A Mean/Variance Analysis of Tracking Error

Minimizing the volatility of tracking error will not produce a more efficient managed portfolio.

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Today's professional money manager is often judged by total return performance relative to a prespecified benchmark, usually a broadly diversified index of assets. At each performance review, the fund sponsor and manager discuss recent month-to-month total returns, net of fees and expenses, and compare them to the published returns of the benchmark.

This is a sensible approach because the sponsor's most direct alternative to an active manager is an index fund matching the benchmark. An index fund will indeed provide returns very close to those of the benchmark because both fees and transaction costs are low for index funds. Thus, an active manager is worth retaining only if performance is on average positive.

Unfortunately, asset returns are exceedingly noisy, and a long time can elapse before the average performance is known and the fund sponsor can be statistically assured that the manager is adding (or subtracting) value. This has led many sponsors to focus on the *volatility* of tracking error, i.e., the month-to-month variability of the difference between the manager's return and the benchmark's return. Minimization of tracking error volatility has become an important criterion for assessing overall manager performance. Lower tracking error volatility implies fewer months of abysmal performance but also, of course, fewer months of relatively superb results.

Low tracking error volatility seems a sensible goal for most plan sponsors for at least two reasons. First, *ideal* active management would outperform the benchmark every single month by a fixed amount net of fees and expenses. This implies *zero* tracking error volatility. It is ideal because the fund sponsor could ascertain with complete statistical reliability that the manager is adding value over an index fund alternative.

Second, executives in charge of investments for fund sponsors are themselves reviewed at least annually on how well their managers have performed. If a manager has beaten broad market indexes, the executive is congratulated on a superior job of selecting managers a priori. On the other hand, if a manager has had a run of bad luck, say six consecutive months of underperformance, the executive is questioned as to whether a new manager should be chosen, regardless of whether the review period is of sufficient length to allow reliable assessment of manager quality. (It almost never *is* long enough.)

Thus, current professional investment practice follows a two-dimensional approach to performance. Beating the benchmark on average is tantamount to having a positive expected tracking error. Reducing the volatility of tracking error is tantamount to minimizing the variance of the *difference* between managed portfolio returns and benchmark returns. This is mean/variance analysis but instead of finding the portfolio with the smallest *total return* volatility for a given expected total return, a Markowitz [1959] EV efficient portfolio, money managers are obliged to find the portfolio with minimum tracking error variance for a given expected performance relative to the benchmark. We shall call this the *TEV Criterion*: minimization of tracking error variance for a given expected tracking error.

The purpose of this article is to formalize this TEV performance criterion and describe its inevitable consequences for overall portfolio efficiency, i.e., for full-blown EV optimization. We show below that a manager successfully pursuing the TEV tracking error criterion will *intentionally* not produce a mean/variance Markowitz efficient portfolio under most circumstances. Even with perfect expected return information, the manager will select a portfolio dominated by other portfolios with higher average returns and lower volatilities (although not lower tracking error volatilities).

The TEV Criterion requires every manager with the same beliefs and the same asset universe to conduct the same trades, *regardless of the benchmark*. Imagine two active managers A and B who assume discretionary control of what had previously been index funds. Manager A receives an S&P 500 index fund, B a NASDAQ 100 index fund; their benchmarks henceforward are the S&P 500 and the NASDAQ 100 indexes, respectively. Managers A and B have identical beliefs about individual asset expected returns, variances, and covariances. The fund sponsor targets the expected performance of each manager at 200 basis points annually over the respective benchmark.

If A and B are allowed to choose any domestic exchange-listed or NASDAQ equity, both would immediately conduct the same trades. Their resulting managed portfolios would be different but only because their original endowments were different (they would not sell everything). *Alterations* to their original endowments would be identical.

The actively managed TEV portfolio will lie exactly as far away from the global EV efficient frontier as the original benchmark. If the benchmark happens to be EV efficient, the TEV portfolio will also be on the global efficient frontier; but if the benchmark is inefficient (as many surely are), the TEV managed portfolio will be inefficient by exactly the same amount in volatility (variance), *regardless of the targeted average performance.*

Managers are sometimes asked to produce portfolios whose "betas" on the benchmark lie within a given range.[1] A fund sponsor might very well ask for a beta of 1.0, recognizing from financial theory that a portfolio with a higher (lower) beta "ought" to have a higher (lower) return than the implicit alternative, the index fund. In general, such a demand is inconsistent with TEV optimization! It is impossible to produce a TEV portfolio that has minimum (unconstrained) tracking error variance, a given expected performance, and *also* a specified beta. In fact, if the benchmark's average return exceeds the mean return on the global minimum variance portfolio, the TEV portfolio's beta will always exceed unity.

This is a particularly troubling result because we also prove here that all portfolios that dominate the benchmark by possessing both a higher mean return and a lower total volatility have betas on the benchmark *less than 1.0.* This implies that a positive performance TEV portfolio can never be a dominating portfolio in the mean/variance sense. Indeed, its characteristics are fundamentally different because it will have more "market risk"[2] than the benchmark while all dominating portfolios have less market risk.

