INVESTMENT DIVERSIFICATION AND BOND MATURITY

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I. THE TERM STRUCTURE AS A CAPITAL ASSET PRICING PROBLEM

WHAT DETERMINES the shape of the term structure of interest rates? What causes its shape to change over time? Answers to these questions are needed by monetary policy makers, by corporate treasurers, by speculators, and by economists studying multi-period consumption-investment choices.

Risk, the central feature of the term structure problem, has been attributed to four sources: (a) default; (b) differences in liquidity or money substitutability among various maturities; (c) uncertain rates of inflation; and (d) incompatibility between bond maturities and investor horizons. We shall introduce another risk component, portfolio interaction, which has been widely discussed in trade and academic literature as a major determinant of common stock prices but has been neglected as an explanatory element of the term structure.

Portfolio diversification has obvious application to the bond-holder and the bond-issuer. A lender with a given horizon period may find that his optimum portfolio contains short-term *and* long-term bonds. This would be a rather trite contention were it not for the many statements in term structure literature about investors plunging into either long or short maturities.[1] It is less obvious than for the lender, but the borrower may also choose to diversify his portfolio of outstanding obligations over various maturities. For example, consider a government agency borrowing for a specific long-term project at current high rate levels. It might be able to reduce total expected interest payments (and expected taxes) by financing the project partly with short-term bonds rather than entirely with bonds whose term-to-maturity matches the project's life. On the other hand, even though the agency expects lower rates in the future, it would not feel secure in funding the entire project with short-term bonds that would require a later refinancing. It would prefer to pay the higher expected rate on a portfolio of long- and short-term bonds rather than accept the risk that rates will go higher contrary to expectations.

Unfortunately, these simple illustrations of maturity diversification fail to indicate the exact economic benefits of such action. Just what are the quantitative links between maturity and risk? The remainder of the paper will attempt to describe them. Section II will review the theory of the term structure of interest rates under uncertainty. Section III will outline the Sharpe-Lintner theory of capital asset pricing and show the relation between liquidity premiums (from term structure theory), and risk premiums (from capital asset

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1. Cf. [2, p. 274], [10, p. 514], [19, p. 524].

pricing theory). Section IV will test the combined theories with U.S. Treasury bill rates and Section V will present conclusions.

II. Theory of the Term Structure Under Uncertainty

The notation and algebra of term structure theory is one of its most confusing qualities. (Some have lamented the infeasibility of printing a cubic figure since a three-dimensional symbol would permit a super- or sub-script at all eight corners rather than at only four.) We shall attempt to alleviate these difficulties by only discussing the simplest debt instruments, the bill, and by using the shortest compounding interval, zero. A bill pays a fixed amount at maturity and nothing before maturity.[2] For convenience, we assume the fixed payment is one dollar. The current market price of a one-dollar bill defines its continuously compounded yield-to-maturity as shown in Table 1. Prices of bills adjacent in maturity define market "forward rates" (also shown in Table 1).

TABLE 1
Continuously Compounded Interest Rate Definitions

$p_{n,t}$	The market price, at the beginning of period t, of a bill with n periods until maturity. (The value at maturity is normalized to \$1 and $p_{n,t}$ is less than \$1 if the yield is positive.)
$R_{n,t} \equiv -\frac{1}{n}\log_e(p_{n,t})$:	the continuously compounded internal rate of return on an n-period bill at period t. This is also called the "yield-to-maturity" and the "n-period spot rate."
$r_{k,t} \equiv kR_{k,t} - (k-1)R_{k-1,t}$ $\equiv \log_e\left(\frac{p_{k-1,t}}{p_{k,t}}\right)$; $(k \geqslant 1)$:	the one-period forward rate at period t which is applicable k periods after t. $r_{k,t}$ is the yield on a current "futures contract" to loan money for one period starting at the beginning of period $t+k-1$. This contract is also called a "forward loan."[a,b]

[a] To make a one-period "forward loan" in a perfect capital market, one simply sells $(k-1)$-period bills short and uses the proceeds to buy k-period bills. At the end of $k-1$ periods, the short sale matures and the trader is left long in one-period loans. Before $k-1$ periods have elapsed, the trader is neither a net borrower nor a lender. In the absence of default risk, the forward rate is a perfectly certain nominal return.

[b] Forward rates applicable over more than one future period are defined analogously. The j-period forward rate, observed at period t, to begin at $t+k-1$ is

$$ {}_jr_{k,t} \equiv \frac{1}{j}[kR_{k,t} - (k-j)R_{k-j,t}] \equiv \frac{1}{j}\log_e\left(\frac{p_{k-j,t}}{p_{k,t}}\right);\ (k \geqslant j) $$

When the preceding subscript, j, equals unity, it will be deleted.

Prominent term structure theories assert relations between each market forward rate and a corresponding future spot rate "expected" by the market.

2. All other types of debt obligations make intermediate "coupon" payments before a final repayment of "principal." These coupon payments cloud the true maturity of the security since they may be regarded as part of the total debt and are paid at many different maturities. This has given rise to the concept of "duration" [12, pp. 44-53], which is an attempt to adjust a bond's maturity downward if its coupon is high and upward if its coupon is low relative to the current yield-to-maturity.

An understanding of this natural relation between forward rates and future spot rates goes back at least to Irving Fisher[3] and perhaps even to ancient times.[4]

The general form of the asserted market equilibrium relation is

$$r_{j,t} = E_t(\tilde{R}_{1,t+j-1}) + L_{j,t} \tag{1}$$

where $r_{j,t}$ is the one-period forward rate currently calculated from market prices,[5] $R_{1,t+j-1}$ is the one-period market spot rate at the beginning of $t + j - 1$ (a random variable in t), E_t is the mathematical expectations operator at the beginning of t, and $L_{j,t}$ is a "liquidity premium." Since (1) is a market equilibrium equation, a question arises concerning $E_t(\tilde{R}_{1,t+j-1})$. If we do not exclude differences of opinion among market participants, whose expectation is this? The answer, provided by [15, pp. 26-32], shows E_t to be a weighted average of individual expectations, the weights depending on individual endowed resources, risk preferences, consumption plans, and levels of confidence. The precise composition of each liquidity premium, $L_{j,t}$, in terms of individual investor characteristics, is also given in [15, p. 32].

