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Conditional Predictions of Bond Prices and Returns

MICHAEL J. BRENNAN and EDUARDO S. SCHWARTZ*

MODERN THEORIES of the term structure of interest rates posit that the values of default-free discount bonds of all maturities may be expressed as deterministic functions of a small number of state variables, which follow a continuous diffusion process. Arbitrage arguments of the type familiar in the options pricing literature may then be employed to derive a single partial differential equation which must be satisfied by the values of all such bonds. This differential equation typically involves as many unknown parameters as there are stochastic state variables, each of these parameters representing the market price of the risk associated with one of the stochastic state variables.

The power of such arbitrage theories depends upon the number of relevant stochastic state variables being manageably small.[1] Models with a single state variable have been studied by Brennan and Schwartz [1977], Cox, Ingersoll and Ross [1978], Dothan [1978], and Vasicek [1977], all of these authors taking the instantaneously riskless interest rate as the relevant state variable. Two state variable models, in which the second state variable is the exogenously determined stochastic rate of inflation or the price level, have been developed by Cox, Ingersoll and Ross [1978], and Richard [1978]. The Cox, Ingersoll and Ross model of the term structure is distinguished from all of the other work in this area by being derived within a general equilibrium framework.

Cox, Ingersoll and Ross have pointed out that yields on bonds of different maturities are deterministic functions of the underlying state variables, so that if it is possible to invert this system and thereby express the state variables as twice differentiable functions of a vector of interest rates, then the vector of interest rates may be used as instruments for the state variables. Brennan and Schwartz [1979] have taken this approach in expressing the whole term structure of yields as a deterministic function of the instantaneous riskless interest rate, r, and the consol yield, l. While this leaves the issue of the identity of the underlying state variables unresolved, it has the advantage for empirical purposes of eliminating from the partial differential equation one of the utility dependent market price of risk parameters.[2]

In Brennan and Schwartz [1979] preliminary evidence was presented of the ability of such a two-factor model to price a sample of Government of Canada bonds conditional on r and l. In this paper the results of a more detailed empirical analysis of the model are presented: the intertemporal stability and predictive

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[1] This is true also of the Ross [1976] arbitrage model.

[2] See Brennan and Schwartz [1979] Appendix A1.

ability of the assumed stochastic process for interest rates are examined. Maximum likelihood procedures are used to estimate the market price of instantaneous interest rate risk, and confidence intervals are derived for this parameter. By factor analyzing the bond pricing errors it is shown that bond prices may be explained by at most a three-state variable model. Finally, the ability of the model to make predictions of bond prices and rates of return conditional on future values of r and l (and the omitted third state variable) is evaluated. It is hoped that conditional prediction models such as this will play the same role in bond portfolio management as Sharpe's [1963] diagonal model and subsequent elaborations thereof have played in the management of stock portfolios.

The model is described briefly in Section I and in Section II the assumed stochastic process for r and l is estimated using Canadian data. In Section III the market price of short-term interest rate risk is estimated by a non-linear maximum likelihood procedure, using data on Government of Canada bonds. The model errors are factor analyzed in Section IV to indicate the presence of at most one state variable in addition to r and l. The final section of the paper is concerned with the ability of the model to generate conditional predictions of quarterly rates of return on bond portfolios of different maturities.

I. The Model[3]

The two interest rates are assumed to follow a joint diffusion process which is specified as

$$d \ln r = \alpha[\ln l - \ln p - \ln r]\, dt + \sigma_1\, dz_1 \tag{1}$$

$$dl = l[l - r + \sigma_2^2 + \lambda_2\sigma_2]\, dt + l\sigma_2\, dz_2 \tag{2}$$

dz_1 and dz_2 are correlated linear processes, and $dz_1\, dz_2 = \rho\, dt$, where t denotes calendar time. The form of (1) is motivated by the assumptions that the standard deviation of the instantaneous change in r is proportional to its current value, and that r stochastically regresses towards a function of the current consol rate, l, at a rate which is determined by the adjustment coefficient, α; p is an undetermined 'liquidity premium'. The stochastic process for the consol rate, (2), specifies also that the standard deviation of instantaneous changes is proportional to the current value of the rate. It can be shown that λ_2 is the risk premium per unit of risk of the consol bond, or the market price of consol rate risk. Define λ_1, the market price of instantaneous interest rate risk, as the risk premium per unit of risk of a portfolio of bonds whose instantaneous rate of return is perfectly correlated with changes in r.

