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ORTHOGONAL PORTFOLIOS

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I. Motivation for Studying Orthogonal Portfolios

Orthogonal portfolios are prominent in the economics of asset pricing. The minimum-variance "zero-beta" portfolio to the market index is one of the two determinants of individual expected asset returns in Black's [1] theory. "Zero-beta" is simply another way to express the absence of linear correlation, or the "orthogonality," of two portfolios' returns.

It is widely believed that the minimum-variance zero-beta portfolio is unique for a given market index and that all such zero-beta portfolios, whether minimum-variance or not, have the same expected return; cf., for example, Fama [4, pp. 285-286]. As proven below, however, this is true *only* when the market index is mean/variance efficient. When the index is not efficient, there are zero-beta portfolios at *all* levels of expected return.

This result is more than a mere mathematical curiosity. It has significant implications for asset pricing theory and tests thereof. To see why, consider Black's [1] expected return model, suitably generalized to allow for the possibility of an *inefficient* index. It would be

$$\alpha_j = E(r_j) - E(r_z) - \beta_j\, E(r_m - r_z)$$

where the "r" refers to return, "E" to expectation and the subscripts indicate

j, an arbitrary asset

m, the market index

z, a zero-beta portfolio for the index.

The "systematic risk" coefficient is given by $\beta_j \equiv \text{cov}(\tilde{r}_j, \tilde{r}_m)/\text{var}(\tilde{r}_m)$. The

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α_j term is essentially a measure of the index, mean/variance efficiency. As proven by Ross [10], α_j will be identically zero *for all j* if and only if the index m is efficient (*ex ante*). This is the case invariably discussed in the theoretical literature, for m *should* be efficient if it is indeed the market portfolio of all assets; if investors have homogeneous beliefs; if returns are generated by a stationary multivariate normal process; if there are no trading costs nor short-selling restrictions; and if the market is in equilibrium.

If the index m is efficient, not only will α_j vanish for every j, but since all zero-beta portfolios have the same expected return, there will be only one possible value for $E(r_z)$.

Suppose to the contrary that an index used for empirical work is not *ex ante* mean/variance efficient. This could happen in the theory is *incorrect* and if the "true" value-weighted market portfolio is actually measured and employed as an index. It could happen also if the theory is correct, but if a mistake were made in measuring the index. As mentioned above, one now well-known consequence of an inefficient index is that some asset or assets *must* display $\alpha_j \neq 0$. A lesser well-known consequence is that $E(r_z)$ can conceivably take on any value. For (an extreme) example, a minimal[1] variance zero-beta portfolio can be found such that $E(r_z) = E(r_m)$, in which case Black's model reduces to

$$E(r_j) = E(r_m) + \alpha_j.^{2}$$

Clearly, this particular choice of z would be less than best for testing the "risk" structure implied by the theory. The "beta" has completely disappeared.

To repeat, differing expected returns on zero-beta portfolios occur for a given choice of market index. The casual reader may know already that $E(r_z)$ changes as the identity of the index changes; but the point here is that $E(r_z)$ can take on any value *even for a fixed index*. If the index is not mean/variance efficient, it possesses orthogonal portfolios at all levels of expected return. Thus, the Black model which relates mean returns to "betas" can have literally any values for its intercept, $E(r_z)$, and its slope, $E(r_m - r_z)$, provided that the slope and intercept sum to $E(r_m)$.

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[1]The word "minimal-variance" will be used here to denote the minimum-variance at a given level of mean return.

[2]Notice that this equation implies $\Sigma\, x_j \alpha_j = 0$ where $r_m \equiv \sum_j x_j r_j$, i.e., x_j is the proportion of the index represented by asset j. This weighted average α is always zero, even when $E(r_z) \neq E(r_m)$.

What have been the choices of z actually employed in empirical work? Some studies that have reported time series of returns on measured zero-beta portfolios are: Fama and MacBeth [5]; Black and Scholes [2]; and Morgan [8]. They employed slightly different computational methods. Similar methods are being adopted or have been used in numerous other scholarly papers and in doctoral dissertations. The Fama/MacBeth time series has been used directly in many other papers (e.g., Charest [3]). No paper has attempted to test whether the portfolio used as a market index was actually *ex ante* mean/variance efficient. There must, therefore, be some nonnegligible chance that differing computational methods, even when used with exactly the same index over exactly the same time period, produced different zero-beta portfolios. The parameters of such orthogonal portfolios can differ and the time series histories can be distinct. For example, it is even possible that two minimal-variance portfolios with different mean returns, both zero-beta to the same index, are themselves mutually orthogonal. Their time series would therefore be linearly unrelated. Without analysis of the data, there is no way to ascertain whether such extreme pathological consequences actually occurred. It seems unlikely. Nevertheless, the mere possibility of such events argues for a more detailed investigation of the general properties of orthogonal portfolios than has heretofore been presented. The remainder of this paper attempts to provide a unified treatment of these properties.