While much controversy in recent years has involved the sign and temporal behavior of liquidity premiums, most economists are willing to accept the basic "expectations hypothesis" (of which equation (1) is an algebraic depiction). Many are still justifiably unsure about the three variants[6] of the hypothesis, however, and a primary goal of the present endeavor is to provide a more appealing theoretical and empirical explanation of liquidity premiums via an "expectations-portfolio hypothesis" which contains all three current term structure theories as special cases.

The theory of efficient securities markets can be combined with (1), the static equilibrium equation, to produce a dynamic term structure theory. In

3. Fisher did not mention forward rates by name but clearly recognized the imputed forward rate when he said, "Such an investor [who has a horizon of 50 years and is contemplating the purchase of a 25-year bond yielding five per cent], if he expected the rate of interest at the end of 25 years to be 2 per cent, would, in purchasing the above-mentioned bond, be getting $5 a year for 25 years and $2 a year for the next 25 years. Under these conditions, if he could buy a 50-year bond at 4 per cent, he would prefer to do so." [2, p. 274]. Fisher ignores the possibility of risk aversion in this example even though it is contained in a chapter entitled, "The Risk Element." In 1907, Fisher represented a formal analysis of multi-period lending in [3]. His "rates of interest for successive years" were actually one-year forward rates [3, pp. 355-417].

4. Cf. "Mesopotamia: Sumer, Babylonia, and Assyria," Chapter II of [5].

5. See Table 1.

6. The three variants are: the pure expectation hypothesis associated with Meiselman [13], the liquidity preference hypothesis, Kessel [7], and the market segmentation hypothesis. The pure expectations hypothesis asserts that $L_{j,t}$ is zero for all j and t. Supposedly, this is accomplished by risk- *and* maturity-indifferent speculators who arbitrage among securities until forward rates equal expected spot rates. The liquidity preference hypothesis, originally suggested by Hicks [4], asserts $L_{j,t}$ is uniformly positive for all j and t. This is supposed to be due to either (a) a continually weak supply of long-term relative to short-term funds Hicks, [4, pp. 146-47], or (b) a higher quality of liquidity (or moneyness) in short-term bonds Kessel [7, p. 45]. The market segmentation hypothesis has served as straw man for many without having received a rigorous exposition. (However, a good summary is provided by Malkiel [11, pp. 28-30].) In its most generous form, Modigliani and Sutch, [14], it asserts that bonds of different maturities are imperfect substitutes. In its strictest form (as straw man), it claims that no trader will deviate from his préferred maturity.

an efficient market, all available information about a security is very rapidly incorporated into its market price by traders who have a great incentive to act on new information quickly. Under the most severe assumptions,[7] the theory concludes that no trading rule based on *any* past information will earn a profit greater than what one could obtain by investing in securities at random.[8] If information is costly to obtain and evaluate, however, a less strict form of the hypothesis asserts that no *economic* profit can be obtained by using a price-predicting model or trading rule. In this case, trading rules based on non-price data may be better than random selection, but they may also be more costly to develop and implement. Competition among traders should ensure an identity between their marginal trading revenues and their marginal implementation costs.

For the term structure, the theory of efficient markets requires forward rates to evolve over time according to

$$E_{t-H}(\tilde{r}_{j,t} - \tilde{L}_{j,t} \mid B_{t-H}) = r_{j+H,t-H} - L_{j+H,t-H} \tag{2}$$

where B_{t-H} represents a union of the individual information sets available to traders in period $t - H$.[9] Over time, $r_{j+1,t-1}$, $r_{j,t}$, $r_{j-1,t+1}$, etc., are sequential market forward rates applicable to the same future one-period spot rate, $R_{1,t+j-1}$. By substituting into (2) from equation (1) for static equilibrium we find:

$$E_{t-H}[\tilde{E}_t(\tilde{R}_{1,t+j-1})] = E_{t-H}(\tilde{R}_{1,t+j-1}) \tag{3}$$

which records a fundamental characteristic of an efficient bill market: in every period (e.g., in $t - H$), the market expects to hold identical expectations in the future (e.g., in t) to those it now holds. From another viewpoint, equation (2) specifies the required expected return demanded by the market for holding bills. For example, to obtain the one-period expected return in $t - 1$ for $(n + 1)$-period bills, we sum both sides[10] of (2) over j and use identities from Table 1 to obtain

$$E_{t-1}\left[\log_e\left(\frac{\tilde{P}_{n,t}}{P_{n+1,t-1}}\right)\right] = R_{1,t-1} + \sum_{j=1}^{n} [L_{j+1,t-1} - E_{t-1}(\tilde{L}_{j,t})]. \tag{4}$$

The left side of (4) is, of course, the expected return from holding the bill[11] for one-period (continuously compounded), and the right side is just the cur-

7. These assumptions are:
a) zero transactions costs;
b) free information, becoming available to everyone at the same time, and evaluated identically by everyone;
c) everyone acting rationally and believing others do the same.

8. A complete summary of efficient market theory including its historical development and its variants is provided by Fama [1].

9. A mathematical derivation of equation (2) is reported in [15]. It is based on a similar result for commodity futures contracts given by Samuelson [16].

10. After setting $H = 1$.

11. $p_{n,t}$ and $p_{n+1,t-1}$ are subsequent prices of the *same* bill. See Table 1.

rent one-period market spot rate plus a summation that contains aggregated risk preferences, consumption plans, and confidence levels. Equation (4) is a general specification of market price behavior over time. It is similar to the random walk hypothesis[12] and to martingale sequences specified by efficient market theory for common stock prices.[13] It contains, however, one crucial additional feature: the expected log price relative is not a constant because the right side of (4) contains variables with t subscripts. In the random walk hypothesis, a price change (a log price relative), is asserted to be drawn each period from a distribution with constant parameters. Equation (4) is analogous. Its price change is also a random draw each period, but the distribution's mean can shift temporarily. Moreover, we are so far unable to state how the mean will shift from one period to the next.[14]

This is an embarassing situation for a theory since the shifting mean precludes any possibility of empirical refutation. We are therefore forced to look elsewhere for supplements to the theory and for an explanation of the temporal evolution of expected price changes. In the next section, capital asset pricing theory will be enlisted for this purpose.