Then it can be shown by an arbitrage argument that, if λ_1 is at most a function of r and l, then the whole term structure of interest rates is a deterministic function of the current values of r and l, and the value of a unit discount bond may be written as $B(r, l, \tau)$, where τ is time to maturity, and $B(\cdot)$ satisfies the

[3] This is described in more detail in Brennan and Schwartz [1979].

partial differential equation

$$\tfrac{1}{2}B_{11}r^2\sigma_1^2 + B_{12}rl\rho\sigma_1\sigma_2 + \tfrac{1}{2}B_{22}l^2\sigma_2^2 + B_1r[\alpha \ln(l/pr) + \tfrac{1}{2}\sigma_1^2 - \lambda_1(r, l)\sigma_1] + B_2l[\sigma_2^2 + l - r] - B_3 - Br = 0 \quad (3)$$

with the terminal boundary condition

$$B(r, l, 0) = 1 \quad (4)$$

All of the parameters of this differential equation may be estimated from the stochastic process, except for $\lambda_1(\cdot)$ which will be estimated in Section III, on the assumption that it is a constant independent of r and l.

II. Estimation of Stochastic Process

To estimate the non-linear joint stochastic process (1) and (2) it is necessary first to linearize it, and then to evaluate the corresponding exact discrete model, since observations on r and l are available only at discrete intervals. The resulting exact discrete model is of the form

$$\ln r_t = a_{11}(\alpha, p, q)\ln r_{t-1} + a_{12}(\alpha, p, q)\ln l_{t-1} + b_1(\alpha, p, q) + \xi_{1t} \quad (5)$$

$$\ln l_t = a_{21}(\alpha, p, q)\ln r_{t-1} + a_{22}(\alpha, p, q)\ln l_{t-1} + b_2(\alpha, p, q) + \xi_{2t} \quad (6)$$

where the coefficients are known non-linear functions of the parameters α, p, q. ξ_{1t} and ξ_{2t} are homoscedastic, serially independent error terms, and the elements of their covariance matrix, $\sum$, are approximately equal to σ_1^2, σ_2^2, $\rho\sigma_1\sigma_2$. q is a parameter arising form the linearizing and does not enter the differential equation.

The estimation was carried out using a non-linear procedure described by Malinvaud [1966]. The data for the instantaneous rate of interest were the yields on 30-day Canadian Bankers' Acceptances converted to an equivalent continuously compounded annual rate of interest, while the consol rate was approximated by the continuously compounded equivalent of the yield to maturity on the Government of Canada bond with a maturity in excess of 25 years which was selling closest to par.[4] Both interest rate series are the mid-market closing rates on the last Wednesday of each month from January 1964 to April 1979.

The parameter estimates of the stochastic process for different subperiods are presented in Table 1.[5] The estimates of α and q are reasonably stable and are significantly different from zero on the usual criteria: on the other hand, the estimate of $\ln p$ is quite unstable and nowhere significant. The estimates of σ_1 and σ_2 are quite similar for the two semi-periods, although the shorter period estimates are more variable: roughly speaking, the estimates suggest that there is three times as much uncertainty about the proportionate change in the short rate as there is about the consol rate proxy. The unanticipated changes in the two rates

[4] This is a different rate from that used in the Brennan and Schwartz [1979].

[5] In comparing these estimates with those in Brennan and Schwartz [1979], it should be recalled that the data are somewhat different: furthermore, due to a programming error, the estimated standard errors were understated in the previous study.