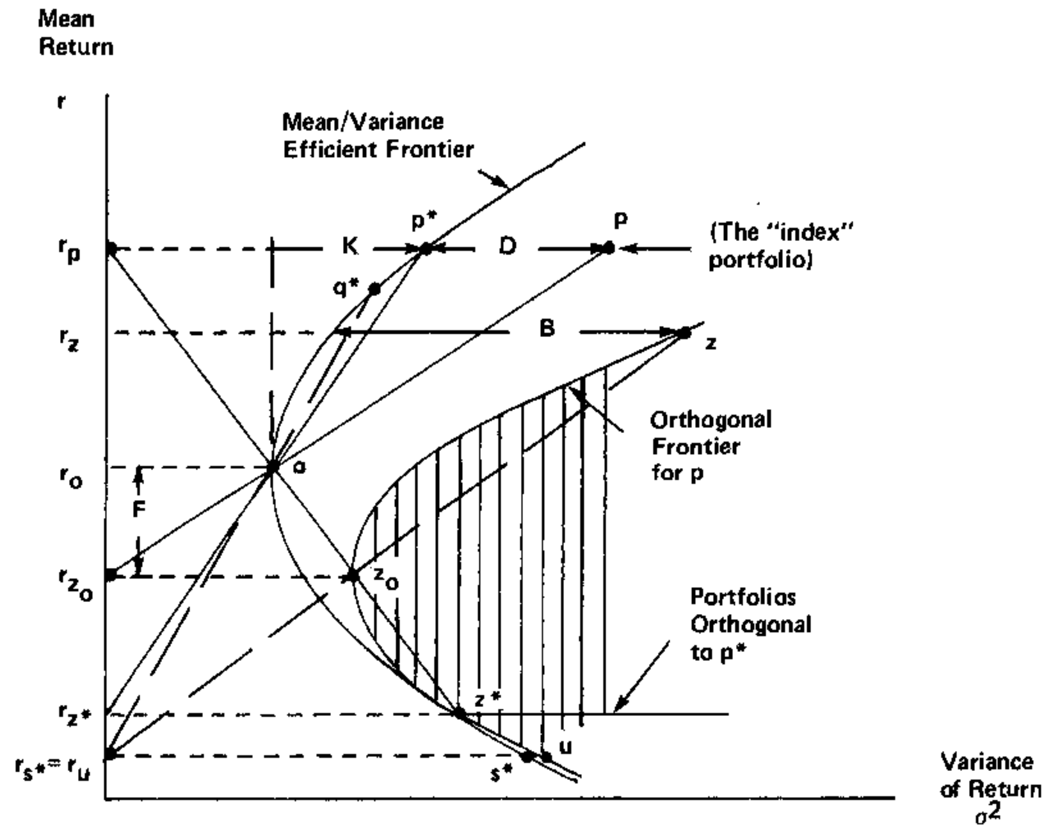
II. General Properties of Minimal-Variance Orthogonal Portfolios

Most of the properties are illustrated in Figure 1, a diagram in the mean/*variance* space. The figure is based on a given portfolio (p) that is not mean/variance efficient. The shaded region of Figure 1 contains portfolios orthogonal to p. It is bounded by a quadratic function, the locus of minimal-variance orthogonal portfolios (1.a).[3] I shall call this locus the "orthogonal frontier for p." Symbol z indicates some particular portfolio on the orthogonal frontier.

There is a minimum-variance member of the orthogonal frontier, z_0 in Figure 1, and like p, it is not mean/variance efficient (1.b). It lies closer to p in the mean return dimension than the unique orthogonal portfolio (z*) that *is* mean/variance efficient (1.e). This latter portfolio lies at the tangency of the efficient frontier and the orthogonal frontier (1.e).[4] The return level of

[3] Proofs of all statements are given in the Appendix. A number and letter such as "(2.b)" following each statement refers to the (corollary number, implication) in the Appendix.

[4] Note that mean/variance efficient portfolios are defined to include those on the negatively-sloped frontier. "Efficiency" in the present usage is a euphemism for "minimum-variance" (for a given mean return). An asterisk denotes an efficient portfolio.



The shaded region, containing portfolios that are uncorrelated with p , is bounded by a quadratic function of r , the orthogonal frontier. As p approaches efficiency at p* ($D \to o$), the orthogonal frontier collapses to a horizontal line at the level r_{z*}. As σ_p^2 increases, the orthogonal frontier approaches coincidence with the efficient frontier. At any return level r_z, the horizontal distance between the two frontiers is $B = [\sigma_0^2 (r_z - r_{z*})/(r_0 - r_{z*})]^2 /D$. The vertical distance between the minimum-variance portfolios, o and z_o , is $F = (r_o - r_{z*})/(D/K + 1)$. Portfolio u is uncorrelated with both z and p . The efficient portfolio q* is a linear combination of the inefficient portfolios z and p.

the tangency is located at the intersection of the return axis and a line drawn from p* through o, the portfolio with global minimum variance.