III. Capital Asset Pricing Theory

The Sharpe [17]-Lintner [9] theory of capital asset pricing under conditions of risk concludes that a market equilibrium is reached when

$$E_{t-H}\left[\log_e\left(\frac{\tilde{P}_{k,t}}{P_{H+k,t-H}}\right)\right] = HR_{H,t-H} + H\beta_k E_{t-H}(\tilde{M}_t - \tilde{R}_{H,t-H})\ (k \geqslant 0). \quad (5)$$

Here $P_{k,t}$ again is defined as the price of a bill with k periods to maturity at the beginning of period t (and the log price relative above is an expected return over H periods), H is an "investor horizon period" which is assumed equal for all investors (hence the H-period spot rate, $R_{H,t-H}$, is a certain return when we neglect default and inflation risks),[15] M_t is a return on a "market" portfolio, and β_k is a market response coefficient.[16] This model assumes

12. Cf. [1].

13. Cf. [1].

14. Note that changes may also occur in other parameters of the distrtibution. Nothing in equation (4) precludes it.

15. In the remainder of the paper, we will assume that bills contain no risks of default or inflation.

16. Theoretically, the "market portfolio contains all assets in the economy weighted in proportion to value. Also, β_k is given by

$$\beta_k = \frac{\mathrm{Cov}\left[\tilde{M}_t; \log\left(\frac{\tilde{P}_{k,t}}{\tilde{P}_{k+H,t-H}}\right)\right]}{\mathrm{Var}\,(\tilde{M}_t)}.$$

Note that k is the maturity remaining at t (the end of the horizon period) on the bill originally purchased. Equation (5) only admits the case of a bill whose time-to-maturity at purchase exceeds or equals the investment horizon. If $k = 0$, the bill originally purchased has maturity H and the investment is riskless. In this case, of course, $\beta_0 = 0$.

Also, (5) assumes the initial purchase is held until the horizon period ends. We will later conthat diversification (through portfolio holding) is a dominant market trait.[17] Every individual is assumed to care only for the risk and return on his *entire* portfolio, and individual securities are only important as contributors to the portfolio's character. This is the well-known raison d'etre for β_k, which measures the co-movement of a security's return with the returns of all other securities. If β_k is positive and large, the security tends to accentuate fluctuations in its portfolio's value. Such is the essence of risk. Securities with large β's will only be held by risk averters if their expected returns are also large, and securities with small β's will be held even when their expected returns are low.[18]

Sharpe-Lintner risk coefficients are related somehow to liquidity premiums. To derive this relation, we first rewrite the static equilibrium term structure equation (1), as

$$E_{t-H}(R_{1,t}) = r_{H+1,t-H} - L_{H+1,t-H}. \tag{6}$$

Letting $k = 1$ in model (5), noting that

$$\log_e (P_{1,t}/P_{H+1,t-H}) \equiv (H + 1)R_{H+1,t-H} - R_{1,t},$$

and combining (5) and (6) we obtain

$$(H + 1)R_{H+1,t-H} - r_{H+1,t-H} + L_{H+1,t-H} = HR_{H,t-H} + H\beta_1 E_{t-H}(\tilde{M}_t - R_{H,t-H}),$$

and by noting that

$$r_{H+1,t-H} \equiv (H + 1)R_{H+1,t-H} - HR_{H,t-H},$$

we obtain

$$L_{H+1,t-H} = H\beta_1 E_{t-H}(\tilde{M}_t - R_{H,t-H}). \tag{7}$$

In the context of the Sharpe-Lintner model the right side of (7) is the *total cost of risk* associated with a bill whose time-to-maturity is just one period longer than the horizon period. It seems appropriate that the liquidity premium associated with a commitment of funds for one period past the horizon should equal this total cost of portfolio risk. Similar algebraic manipulation will provide a second liquidity premium,

$$L_{H,t-H} = -H\beta_{-1} E_{t-H}(\tilde{M}_t - R_{H,t-H}). \tag{8}$$

This risk cost is associated with a bill whose maturity is just one period shorter than the horizon period. A one-period reinvestment (at an uncertain rate) is necessary after $H - 1$ periods when this bill is purchased. Unfortunately, due to the static nature of both capital asset pricing theory and the

sider the possibility of initially purchasing a bill whose maturity is shorter than the horizon and the related possibility of intermediate trading (before the horizon's end).

A complete treatment of the horizon problem in the Sharpe-Lintner model is given by Jensen [6, pp. 186-191].

17. The original theory also assumed homogeneous expectations and investor utility functions which only depend on the mean and dispersion (variance) of rates of return.

18. The precise trade-off between portfolio risk contribution and expected return is established in equilibrium by the preferences of traders. Equilibrium price per unit of risk is $E_{t-H}(\tilde{M}_t - R_{H,t-H})/\mathrm{Var}(\tilde{M}_t)$. See Jensen [6, pp. 175-76].

term structure equilibrium equation (1) used thus far, L_H and L_{H+1} are the only liquidity premiums that adhere to simple relations with Sharpe-Lintner risk coefficients.

