:							
Total					2.78	2.33	1.79
Net profit					1.00	.54	

NOTE.—See text for complete explanation.

TABLE 5 **Sensitivity of the Critical Value of ϵ in Values to Stochastic Process Parameter Estimates (Critical Value of ϵ to Initiate an Arbitrage Position)**

	Parameter Estimates		Calendar Time (in Days)										
Contract	μ	γ	0	10	20	30	40	50	60	70	80	90	99
September 1983	3.51	.25	1.35	1.35	1.35	1.30	1.30	1.30	1.30	1.30	1.25	1.25	1.20
December 1983	.82	.19	1.70	1.70	1.65	1.60	1.55	1.55	1.50	1.45	1.35	1.30	1.20
March 1984	1.13	.21	1.65	1.60	1.60	1.55	1.50	1.50	1.45	1.40	1.35	1.30	1.20
June 1984	5.03	.21	1.25	1.25	1.25	1.25	1.25	1.25	1.25	1.25	1.25	1.20	1.20
September 1984	2.04	.25	1.55	1.50	1.45	1.45	1.40	1.40	1.35	1.35	1.30	1.25	1.20
December 1984	.55	.22	2.00	1.95	1.90	1.85	1.80	1.75	1.65	1.60	1.50	1.35	1.20
March 1985	.98	.22	1.75	1.70	1.65	1.65	1.60	1.55	1.50	1.45	1.40	1.30	1.20
June 1985	2.60	.18	1.35	1.35	1.35	1.30	1.30	1.30	1.30	1.25	1.25	1.25	1.20
September 1985	1.22	.17	1.50	1.50	1.45	1.45	1.40	1.40	1.35	1.35	1.30	1.25	1.20
December 1985	1.79	.21	1.50	1.45	1.45	1.40	1.40	1.40	1.35	1.30	1.30	1.25	1.20
March 1986	3.59	.32	1.45	1.45	1.40	1.40	1.35	1.30	1.30	1.30	1.30	1.25	1.20
June 1986	3.12	.35	1.60	1.55	1.50	1.45	1.40	1.40	1.35	1.30	1.30	1.25	1.20
September 1986	1.41	.36	2.00	1.95	1.90	1.80	1.75	1.70	1.60	1.55	1.45	1.35	1.20
December 1986	3.91	.36	1.50	1.50	1.45	1.40	1.35	1.35	1.30	1.30	1.25	1.25	1.20
March 1987	14.37	.52	1.30	1.25	1.25	1.20	1.20	1.15	1.15	1.15	1.20	1.25	1.20
June 1987	10.27	.51	1.40	1.35	1.35	1.30	1.25	1.20	1.20	1.20	1.20	1.25	1.20
All	2.28	.30	1.60	1.55	1.55	1.50	1.45	1.40	1.40	1.35	1.30	1.25	1.20

in table 3. In addition to sampling error, the discrepancy may be attributed to the discreteness of the price observations, as well as to possible misspecification of the stochastic process.

The theoretical model assumes that the value of the simple arbitrage profit may be monitored continuously and that arbitrage positions may be taken at any time. Also, the optimal policy was derived using the stochastic process parameters estimated from the observations for all contracts despite the fact that we could reject the constancy of the mean reversion parameter across contracts. Table 5 shows the sensitivity of the optimal policy to the estimation period in terms of the critical value of ϵ at which it is optimal to initiate an arbitrage position. These results suggest that, in particular for days that are far away from maturity, the critical ϵ values are sensitive to the parameter estimates.

VI. Conclusion

In this article we have developed the optimal strategy for a program trader or arbitrageur in stock index futures contracts and have simulated the performance of the strategy on 16 futures contracts maturing from 1983 to 1987. The optimal policy depends on the stochastic process that describes the evolution of the simple arbitrage opportunity. We assumed that the simple arbitrage opportunity follows a Brownian Bridge process whose parameters we estimated. The parameters were not stationary across contracts, possibly because the contract is still maturing.

The real challenge, however, remains to endogenize the stochastic behavior of the simple arbitrage opportunity given the nature of transaction costs and the structure of the market.