In the analysis of Long [6], the tangency portfolio z* is the zero-beta portfolio for p using his first method and z_o is the zero-beta portfolio for p using his second method. As explained in Morgan [8, pp. 363-364], the first method entails choosing an orthogonal portfolio whose covariance with asset j is proportional to $1 - \beta_j$. This will be true only if z is mean/variance efficient. Hence, the portfolio must be z*. The second method entails minimizing the variance of the orthogonal portfolio and the solution to this problem is clearly z_o.

A line drawn from z* through o intersects the return axis at r_p. This line passes also through z_o (3.b). Furthermore, a line from p through o intersects the return axis at r_{z_o} (1.b). By using these lines and the method of similar triangles, many interesting relationships can be obtained, e.g., the vertical distance from r_o to r_{z_o}.

The shape and position of the orthogonal frontier are functionally related to the horizontal distance (D) from the efficient frontier of portfolio p's position (1.d). As p's position approaches the efficient frontier (at p*), the orthogonal frontier collapses to a horizontal line emanating from r_{z*}. Thus, all portfolios orthogonal to a mean/variance efficient portfolio have the same mean return. As p becomes "less efficient," i.e., as its location moves further to the right in Figure 1, the orthogonal frontier widens toward the left. The two frontiers coincide for infinite σ_p^2. Thus, extremely high variance inefficient portfolios must be approximately orthogonal to *all* efficient portfolios (1.f).

Any arbitrary minimal-variance orthogonal portfolio (like z) lies to the right of the efficient frontier by the distance $B = (\sigma_{p*z})^2/D$, where σ_{p*z} is the covariance between z and the efficient portfolio p* which has the same mean return as p (1.d). This quantity can also be expressed as $B = [\sigma_o^2 (r_z - r_{z*})/(r_o - r_{z*})]^2/D$. These relations are useful in understanding the movement of the orthogonal frontier with respect to changes in the mean return, r_p, of the inefficient base portfolio, p. The geometry makes it obvious that a portfolio whose mean return is less than that of portfolio o, (the global minimum variance portfolio), will have an orthogonal frontier whose tangency z* lies *above* o. For a portfolio p whose return is exactly r_o, the orthogonal frontier is somewhat more difficult to visualize. It will be positioned as a horizontal translation of the efficient frontier; so that $r_{z_o} = r_o$, $B = \sigma_o^4/D$ (a constant over all levels of return), and so that the tangency vanishes (i.e., $r_{z*} = \pm \infty$).

Since the orthogonal frontier extends indefinitely with nonzero slope, every nonefficient portfolio has an orthogonal portfolio with the same mean return. In fact, if p lies sufficiently far to the right, it can have orthogonal portfolios with the same mean return and the same variance! For example, if p happens to be located just twice as far from the return axis as the efficient frontier ($\sigma_p^2 = 2\sigma_{p*}^2$), then the orthogonal frontier passes through p's location, thereby implying the existence of a minimal-variance zero-beta portfolio with the same mean and same variance as p itself (1.g).

The vector of investment proportions that defines a member of the orthogonal frontier can be written as a linear combination of three "funds," portfolio p and two (different) mean/variance efficient portfolios (2.). This implies that every nonefficient index, if combined in the correct proportions with any of its minimal-variance orthogonal portfolios, will yield an efficient portfolio (3.). Unfortunately, the correct proportions turn out to be rather complex in the general case, but they are given in the Appendix, Corollary 3.

If q* is the efficient portfolio constructed from p and one of its orthogonal partners z, then q* and z possess orthogonal portfolios with the same mean return, (3.a). This implies a pedagogically-useful geometric property illustrated in Figure 1. Portfolio a* is positioned such that s* is its (efficient) orthogonal partner. Portfolio z has an orthogonal portfolio, u, which is orthogonal also to p. The mean returns are equal for u and s*. Thus, q* can be located by passing a line from z through z_o to the return axis, then passing a second line from this point through o until it intersects the efficient frontier.

The three-fund property of orthogonal portfolios also implies a functional relationship between the vector of investment proportions of the orthogonal portfolio and the vector α of deviations[5] about the return/beta securities market line (4.). The deviations turn out to be closely related to the vector (C) of covariances between individual assets and the orthogonal portfolio. Both C and α are linear combinations of the same three vectors. However, this relation tends to be obscured by a further property that the inner product of α is exactly zero with both the investment proportions vectors for p and for z, (4.).