Adding the dynamic term structure equation (2) will provide some additional information on the relation between risk coefficients and liquidity premiums. First, we let $k = j - 1$ in the Sharpe-Lintner model (5) and subtract the result from (5) with $k = j$. This provides

$$E_{t-H}(r_{H+j,t-H} - \tilde{r}_{j,t}) = H(\beta_j - \beta_{j-1})E_{t-H}(\tilde{M}_t - R_{H,t-H}) \tag{9}$$

for $j > 0$. According to the dynamic equation (2), the left side of (9) is $E_{t-H}(L_{H+j,t-H} - \tilde{L}_{j,t})$ and thus

$$E_{t-H}(L_{H+j,t-H} - \tilde{L}_{j,t}) = H(\beta_j - \beta_{j-1})E_{t-H}(\tilde{M}_t - R_{H,t-H}) \tag{10}$$

for $j > 0$. Equation (10) provides some insight about long-term averages of liquidity premiums and thus about "biases" in forward rate forecasts of future spot rates.[19] For example, if the market horizon is very short, say $H = 1$, equations (10) and (7) can be combined with $\beta_0 \equiv 0$ to obtain

$$\bar{L}_j = \beta_{j-1}(\bar{M} - \bar{R}_1) \tag{11}$$

where the bars indicate long-term averages.[20] The mean forward rate in this case is

$$\bar{r}_j = \bar{R}_1 + \beta_{j-1}(\bar{M} - \bar{R}_1). \tag{12}$$

Thus with a one-period horizon prevailing in the market, the term structure will be monotonically upward-sloping on average if and only if the β's (the risk coefficients) increase monotonically with maturity,[21] j. This will be true, of course, if longer-term bonds tend to fluctuate in price with general economic conditions more than shorter-term bonds. Nothing in portfolio theory, however, asserts whether longer- or shorter-term bonds have greater price variability and we are obliged to consult data for an answer.

To obtain empirically testable models, we will use equation (9). Setting $j = 1$ in (9), setting $j = k$ in a repetition of (9), and jointly eliminating $E_{t-H}(\tilde{M}_t - R_{H,t-H})$ results in

$$E_{t-H}(\tilde{r}_{k,t} - r_{H+k,t-H}) = \frac{\beta_k - \beta_{k-1}}{\beta_1} E_{t-H}(\tilde{R}_{1,t} - r_{H+1,t-H}) \tag{13}$$

for $H,k > 0$. Assume now that these expectations can be replaced by an additive error term, i.e., that

$$r_{k,t} - r_{H+k,t-H} = \frac{\beta_k - \beta_{k-1}}{\beta_1}(R_{1,t} - r_{H+1,t-H}) + \varepsilon_{H,k,t} \tag{14}$$

for $H,k > 0$. When $H = 1$, the shortest possible horizon period, equation (14) becomes

19. Forward rate bias is the main issue of contention between the "pure" expectations hypothesis and the liquidity preference hypothesis of the term structure (Cf. footnote 6).

20. Equation (11) is obtained by deleting the expectations operators and adding a disturbance term with sample mean zero.

21. When $H \neq 1$, however, the term structure's average shape is a much more complex function of the β's.

$$r_{k,t} - r_{k+1,t-1} = \frac{\beta_k - \beta_{k-1}}{\beta_1}(R_{1,t} - r_{2,t-1}) + \varepsilon_{1,k,t}, \tag{15}$$

which should be familiar to readers of term structure literature because it is formally identical to Meiselman's "error-learning model" [13, p. 20]. It even lacks a constant term, a fact used by Meiselman to infer that all liquidity premiums are zero[22] [13, pp. 45-47]. The error-learning "response coefficient" turns out to be $\frac{\beta_k - \beta_{k-1}}{\beta_1}$, a simple function of the risk coefficients from the Sharpe-Lintner model. Although (15) and the "error-learning" model are formally identical, the economic theory behind them is much different. While the error-learning model asserts that short-term forecast errors generate revisions throughout the term structure, equation (15) has nothing to do with forecast errors but is the direct result of a propensity to hold diversified portfolios. Another contrast between the error-learning model and the theory leading to (15) involves the differencing interval. In published empirical work with the error-learning model, the differencing interval was chosen arbitrarily as the calendar time between successive observations of collected data. Equation (14), however, indicates the differencing interval is exactly H, the "market horizon," and this means that H is another parameter to be estimated empirically, not just selected conveniently.[23]

To accompany (14), we shall now derive a testable model for the situation when a reinvestment is necessary *before* the horizon period expires.[24] In this case, the maturity, k, left on the original bill at the end of the horizon, may be negative. First, we assume that an initial purchase is an $(H + k)$-period bill in $t - H$ at a price $P_{H+k,t-H}$ $(k \leqslant 0)$. After $H + k$ periods this bill matures and a reinvestment must be made for the remaining $-k$ periods of the horizon. For simplicity, assume that the reinvestment is riskless, i.e., a $(-k)$-period bill that will mature exactly at the end of the horizon is purchased as the reinvestment. The total expected return to this strategy is

$$E_{t-H}\left[\log_e\left(\frac{1/\tilde{P}_{-k,t+k}}{P_{H+k,t-H}}\right)\right] = HR_{H,t-H} + H\beta_k E_{t-H}[\tilde{M}_t - R_{H,t-H}];\ (H + k > 0), (k \leqslant 0). \tag{16}$$

Second, consider the strategy of initially purchasing an $(H + k + 1)$-period bill which is sold in $t + k$. (It still has one period to maturity in $t + k$.) The expected return to this strategy is

$$E_{t-H}\left[\log_e\left(\frac{\tilde{P}_{1,t+k}/\tilde{P}_{-k,t+k}}{P_{H+k+1,t-H}}\right)\right] = HR_{H,t-H} + H\beta_{k+1}E_{t-H}(\tilde{M}_t - R_{H,t-H}) \tag{17}$$

22. Of course, liquidity premiums are not necessarily zero in the theory leading to (15).

23. The linear form was also an assumption in Meiselman's theory that had to be justified empirically. Here the linear form is a conclusion.

24. Recall that (14) was developed from (5), a form of the Sharpe-Lintner model that requires the originally purchased bill to have a maturity *longer* than the horizon. Cf. footnote 16.

for $k \leqslant 0$ and $H + k > 0$. Note that when $k = 0$, $\beta_{k+1} = \beta_1$ and (17) is equivalent to (5) with $k = 1$. Algebraic manipulation with (16) and (17) provides

$$E_{t-H}(\tilde{R}_{1,t+k-1} - r_{H+k,t-H}) = \frac{\beta_k - \beta_{k-1}}{\beta_1} E_{t-H}(\tilde{R}_{1,t} - r_{H+1,t-H}) \qquad (18)$$

for $k \leqslant 0$ and $H + k > 0$. Replacing the expectations with an additive error results in a testable model,

$$R_{1,t+k-1} - r_{H+k,t-H} = \frac{\beta_k - \beta_{k-1}}{\beta_1}(R_{1,t} - r_{H+1,t-H}) + \varepsilon_{H,k,t} \qquad (19)$$

which corresponds for $k \leqslant 0$ and $H + k > 0$ to (14) for $H > 0$ and $k > 0$.