Appendix

Standard and Poor's 500 Stock Index Futures Contracts, September 1983–June 1987

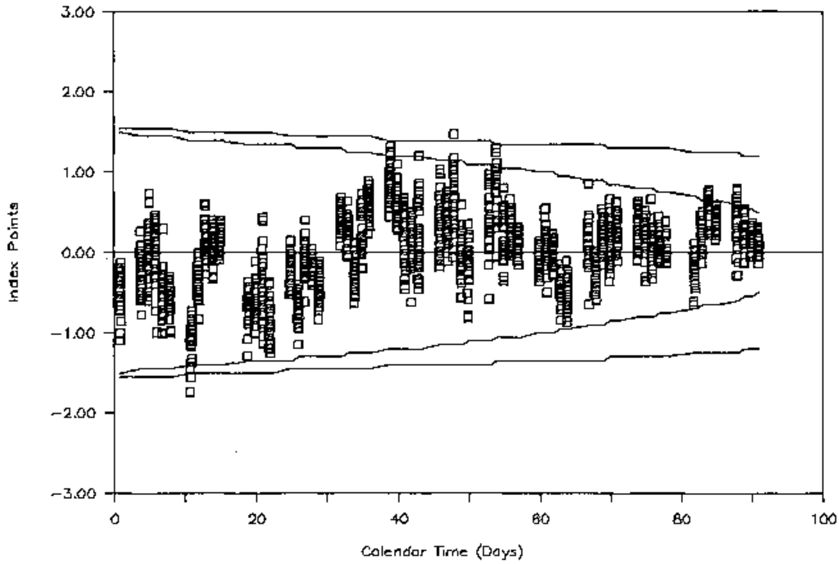

FIG. A1.—September 1983 contract

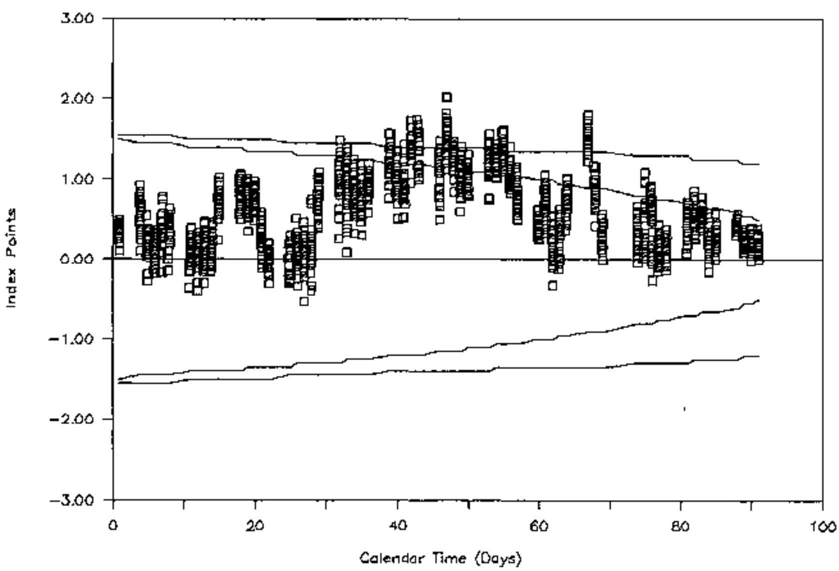