The deviations (α's) can be expressed as *analytic* functions of the mean and variance of the index p, of the distance (D) of p's location from the efficient frontier, and of the mean and variance of the chosen zero-beta

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[5]For an individual asset j, the deviation α_j is $\alpha_j \equiv r_j - r_z - \beta_j(r_p - r_z)$ where p is the index, z is one of its orthogonal partners, and β_j is the covariance of j and p divided by p's variance. α(without subscript) is the vector of α_j's.

portfolio z. Since both p and z can be freely selected, it is easy to see that algebraic values of α_j's have very little to do with the relative desirabilities of different individual assets.

III. Summary

There is a false, but widely-held belief about orthogonal ("zero-beta") portfolios: for a *given* market index, all zero-beta portfolios have the same expected return and the minimal-variance, zero-beta portfolio is unique. This is true only when the index is mean/variance efficient. Every nonefficient index possesses zero-beta portfolios at all levels of expected return. For a given index, minimal-variance zero-beta portfolios corresponding to different expected returns lie along an "orthogonal frontier" in the mean/variance space. The frontier has some unusual properties which turn out to be relevant for empirical work on asset pricing. It is functionally related to deviations about the "securities market line."

REFERENCES

[1] Black, F. "Capital Market Equilibrium with Restricted Borrowing." *Journal of Business*, Vol. 45 (1972), pp. 444-454.

[2] Black, F., and M. Scholes. "The Effects of Dividend Yield and Dividend Policy on Common Stock Prices and Returns." *Journal of Financial Economics*, Vol. 1 (1974), pp. 1-22.

[3] Charest, G. "Split Information, Stock Returns and Market Efficiency - I." *Journal of Financial Economics*, Vol. 6 (1978), pp. 265-296.

[4] Fama, E. F. *Foundations of Finance*. New York: Basic Books (1977).

[5] Fama, E. F., and J. D. MacBeth. "Risk, Return, and Equilibrium: Empirical Tests." *Journal of Political Economy*, Vol. 61 (1973), pp. 607-636.

[6] Long, J. B. "Notes on the Black Valuation Model for Risky Securities." Unpublished manuscript (1971). Rochester, New York: University of Rochester.

[7] Merton, R. C. "An Analytic Derivation of the Efficient Portfolio Frontier." *Journal of Financial and Quantitative Analysis*, Vol. 7 (1972), pp. 1851-1872.

[8] Morgan, I. G. "Prediction of Return with the Minimum-Variance Zero-beta Portfolio." *Journal of Financial Economics*, Vol. 2 (1975), pp. 361-376.

[9] Roll, R. "A Critique of the Asset Pricing Theory's Tests." *Journal of Financial Economics*, Vol. 4 (1977), pp. 129-176.

[10] Ross, S. "A Note on the Capital Asset Pricing Model (CAPM), Short-selling Restrictions and Related Issues." *Journal of Finance*, Vol. 32 (1977), pp. 177-183.

APPENDIX

PROVING THE STATEMENTS IN THE TEXT

All results assume a finite set of N individual assets and use the following notation:

V, the $(N \times N)$ covariance matrix of returns on individual assets, assumed to be nonsingular,

R, the $(N \times 1)$ column vector of mean returns, assumed to contain some differing entries,

Note: V and R can be either subjective *ex ante* estimates or *ex post* statistical estimates,

ι, the $(N \times 1)$ unit vector,

X_j, an $(N \times 1)$ column vector of investment proportions whose sum is unity, defining an arbitrary portfolio j,

Z_j, an $(N \times 1)$ column vector of investment proportions defining an orthogonal portfolio j,

r_j, the (scalar) mean return on portfolio j, and

σ_{ij}, the (scalar) covariance of portfolios i and j, where the i and j subscript indicates

$$i,j = \begin{cases} p \text{ an arbitrary portfolio} \\ z \text{ an orthogonal portfolio} \\ o \text{ the global minimum-variance portfolio} \\ * \text{ appended to any symbol indicates a mean/variance efficient portfolio.} \end{cases}$$

The following properties are stated without proof. (Proof can be found in Merton [7] or Roll [9]):

(1) Portfolio additivity condition: $X_j'\iota = 1$ (also $Z_j'\iota = 1$).

(2) Covariance between portfolios: $\sigma_{ij} = X_i'VX_j$ (and $\sigma_j^2 = X_j'VX_j$).

(3) Mean return of portfolio: $r_j = X_j'R$.

(4) Covariance between portfolios when one is mean/variance efficient: $\sigma_{p*j} = (r_{p*}1)\, A^{-1}(r_j 1)'$

where $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$, (2x2),

is the "efficient set information matrix" whose elements are $a = R'V^{-1}R$, $b = R'V^{-1}\iota$, $c = \iota'V^{-1}\iota$.[6]

(5) Definition of orthogonal portfolios: $Z'VX = 0$.[7]

(6) Mean/variance efficient portfolios: $X_{p*} = V^{-1}(R\iota)\ A^{-1}(r_{p*}1)'$.