Table 1

Estimates of the Stochastic Process (Standard Error in Parentheses)

Period	α	lnp	q	σ_1	σ_2	ρ	$\rho(\xi_{1t}\xi_{1t-1})$	$\rho(\xi_{2t}\xi_{2t-1})$	$D - W_1$	$D - W_2$	γ_1	γ_2
Jan 64–Apr 79	.0722	.0495	.0066	.0696	.0238	.29	.19	.14	1.62	1.72	−.65	1.43
	(.0286)	(.0806)	(.0018)								(.48)	(.92)
Jan 64–Aug 71	.0705	.1128	.0063	.0722	.0223	.20	.13	.10	1.73	1.75	−.08	3.78
	(.0432)	(.1124)	(.0024)								(.79)	(1.74)
Sep 71–Apr 79	.0586	−.0251	.0073	.0660	.0243	.42	.27	.13	1.46	1.72	−1.66	1.89
	(.0364)	(.1518)	(.0026)								(.87)	(1.96)
Jan 64–Oct 67	.1089	.1030	.0076	.0605	.0163	.42	−.18	.13	2.34	1.69	−2.37	13.20
	(.0664)	(.0995)	(.0024)								(2.44)	(5.67)
Nov 67–Aug 71	.0402	.2653	.0047	.0813	.0284	.10	.30	.09	1.42	1.76	−1.33	−3.12
	(.0588)	(.3915)	(.0042)								(1.59)	(3.35)
Sep 71–Jun 75	.0687	.0486	.0105	.0847	.0277	.46	.29	.19	1.43	1.60	−1.07	6.47
	(.0513)	(.2052)	(.0041)								(1.09)	(2.12)
Jul 75–Apr 79	.0587	−.0575	.0030	.0393	.0191	.34	.18	−.08	1.57	2.11	−2.45	5.31
	(.0583)	(.1765)	(.0029)								(3.26)	(8.39)

are positively correlated, although the estimated degree of correlation is different for the different subperiods.

As stated above, if the stochastic process (1), (2) is the correct one, then the error terms from (5), (6) should be serially independent. The Durbin-Watson statistics from the two equations suggest that this assumption is violated: moreover, the estimated serial correlation of .19 and .14 for the instantaneous and consol rate equations respectively is likely to be an underestimate of the true serial correlation because of the lagged dependent variables in the equations. Serial correlation in the errors from the stochastic process implies either that the functional form of the stochastic process is mis-specified, or that the current values of r and l are not sufficient statistics for the joint distribution of future values of r and l, and that therefore if the true stochastic process is to be represented in Markov form as is necessary for the derivation of the partial differential equation, at least one other state variable must be introduced.[6] The practical importance for bond pricing of omitting these state variables (or of mis-specifying the stochastic process) is an empirical issue which will be taken up below. For the moment it will be assumed that the two state variable representation is adequate.

To test the assumption of homoscedastic errors, the logarithm of the squared error was regressed on the logarithm of the two interest rates, following a procedure suggested by Park [1966]:

$$\ln \xi_{1t}^2 = \delta_1 + \gamma_1 \ln r_{t-1} \tag{7}$$

$$\ln \xi_{2t}^2 = \delta_2 + \gamma_2 \ln l_{t-1} \tag{8}$$

The estimated values of γ_1 and γ_2 are reported in Table 1. Under the null hypothesis of homoscedastic errors the estimates of γ_1 and γ_2 should be insignificantly different from zero, as they are for the whole period and most of the subperiods. The fact that $\hat{\gamma}_1$, although not statistically significant, is negative for each subperiod provides weak evidence that the true stochastic process for r has a diffusion variance elasticity[7] of less than 2, which is the value assumed in our process. Similarly, it appears that the corresponding elasticity for l may be greater than 2. However, given the statistical insignificance of $\hat{\gamma}_1$ and $\hat{\gamma}_2$ for the whole period, the assumption of an elasticity of 2 is maintained, and the parameter values used for the partial differential equation (3) are those estimated from the whole sample period: the remaining unknown parameter in the equation is λ_1, the market price of instantaneous interest rate risk: estimation of this is taken up in the following section.

To gain some insight into the power of the stochastic process to predict future values of r and l, forecasts derived from the process were compared with those derived from a naive no-change or Martingale model.[8] Writing the equation

[6] Strictly speaking, it would still be possible to derive the differential equation even if $\lambda_2(\cdot)$ depended on other state variables.

[7] See Cox [1975].