Of course, it *is* possible to minimize tracking error variance *conditional* on a specified beta. We show below that in some cases the resulting portfolio can have a dramatically greater tracking error volatility than the unconstrained optimum TEV portfolio. However, there are other cases for which the beta-constrained managed portfolio actually has a lower overall volatility (although it always has a higher tracking error volatility). Minimizing tracking error while constraining the beta on the benchmark can actually produce dominating portfolios. Thus, it has some attractive features.

THE TEV TRACKING ERROR FRONTIER

Assume that a professional investment manager has been obliged to pursue the following objective: Minimize the variance of tracking error conditional on a given level of expected performance relative to a specified benchmark. This is a straightforward optimization problem that can be solved analytically when there are no short-selling constraints.

The algebraic solution employs the notation following:[3]

N = The number of individual assets in the manager's universe. This does not have to be comprehensive; the manager might be restricted to equities, for instance, or to growth stocks.

q = An (N × 1) vector representing a portfolio. The kth entry is the portfolio's proportion invested in asset k. Subscripts identify the portfolio; e.g., $\mathbf{q}_B$ is the benchmark. In all cases, the portfolio weights sum to unity; i.e., using standard matrix transpose notation, $\mathbf{q}'\mathbf{1} = 1$, where **1** is an (N × 1) vector of 1's.

x = An (N × 1) vector representing a portfolio alteration (or exchange) that is self-financing; i.e., $\mathbf{x}'\mathbf{1} = 0$. If $\mathbf{q}_P$ is the actively managed portfolio P, $\mathbf{x} = \mathbf{q}_P - \mathbf{q}_B$ represents the difference, stock by stock, between the managed portfolio and the benchmark. Note that **x** can contain negative entries whenever the managed portfolio contains less of an asset than the benchmark. Note also that **x**, $\mathbf{q}_P$, and $\mathbf{q}_B$ could conceivably have many zero entries. If, for example, the benchmark were the S&P 500, while the universe consisted of the 5,000 or so stocks listed on exchanges or traded through NASDAQ in the U.S., there would be about 4,500 zero entries in $\mathbf{q}_B$.

R = An (N × 1) vector of *expected* returns on all assets in the universe (about which everyone agrees).

R_j = The expected return of individual asset or portfolio j. This is always denoted with a subscript and is not boldface; for example, R_B is the expected return of the benchmark.

r_{jt} = The actual return on asset or portfolio j in time period t; indicated by a lower case letter and a "t" subscript.

V = The (N × N) covariance matrix of individual asset total returns (also agreed upon by everybody).

σ_j = The volatility (standard deviation) of the returns on asset or portfolio j; σ_j^2 is the variance.

G = The manager's target or expected performance relative to the benchmark; "G" stands for average "Gain" over the benchmark's return. $G = R_P - R_B$.

The manager's objective is to minimize the volatility of tracking error conditional on a target expected performance relative to the benchmark. The tracking error's expected value is

$$G = (\mathbf{q}_P - \mathbf{q}_B)'\mathbf{R} = \mathbf{x}'\mathbf{R}$$

and its variance is

$$(\mathbf{q}_P - \mathbf{q}_B)'\mathbf{V}(\mathbf{q}_P - \mathbf{q}_B) = \mathbf{x}'\mathbf{V}\mathbf{x}.$$

Thus, the formal TEV optimization problem can be stated as follows:

With respect to **x**, minimize $\mathbf{x}'\mathbf{V}\mathbf{x}$
subject to the constraints,

$$\mathbf{x}'\mathbf{1} = 0$$

and

$$\mathbf{x}'\mathbf{R} = G.$$

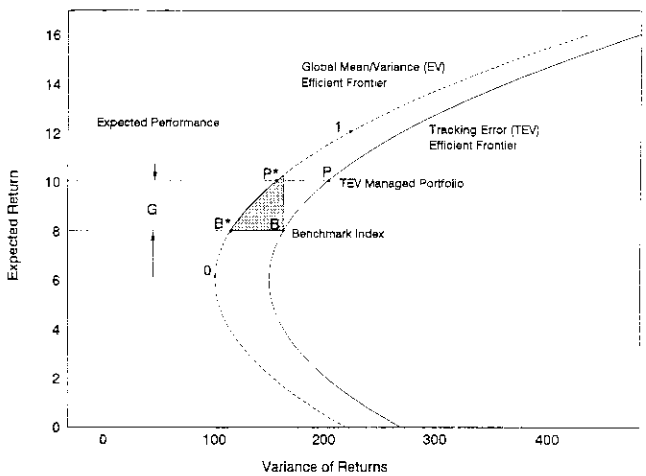
The appendix (Proof A.1) derives the solution, which is

$$\mathbf{x} = D(\mathbf{q}_1 - \mathbf{q}_0) \quad (1)$$

where $D \equiv G/(R_1 - R_0)$ and portfolios designated by subscripts "1" and "0" are particular members of the global efficient frontier that are positioned as shown in Figure 1. Portfolio "0" is the global minimum-variance portfolio while portfolio "1" is located where a ray from the origin through the global minimum variance portfolio intersects the efficient frontier.[4]

Notice in the solution (1) that **x** is completely

FIGURE 1
MEAN/VARIANCE TRACKING ERROR



independent of the original benchmark. This implies that the exact same trades from an initial position in the benchmark will be conducted by any manager who has the same universe of assets and the same expectations, *regardless* of the particular benchmark against which the performance is being measured. Of course the actual managed portfolio will depend on the benchmark because it is simply the benchmark *plus* **x**; but the elements of **x** themselves, the alterations in the benchmark undertaken to form the managed portfolio, do not depend on the benchmark's composition.[5]

The parameter D represents a *relative* performance target. It is the targeted expected excess return G over the benchmark divided by the difference in expected returns of special efficient portfolios "1" and "0." The larger the target performance, the greater the alterations represented by **x**, but the possibilities for performance are improved when R_1 and R_0 are far apart. Referring to Figure 1, a larger distance between R_1 and R_0 implies that the global efficient frontier is more steeply sloped and less curved. Clearly, this permits the production of higher performance, ceteris paribus.