Note: mean/variance efficient portfolios are defined to include those on the negatively-sloped segment of the efficient frontier.

As equation (5) indicates, an orthogonal pair of portfolios, with investment proportions vectors Z and X, has zero covariance. For a given X, there is a continuum of solutions Z to (1) and (5), but only some of these solutions have both desirable properties of minimal variance and orthogonality.

Definition. The orthogonal frontier for p is the set of minimal variance portfolios orthogonal to portfolio p.

For an efficient portfolio p*, the orthogonal frontier for p* can be obtained easily from (4). Setting $\sigma_{p*z} = 0$ in (4), we obtain the result,

(7) $$r_{z*} = (br_{p*} - a)/(cr_{p*} - b).$$

Since the right side of (7) is a scalar, all portfolios orthogonal to an efficient portfolio must have the same return. Clearly, one of these is also efficient and its weights are given from (6) as

(8) $$Z* = V^{-1}(R\iota)\ A^{-1}(r_{z*}1)'\ .$$

Thus, there is a unique minimum-variance portfolio which is orthogonal to the efficient portfolio X_{p*}.

[6] Cf. Roll [9, Appendix], for proofs of these results.

[7] Note that the vector of individual asset "betas" computed against portfolio p is given by $\beta = VX_p/\sigma_p^2$. Thus $Z'\beta = \beta_z = Z'VX/\sigma_p^2 = 0$.

When the initial portfolio p is not mean/variance efficient, a modified approach is required:

<u>Theorem</u>. Let the covariance matrix of individual asset returns be nonsingular and let at least two assets have different mean returns. Let X be the investment proportions vector of a nonefficient portfolio. Then the portfolio orthogonal to X, whose variance is minimum for the return level r_z, has investment proportions

$$Z = [X \vdots V^{-1}R \vdots V^{-1}\iota]\, H^{-1}\, (0\ r_z\ 1)' \tag{9}$$

where H, a (3x3) symmetric matrix of constants, is given in (11) below.

<u>Proof</u>. To find minimal-variance orthogonal portfolios when X is not efficient, we must minimize $\sigma_z^2 = Z'VZ$

$$\text{subject to}\quad \begin{aligned} Z'VX &= 0, \\ Z'R &= r_z, \text{ and} \\ Z'\iota &= 1. \end{aligned}$$

The Lagrangian is

$$= Z'VZ - \lambda_1(Z'VX) - \lambda_2(Z'R - r_z) - \lambda_3(Z'\iota - 1)$$

and the first-order conditions are

$$2VZ = (VX \vdots R \vdots \iota)(\lambda_1\lambda_2\lambda_3)'\ .$$

Pre-multiplying by $(VX \vdots R \vdots \iota)'\, V^{-1}/2$ gives

$$(0 \vdots r_z \vdots 1)' = H(\lambda_1\lambda_2\lambda_3)'/2 \tag{10}$$

where the (3x3) matrix H is defined as

$$H \equiv (VX \vdots R \vdots \iota)'\, V^{-1}(VX \vdots R \vdots \iota) \tag{11}$$

$$= \left[\begin{array}{c|cc} \sigma_p^2 & r_p & 1 \\ \hline r_p & & \\ & \multicolumn{2}{c}{A} \\ 1 & & \end{array}\right]$$

and where A is the efficient set information matrix, (cf., (4)). Eliminating

$1/2(\lambda_1\lambda_2\lambda_3)'$ from the first-order conditions by substituting from (10) gives (9).

Q.E.D.

The solution to (9) and many of the further results below require the inverse of H. It is convenient to compute this inverse by the partition method, using the structure displayed just below (11). The result can be used to obtain

$$H^{-1}\begin{bmatrix}0\\ r_z\\ 1\end{bmatrix} = \begin{bmatrix}-\sigma_{p^*z}/D\\ G(r_z 1)'\end{bmatrix}^{8}$$

where

p* is the mean/variance efficient portfolio with the same mean return as p itself; N.B.: although the covariance between z and p is zero, the covariance is not generally zero between z and an efficient portfolio with the same return as p;

$D \equiv \sigma_p^2 - \sigma_{p^*}^2$,

measures the degree of p's inefficiency by the horizontal distance (in the variance direction) of p's location from the efficient frontier;

and

(14) $$G \equiv A^{-1}[I + (r_p\ 1)'(r_p\ 1)A^{-1}/D],$$

the lower right (2x2) submatrix of H^{-1}, is symmetric and constant for given p. (I is the 2 x 2 identity matrix.)