[8] Pesando [1978] has suggested that long term bond yields should follow an approximate Martingale; Ayres and Barry [1979] incorrectly assert that they must follow an exact Martingale.

system (5), (6) as

$$X_t = AX_{t-1} + B + \xi_t \tag{9}$$

$$E[X_{t+n} | X_t] = A^n X_t + (I + A + \cdots + A^{n-1})B \tag{10}$$

and the variance-covariance matrix of X_{t+n} is

$$\Omega_{t+n} = \Sigma + A\,\Sigma\,A' + \cdots A^{n-1}\,\Sigma\,A^{n-1'} \tag{11}$$

The vector of forecast interest rates is then lognormally distributed with mean and covariance matrix given by (10) and (11) respectively. The n-period forecasts are then calculated using the formula for the mean of a lognormal variable. The root mean square error of the forecasts generated from the estimated stochastic process and from the Martingale model are compared in Table 2. The stochastic process outperforms the Martingale forecasts by only a small margin, although this is an increasing function of the forecast interval. The improvement of the stochastic process forecast over the Martingale is substantially better for r than for l, and the generally small improvement achieved by the stochastic process is indicative that the stochastic element is large relative to the trend of the stochastic process.

III. Estimation of λ_1

Having estimated the parameters of the stochastic process, the valuation equation (3) may be solved numerically for a given value of λ_1 (henceforth the subscript will be omitted). The resulting values of $B(r, l, \tau; \lambda)$ are the present values of \$1 receivable in τ periods when the two interest rates take on the values of r and l. For each of several values of λ, a sample of 126 Government of Canada bonds was valued on the last Wednesday of each quarter from January 1964 to April 1979 by applying the present value factor appropriate to the prevailing instantaneous and consol rates to the promised stream of coupon and principal payments. The resulting predicted bond values are written $B_{it}(\lambda)(i = 1, \cdots, 126; t = 1, \cdots, 62)$, and the actual bond value, y_{it}, may be expressed as

$$y_{it} = B_{it}(\lambda) + u_{it} \tag{12}$$

where u_{it} is the valuation error. Consider an estimator of λ, $\lambda(S)$, which minimizes

$$L(\lambda, S) = \textstyle\sum_t (y_t - B_t(\lambda))'S(y_t - B_t)\lambda)) \tag{13}$$

Table 2

Root Mean Square Error (% per year) of Model and Martingale Forecasts of Interest Rates

Forecast Interval (months)	1		3		6		12	
	r	l	r	l	r	l	r	l
Stochastic Process	.45	.19	.87	.37	1.30	.56	1.72	0.71
Martingale	.45	.19	.90	.38	1.41	.57	2.03	0.75

where S is a positive definite matrix and y_t and $B_t(\lambda)$ are vectors. Then $\lambda(S)$ satisfies the necessary conditions for a minimum:

$$H(S, \lambda) = \sum_t Q_t(\lambda)'S(y_t - B_t(\lambda)) = 0 \tag{14}$$

where $Q_t(\lambda) = \left[\frac{\partial B_{it}(\lambda)}{\partial \lambda}\right]$ is a vector. Starting from a particular value of λ, λ^0, $H(S, \lambda)$ can be approximated by

$$H(S, \lambda) = H(S, \lambda^0) + \frac{\partial H(S, \lambda^0)}{\partial \lambda}(\lambda - \lambda^0) \tag{15}$$

and setting (15) equal to zero yields $\lambda(S)$. In the present context no analytic expressions are available for $Q_t(\lambda)$ *or* $\partial H/\partial \lambda$, and therefore $H(S, \lambda^0)$ and its partial derivative were calculated using values of $B_{it}(\lambda)$ obtained from adjacent values of λ.

The estimator $\lambda(S)$ will be a maximum likelihood estimator if the errors in (12) are $N(0, \Omega)$, and $S = \Omega^{-1}$. Since Ω is unknown, an asymptotically MLE is obtained by the following iterative procedure described by Malinvaud [1966]:

(i) Calculate $\lambda^* = \lambda(I)$ by the above procedure: this is the (non-linear) OLS estimator:

(ii) Calculate the residuals $\epsilon_{it} = y_{it} - B_{it}(\lambda^*)$, and their covariance matrix, Ω^*.