It is straightforward to derive[6] from (1) the properties of the managed portfolio, P, whose investment proportions are $\mathbf{q}_P = \mathbf{q}_B + \mathbf{x}$:

Tracking Error Variance, T:

$$T = D^2(\sigma_1^2 - \sigma_0^2) \qquad (2)$$

Total Variance:

$$\sigma_p^2 = \sigma_B^2 + T + 2D\,\sigma_0^2(R_B/R_0 - 1) \qquad (3)$$

Beta of P on B,

$$\beta_{P|B} = 1 + D(\sigma_0^2/\sigma_B^2)(R_B/R_0 - 1) \qquad (4)$$

As Equation (2) shows, the variance of the tracking error does not depend on the benchmark. There is, however, a trade-off between the expected performance and the volatility of tracking error, a result that is intuitively understood by most managers and sponsors. Indeed, if the manager has pursued the TEV policy, the trade-off describes a quadratic function in the mean/variance plane. This is easy to discern in Figure 1 where Equation (3) is plotted. It is the locus of minimum tracking error variance portfolios for different expected levels of performance.

This locus passes through the position of the benchmark, indicated by a "B." Figure 1 shows the benchmark as a globally *inefficient* portfolio, which is probably the situation for most commonly used benchmarks. Note that the TEV minimum tracking error managed portfolio is also EV *in*efficient, regardless of the targeted performance level G. This implies that TEV optimization is suboptimal in the sense that the resulting managed portfolio (an example with G = 200 basis points is plotted as P in Figure 1) is dominated by feasible portfolios with higher average return and lower volatility.[7]

At the point where the TEV locus passes through B, tracking error volatility is zero. The managed portfolio in such a position is a perfect index fund. However, unless the benchmark happens to have the exact same expected return as the global minimum variance portfolio, there exist TEV portfolios with larger tracking error variances than the index fund but with *smaller* total variances.

The appendix (Proof A.4) proves that the locus of TEV portfolios is *at every level of expected return* the same distance from the global efficient frontier. As B is on the TEV locus, this distance is the difference between the benchmark's variance and the variance of an efficient portfolio B^* with the same expected return. The managed portfolio P will be inefficient by the same amount as B, measured along the volatility dimension. In Figure 1, the EV and TEV frontiers are equidistant apart at every expected return level (although an optical illusion makes it appear that they are farther apart near the global minimum variance portfolio, 0). Actually, $\sigma_B^2 - \sigma_{B*}^2 = \sigma_P^2 - \sigma_{P*}^2$, for any choice of P.

From the geometry, when R_B exceeds R_0 and when expected performance, G, is positive, the managed portfolio P will have both a higher expected return *and* a higher volatility than the benchmark. In the mean/variance diagram, the managed portfolio will lie above and to the right of the benchmark. However, if B is grossly inefficient to the extent that its expected return is less than R_0, the managed portfolio could conceivably dominate B. If G is not too large and $R_B < R_0$, P could have both a higher expected return and a *lower* overall volatility.

Turning now to examine the "beta" of the TEV managed portfolio computed against the benchmark, Equation (4) shows that it is either above or below 1.0, depending on the magnitude of the inefficiency in the benchmark (again assuming that targeted performance G is positive). When $R_B > R_0$, the beta of the managed portfolio on the benchmark exceeds unity. Thus, a manager who successfully satisfies clients by minimizing tracking error volatility while maximizing expected performance will generally do better in up markets (defined as periods with positive benchmark returns) than in down markets.

If targeted performance (G) is not zero, the beta of the managed portfolio on the benchmark *cannot* be unity except under the fortuitous circumstance that the benchmark's expected return is exactly equal to the expected return of the global minimum variance portfolio. This unlikely condition implies that most TEV managers will not have unitary betas even if they are actually performing perfectly in accordance with their client's instructions!

This points out one of the major difficulties in adopting a minimum variance tracking error (TEV) criterion. Under TEV, all managed portfolios with positive expected performance will have betas greater than 1.0. In contrast, all portfolios that dominate the benchmark, i.e., that have higher expected return and lower *total* volatility, have betas on the benchmark *less* than 1.0! This assertion is not difficult to prove. See the appendix, Proof A.5.

In Figure 1, the shaded area contains portfolios that dominate the benchmark, B. Only the northwest boundary of this area is populated with EV efficient portfolios, but every portfolio in the shaded area, whether or not EV efficient, beats the benchmark by having a lower total volatility and a higher expected return. All such portfolios have betas less than unity on the benchmark.