Equations (14) and (19) will be empirically supported if money market participants value bills as portfolio components just as stock market participants value common stock. This is not to say that all investors who hold portfolios of common stock need to enter the money market or vice versa. In fact, major holders of bills such as banks and corporate treasurers may not hold common stocks at all, and many mutual funds may not hold bills. Their specialized investment policies can be regarded, however, as a reliance on arbitrageurs whose livelihood is secured by equating money market returns with stock market returns. In the next section, we will examine the extent of such activity by testing equations (14) and (19).

IV. Empirical Tests

For convenience, the two testable equations developed in Section III are reiterated below:

$$r_{k,t} - r_{H+k,t-H} = \frac{\beta_k - \beta_{k-1}}{\beta_1}(R_{1,t} - r_{H+1,t-H}) + \varepsilon_{H,k,t} \qquad (14)$$

for $k, H > 0$; and

$$R_{1,t+k-1} - r_{H+k,t-H} = \frac{\beta_k - \beta_{k-1}}{\beta_1}(R_{1,t} - r_{H+1,t-H}) + \varepsilon_{H,k,t} \qquad (19)$$

for $k \leqslant 0$ and $H + k > 0$. Data, described in the Appendix, are weekly observations of U.S. Treasury Bill yield curves during the period October, 1949, to December, 1964. Time subscripts in (14) and (19) must be measured in weeks and the maximum maturity is 26 weeks. For equation (14), the maturity, $H + k$, on the bill when originally purchased is constrained to the interval $0 < H + k \leqslant 26$. For (19), the limiting subscript, $H + 1$, must remain in $1 < H + 1 \leqslant 26$.

Our data analysis objective is to infer the most likely value for the horizon, H, while simultaneously testing the adequacy of the models represented by (14) and (19). This is a very difficult econometric problem because little is known about suitable methods for comparing alternate models[25] when the

25. Equations (14) and (19) constitute a different model for each horizon value, H. Each model is a set of 24 equations as $H + k = 2, \ldots, H, H + 2, \ldots, 26$ for each given H. Note that when $H + k = H + 1$, (14) is a trivial equation.

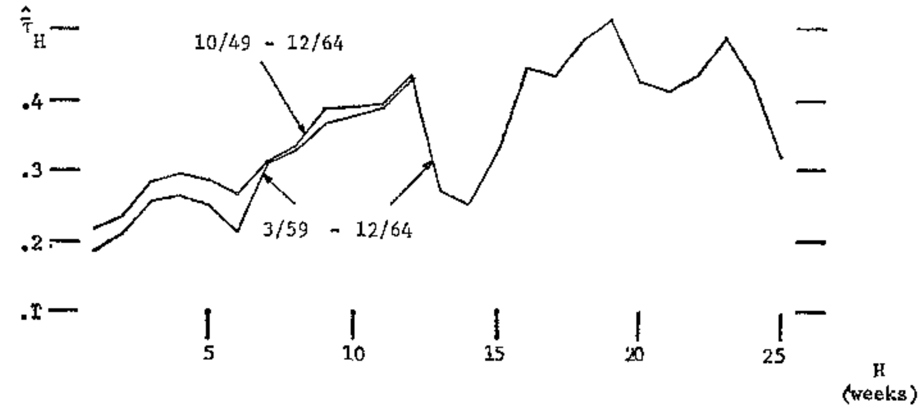
number of sample observations differs among them. Note that the number of observations available for each equation is constrained by a data sample of fixed calendar length because the differencing interval spans H periods. Since sample intervals must not overlap, longer differencing intervals have smaller numbers of observations.

An Attempt to Measure the Market Horizon Period

The strength of association between variables on the left and right sides of equations (14) and (19) will provide an empirical basis for inferring the most likely value of the market horizon period. For each value of the horizon ($H = 1, \ldots, 25$ weeks), we shall measure association by Kendall's tau,[26] $\tau_{H,H+k}$, calculated between the two variables[27] in equation (14) and between the two variables[28] in (19). This is a total of 576 sample values of Kendall's tau, but since we are only interested in estimating the horizon, H, in this subsection, we shall average $\hat{\tau}_{H,H+k}$ over the nuisance parameter k (the maturity remaining on the initial purchase at the horizon's end), to obtain

$$\hat{\bar{\tau}}_H = \left[\sum_{H+k=2}^{H} \hat{\tau}_{H,H+k} + \sum_{H+k=H+2}^{26} \hat{\tau}_{H,H+k} \right] \Big/ 24.$$

These means[29] are presented in Table 2 along with the minimum and maximum sample sizes for each H and the standard deviations[30] of $\tau_{H,H+k}$ calculated over $H + k$ given H. $\hat{\bar{\tau}}_H$ is plotted against H in Figure 1.



26. This is a well-know ordinal measure of association. For a bivariate population on the random variable y and x, let (y_1,x_1) and (y_2,x_2) be a two-fold sample. Kendall's tau is $\tau =$ concordance $-$ discordance $= \Pr(y_1 > y_2 \text{ and } x_1 > x_2) + \Pr(y_1 < y_2 \text{ and } x_1 < x_2) - \Pr(y_1 < y_2 \text{ and } x_1 > x_2) - \Pr(y_1 > y_2 \text{ and } x_1 < x_2)$. See [8, pp. 822, 836-839].

27. These variables are $(r_{k,t} - r_{H+k,t-H})$ and $(R_{1,t} - r_{H+1,t-H})$ for $H + k = H + 2, \ldots, 26$; $H = 1, \ldots 24$.