(iii) Calculate the estimator $\lambda^{**} = \lambda(\Omega^{*-1})$. This estimator is asymptotically normal with variance

$$\left[\sum_t \left[\frac{\partial B_{it}(\lambda^{**})}{\partial \lambda}\right]' \Omega^{*-1} \left[\frac{\partial B_{it}(\lambda^{**})}{\partial \lambda}\right]\right]^{-1} \tag{16}$$

Estimation of the covariance matrix Ω^* requires prior restrictions on the covariance structure of the valuation errors u_{it}. It was assumed that the covariance of the valuation errors of any two bonds depends only on their maturities. Therefore, each quarter the outstanding bonds were assigned to one of ten equally weighted portfolios ($j = 1, \cdots, 10$) depending on their maturities: the first portfolio consisting of all bonds with maturities of less than one year, the second of all bonds with maturities between one and two years etc. The valuation error for portfolio j in period t is assumed to be given by

$$u_{jt} = \rho_j u_{jt-1} + v_{jt} \tag{17}$$

where $E[v_{jt}] = E[v_{jt}v_{jt'}] = 0$; $E[v_{jt}v_{kt}] = w_{jk}$, an element of the (10×10) matrix Ω.

The OLS estimator, λ^*, was computed using the portfolio data described above. Not all bond maturities were represented in each quarter of the sample period: the potential number of observations for each portfolio is 62, and the actual numbers are given in the first row of Table 3. To take account of the unequal numbers of observations by period, matrix S of equations (13)–(14) was indexed: $S_t = I_{n_t}$, where n_t is the number of observations in period t.

The serial correlation of the errors of each of the 10 portfolios was estimated from the residuals of this regression according to equation (17) and the resulting

Table 3

Numbers of Observations and Estimated Serial Correlation of Errors ($\hat{\rho}_j$) by Portfolio Maturity

Portfolio Maturity (Year)	1	2	3	4	5	6	7	8	9	10
NOBS	60	62	62	62	62	60	53	51	48	42
$\hat{\rho}_j$	.65	.66	.66	.68	.74	.65	.67	.79	.87	.78

Table 4

Ordinary Least Squares Estimator (λ^*) and Maximum Likelihood Estimator (λ^{**}) of the Market Price of Instantaneous Interest Rate Risk (λ_1)

Period	λ^*	λ^{**}	SE(λ^{**})	N
Jan 64–Apr 79	.0050	.0169	.0163	538
Jan 64–Aug 71	−0.135	−.0192	.0248	266
Sep 71–Apr 79	.1278	.0682	.0220	262
Jan 64–Oct 67	−.0177	−.0023	.0391	127
Nov 67–Aug 71	−.0107	−.0189	.0336	129
Sep 71–Jun 75	.1248	.0309	.0345	126
Jul 75–Apr 79	.1418	.1216	.0298	128

estimates are given in the second row of Table 3. To take account of both serial and contemporaneous correlation of the errors, an asymptotically efficient estimator proposed by Parks [1967] was developed: the actual and predicted portfolio values were transformed according to

$$\begin{aligned} \tilde{y}_{jt} &= y_{jt} - \hat{\rho}_j y_{jt-1} \\ \tilde{B}_{jt}(\lambda) &= B_{jt}(\lambda) - \hat{\rho}_j B_{jt-1}(\lambda) \end{aligned} \tag{18}$$

Then equation (12) can be written as

$$\tilde{y}_{jt} = \tilde{B}_{jt}(\lambda^*) + \hat{v}_{jt} \tag{20}$$

The errors $\hat{v}_{jt}$, now serially uncorrelated, were used to estimate Ω^*, and the MLE, λ^{**} was computed by minimizing

$$L(\lambda, \hat{\Omega}^{*-1}) = \sum_t (\tilde{y}_{jt} - \tilde{B}_{jt}(\lambda))' \hat{\Omega}_t^{*-1} (\tilde{y}_{jt} - \tilde{B}_{jt}(\lambda)) \tag{21}$$

where $\hat{\Omega}_t^*$ is the relevant submatrix of $\hat{\Omega}^*$, taking account of missing observations.