In contrast, a manager induced to minimize the volatility of tracking error must choose a portfolio with beta greater than unity. If the benchmark's mean return R_B exceeds the mean return R_0 of the global minimum variance portfolio, the managed portfolio P *cannot* dominate the benchmark. Even worse, there are portfolios with the same volatility as P but with higher mean returns.

THE TEV TRACKING ERROR FRONTIER WITH A CONSTRAINT ON BETA

Many fund sponsors are concerned that their managers not display systematic biases relative to the benchmark in up versus down markets. With growing frequency, this desire is actually quantified in a specific instruction to maintain the managed portfolio's "beta" on the benchmark within certain bounds; e.g., from 0.95 to 1.05. A manager in this position must minimize the volatility of tracking error for a given target level of performance while selecting a portfolio with a specified beta.

Using the notation as defined, the manager's problem is:

$$\text{With respect to } \mathbf{x}, \text{ minimize } \mathbf{x}'\mathbf{V}\mathbf{x}$$
$$\text{subject to the constraints}$$
$$\mathbf{x}'\mathbf{1} = 0$$
$$\mathbf{x}'\mathbf{R} = G$$
$$\text{and} \quad \mathbf{q}_P'\mathbf{V}\mathbf{q}_B / \sigma_B^2 = \beta,$$

where β is the sponsor-specified level of market risk (relative to the benchmark). The third constraint can be expressed in terms of the alteration vector $\mathbf{x}$ by noting that $\mathbf{q}_B + \mathbf{x} = \mathbf{q}_P$ and simplifying to

$$\mathbf{x}'\mathbf{V}\mathbf{q}_B = \sigma_B^2(\beta - 1).$$

The appendix (Proof A.6) proves that the solution to this problem is

$$\mathbf{x} = \gamma_1\mathbf{q}_1 + \gamma_0\mathbf{q}_0 + \gamma_B\mathbf{q}_B, \qquad (5)$$

where $\mathbf{q}_1$ and $\mathbf{q}_0$ are special EV efficient portfolios located as indicated in Figure 1, $\mathbf{q}_B$ is the benchmark, and the scalar constants, γ_1, γ_0, and γ_B, satisfy the constraints of the optimization problem.[8]

This beta-constrained problem is algebraically rather messy. Except in some special cases, the three "γ" coefficients are not that easy to interpret. (See their exact formulas in the appendix, Proof A.6.) The alteration vector **x** in Equation (5) now depends on the benchmark's composition, in contrast to the alteration vector from TEV optimization without a beta constraint. Thus, if fund sponsors ask for tracking error volatility minimization and also insist on a particular beta, portfolio managers would not all make the same trades. Their optimum trades depend on their particular benchmark.

In order to examine the properties of beta-constrained TEV optimization, we resort to some examples. Figure 2 shows an inefficient benchmark and a managed portfolio with expected performance of 200 basis points and $\beta = 1.0$. Also plotted are three loci, the EV efficient frontier, the unconstrained TEV minimum tracking error volatility frontier, and the TEV minimum tracking error volatility frontier on which all portfolios have $\beta = 1.0$. Paradoxically, the constrained TEV frontier actually dominates the unconstrained frontier at positive levels of performance. For a benchmark in the position shown, it would actually be better to constrain the beta (to 1.0) and then to minimize tracking error volatility rather than simply to minimize tracking error volatility without paying any attention to the resulting portfolio's beta.

At first, it might seem puzzling that a constrained optimization can actually be better than an unconstrained optimization. But remember that the parameter being minimized in both cases is *tracking error* volatility, not *total* volatility. It turns out that the managed portfolio with a beta of 1.0 has a higher tracking error volatility than an unconstrained TEV

FIGURE 2
MEAN/VARIANCE TRACKING ERROR WITH BETA=1 CONSTRAINT

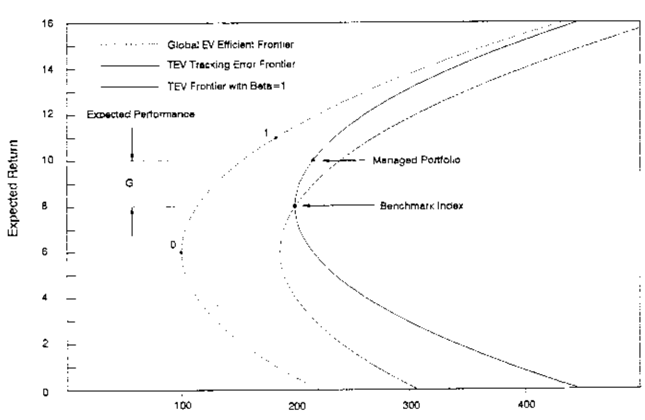
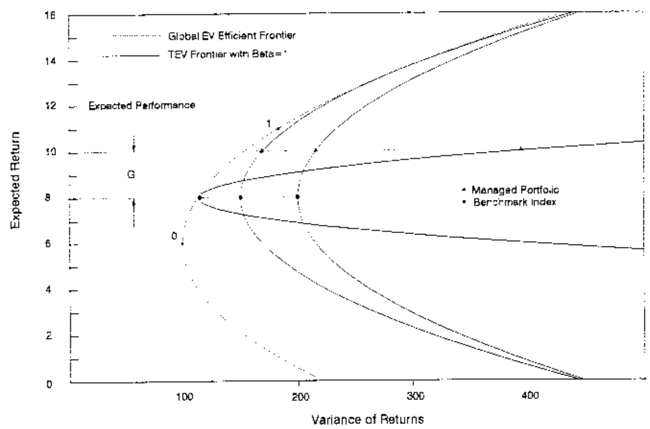


FIGURE 3
MEAN/VARIANCE TRACKING ERROR WITH BETA=1 CONSTRAINT AND THREE DIFFERENT LEVELS OF BENCHMARK VOLATILITY



portfolio with the same performance G, but it has a lower total volatility.