Using the notation just presented, the covariance of a (minimal-variance) orthogonal portfolio with any arbitrary portfolio, say q, can be obtained from (2) and (9) as

(15) $$X_q VZ = (\sigma_{qp}\ r_q\ 1)\ H^{-1}(0\ r_z\ 1)\ .$$

If q also is orthogonal, then $\sigma_{qp} = 0$ and only G, the lower right (2x2) submatrix of H^{-1}, needs to be considered; i.e., if Z_1 and Z_2 are both orthogonal

[8]The development of $H^{-1}(0\ r_z\ 1)'$ into its individual elements yields,

$$H^{-1}(0\ r_z\ 1)' = \frac{1}{D|A|}\begin{bmatrix} r_z(b - cr_p) + br_p - a \\ r_z(c\sigma_p^2 - 1) + r_p - b\sigma_p^2 \\ r_z(r_p - b\sigma_p^2) + a\sigma_p^2 - r_p^2 \end{bmatrix}.$$

to p and one of them is on the orthogonal frontier, then

$$\sigma_{z_1 z_2} = Z_1 V Z_2 = (r_{z_1} \; 1) G (r_{z_2} \; 1)' . \tag{16}$$

Corollary 1. Any property of efficient portfolios which depends only upon their means, variances, and covariances is true also for orthogonal portfolios.

Proof. The covariance between two efficient portfolios, say q* and p* is, from (4),

$$\sigma_{p^*q^*} = (r_{p^*} \; 1) \; A^{-1} (r_{q^*} \; 1)' . \tag{17}$$

Since the two matrices G in (16) and A^{-1} in (17) are both 2 x 2 and symmetric, the structures of (16) and (17) are identical. Thus their means, variances and covariances are related by identical functions (but with different parameters).

Q.E.D.

This implies that much of the efficient set mathematics can be applied directly to the orthogonal set. In particular, the geometric form and the position of the orthogonal frontier can be obtained easily from (16).

Implications of Corollary 1.

(a) The orthogonal frontier is a quadratic function of r_z in the mean/variance space. The scalar formula for the orthogonal frontier is $\sigma_z^2 = [(a_z - 2b_z r_z + c_z r_z^2)/(a_z c_z - b_z^2)]\sigma_p^2$, where $a_z \equiv a\sigma_p^2 - r_p^2$, $b_z \equiv b\sigma_p^2 - r_p$, and $c_z \equiv c\sigma_p^2 - 1$. This formula is obtained most easily from (16) and the results in note 8 by using the fact that $a_z c_z - b_z^2 = D|A|\sigma_p^2$. Recall the analogous scalar formula for the efficient frontier, $\sigma_{p^*}^2 = (a - 2br_{p^*} + cr_{p^*}^2)/(ac - b^2)$.

(b) The minimum-variance orthogonal portfolio (analogous to the global minimum-variance portfolio) has a return, r_{z_o}, which is the solution for the first row of $G(r_{z_o} \; 1)' = 0$. This is analogous to the first row of $A^{-1}(r_o 1)' = 0$, whose solution is $r_o = b/c$. The solution for r_{z_o} is

$$r_{z_o} = (r_o \sigma_p^2 - r_p \sigma_o^2)/(\sigma_p^2 - \sigma_o^2) = b_z/c_z \tag{18}$$

where r_o and σ_o^2 are the mean and variance of the global minimum-variance portfolio. From (18) it is easy to prove that r_o always falls between r_p and r_{z_o}.[9] Furthermore, (18) can be rewritten $r_p = r_{z_o} + (\sigma_p^2/\sigma_o^2) \cdot (r_o - r_{z_o})$ which shows that a line from p through o intersects the return axis at r_{z_o}. The vertical distance $F = r_o - r_{z_o}$ is, from (18), $\sigma_o^2(r_p - r_o)/(\sigma_p^2 - \sigma_o^2)$. But from the efficient set geometry, $(r_p - r_o)/(\sigma_{p*}^2 - \sigma_o^2) = (r_o - r_{z*})/\sigma_o^2$ where z* is the efficient portfolio orthogonal to p*, the efficient portfolio with return r_p. Thus, $F = r_o - r_{z_o} = (r_o - r_{z*})/(D/K+1)$ where $K = \sigma_{p*}^2 - \sigma_o^2$ and $D = \sigma_p^2 - \sigma_{p*}^2$. See Figure 1.

(c) A line drawn in the mean/variance space from any orthogonal portfolio, z, through the position of z_o, the minimum-variance orthogonal portfolio, intersects the return axis at the level of another portfolio that is orthogonal to both z and to p. In Figure 1, for example, the portfolios z and u are uncorrelated. Both are uncorrelated with p.