Table 4 reports both the OLS estimator, λ^*, and the MLE, λ^{**}, together with the asymptotic standard error of λ^{**} computed from expression (16). λ^{**} is unstable, being positive in some periods and negative in others; while the estimate is several times its standard error for the final quarter, the evidence for the total period is consistent with the hypothesis that $\lambda_1 = 0$.[9] It is also possible that, contrary to what has been assumed in the estimation, λ_1 is a function of r and l and perhaps of other state variables also. To put the range of estimates of λ_1 in perspective, it may be noted that a change in λ_1 from 0.0 to 0.10 changes the

[9] The full period estimate is within one standard error of the value reported by Brennan and Schwartz [1979].

predicted value of the average bond in the sample by about 1.2% $\left(\frac{\partial B_{it}(\lambda)}{\partial\lambda} > 0 \text{ for all bonds}\right)$.

IV. Factor Analysis

The contemporaneous correlations between valuation errors for the different portfolios were computed from the transformed residuals, $\tilde{y}_{jt} - \tilde{B}_{jt}(\lambda^{**})$, and are reported in Table 5. If the assumptions of the pricing model discussed in this paper were correct, then the errors would be independent. Thus the strong contemporaneous and serial correlations reported in Tables 5 and 3 offer strong support for the existence of relevant state variables in addition to those represented by the r and l: the possible existence of such state variables has already been suggested by the serial correlation found in the errors from the stochastic process for r and l. Assuming that such state variables affect bond prices in a linear fashion, they may be extracted by factor analysis of the untransformed error terms, $\hat{u}_{jt} = y_{jt} - B_{jt}(\lambda^{**})$.

The factor analysis model assumes that these errors may be expressed as

$$\hat{u}_{jt} = \sum_k \lambda_{jk} f_{kt} + e_{jt} \tag{22}$$

where e_{jt} is a residual error which is assumed to be serially and cross-sectionally independent; f_{kt} is the value of factor k in period t and λ_{jk} is the loading of the valuation error for portfolio j on factor k. By factor analyzing the valuation errors it is possible to estimate the number of common factors affecting the errors, and therefore, the number of relevant state variables which have been omitted from the model.

Factor analysis revealed the existence of only one important factor, which accounted for 83.5% of the total variance: the next most important factor contributed only a further 2.4%. This constitutes strong evidence for the existence of at

Table 5

Contemporaneous Correlation of Portfolio Valuation Errors

Portfolio Maturity (years)	1	2	3	4	5	6	7	8	9	10
1	1.00									
2	.82	1.00								
3	.76	.95	1.00							
4	.67	.90	.95	1.00						
5	.60	.83	.88	.92	1.00					
6	.43	.70	.77	.87	.89	1.00				
7	.43	.60	.71	.74	.76	.80	1.00			
8	.16	.40	.49	.55	.62	.64	.57	1.00		
9	.54	.70	.76	.79	.85	.80	.79	.61	1.00	
10	.53	.77	.79	.79	.78	.72	.59	.48	.79	1.00

most one relevant state variable in addition to r and l. Factor loadings for the 10 portfolios are given in Table 6. The factor score in each period was estimated by regressing the errors for that period in the factor loadings: the factor scores were found to have a serial correlation of 0.82.

If the assumption that λ_1 and λ_2 are constant is correct, then the dependence of the valuation errors can arise only from the mis-specification of the stochastic process for r and l. If the problem is not due simply to selecting an incorrect functional form for this process, but is due to omitting a state variable from the joint Markov process of relevant state variables, then the state variable identified by factor analysis of the valuation errors should enter the stochastic process for r and l. To test this hypothesis, the errors obtained from the nonlinear regression (5) and (6) were regressed on the lagged values of the factor scores for the months immediately following the quarterly calculations of the factor scores. In both cases the lagged factor score explained about 8% of the variance of the errors: this is significant at the 5% level and is evidence that the true stochastic process for r and l contains an omitted state variable. The question of the relative importance of this state variable for bond pricing is taken up in the next section.