28. These variables are $(R_{1,t+k-1} - r_{H+k,t-H})$ and $(R_{1,t} - r_{H+1,t-H})$ for $H + k = 2, \ldots, H - 1, H$; $H = 2, \ldots, 25$.

29. The complete 576-element table of $\hat{\tau}_{H,H+k}$ is available from the author on request.

30. This measure of dispersion cannot be used in the ordinary way to calculate the standard error of the mean because there is no reason to suppose that the $\hat{\tau}_{H,H+k}$ are independent.

TABLE 2

SUMMARY MEASURES OF ASSOCIATION BETWEEN CHANGES IN FORWARD RATES
(U.S. Treasury Bills, 1949-64)

Horizon H (weeks)	$\hat{\tau}_H$	Standard* Deviation of $\hat{\tau}_{H,H+k}$	Sample Sizes† Minimum	Sample Sizes† Maximum	Number of Negative $\hat{\tau}_{H,H+k}$ Out of 24
		October, 1949—December, 1964			
1	.119	.0855	291	795	1
2	.138	.0972	146	397	0
3	.181	.103	96	265	1
4	.195	.109	73	198	0
5	.184	.148	57	159	4
6	.164	.146	47	132	3
7	.213	.134	39	113	0
8	.235	.174	37	99	2
9	.283	.153	31	88	1
10	.284	.144	28	79	0
11	.290	.149	25	72	1
12	.336	.151	24	65	1
		March, 1959—December, 1964			
1	.0854	.0681	291	304	1
2	.110	.0910	146	151	1
3	.158	.0838	96	101	1
4	.163	.0888	73	75	0
5	.154	.140	57	60	5
6	.114	.112	47	50	4
7	.213	.125	39	42	0
8	.223	.167	37	37	2
9	.262	.149	31	33	2
10	.279	.145	28	29	0
11	.287	.175	25	27	0
12	.330	.168	24	25	1
13	.171	.125	23	23	2
14	.156	.187	20	20	5
15	.229	.170	20	20	3
16	.344	.146	18	18	0
17	.338	.108	16	17	0
18	.382	.176	14	16	0
19	.416	.159	14	15	0
20	.325	.175	13	14	2
21	.313	.272	12	13	4
22	.337	.195	11	13	1
23	.388	.172	12	12	0
24	.321	.169	12	12	1
25	.220	.208	10	10	4

$$* \left[\sum_{k=2-H}^{0} (\hat{\tau}_{H,H+k} - \hat{\bar{\tau}}_H)^2 \Big/ 24 + \sum_{k=2}^{26-H} (\hat{\tau}_{H,H+k} - \hat{\bar{\tau}}_H)^2 \Big/ 24 \right]^{1/2}$$

† Minimum sample sizes correspond to longest differencing intervals and maximum sample sizes to shortest differencing intervals in equations (14) and (19).

The results indicate a positive association among the forward rate changes in equations (14) and (19).[31] Estimated means of $\hat{\hat{\tau}}$ are uniformly positive over all values of the horizon parameter, H, and only 36 of the 576 $\hat{\tau}$'s are negative.[32] However, the results are far from conclusive concerning the horizon parameter, H. One would like to select the value of H that results in the best fit of equations (14) and (19) as evidenced by the largest absolute magnitude of $\hat{\tau}_H$. Unfortunately there is no clear-cut winner. If there is a unique market horizon, the results only suggest that it is not very short term, say 2-6 weeks, nor does it correspond to the issues of new maturities of Treasury bills, 13 and 26 weeks, because the degrees of association between the variables in (14) and (19) are close to zero in these ranges.[33] Portfolio considerations seem rather weak over the shortest horizons because the model does not explain much of the short-term variation in forward rate changes.

Risk Coefficients for Treasury Bills

Estimated values of Kendall's tau, measuring association between forward rate changes, are largest in the horizon ranges of 8-12 and 16-23 weeks. If portfolio considerations motivate money market investors, these periods are the most likely candidates, among those considered, for investor horizons. After choosing one of these as the "market's" horizon, estimates of Sharpe-Lintner risk coefficients can be calculated for Treasury bills. These estimates should measure the value of bills as portfolio components and also provide evidence on the structure of liquidity premiums.

To conserve space, estimates for only one horizon are reported below.[34] $H = 8$ weeks was chosen because its mean degree of association, $\hat{\hat{\tau}}_8$, was a median value of the $\hat{\tau}$'s and thus seems representative, being biased in neither direction with respect to the sample of $\hat{\hat{\tau}}$'s. The quantity

$$\gamma_k \equiv \frac{\beta_k - \beta_{k-1}}{\beta_1}$$

is the coefficient of forward rate change on the right sides of (14) and (19). Table 3 presents unconstrained least-squares estimates of $\gamma_k (k = -6, -5, \ldots, 0, 2, \ldots 18)$ and calculated standard errors.[35] Since $\beta_o \equiv 0$, estimates of

31. Of course, this results might have been anticipated because past empirical fits of Meiselman's error-learning model, equation (14) with $H = 1$, have generally shown strong positive correlation between forward rate changes [13, p. 22]. Note, however, that $H = 1$ provides the weakest degree of positive association with these data.

32. In the sample period from March, 1959, on.

33. Treasury bill maturities near 13 and 26 weeks are peculiar in several respects. Transaction costs are considerably lower at these maturities than at adjacent ones. Also, the term structures of yields and forward rates have discontinuities at these points. [See 15, ch. 5]. New issues of Treasury bills evidentally affect the patterns of yields and prices in a manner admittedly unexplained by the current theory.