(d) The difference in the variance direction between the efficient and orthogonal frontiers is $B \equiv (\sigma_{p*z}^2)^2/D$. This is obtained from (14) and (16) by noting that

$$\sigma_z^2 = (r_z\ 1)A^{-1}(r_z\ 1)' + (r_z\ 1)A^{-1}(r_p\ 1)'(r_p\ 1)A^{-1}(r_z\ 1)'/D. \tag{19}$$

But from (4), the first term is the variance of an efficient portfolio whose return is r_z. Furthermore, $\sigma_{p*z} = \sigma_{zp*} = (r_z\ 1)A^{-1}(r_p\ 1)'$, and so the second term is $(\sigma_{p*z})^2/D$. For a portfolio z* which is orthogonal to p*, we must have

$$\sigma_{p*z} = \sigma_{p*}^2(r_z - r_{z*})/(r_p - r_{z*})$$

[9] Rearranging (18) we have $r_0 = (1 - \gamma)r_{z_0} + \gamma r_p$ where $\gamma = \sigma_0^2/\sigma_p^2$. Since $\sigma_0^2 < \sigma_p^2$ always, $0 \le \gamma < 1$ and thus r_0 is always between r_{z_0} and r_p.

and thus

$$B = [\sigma_o^2(r_z - r_{z*})/(r_o - r_{z*})]^2/D .$$

(e) We observe that $B = 0$ for $r_z = r_{z*}$ (and for $\sigma_{p*z} = 0$). Thus, the efficient and orthogonal frontiers must be tangent at z*. Since the global minimum-variance portfolio return r_o always falls between r_p and r_{z_o}, (implication b above), the tangency portfolio z* always lies farther from p in the return dimension than does the minimum-variance orthogonal portfolio z_o.

(f) As p's variance increases, $(D \to \infty)$, the orthogonal frontier approaches coincidence with the efficient frontier. This result can be obtained by noting that the second term in (19) vanishes as D increases. As p approaches efficiency, $(D \to 0)$, σ_z^2 in (19) grows without bound when $\sigma_{p*z} \neq 0$. We know already, however, that there is a unique orthogonal portfolio z* when p = p*. Thus, the orthogonal frontier collapses to the line $r_z = r_{z*}$ as p approaches p*.

(g) Every inefficient portfolio has an orthogonal portfolio with the same mean return and some have orthogonal portfolios with the same mean return and the same variance. From (19), the minimal-variance portfolio z' orthogonal to p and with the same return as p has variance

$$\sigma_{z'}^2 = \sigma_{p*}^2(1 + \sigma_{p*}^2/D)$$

$$= \sigma_p^2\, \sigma_{p*}^2/(\sigma_p^2 - \sigma_{p*}^2) .$$

When $\sigma_{z'}^2 < \sigma_p^2$, portfolio p lies within the space bounded by the orthogonal frontier. Thus for $\sigma_p^2 > 2\sigma_{p*}^2$, p has an orthogonal portfolio with the same mean and variance. For $\sigma_p^2 = 2\sigma_{p*}^2$, this is the minimal-variance orthogonal portfolio with return r_p.

The orthogonal vector Z in (9) is a weighted sum of the three other vectors: X_p (the original portfolio); $V^{-1}R/b$, the mean/variance efficient vector whose mean return is a/b and whose orthogonal portfolios have zero mean return; and $X_o = V^{-1}\iota/c$, the global minimum-variance portfolio vector.[10] The well-known two-fund property of efficient vectors implies that any two (different) efficient portfolios could also be used to obtain the vector Z in (9). We have therefore established the following:

[10] Cf. Roll [9, pp. 161, 165].

Corollary 2. ("Three-Fund Theorem"). The investment proportions vectors of portfolios with minimal-variance (for a given mean) that are orthogonal to an arbitrary portfolio p, are spanned by p's vector of investment proportions plus any two (differing) mean/variance efficient investment proportions vectors.

Proof. See above.

Using the two convenient efficient portfolios X_1 and X_o, (9) can be expressed as

(20) $$Z = -hX_p + (X_1 b : X_o c)\ G(r_z\ 1)'$$

where $h \equiv \sigma^2_{p^*z}/D$. By applying the portfolio additivity condition (9), we obtain $1 = h + (b\ c)\ G(r_z\ 1)'$, and thus Z is a weighted average of X_p, X_1 and X_o.

The following implications are obtained readily from (20).

Implications of Corollary 2.

(a) The minimum-variance orthogonal portfolio, z_o, is a weighted average of only X_p and of X_o. This follows from (20) and implication (b) of Corollary 1 (the first row of $G(r_z\ 1)'$ is zero for z_o).

(b) There is a unique member of both the orthogonal and efficient frontiers. When $\sigma_{p^*z} = 0$ in (20), Z is a weighted average of the efficient portfolios X_1 and X_o alone. Therefore, Z also is efficient. This is portfolio **z*** in Figure 1 and its return is, from (7), $r_{z^*} = (br_p - a)/(cr_p - b)$.

Corollary 3. There is a weighted average of portfolio p with its minimal-variance orthogonal partner **z** that produces an efficient portfolio, q*. Furthermore, q* and z have orthogonal portfolios with the same mean return.