FIG. A2.—December 1983 contract

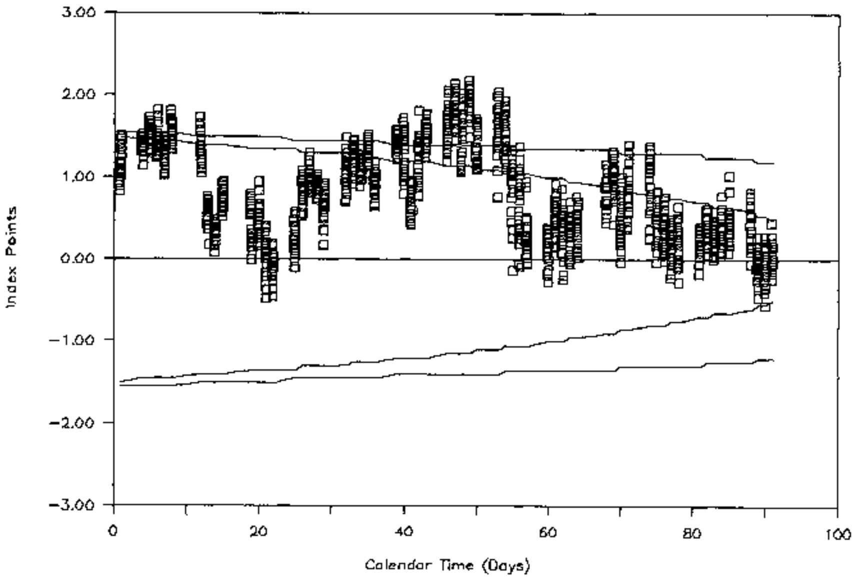

FIG. A3.—March 1984 contract

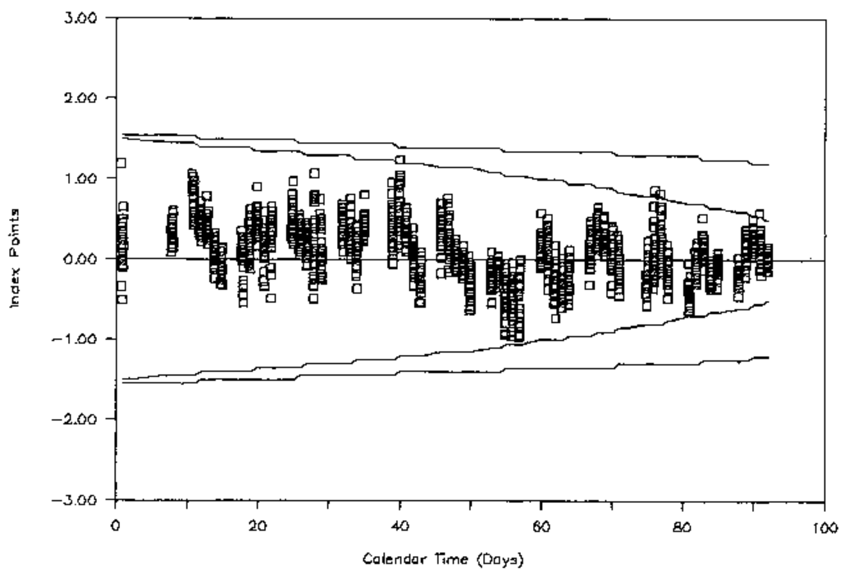

FIG. A4.—June 1984 contract

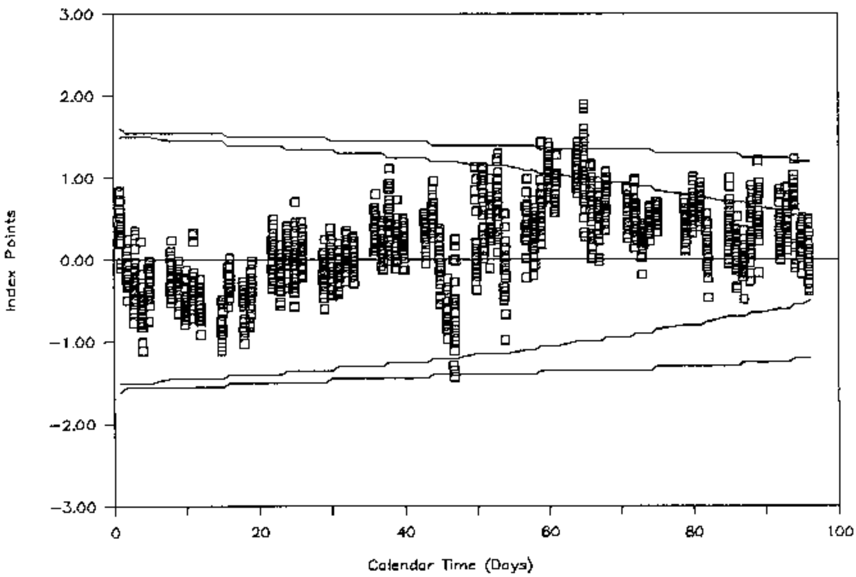