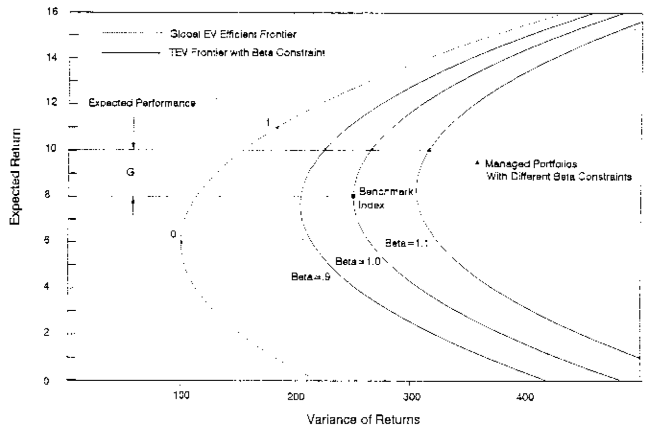
To see that this is indeed possible, write the constrained and unconstrained TEV portfolios as linear generating functions of the benchmark's return. Denote the constrained TEV portfolio as P and the unconstrained TEV portfolio as P′ and denote actual (not expected) returns by lower case letters, $r_{P,t} = r_{B,t} + e_{P,t}$ and $r_{P',t} = \beta_{P'} r_{B,t} + e_{P',t}$ where the e's are perturbations that are assumed uncorrelated with the benchmark. Then *if* $\sigma_P < \sigma_{P'}$ *and* $\text{var}(r_{P,t} - r_{B,t}) > \text{var}(r_{P',t} - r_{B,t})$, it is straightforward to show that $\beta_{P'} > 1$. The converse is *not* true: $\beta_{P'} > 1$ does not guarantee that the constrained TEV portfolio has lower variance.

In fact, as the benchmark's position moves closer to the global EV efficient frontier, the volatilities of beta-constrained TEV portfolios grow indefinitely large. This is illustrated in Figure 3, which plots three different possible positions of the benchmark and three corresponding constrained TEV portfolios, each with the same level G of target performance and each with beta of 1.0.

At first, as the benchmark gets closer to EV efficiency, the TEV portfolio also becomes closer to the EV frontier. Thus, the illustrated middle position for B in Figure 3 produces a locus of $\beta = 1$ TEV portfolios that lies quite close to the EV frontier at higher levels of G.

But notice the impact on the total volatility of the managed portfolio as B gets very close to the EV

FIGURE 4
MEAN/VARIANCE TRACKING ERROR
WITH THREE DIFFERENT BETA CONSTRAINTS



frontier. The locus becomes much more acutely curved, and the managed portfolio's variance becomes extremely large. If the benchmark happened to lie exactly *on* the EV frontier, there would be no managed portfolio with finite variance, positive target performance, and $\beta = 1.0$.[9]

The impact of the beta constraint is illustrated in Figure 4. For a given benchmark positioned as shown in the Figure, three separate loci are plotted for TEV portfolios with betas of 0.9, 1.0, and 1.1. For a given target level of performance G, there would be three different managed portfolios as shown. Note that betas greater than or equal to 1.0 never produce dominating portfolios. For positive target performance, the benchmark always has a lower volatility.

However, if the beta is constrained to be less than 1.0, it is quite possible to produce a dominating portfolio. The illustrated managed portfolio with $\beta = 0.9$ and G = 2% actually has both higher average return and *lower* volatility. Of course, the benchmark must be considerably inside the EV efficient frontier for this to be possible. If it is near the frontier, a 2% target performance may require the TEV managed portfolio to have higher total volatility. It may not even be possible to have a beta as low as 0.9 with the specified level of target performance.

SUMMARY AND IMPLICATIONS OF TRACKING ERROR MANAGEMENT FOR PORTFOLIO TOTAL RETURN PERFORMANCE

This article derives the composition of a portfolio that maximizes average performance over a benchmark for a given *volatility* of tracking error. The "TEV frontier" is composed of all such portfolios, one at each level of tracking error volatility. TEV portfolios are not total return mean/variance "EV" efficient if the benchmark is not EV efficient. In other words, TEV managed portfolios are dominated by other feasible portfolios that have both lower volatility and higher average total return.

TEV managed portfolios have betas greater than 1.0 on the benchmark if the benchmark has a mean return greater than that of the global minimum variance portfolio. Thus, the unconstrained TEV criterion will usually result in a managed portfolio with greater "market risk" than the benchmark. This is unfortunate because any portfolio that dominates the benchmark by possessing lower volatility and higher mean return *must* have a beta less than 1.0.

It is possible, however, to minimize tracking error volatility subject to a constraint on beta. This is a superior total return mean/variance strategy for benchmarks that are not particularly close to the efficient frontier. With a beta-constrained strategy, it is usually better to select a beta less than 1.0.