Proof. Let the vector of investment proportions of q* be denoted X_{q^*}. Then from (20), $X_{q^*} = (Z + hX_p)/(1 + h)$. X_{q^*} is efficient because it is a linear combination of X_1 and X_o. Now rewrite (20) as

$$X_{q^*} = (X_1 b : X_o c)\ G(r_z 1)'/(1 + h)\ .$$

Then pre-multiplying both sides by $(R\ \iota)'$ and recalling that $r_1 = a/b$, $r_o = b/c$, we obtain

$$(r_{q^*}\ 1)' = AG(r_z\ 1)'/(1 + h)\ .$$

Since q^* is efficient, its investment proportions can also be expressed from (6), as

$$X_{q^*} = V^{-1}(R\ 1)\ G(r_z\ 1)'/(1+h)\ .$$

Also, if s^* is a portfolio which is orthogonal to q^*, then

$$\sigma_{s^*q^*} = (r_{s^*}\ 1)\ A^{-1}(r_{q^*}\ 1)' = 0$$

$$= (r_{s^*}\ 1)\ G(r_z\ 1)' \quad = 0\ .$$

If u is a portfolio orthogonal to z then from (16)

$$\sigma_{uz} = (r_u\ 1)\ G(r_z\ 1)' = 0.$$

Thus $\qquad r_u = r_{s^*}. \qquad$ *Q.E.D.*

Implications of Corollary 3.

(a) There is a useful geometric property, illustrated in Figure 1, that lines passed from $r_u = r_{s^*}$ on the return axis through o and z_o intersect respectively the efficient frontier at q^* and the orthogonal frontier at z.

(b) A special case of this geometric property occurs when portfolios orthogonal to q^* and z have the return r_p. Then a line from r_p on the return axis through o must pass also through z^*. (See Figure 1.) But z^* is on both orthogonal and efficient frontiers, so the same line must pass also through z_o!

Corollary 4. Let C = VZ denote the vector of covariances between individual assets and the orthogonal portfolio z. Let α denote the vector of deviations about the securities market line (cf., note 5). Then C and α are spanned by the same three vectors, (VX, R and 1). Also $Z'\alpha = X_p'\alpha = o$ for every Z, Xp, and α.

Proof. Actually, when p is nonefficient, there are two different definitions of deviations from the securities market line. The simplest is

$$\alpha_1 \equiv R - r_z 1 - \beta[r_p - r_z] \tag{21}$$

where the beta vector is $\beta = VX_p/\sigma_p^2$. Equation (21) can be rewritten as

$$\alpha_1 = (VX\ R\ \iota)[(r_z - r_p)/\sigma_p^2 : 1 : -r_z]'. \tag{22}$$

From (9) or (20), we can obtain the covariance vector of Z with individual assets as

$$C = VZ = (VX\ R\ \iota)[-h : G(r_z\ 1)]'. \tag{23}$$

Clearly both α_1 and C are spanned by VX, R, and ι.

A second definition of deviations from the securities market line uses the vector of individual betas computed against the zero-beta portfolio.

$$\alpha_2 \equiv R - VZ(r_z/\sigma_z^2) - VX(r_p/\sigma_p^2). \tag{24}$$

(The vectors α_1 from (21) and α_2 from (24) are equal when p is mean/variance efficient and only then.) Equation (24) contains VZ on the right side. But from (23), we note that VZ is just a linear combination of VX, R and ι. Thus, α_2 also is spanned by the same vectors as α_1. This demonstrates the first part of the corollary.

The last sentence is demonstrated simply by carrying out the vector products. From (22), we obtain $X_p'\alpha_1 = 0$ and from (24), $X_p'\alpha_2 = 0$, (because $X_p'VZ = 0$). For the products of Z and α, we have

$$Z'\alpha_1 = (0\ r_z\ 1)\ H^{-1}(VX\ R\ \iota)'\ V^{-1}(VX\ R\ \iota)[(r_z - r_p)/\sigma_p^2 : 1 : -r_z]'$$

$$= (0\ r_z\ 1)[(r_z - r_p)/\sigma_p^2)/\sigma_p^2 : 1 : -r_z]' = 0$$

and similarly,

$$Z'\alpha_2 = (r_z\ 1)[(1\ 0)' - G(r_z\ 1)'(r_z/\sigma_z^2)]$$

$$= r_z - r_z(r_z\ 1)\ G(r_z\ 1)'/\sigma_z^2 = 0\ . \qquad \textit{Q.E.D.}$$

Implications of Corollary 4.

The deviations about the securities market line are commonly considered as measures of the qualities of individual assets. We observe from this corollary, however, that the α's are strictly functions of the portfolios p and z chosen to obtain the securities market line. Unfortunately, the deviations will generally appear to be unrelated to their determining portfolios because their vector inner products, $\alpha'X$ and $\alpha'Z$, which are simply the weighted averages of the α's with respect to the investment proportions of z and p, are constrained mathematically to zero.