FIG. A5.—September 1984 contract

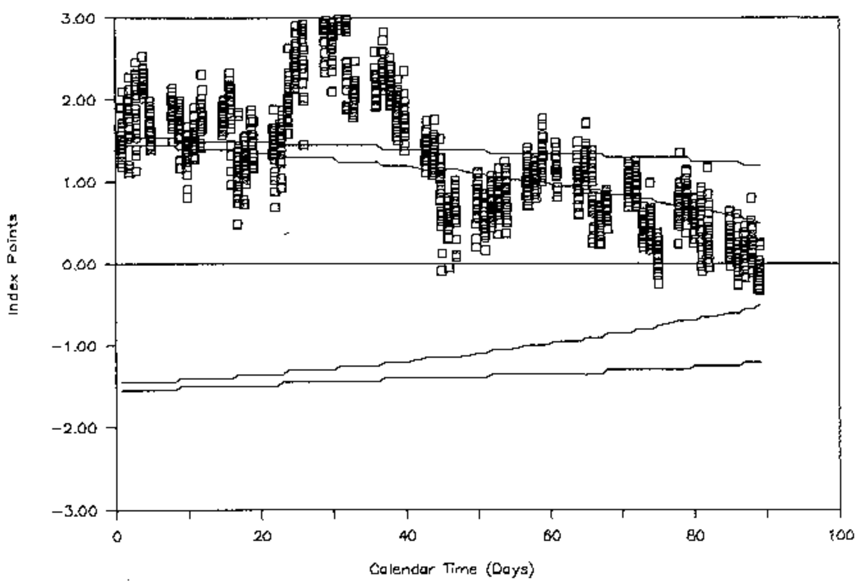

FIG. A6.—December 1984 contract

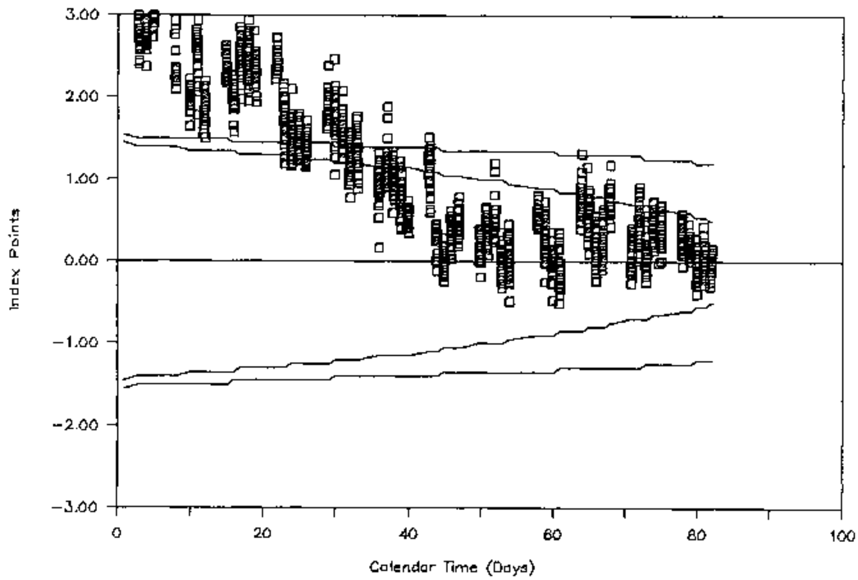

FIG. A7.—March 1985 contract

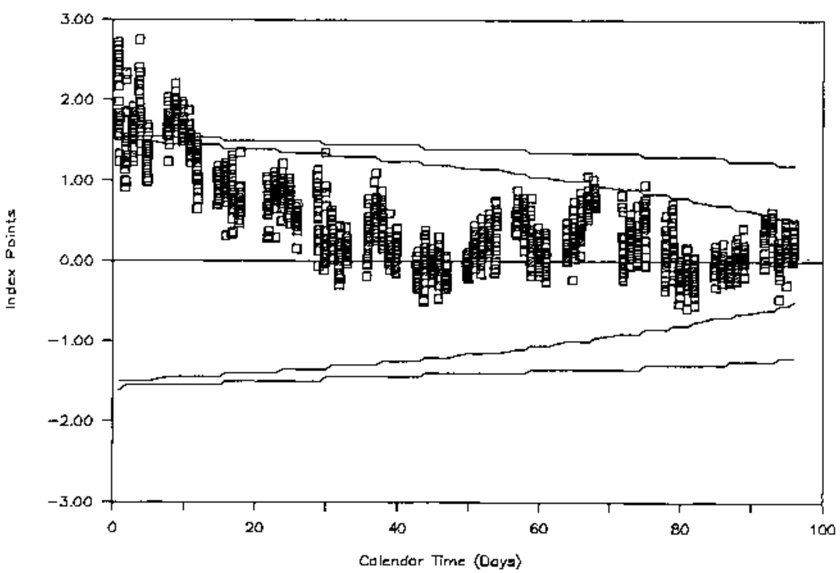