V. Conditional Predictions of Returns

To evaluate the ability of the model described in this paper to make predictions of bond returns conditional on forecasts of the exogenous state variables, r and l, rates of return were calculated for each quarter from 1964:I to 1979:I for each of ten equally weighted portfolios of bonds of different maturities: each portfolio, n, ($n = 1, \cdots, 10$) consisted of all outstanding bonds with a time to maturity of between n and $(n - 1)$ years at both the beginning and end of the quarter under consideration.

The predicted rate of return of portfolio n in quarter t, $\hat{R}_{nt}$, is given by

$$\hat{R}_{nt} = (\hat{y}_{nt} + c_{nt} - y_{nt-1})/y_{nt-1}$$

where $\hat{y}_{nt}$ is the predicted value of the portfolio at the end of the quarter, c_{nt} is the coupon during the quarter, and y_{nt-1} is the value at the beginning of the quarter.

Four different rules were used to obtain $\hat{y}_{nt}$. It should be noted that none of the prediction rules are true ex-ante forecasts since they rely on estimates of the parameters of the stochastic process and λ_1 which were derived from the data available for the whole period: nevertheless, these predictions should provide an indication of the ability of the model to make such true ex-ante forecasts.

Define $B_{nt}(r, l; \lambda^{**})$ as the model value (based on $\lambda_1 = \lambda^{**}$) of portfolio n at the end of quarter t, when the interest rates are r and l. Then,

Table 6

Factor Loadings of Portfolio Errors

Portfolio	1	2	3	4	5	6	7	8	9	10
Loading	0.15	0.55	1.00	1.35	1.77	1.96	1.73	1.86	2.40	2.39

Table 7

Actual and Predicted Returns for Portfolios of Different Maturities

Prediction Rule		(1) Unadjusted Model Value				(2) Model Value With Factor Score Serial Correlation Correction				(3) Model Value With Individual Serial Correlation Correction				(4) Model Value Adjusted For Actual Factor Score			
Maturity	Standard Deviation of Returns	RMSE	α	β	R^2	RMSE	α	β	R^2	RMSE	α	β	R^2	RMSE	α	β	R^2
1	.47%	.32%	.0010	1.095	.75	.30%	.0027	.935	.74	.21%	.0020	.863	.81	.26%	.0029	.906	.82
			(.0012)	(.086)			(.0011)	(.076)			(.0009)	(.058)			(.0008)	(.059)	
2	.81	.89	.0086	.753	.42	.72	.0066	.806	.53	.52	.0032	.885	.64	.56	.0060	.823	.79
			(.0013)	(.116)			(.0013)	(.100)			(.0014)	(.086)			(.0008)	(.055)	
3	1.42	1.51	.0013	.785	.40	1.13	.0060	1.100	.61	.90	.0010	1.166	.65	.79	.0062	.967	.86
			(.0016)	(.125)			(.0015)	(.116)			(.0018)	(.111)			(.0008)	(.050)	
4	1.88	1.85	.0123	.777	.39	1.27	.0036	1.331	.72	1.15	.0003	1.326	.72	.84	.0057	.988	.88
			(.0021)	(.128)			(.0017)	(.111)			(.0019)	(.110)			(.0010)	(.048)	
5	2.22	2.10	.0110	.646	.37	1.30	.0003	1.237	.70	1.20	−.0028	1.358	.76	.66	.0021	.983	.92
			(.0025)	(.114)			(.0021)	(.110)			(.0019)	(.102)			(.0010)	(.040)	
6	3.25	2.72	.0125	.771	.45	1.88	.0012	1.359	.74	1.90	−.0002	1.442	.75	.98	.0027	1.040	.92
			(.0038)	(.134)			(.0030)	(.127)			(.0030)	(.132)			(.0016)	(.048)	
7	2.87	2.23	.0068	.696	.50	1.54	.0006	.982	.70	1.47	(.0009)	1.139	.75	.82	.0002	.901	.93
			(.0035)	(.118)			(.0029)	(.108)			(.0027)	(.113)			(.0014)	(.042)	
8	2.50	2.30	.0085	.556	.39	1.14	−.0008	.986	.79	1.01	−.0012	1.162	.85	1.06	−.0002	.838	.87
			(.0036)	(.115)			(.0024)	(.085)			(.0020)	(.081)			(.0018)	(.055)	
9	2.70	2.78	.0105	.520	.41	1.56	−.0034	1.057	.67	1.37	−.0028	1.306	.78	.92	−.0003	.830	.93
			(.0037)	(.114)			(.0033)	(.134)			(.0026)	(.125)			(.0014)	(.041)	
10	4.51	3.34	.0198	.852	.65	1.86	−.0012	1.163	.84	2.04	.0029	1.224	.84	1.08	−.0026	.922	.96
			(.0048)	(.115)			(.0037)	(.092)			(.0036)	(.099)			(.0019)	(.035)	