It may be conjectured that a tracking error goal is appropriate for individual portfolio managers because fund sponsors usually employ an entire stable of managers. Perhaps an optimal overall portfolio can be achieved by directing each manager independently to follow a TEV strategy. This is an important possibility that remains for future research to prove one way or the other.

But, a priori, remember that each TEV manager in the sponsor's stable could conceivably produce a portfolio with higher mean return and lower volatility. For the sponsor's overall portfolio to be optimal, these individual manager disadvantages must be offset by substantially lower cross-manager correlations when they adhere to the TEV strategy than when they strive for total return EV efficiency. The geometry of mean/variance analysis makes it appear that such a result would indeed be fortuitous.

There remains, however, one other possible recommendation for the TEV strategy: Estimation error is severe in portfolio analysis. No one knows where the global total return efficient frontier is really located. Its position depends, inter alia, on individual asset expected returns, which can be estimated only with substantial error because of the large component of noise in observed returns.

Perhaps expected *differences* between portfolio returns can be estimated more precisely.[10] If this be true, a TEV policy may induce portfolio managers to place less emphasis on estimates of individual expected returns, and, because such estimates are largely figments of manager imagination anyway, it may be better to relegate them to obscurity. Using them in an attempt to form a more efficient portfolio may simply add noise and volatility to the process without achieving any significant improvement in performance.

Estimating the expected tracking error may be a more feasible manager goal. It is certainly easier for the sponsor to monitor.

APPENDIX: PROOFS OF PROPOSITIONS

A.1. THE TEV MINIMUM VARIANCE TRACKING ERROR PORTFOLIO

Using notation as defined in the text, the TEV portfolio manager's problem can be stated mathematically as:

$$\min_{\mathbf{x}} \mathbf{x}'\mathbf{V}\mathbf{x}$$

subject to $\mathbf{x}'\mathbf{R} = G$
$\mathbf{x}'\mathbf{1} = 0$

The solution must satisfy the LaGrangian equation

$$\mathbf{V}\mathbf{x} - \lambda_1\mathbf{R} - \lambda_2\mathbf{1} = 0$$

where λ_1 and λ_2 are undetermined LaGrange multipliers.

Solving for $\mathbf{x}$,

$$\mathbf{x} = \mathbf{V}^{-1}[\mathbf{R}\ \mathbf{1}]\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} \qquad \text{(A-1)}$$

Premultiplying (A-1) by $[\mathbf{R}\ \mathbf{1}]'$, and rearranging gives

$$\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} = \mathbf{A}^{-1}\begin{pmatrix}G\\ 0\end{pmatrix} \qquad \text{(A-2)}$$

where $\mathbf{A}$ is the efficient set information matrix that plays a key role in EV set algebra,

$$\mathbf{A} = \begin{pmatrix}\mathbf{R}'\mathbf{V}^{-1}\mathbf{R} & \mathbf{R}'\mathbf{V}^{-1}\mathbf{1}\\ \mathbf{R}'\mathbf{V}^{-1}\mathbf{1} & \mathbf{1}'\mathbf{V}^{-1}\mathbf{1}\end{pmatrix} \equiv \begin{pmatrix}a & b\\ b & c\end{pmatrix} \qquad \text{(A-3)}$$

It can be shown that the three parameters a, b, and c are related to the means and variances of two particular efficient portfolios, the global minimum variance portfolio, denoted with subscript "0," and portfolio "1," which lies where a ray from the origin through the global minimum variance portfolio intersects the efficient frontier. The positions of these portfolios are shown in Figure 1 of the text. They have the following properties:

Portfolio	Mean	Variance	Proportions
0	$R_0 = b/c$	$\sigma_0^2 = 1/c$	$\mathbf{q}_0 = \mathbf{V}^{-1}\mathbf{1}/c$
1	$R_1 = a/b$	$\sigma_1^2 = a/b^2$	$\mathbf{q}_1 = \mathbf{V}^{-1}\mathbf{R}/b$

Eliminating the LaGrange multipliers by combining (A-1) and (A-2), and using the definitions of portfolios "0" and "1," we obtain,

$$\mathbf{x} = \frac{G}{R_1 - R_0}(\mathbf{q}_1 - \mathbf{q}_0) \qquad \text{(A-4)}$$

Equation (A-4) describes the TEV minimum variance tracking error set. The investment alterations, $\mathbf{x}$, required to minimize tracking error volatility for a given expected performance (G) are proportional to the difference between the investment proportions of two particular total return mean/variance (EV) efficient portfolios, 1 and 0. $\mathbf{x}$ is independent of the original benchmark.