FIG. A8.—June 1985 contract

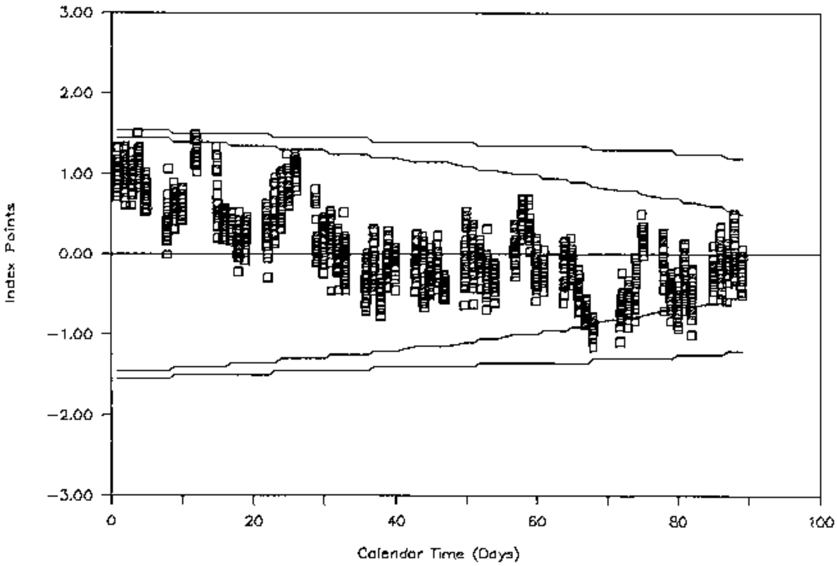

FIG. A9.—September 1985 contract

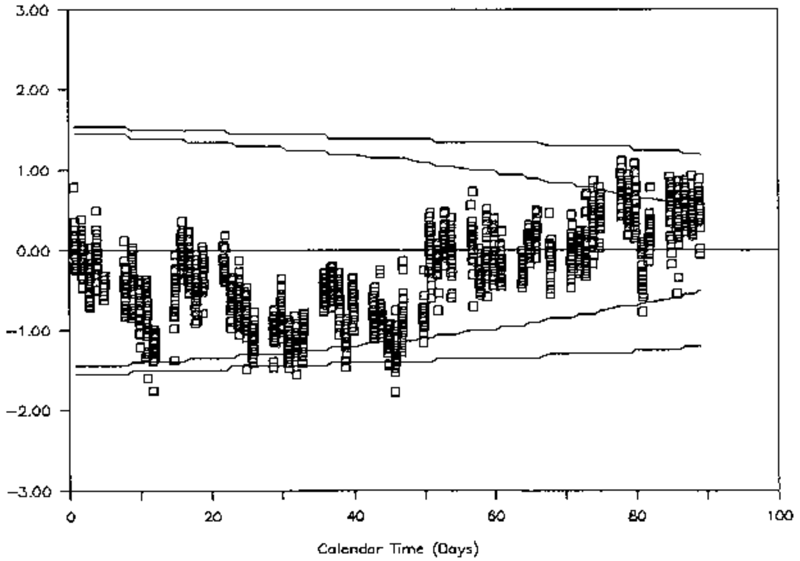

FIG. A10.—December 1985 contract

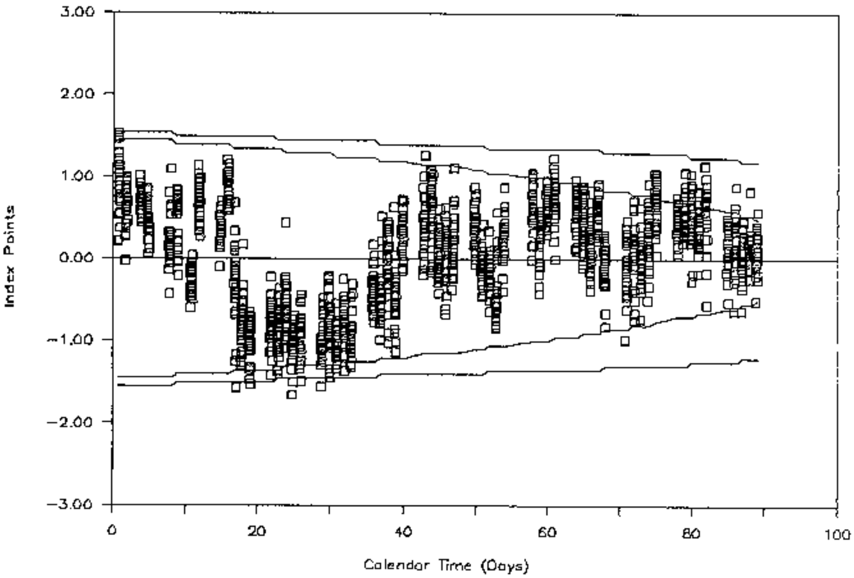