Rule 1: $\hat{y}_{nt} = B_{nt}(r_t, l_t; \lambda^{**})$

The predicted value is simply the model value at the end of quarter using the contemporaneous values of r and l. This rule takes no account of the known serial correlation in the valuation errors.

Rule 2: $\hat{y}_{nt} = B_{nt}(r_t, l_t; \lambda^{**}) + \lambda_n \rho_f f_{t-1}$

Where ρ_f is the serial correlation in the factor score, λ_n is the factor loading and f_{t-1} is the factor score at the beginning of the quarter. This rule takes account of the serial correlation in the errors arising from the influence of the common factor.

Rule 3: $\hat{y}_{nt} = B_{nt}(r_t, l_t; \lambda^{**}) + \hat{\rho}_n u_{nt-1}$

This rule is similar to Rule 2 but adjusts for serial correlation on an individual portfolio basis.

Rule 4: $\hat{y}_{nt} = B_{nt}(r_t, l_t; \lambda^{**}) + \lambda_n f_t$

Under this rule the model value is adjusted by the factor loading times the actual factor score.

For each of the four rules, the Root Mean Square Error of the quarterly rate of return prediction was calculated, and the actual rate of return was regressed on the predicated rate of return:

$$R_{nt} = \alpha + \beta \hat{R}_{nt} + \epsilon_{nt}$$

The results are reported in Table 7 along with the standard deviations of the quarterly rates of return. Rule 1 performs poorly, which is not surprising in view of the substantial serial correlation observed in the errors. Rules 2 and 3 which take account of the serial correlation do substantially better, the RMSE of the forecast being of the order of 1½% of the value of the portfolios. When the actual factor score is employed (Model 4) the RMSE drops further to about 1%.

It seems clear from the above that, while the posited two factor model is only moderately successful in predicting future bond values, a three factor model is quite adequate. This suggests that the expanded three factor model could provide the basis for a usable bond portfolio management model. First, the variance-covariance matrix of bond returns may be inferred from the covariance structure of r, l and the third factor using the conditional valuation predictions of the model. Secondly, given predictions of r, l and the third factor, fairly accurate assessments may be made of the expected returns on the bonds: while it is unrealistic to expect portfolio managers to predict the unknown third factor directly, this may be done indirectly by predicting the value of one particular maturity bond: the value of the third factor may then be inferred from this and the predicted values of r and l.

Finally, there remains the intriguing question of whether the valuation errors obtained from Model 4 using the actual factor score represent transitory errors which may be attributed to market inefficiency, or whether they are due to further model limitations.

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DISCUSSION:

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Progress on the problem of the term structure since the 1940's has been meagre. In contrast, the work of the present authors, and the related work referenced in their paper, represents a highly promising line of thought. In these models, as the authors explain, the value of a bond of any maturity is expressed as a deterministic function of a number of state variables. In the present paper, the state variables are taken to be two interest rates—a short rate and a long rate. The authors suggest—and I fully agree with this—that models of this type may be of practical use in bond portfolio management, perhaps the most obvious example being performance measurement.

Before passing on to the details of the paper, we might consider for a moment the question of whether models of this kind are properly viewed as a "theory" of the term structure. After all, to put it simply, the theory of the term structure concerns the economics of the relationship between the long rate and the short rate. However, the Brennan and Schwartz model determines interest rates on intermediate maturity instruments *given* the long and short rates. Thus the present paper might be viewed as presenting a theory which reduces the dimensionality of the term structure problem, rather than providing a theory of the

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