A.2. TRACKING ERROR VARIANCE

The managed portfolio P has investment proportions $\mathbf{q}_P = \mathbf{q}_B + \mathbf{x}$. Thus, tracking error volatility is

$$(\mathbf{q}_P - \mathbf{q}_B)'\mathbf{V}(\mathbf{q}_P - \mathbf{q}_B) = \mathbf{x}'\mathbf{V}\mathbf{x}. \qquad \text{(A-5)}$$

Denoting the *relative* target performance as $D \equiv G/(R_1 - R_0)$ and using (A-4), Equation (A-5) can be expanded to

$$D^2(\mathbf{q}_1 - \mathbf{q}_0)'\mathbf{V}(\mathbf{q}_1 - \mathbf{q}_0) = D^2[\mathbf{q}_1'\mathbf{V}\mathbf{q}_1 + \mathbf{q}_0'\mathbf{V}\mathbf{q}_0 - 2\mathbf{q}_1'\mathbf{V}\mathbf{q}_0].$$

But because $\mathbf{q}_0 = \mathbf{V}^{-1}/c$,

$$\mathbf{q}_1'\mathbf{V}\mathbf{q}_0 = \mathbf{q}_0'\mathbf{V}\mathbf{q}_0 = 1/c = \sigma_0^2,$$

and thus,

$$\mathbf{x}'\mathbf{V}\mathbf{x} = D^2[\sigma_1^2 - \sigma_0^2] \equiv T, \qquad \text{(A-6)}$$

where T denotes tracking error variance.

A.3. THE VARIANCE AND "BETA" OF THE MANAGED TEV PORTFOLIO

The variance of the managed portfolio (P) is

$$(\mathbf{q}_B + \mathbf{x})'\mathbf{V}(\mathbf{q}_B + \mathbf{x}) = \sigma_B^2 + T + 2\mathbf{q}_B'\mathbf{V}\mathbf{x}.$$

But $\quad \mathbf{q}_B'\mathbf{V}\mathbf{x} = \mathbf{q}_B'\mathbf{V}(\mathbf{q}_1 - \mathbf{q}_0)D = D\,\sigma_0^2\,(R_B/R_0 - 1).$

Thus, $\quad \sigma_P^2 = \sigma_B^2 + T + 2D\,\sigma_0^2\,(R_B/R_0 - 1) \qquad \text{(A-7)}$

Similarly, the beta of P computed against the benchmark B is

$$\beta = (\mathbf{q}_B + \mathbf{x})'\mathbf{V}\mathbf{q}_B/\sigma_B^2 = 1 + D(\mathbf{q}_1 - \mathbf{q}_0)'\mathbf{V}\mathbf{q}_B/\sigma_B^2$$
$$= 1 + D(\sigma_0^2/\sigma_B^2)(R_B/R_0 - 1) \qquad \text{(A-8)}$$

When the benchmark's return exceeds the return on the global minimum variance portfolio, every TEV portfolio with positive expected performance has a beta greater than unity.

A.4. THE TEV FRONTIER IS A VARIANCE TRANSLATION OF THE EV FRONTIER

Let P^* denote an EV efficient portfolio with the same expected return as the managed portfolio P. Similarly, portfolio B^* has the same expected return as the benchmark B.

Equation (A-7) applies to any portfolio, including B^* and P^*,

$$\sigma_{P*}^2 = \sigma_{B*}^2 + T + 2D\,\sigma_0^2(R_B/R_0 - 1). \qquad \text{(A-9)}$$

The second two terms on the right are the same in (A-7) and (A-9), whether or not we use B and P or B^* and P^*. Thus, subtracting (A-7) from (A-9),

$$\sigma_P^2 - \sigma_{P*}^2 = \sigma_B^2 - \sigma_{B*}^2.$$

The situation is illustrated in Figure 1 of the text.

A.5. PORTFOLIOS THAT DOMINATE THE BENCHMARK (HIGHER EXPECTED RETURN AND LOWER TOTAL RETURN VOLATILITY) ALL HAVE BETAS ON THE BENCHMARK LESS THAN UNITY

The beta of any portfolio P on the benchmark B is

$$\beta_{P|B} = \frac{\mathrm{Cov}(R_P, R_B)}{\sigma_B^2} = \frac{\rho_{PB}\sigma_P\sigma_B}{\sigma_B^2}$$

where ρ_{PB} is the correlation coefficient between the returns of P and B.

But because all portfolios that dominate B have lower volatilities ($\sigma_P < \sigma_B$) as well as higher mean returns, then assuming that ρ_{PB} is positive,

$$\beta_{P|B} = \rho_{PB}\frac{\sigma_P}{\sigma_B} < \rho_{PB} \le 1.$$

If ρ_{PB} happens to be negative, then of course $\beta_{P|B} < 0$ as well.

A.6. THE TEV MINIMUM VARIANCE TRACKING ERROR FRONTIER WITH A BETA CONSTRAINT

Assume that the manager is instructed to form a portfolio P that minimizes the variance of tracking error, produces an expected performance G, and also maintains a specified beta (β) against the benchmark portfolio B. Using the same notation, this optimization problem can be expressed as

$$\min_{\mathbf{x}} \ \mathbf{x}'\,\mathbf{V}\,\mathbf{x}$$

Subject to:

$$\mathbf{x}'\,\mathbf{R} = G$$
$$\mathbf{x}'\,\mathbf{1} = 0$$
$$\mathbf{q}_P'\mathbf{V}\mathbf{q}_B = \beta\,\sigma_B^2$$