FIG. A11.—March 1986 contract

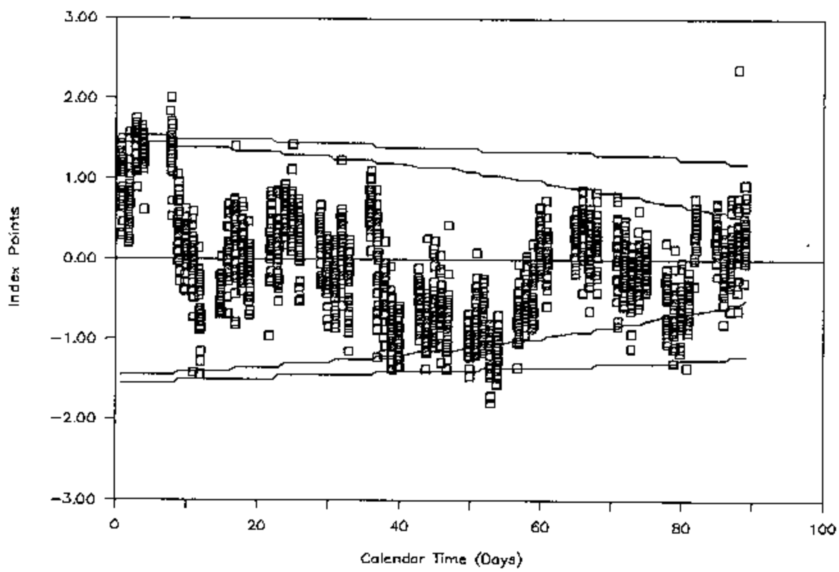

FIG. A12.—June 1986 contract

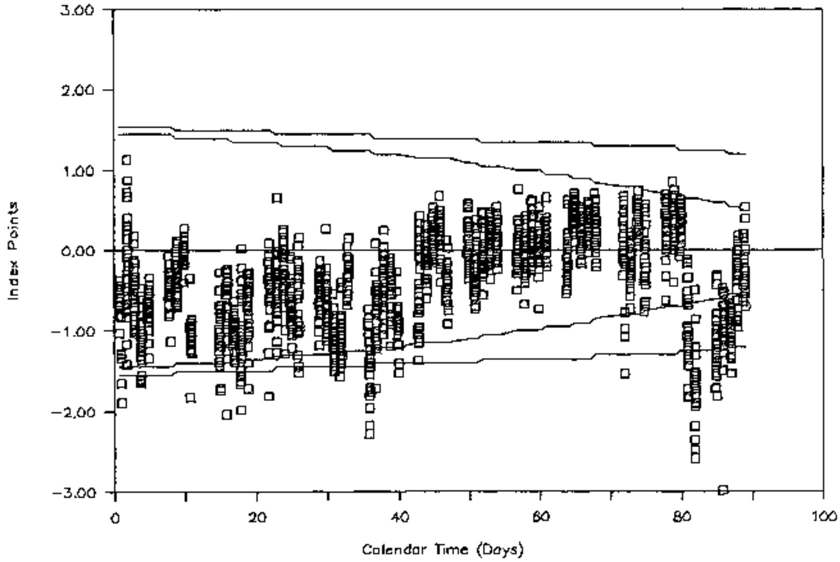

FIG. A13.—September 1986 contract

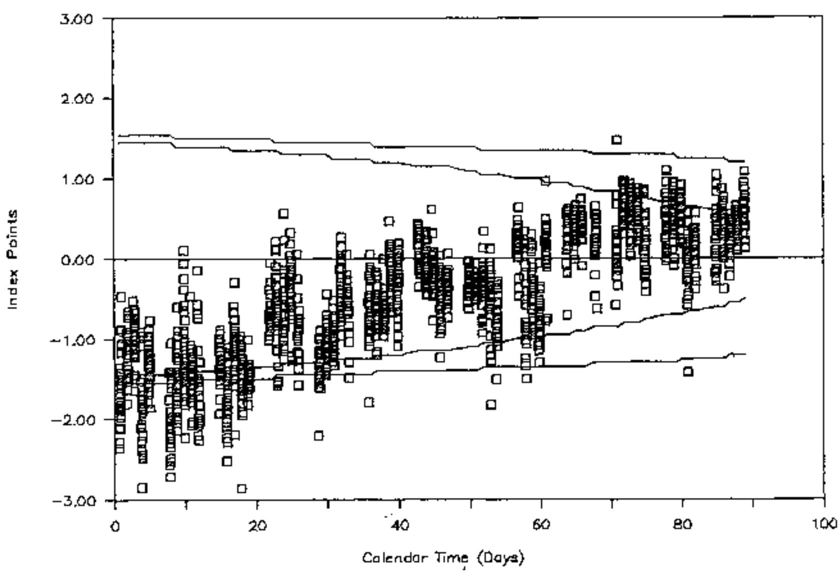