34. Results for all values of H are available on request.

35. A note of caution: We have used non-parametric methods here when possible because the distributions of forward rate changes seem non-Gaussian. The distributions are symmetric but have fat tails and the standard errors in Table 3 should not be used for Gaussian-based tests of hypotheses. They are only reported for completeness. For more information on the distribution of forward rate changes, see [15].

individual β's (as proportions of β_1) can be obtained from estimated γ's by using

$$\frac{\beta_k}{\beta_1} = 1 + \sum_{j=2}^{k} \hat{\gamma}_j \qquad (k > 1)$$

and

$$\frac{\beta_k}{\beta_1} = -\sum_{j=k+1}^{0} \hat{\gamma}_j \qquad (k \leqslant 0).$$

Estimated values of β_k/β_1 are reported in Table 3 and are plotted in Figure 2. Sample sizes are smaller for $k > 5$ (or $H + k > 13$) because maturities greater than 13 weeks were only issued by the Treasury in the latter part of

TABLE 3

LEAST SQUARES ESTIMATES FOR RISK COEFFICIENTS, HORIZON-EIGHT WEEKS

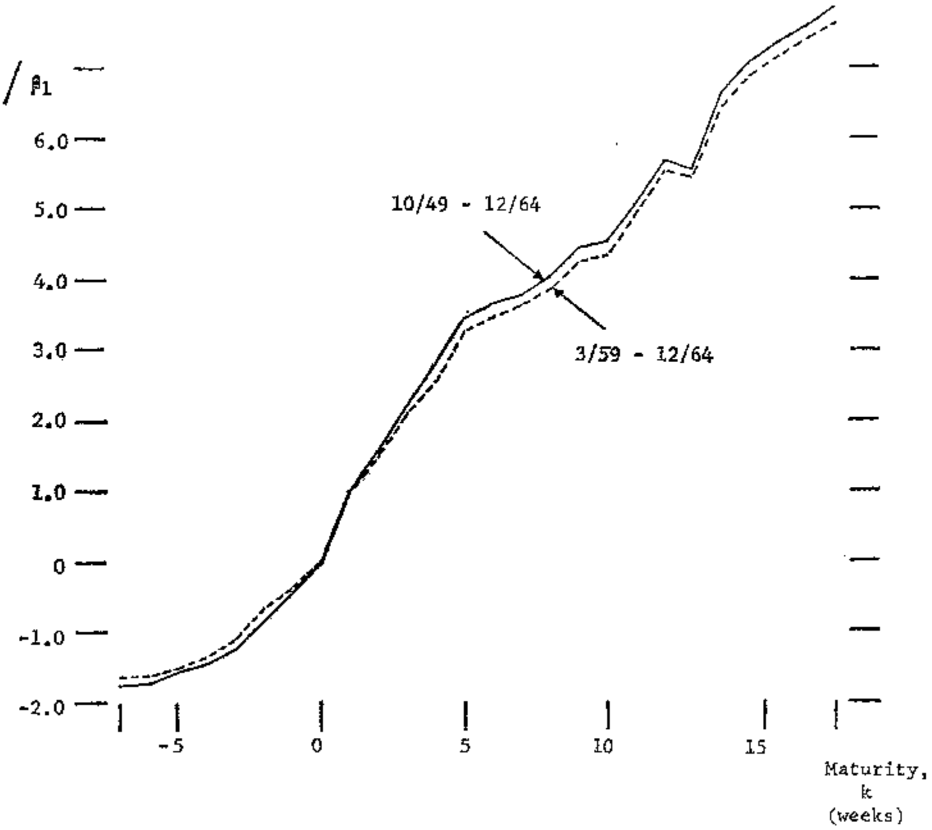
(U.S. Treasury Bills, 1949-64*)

Maturity, k (weeks)	$\hat{\gamma}_k$	Standard Error, $\hat{\gamma}_k$	$\hat{\beta}_k/\beta_1$	Sample Size
−7			−1.71	
−6	.0250	.0295	−1.69	99
−5	.116	.0443	−1.57	99
−4	.124	.0557	−1.45	99
−3	.242	.0681	−1.21	99
−2	.388	.0684	− .819	99
−1	.418	.0950	− .401	99
0	.401	.0556	0.0	99
2	.587	.0997	1.587	99
3	.609	.0558	2.20	99
4	.591	.0600	2.79	99
5	.635	.0844	3.42	97
6	.209	.230	3.63	37
7	.157	.173	3.79	37
8	.240	.122	4.03	37
9	.400	.164	4.43	37
10	.115	.109	4.54	37
11	.523	.209	5.07	37
12	.604	.164	5.67	37
13	−.0755	.154	5.59	37
14	1.037	.293	6.63	37
15	.437	.451	7.07	37
16	.283	.184	7.35	37
17	.230	.204	7.58	37
18	.240	.140	7.82	37

* For $k < 6$ estimates are based on period from October, 1949 to December, 1964. For $k \geqslant 6$, $\hat{\gamma}_k$ is based on March, 1959 to December, 1964.

the sample period.[36] (See Appendix.) Least-squares estimates of $\hat{\gamma}_j$ show considerably more variation for these smaller sample sizes.

36. Although they are not reported in Table 3, Figure 2 also plots, with the dashed curve, $\hat{\beta}_k/\beta_1$ when *all* the least-squares estimates used only the last part of the sample period, from March, 1959, on. The differences are very small.



The pattern in $\hat{\beta}_k/\beta_1$ is at least consistent with capital asset pricing theory. Recall that k is the maturity remaining on a bill at the end of the investor's horizon. Thus, for $k > 0$, the larger k, the less perfectly the security matches the horizon and the greater its risk. Hence, the risk coefficient β_k should increase with k. For $k < 0$, the risk coefficients are negative. This might also be expected because: (a) when $k < 0$ the initially purchased bill matures at a non-stochastic known value[37] *before* the horizon period expires; (b) if asset prices have fallen when the initial purchase matures, the investor can buy a larger quantity of bills for reinvestment, and vice versa; (c) therefore the holding period return on this strategy is negatively correlated with the market return over the total horizon. This negative relation will be less pronounced as k approaches zero, i.e., as the initial purchase approaches the horizon period in maturity. We thus conclude that bills with maturities shorter than the horizon have *negative* risk. They may be combined in a portfolio with common stock or long-term bonds to reduce the expected variation in return of the total portfolio.