The third constraint can also be written as

$$(\mathbf{q}_B + \mathbf{x})'\mathbf{V}\mathbf{q}_B = \beta\,\sigma_B^2,$$

or
$$\mathbf{x}'\mathbf{V}\mathbf{q}_B = \sigma_B^2(\beta - 1).$$

Using the LaGrange multiplier technique, collecting terms, and simplifying, the solution can be written

$$\mathbf{x} = \gamma_1\mathbf{q}_1 + \gamma_0\mathbf{q}_0 + \gamma_B\mathbf{q}_B, \qquad \text{(A-10)}$$

where

$$\gamma_1 \equiv \frac{G(\sigma_B^2 - \sigma_0^2) + \sigma_B^2(\beta - 1)(R_0 - R_B)}{(R_1 - R_0)(\sigma_B^2 - \sigma_{B*}^2)}$$

$$\gamma_0 \equiv \frac{G(R_B/b - \sigma_B^2) + \sigma_B^2(\beta - 1)(R_B - R_1)}{(R_1 - R_0)(\sigma_B^2 - \sigma_{B*}^2)}$$

$$\gamma_B \equiv \frac{G(\sigma_0^2 - R_B/b) + \sigma_B^2(\beta - 1)(R_1 - R_0)}{(R_1 - R_0)(\sigma_B^2 - \sigma_{B*}^2)}$$

Note that the change portfolio **x** is now a weighted average of the two special EV efficient portfolios, 1 and 0, *and* the benchmark (B). It is easy to ascertain by direct inspection that the γ coefficients sum to zero: thus, two of them must have opposite signs, or they must all be zero. A tedious algebraic reduction proves that $\gamma_1 R_1 + \gamma_0 R_0 + \gamma_B R_B = G$ and that $\mathbf{x}'\mathbf{V}\mathbf{q}_B = \sigma_B^2 (\beta - 1)$ as required.

The denominator of each γ coefficient becomes arbitrarily small as the benchmark B approaches the EV efficient frontier, i.e., as $\sigma_B^2 \rightarrow \sigma_{B*}^2$. This implies that the γ's grow indefinitely large (in absolute value) as B approaches EV efficiency. Ultimately when B is total return mean/variance efficient, it is impossible to have *both* a prespecified performance G and an *arbitrary* prespecified β; i.e., every choice of G implies a value of β.

When β is 1.0, the γ coefficients simplify considerably. In this case, $\mathbf{x}$ always has a positive weighting in special efficient portfolio "1" while either γ_0 or γ_B (or both) must be negative, depending on the position of the benchmark; γ_0 is positive only when B lies between the EV efficient frontier and a line connecting portfolios 1 and 0, while γ_B is positive only when $R_0 > R_B$. Otherwise both γ_0 and γ_B are negative. All these statements are reversed if R_0 is negative (which implies that R_1 and b also are negative).

In the case of $\beta = 1$, $\mathbf{q}_1$ and $\mathbf{q}_0$ are weighted to form an EV efficient portfolio whose beta on the benchmark is also 1.0. To see this, note that when $\beta = 1$ and

$$\delta \equiv \gamma_1/(\gamma_1 + \gamma_0) = (\sigma_B^2 - \sigma_0^2)/(R_B/b - \sigma_0^2),$$

then

$$[\delta \mathbf{q}_1 + (1 - \delta)\mathbf{q}_0]'\mathbf{V}\mathbf{q}_B = \delta(R_B/b) + (1 - \delta)\,\sigma_0^2 = \sigma_B^2.$$

ENDNOTES

[1]The beta of a managed portfolio on the benchmark is commonly used as a measure of risk. A beta greater (less) than 1.0 implies that the managed portfolio has a larger (smaller) amplitude than the benchmark in price fluctuations that are correlated with the benchmark.

[2]I.e., a higher beta on the benchmark.

[3]Boldface letters indicate vectors and matrices. Scalars are denoted without boldface.

[4]Those familiar with efficient set algebra will recognize that *any* two (different) efficient portfolios could be employed in place of portfolios 0 and 1. However, the algebra is messier with other choices.

[5]To see more intuitively why the alteration vector cannot depend on the benchmark, note that the actual return on the managed portfolio P can be expressed as the return on the benchmark B plus the alteration vector's return: i.e., $r_{P,t} = r_{B,t} + r_{x,t}$. TEV optimization requires minimizing the variance of $r_{x,t}$ for a given level of its expected value. Clearly, whatever alteration portfolio minimizes the variance of $r_{x,t}$, it will be the same for any choice of B.

[6]See the appendix, Proofs A.2 and A.3.

[7]The formulas remain valid whenever the benchmark happens to be EV efficient. In that case, however, the TEV minimum tracking error managed portfolio would also be EV efficient.

[8]E.g., $\gamma_1 + \gamma_0 + \gamma_B = 0$ and $\gamma_1 R_1 + \gamma_0 R_0 + \gamma_B R_B = G$.

[9]All EV efficient portfolios that lie above an efficient portfolio B^* with $R_{B*} > R_0$ have betas on B^* greater than 1.0.

[10]The standard error of the sample mean return difference between managed portfolio P and benchmark B will be smaller than the standard error of the sample mean of P alone if their correlation ρ_{PB} exceeds ½ (σ_B/σ_P). For instance, when the volatilities (σ's) of P and B are equal, their mean return difference has a smaller estimation error than the estimation error for P alone whenever their correlation exceeds one-half.

REFERENCE

Markowitz, Harry M. *Portfolio Selection: Efficient Diversification of Investments.* New York: John Wiley & Sons, 1959.