FIG. A14.—December 1986 contract

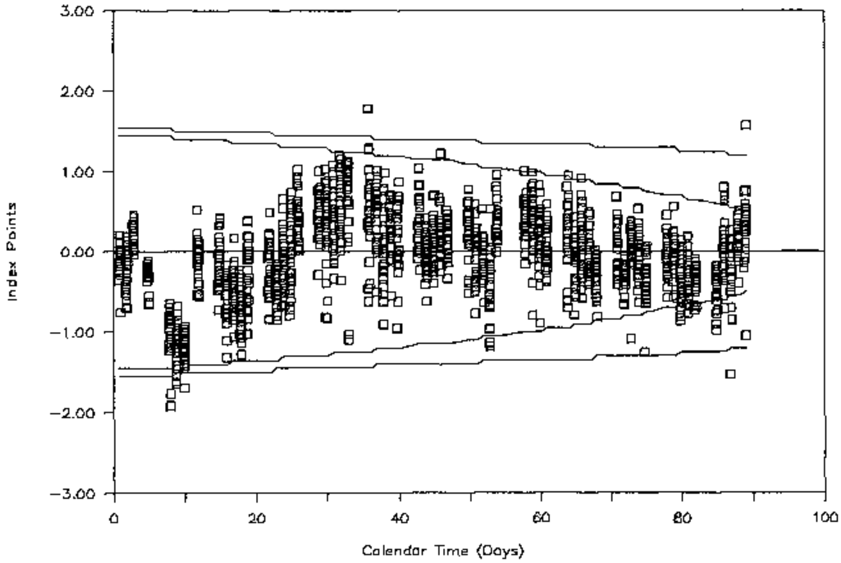

FIG. A15.—March 1987 contract

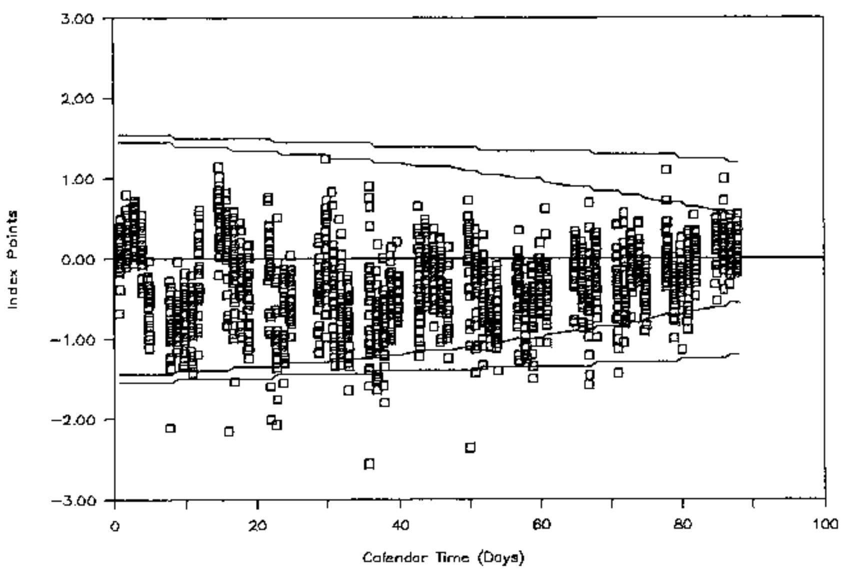

FIG. A16.—June 1987 contract

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