However, results for a single horizon period of eight weeks are only given as an example and are not meant to constitute a favorable test of the theory in Sections I-III. In fact, it seems reasonable to suppose that the horizon

37. Assuming no risk of default or inflation.

period differs among investors and is not even known with precision by many individuals. With the present state of theory, it seems impossible to derive a general model that accounts simultaneously for differing investor risk attitudes and horizon periods. We are only entitled to conclude that a general tendency exists for a positive relation between the Sharpe risk coefficients on a Treasury bill and the bill's maturity. If investors are primarily risk averters, a higher expected return must accompany the higher risk inherent in longer maturities and the term structure must be upward-sloping on average.

V. Conclusion

Theories of the term structure of interest rates have traditionally not mentioned portfolio diversification as a possible motive of bond investors.[38] This paper has attempted to show how Sharpe-Lintner capital asset pricing theory [9, 17], which is based entirely on portfolio motives, is relevant for the pricing of fixed-income securities. A static and dynamic theory of term structure behavior developed in [15] was combined with the Sharpe-Lintner equilibrium equation to derive relations between the portfolio risk of bonds and the term structure's "liquidity premiums."

Theoretical and empirical difficulties arise from the horizon period, i.e., the length of time an investor plans to forego consumption and remain invested. Aggregation problems stem from the possibility that individual horizons may vary and a different set of testable equations is necessary for every choice of the market's horizon. Surprisingly, if there exists a unique market horizon that is very short, portfolio behavior implies a movement of forward rates in formal conformation to Meiselman's [13] error-learning model, even though this theory has nothing to do with the forecast revisions Meiselman postulated.

U.S. Treasury bill rates were used to test operational forms of the theory.[39] The results were inconclusive with respect to measurement of a unique horizon period applicable to all investors. However, the data did indicate that portfolio risk components of Treasury bills, as measured by Sharpe-Lintner β coefficients, increase with term-to-maturity. This implies an upward-sloping term structure on average.

APPENDIX

Bid and asked rates on U.S. Treasury bills were keypunched and verified directly from dealer quote sheets provided by Merrill Lynch, Pierce, Fenner and Smith, government securities division (formerly C. J. Devine and Company). The original rates were "banker's discounts" but were converted to continuously compounded internal rates of return per annum. The arithmetic mean of bid and asked rates was used here.

The sample consists of 796 weekly yield curve observations from October, 1949, through December, 1964. Through February, 1959, the Treasury normally issued 91-day bills only. Beginning in March, 1959, 182-day bills were also issued. This is why the tables contain fewer observations for maturities of greater than 13 weeks.

Ordinarily, Tuesday closing quotations were used. When holidays occurred, Monday closing quotations were used. All reported tests were also conducted excluding holiday-

38. An exception is [18].

39. The data sample included over 15 years of weekly observations.

associated observations and no significant differences were obtained. The data were also filtered in various ways to detect errors. Any errors that remain are due to:

(a) printing mistakes on the quote sheets;
(b) quoted rates that were not firm;
(c) averaging bid and asked rates.

Since these errors are very small or infrequent, they are unlikely to have affected the results.

A more complete description of the data is available from the author on request. It includes the method of data filtering, an extensive historical description, and a discussion of the bid-asked spreads.

REFERENCES

1. Eugene Fama. "Efficient Capital Markets: A Review of Theory and Empirical Work," *Journal of Finance* (May, 1970), pp. 383-423.
2. Irving Fisher. *The Nature of Capital and Income.* New York: The Macmillan Company, 1906.
3. ———. *The Rate of Interest.* New York: The Macmillan Company, 1907.
4. J. R. Hicks. *Value and Capital.* Oxford: Oxford University Press, 1946.
5. Sidney Homer. *A History of Interest Rates.* New Brunswick: Rutgers University Press, 1963.
6. Michael C. Jensen. "Risk, the Pricing of Capital Assets, and the Evaluation of Investment Portfolios," *Journal of Business* XLII (April, 1969), pp. 167-247.
7. Reuben Kessel. *The Cyclical Behavior of the Term Structure of Interest Rates.* New York: National Bureau of Economic Research, 1965.
8. William H. Kruskal. "Ordinal Measures of Association," *Journal of the American Statistical Association* LIII (December, 1958), pp. 814-861.
9. John Lintner. "The Valuation of Risk Assets and the Selection of Risky Investments in Stock Portfolios and Capital Budgets," *Review of Economics and Statistics* XLVII (February, 1965), pp. 13-37.
10. Frederick A. Lutz. "The Structure of Interest Rates," *Quarterly Journal of Economics* LV (1940-41), pp. 36-63. Reprinted in A. E. A. *Readings in the Theory of Income Distribution.* Homewood, Ill.: Richard D. Irwin, Inc., 1951, pp. 499-529.
11. Burton G. Malkiel. *The Term Structure of Interest Rates.* Princeton: Princeton University Press, 1966.
12. Frederick R. Macaulay. *The Movements of Interest Rates, Bond Yields, and Stock Prices in the United States Since 1856.* New York: National Bureau of Economic Research, 1938.
13. David Meiselman. *The Term Structure of Interest Rates.* Englewood Cliffs, N.J.: Prentice-Hall, Inc., 1962.
14. Franco Modigliani and Richard M. Sutch. "Innovations in Interest Rate Policy," *American Economic Review Papers and Proceedings* LVI (May, 1966), pp. 178-197.
15. Richard Roll. *The Behavior of Interest Rates: Application of the Efficient Market Model to U. S. Treasury Bill Rates.* New York: Basic Books, Inc., 1970.
16. Paul A. Samuelson. "Proof that Properly Anticipated Prices Fluctuate Randomly," *Industrial Management Review* VI (Spring, 1965), pp. 41-49.
17. William F. Sharpe. "Capital Asset Prices: A Theory of Market Equilibrium Under Conditions of Risk," *Journal of Finance* XIX (September, 1964), pp. 425-442.
18. Joseph Stiglitz "A Consumption-Oriented Theory of the Demand for Financial Assets and the Term Structure of Interest Rates," *Review of Economic Studies* XXXVII (July, 1970), pp. 321-351.
19. John H. Wood. "Expectations and the Demand for Bonds," *American Economic Review* LIX (September, 1969), pp. 522-530.